CHAPTER 5

FACTORIZATION OF ALGEBRAIC EXPRESSIONS

REVISION OF KEY CONCEPTS AND FORMULAE

We have learnt the use of the following identities for the factorization of algebraic expressions :

(i) $(a+b)^2 = a^2 + b^2 + 2ab$

(ii) $(a-b)^2 = a^2 + b^2 - 2ab$

(iii) $(a^2 - b^2) = (a+b)(a-b)$

(iv) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

(v) $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

(vi) $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

(vii) $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$

(viii) $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$

(ix) $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$

(x) If $a+b+c=0$, then $a^3 + b^3 + c^3 = 3\,abc$

SOLVED EXAMPLES

MULTIPLE CHOICE

EXAMPLE 1 *$(a-b)^3 + (b-c)^3 + (c-a)^3$ is equal to*

(a) $2a^3 + 2b^3 + 2c^3$ (b) $(a-b)(b-c)(c-a)$ (c) 0 (d) $3(a-b)(b-c)(c-a)$

Ans. (d)

SOLUTION We observe that $(a-b)+(b-c)+(c-a)=0$.

$\therefore$ $$(a-b)^3 + (b-c)^3 + (c-a)^3 = 3(a-b)(b-c)(c-a).$$

EXAMPLE 2 *If $x+y=12$ and $xy=27$, then $x^3+y^3=$*

(a) 765 (b) 756 (c) 657 (d) 675

Ans. (b)

SOLUTION We know that

$\therefore$ $$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

$\Rightarrow$ $$x^3 + y^3 = (x+y)\left\{\left((x+y)^2 - 3xy\right)\right\} = 12(12^2 - 3\times 27) = 12(144-81) = 756$$

EXAMPLE 3 *If* $x + y = -4$, *then* $x^3 + y^3 - 12xy + 64 =$

(a) -64 (b) 128 (c) 0 (d) *none of these*

Ans. (c)

SOLUTION We have, $x + y = -4$ or, $x + y + 4 = 0$.

$\therefore \quad x^3 + y^3 + 4^3 = 3xy \times 4$ or, $x^3 + y^3 + 64 = 12xy$ or, $x^3 + y^3 - 12xy + 64 = 0$

EXAMPLE 4 *If* $x = 2y + 6$, *then* $x^3 - 8y^3 - 36xy =$

(a) 216 (b) -216 (c) 36 (d) -36

Ans. (a)

SOLUTION We have, $x = 2y + 6$ or, $x - 2y - 6 = 0$.

$\therefore \quad x^3 + (-2y)^3 + (-6)^3 = 3x(-2y)(-6)$

$\Rightarrow \quad x^3 - 8y^3 - 216 = 36xy \Rightarrow x^3 - 8y^3 - 36xy = 216$

EXAMPLE 5 $(a + b + c)\left\{(c - b)^2 + (b - c)^2 + (c - a)^2)\right\} =$

(a) $a^3 + b^3 + c^3 - 3abc$ (b) $a^3 + b^3 + c^3$ (c) $2(a^3 + b^3 + c^3 - 3abc)$ (d) $3abc$

Ans. (c)

SOLUTION $(a + b + c)\left\{(a - b)^2 + (b - c)^2 + (c - a)^2\right\}$

$= 2(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 2(a^3 + b^3 + c^3 - 3abc)$

EXAMPLE 6 *If* $a^3 + b^3 = 5$ *and* $a + b = 1$, *then* $ab =$

(a) $-\frac{4}{3}$ (b) $\frac{4}{3}$ (c) $\frac{-3}{4}$ (d) $\frac{3}{4}$

Ans. (a)

SOLUTION $(a^3 + b^3) = (a + b)(a^2 - ab + b^2)$

$\Rightarrow \quad a^3 + b^3 = (a + b)\left\{(a + b)^2 - 3ab\right\} \Rightarrow 5 = (1 - 3ab) \Rightarrow ab = -\frac{4}{3}$

EXAMPLE 7 *If* $a^3 + (b - a)^3 - b^3 = k(a - b)$, *then* $k =$

(a) ab (b) $3ab$ (c) $-3ab$ (d) 3

Ans. (b)

SOLUTION We observe that $a + (b - a) + (-b) = 0$.

