16

CONGRUENCE

16.1 INTRODUCTION

As you know that in geometry, we learn about geometrical figures, their properties and the relations between them. Every geometrical figure can be described by its shape, size and position. If we are given two geometrical figures, then by simply looking at them we can determine whether they have the same shape or not. When two figures are of same shape, then we try to know whether they are of the same size or not. In order to determine this, we cut the two figures and put one over the other. If they cover each other exactly, then we say that the two figures are of the same shape and same size. When two figures are of the same shape and same size, we say that they are congruent. In this chapter, we shall study the congruence of line segments, angles, squares, rectangles and triangles.

16.2 CONGRUENT FIGURES

The word 'congruent' means 'same shape and size', that is, equal in every respect. Thus, if two figures have exactly the same shape and size, they are said to be congruent. We can also say that two plane figures are congruent if each is a carbon copy of the other. The relation of two figures being congruent, is called congruence. If two figures are congruent, then without bending or changing their positions, we can superpose one figure on the other so that they completely cover each other.

Thus, we can say that:

Two plane figures are congruent, if each when superposed on the other, covers it exactly.

We shall use the symbol '$\cong$' to indicate 'is congruent to'.

Thus, if there are two figures F_1 and F_2, then $F_1 \cong F_2$ means that figure F_1 is congruent to figure F_2.

CONGRUENCE OF LINE SEGMENTS Let there be two line segments AB and CD as shown in Fig. 1. In order to see whether these two line segments are congruent or not, we take a trace copy of line segment CD and then try to superpose it on the line segment AB. If they cover exactly, the two line segments are congruent, otherwise not. When AB and CD cover each other exactly with C on A and D on B or C on B and D on A, then AB and CD have the same length.

Thus, in case of line segments, if two line segments have the same length, they are congruent and if they are congruent, they have the same length.

Fig. 1

Hence, we obtain the following result:

Two line segments are congruent, if they have the same length.

That is, line segment $AB \cong$ line segment CD, if $AB = CD$.

CONGRUENCE OF TWO ANGLES Consider two angles $\angle BAC$ and $\angle QPR$ as shown in Fig. 2. In order to see the congruence of these two angles we proceed as follows. Take a trace copy of $\angle QPR$ and try to superpose it on $\angle BAC$ such that P falls on A and ray PQ falls along ray AB. If now, ray PR falls along ray AC, then $\angle QPR$ covers $\angle BAC$ exactly and we say that $\angle BAC$ and $\angle QPR$ are congruent.

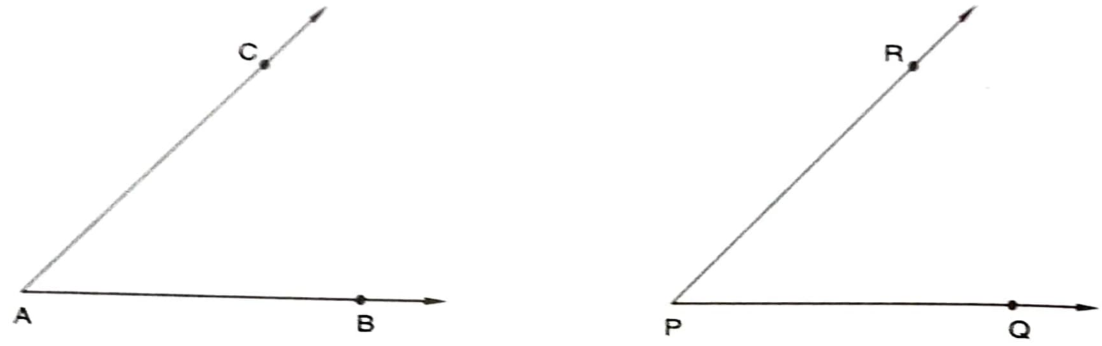


Fig. 2

Clearly, in such a situation $\angle BAC$ and $\angle QPR$ will have the same measure.

Thus, we obtain the following result:

Two angles are congruent, if they have the same measure.

That is,

$\angle BAC \cong \angle QPR$, then $m\ \angle BAC = m\ \angle QPR$

CONGRUENCE OF TWO SQUARES We know that two plane figures are congruent, if each when superposed on the other, covers it exactly, that is, they have the same shape and same size. Since all squares have the same shape and the size of a square is determined by the length of a side. Therefore if we are given two squares $ABCD$ and $PQRS$ as shown in Fig. 3, then to see whether they are congruent or not, we make a trace copy of one of the squares, say, $PQRS$ and try to superpose it on the other square $ABCD$ such that P falls on A and side PQ falls on side AB. We find that the square $PQRS$ will cover square $ABCD$ exactly, if $PQ = AB$. Thus, we have the following fact:

Two squares are congruent, if they have the same side length. That is,

square $ABCD \cong$ square $PQRS$, if $AB = PQ$.