$\therefore \quad a^3 + (b - a)^3 + (-b)^3 = 3a(b - a)(-b)$

$\Rightarrow \quad a^3 + (b - a)^3 - b^3 = 3ab(a - b) \Rightarrow k(a - b) = 3ab(a - b) \Rightarrow k = 3ab$

EXAMPLE 8 *If* $a + b + c = 0$, *then* $\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} =$

(a) 1 (b) 0 (c) -1 (d) 3

Ans. (d)

SOLUTION We have, $a + b + c = 0$

$\Rightarrow \quad a^3 + b^3 + c^3 = 3abc \Rightarrow \frac{a^3 + b^3 + c^3}{abc} = 3 \Rightarrow \frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} = 3$

ASSERTION-REASON

Each of the following examples contains STATEMENT-1 (Assertion) and STATEMENT-2 (Reason) and has following four choices (a), (b), (c) and (d), only one of which is the correct answer. Mark the correct choice.

(a) Statement-1 and Statement-2 are True; Statement-2 is a correct explanation for Statement-1.
(b) Statement-1 and Statement-2 are True; Statement-2 is not a correct explanation for Statement-1.
(c) Statement-1 is True, Statement-2 is False.
(d) Statement-1 is False, Statement-2 is True.

EXAMPLE 9 *Statement-1 (Assertion):* *The value of* $1000^3 - 900^3 - 100^3$ *is* 270,000,000

Statement-2 (Reason): *If* $a + b + c = 0$, *then* $a^3 + b^3 + c^3 = 3abc$

Ans. (a)

SOLUTION We know that

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

If $a + b + c = 0$, then

$$a^3 + b^3 + c^3 - 3\,abc = 0 \times (a^2 + b^2 + c^2 - ab - bc - ca)$$

$$\Rightarrow \quad a^3 + b^3 + c^3 - 3\,abc = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$$

So, statement-2 is true. We find that: $1000 + (-900) + (-100) = 0$

Using statement-2, we obtain

$$1000^3 + (-900)^3 + (-100)^3 = 3 \times 1000 \times (-900) \times (-100)$$

$$\Rightarrow \quad 1000^3 - 900^3 - 100^3 = 270{,}000{,}000$$

So, statement-1 is also true.

We find that statement-2 is a correct explanation for statement-1. Hence, option (a) is correct.

EXAMPLE 10 *Statement-1 (Assertion):* *The value of* $\dfrac{(0.093)^3 + (0.007)^3}{(0.093)^2 - (0.093)(0.007) + (0.007)^2}$ *is* 0.1.

Statement-2 (Reason): $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

Ans. (a)

SOLUTION We observe that statement-2 is true.

Now, $a^3 + b^3 = (a + b)(a^2 - ab + b^2) \Rightarrow \dfrac{a^3 + b^3}{a^2 - ab + b^2} = a + b$

Replacing a by 0.093 and b by 0.007, we obtain

$$\frac{(0.093)^3 + (0.07)^3}{(0.093)^2 - (0.093)(0.07) + (0.07)^2} = 0.093 + 0.07 = 0.1$$

So, statement-2 is also true and statement-2 is a correct explanation for statement-1. Hence, option (a) is correct.

EXAMPLE 11 *Statement-1 (Assertion):* $a^3(b - c)^3 + b^3(c - a)^3 + c^3(a - b)^3 = 3(a - b)(b - c)(c - a)$

Statement-2 (Reason): *If* $a + b + c = 0$, *then* $a^3 + b^3 + c^3 = 3abc$

Ans. (d)

SOLUTION Statement-2 is true.

We observe that $a(b - c) + b(c - a) + c(a - b) = 0$

$\therefore \quad a^3(b-c)^3 + b^3(c-a)^3 + c^3(a-b)^3 = 3abc(a-b)(b-c)(c-a)$

So, statement-1 is not true. Hence, option (d) is correct.

EXAMPLE 12 *Statement-1 (Assertion):* $(a+b+c)\left\{(a-b)^2+(b-c)^2+(c-a)^2\right\} = 2(a^3+b^3+c^3-3abc)$

Statement-2 (Reason): *If* $a+b+c=0$ *then* $(a+b)^3+(b+c)^3+(c+a)^3 = -3abc$

Ans. (b)

SOLUTION $(a+b+c)\left\{(a-b)^2+(b-c)^2+(c-a)^2\right\}$

$$= 2(a+b+c)(a^2+b^2+c^2-ab-bc-ca) = 2(a^3+b^3+c^3-3abc)$$

So, statement-1 is true.

If $a+b+c=0$, then.

$$2(a+b+c) = 0$$

$$\Rightarrow \quad (a+b)+(b+c)+(c+a) = 0$$

$$\Rightarrow \quad (a+b)^3+(b+c)^3+(c+a)^3 = 3(a+b)(b+c)(c+a)$$

$$\Rightarrow \quad (a+b)^3+(b+c)^3+(c+a)^3 = 3(-c)(-a)(-b) = -3abc$$

So, statement-2 is also true. Hence, option (b) is true.

EXAMPLE 13 *Statement-1 (Assertion): The product of* $(x^2+4y^2+z^2+2xy+xz-2yz)$ *and* $(-z+x-2y)$ *is* $x^3-8y^3-z^3-6xyz$

Statement-2 (Reason): $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

Ans. (a)

SOLUTION Statement-2 is true, being a standard formula.

Now,

$$(-z+x-2y)(x^2+4y^2+z^2+2xy+xz-2yz)$$

$$= \left\{x+(-2y)+(-z)\right\}\left\{x^2+(-2y)^2+(-z)^2-x(-2y)-(-2y)(-z)-x(-z)\right\}$$

$$= x^3+(-2y)^3+(-z)^3-3x(-2y)(-z) \qquad \text{[Using statement-2]}$$

$$= x^3-8y^3-z^3-6xyz$$

So, statement-1 is true. Also, statement-2 is a correct explanation for statement-1.
Hence, option (a) is correct.

EXAMPLE 14 *Statement-1 (Assertion):* $a^2+b^2+c^2-ab-bc-ca=0$ *if and only if* $a=b=c$.

Statement-2 (Reason): $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

Ans. (b)

SOLUTION Statement-2 is true.

Now, $\quad a^2+b^2+c^2-ab-bc-ca = 0$

$$\Leftrightarrow \quad 2(a^2+b^2+c^2-ab-bc-ca) = 0$$

$$\Leftrightarrow \quad (a-b)^2+(b-c)^2+(c-a)^2 = 0 \Leftrightarrow a-b=0 \text{ and } b-c=0 \text{ and } c-a=0 \Leftrightarrow a=b=c$$

So, statement 1 is also true. Hence, options (b) is correct.

PRACTICE EXERCISES

MULTIPLE CHOICE

Mark the correct alternative in each of the following:

1. The factors of $x^3 - x^2y - xy^2 + y^3$, are
   (a) $(x+y)(x^2-xy+y^2)$
   (b) $(x+y)(x^2+xy+y^2)$
   (c) $(x+y)^2(x-y)$
   (d) $(x-y)^2(x+y)$

2. The factors of $x^3 - 1 + y^3 + 3xy$, are
   (a) $(x-1+y)(x^2+1+y^2+x+y-xy)$
   (b) $(x+y+1)(x^2+y^2+1-xy-x-y)$
   (c) $(x-1+y)(x^2-1-y^2+x+y+xy)$
   (d) $3(x+y-1)(x^2+y^2-1)$

3. The factors of $8a^3 + b^3 - 6ab + 1$, are
   (a) $(2a+b-1)(4a^2+b^2+1-3ab-2a)$
   (b) $(2a-b+1)(4a^2+b^2-4ab+1-2a+b)$
   (c) $(2a+b+1)(4a^2+b^2+1-2ab-b-2a)$
   (d) $(2a-1+b)(4a^2+1-4a-b-2ab)$

4. $(x+y)^3 - (x-y)^3$ can be factorized as
   (a) $2y(3x^2+y^2)$ (b) $2x(3x^2+y^2)$ (c) $2y(3y^2+x^2)$ (d) $2x(x^2+3y^2)$

5. The expression $(a-b)^3 + (b-c)^3 + (c-a)^3$ can be factorized as
   (a) $(a-b)(b-c)(c-a)$
   (b) $3(a-b)(b-c)(c-a)$
   (c) $-3(a-b)(b-c)(c-a)$
   (d) $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

6. The value of $\dfrac{(2.3)^3 - 0.027}{(2.3)^2 + 0.69 + 0.09}$, is
   (a) 2 (b) 3 (c) 2.327 (d) 2.273

7. The value of $\dfrac{(0.013)^3 + (0.007)^3}{(0.013)^2 - 0.013 \times 0.007 + (0.007)^2}$, is
   (a) 0.006 (b) 0.02 (c) 0.0091 (d) 0.00185

8. The factors of $a^2 - 1 - 2x - x^2$, are
   (a) $(a-x+1)(a-x-1)$
   (b) $(a+x-1)(a-x+1)$
   (c) $(a+x+1)(a-x-1)$
   (d) none of these

9. The factors of $x^4 + x^2 + 25$, are
   (a) $(x^2+3x+5)(x^2-3x+5)$
   (b) $(x^2+3x+5)(x^2+3x-5)$
   (c) $(x^2+x+5)(x^2-x+5)$
   (d) none of these

10. The factors of $x^2 + 4y^2 + 4y - 4xy - 2x - 8$, are
   (a) $(x-2y-4)(x-2y+2)$
   (b) $(x-y+2)(x-4y-4)$
   (c) $(x+2y-4)(x+2y+2)$
   (d) none of these

11. The factors of $x^3 - 7x + 6$, are
   (a) $x(x-6)(x-1)$
   (b) $(x^2-6)(x-1)$

(c) $(x+1)(x+2)(x-3)$ (d) $(x-1)(x+3)(x-2)$

12. The expression x^4+4 can be factorized as

(a) $(x^2+2x+2)(x^2-2x+2)$ (b) $(x^2+2x+2)(x^2+2x-2)$

(c) $(x^2-2x-2)(x^2-2x+2)$ (d) $(x^2+2)(x^2-2)$

13. If $3x=a+b+c$, then the value of $(x-a)^3+(x-b)^3+(x-c)^3-3(x-a)(x-b)(x-c)$, is

(a) $a+b+c$ (b) $(a-b)(b-c)(c-a)$

(c) 0 (d) none of these

14. If $(x+y)^3-(x-y)^3-6y(x^2-y^2)=ky^3$, then $k=$

(a) 1 (b) 2 (c) 4 (d) 8

15. If $x^3-3x^2+3x+7=(x+1)(ax^2+bx+c)$, then $a+b+c=$

(a) 4 (b) 12 (c) -10 (d) 3

16. If $\frac{x}{y}+\frac{y}{x}=-1\ (x,y\neq 0)$, then the value of x^3-y^3 is

(a) 1 (b) -1 (c) 0 (d) $\frac{1}{2}$

17. Which of the following is a factor of $(x+y)^3-(x^3+y^2)$?

(a) x^2+y^2+2xy (b) x^2+y^2-xy (c) xy^2 (d) $3xy$

ASSERTION-REASON

Each of the following questions contains STATEMENT-1 (Assertion) and STATEMENT-2 (Reason) and has following four choices (a), (b), (c) and (d), only one of which is the correct answer. Mark the correct choice.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
(c) Statement-1 is true, Statement-2 is false.
(d) Statement-1 is false, Statement-2 is true.

18. Statement-1 (Assertion): $(a-b)^3+(b-c)^3+(c-a)^3=3(a-b)(b-c)(c-a)$
Statement-2 (Reason): If $a+b+c=0$, then $a^3+b^3+c^3=3abc$

19. Statement-1 (Assertion): If $3x=a+b+c$, then
$(x-a)^3+(x-b)^3+(x-c)^3=3(x-a)(x-b)(x-c)$
Statement-2 (Reason): If $a+b+c=0$, then $a^3+b^3+c^3=3abc$

20. Statement-1 (Assertion): If $a+b+c=5$ and $ab+bc+ca=10$, then $a^3+b^3+c^3-3abc=25$
Statement-2 (Reason): $a^3+b^3+c^3-3abc=(a+b+c)\left\{(a+b+c)^2-3(ab+bc+ca)\right\}$

21. Statement-1 (Assertion): If a, b, c are all non-zero such that $a+b+c=0$, then
$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}=3$
Statement-2 (Reason): If $a+b+c=9$ and $a^2+b^2+c^2=35$, then $ab+bc+ca=23$

22. Statement-1 (Assertion): The value of $\frac{(0.027)^3 + (0.023)^3}{(0.027)^2 - (0.027)(0.023) + (0.023)^2}$ is 0.05

Statement-2 (Reason): $a^3 - b^3 = (a - b)(a^2 - ab + b^2)$

ANSWERS

1. (d)	2. (a)	3. (c)	4. (a)	5. (b)	6. (a)	7. (b)
8. (c)	9. (a)	10. (a)	11. (d)	12. (a)	13. (c)	14. (d)
15. (a)	16. (c)	17. (d)	18. (a)	19. (a)	20. (d)	21. (b)
22. (c)						