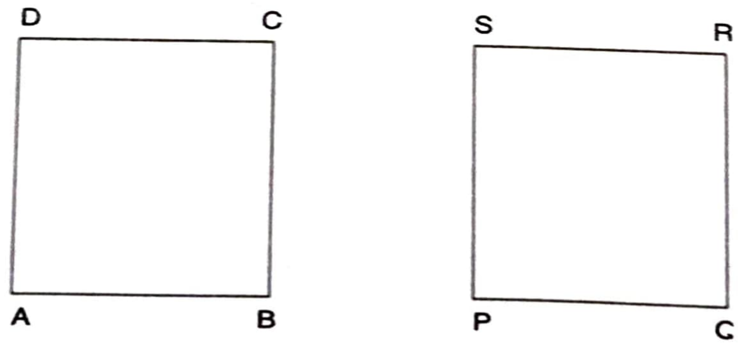


Fig. 3

CONGRUENCE OF TWO RECTANGLES The length and breadth of a rectangle determine its shape and size. In order to see whether two given rectangles $ABCD$ and $PQRS$ are congruent or not, we make a trace copy of rectangle $PQRS$. By superposing it on rectangle $ABCD$, we find that it will cover rectangle $ABCD$ exactly, if $AB = PQ$ and $BC = QR$. That is, rectangles $ABCD$ and $PQRS$ have the same length and breadth. Thus, we have the following fact:

Two rectangles are congruent, if they have the same length and breadth.

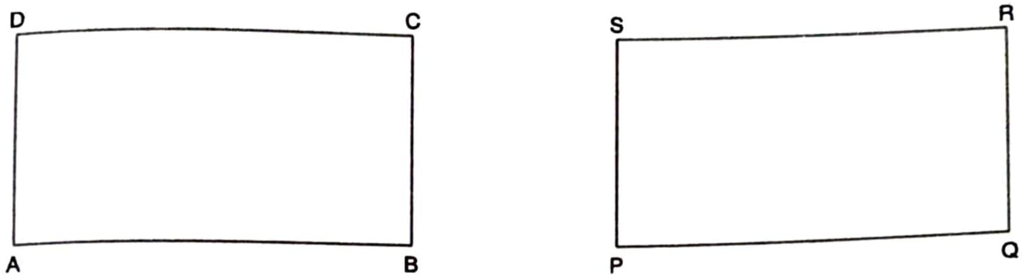


Fig. 4

CONGRUENCE OF TWO CIRCLES We know that all circles have the same shape and the size of a circle is determined by its radius. In order to see whether two given circles C_1 and C_2 (Fig. 5) are congruent or not we make a trace copy of one of the two circles, say C_1, and try to make it cover the second circle. We find that the circle C_2 will cover the circle C_1 exactly, if they are of the same radius. Thus, we have the following fact:

Two circles are congruent if they have the same radius.

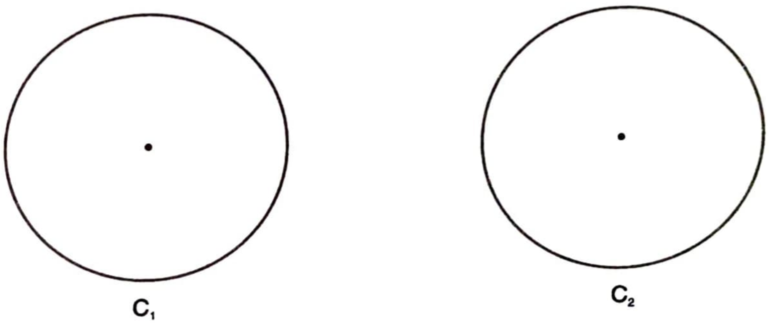


Fig. 5

That is, Circle $C_1 \cong$ Circle C_2, if radius of C_1 = radius of C_2.

EXERCISE 16.1

1. Explain the concept of congruence of figures with the help of certain examples.
2. Fill in the blanks:
   (i) Two line segments are congruent if ....
   (ii) Two angles are congruent if ......
   (iii) Two squres are congruent if ....
   (iv) Two rectangles are congruent if ......
   (v) Two circles are congruent if ....
3. In Fig. 6, $\angle POQ \cong \angle ROS$, can we say that $\angle POR \cong \angle QOS$

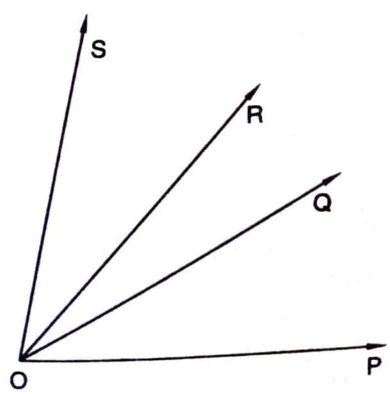


Fig. 6

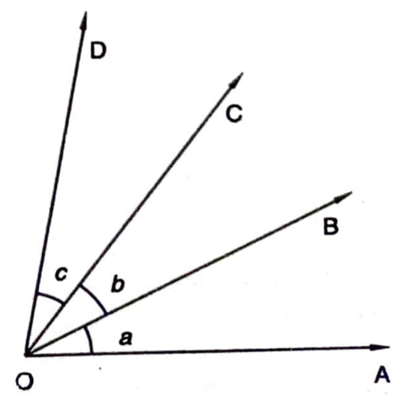


Fig. 7

4. In Fig. 7, $a = b = c$, name the angle which is congruent to $\angle AOC$.
5. Is it correct to say that any two right angles are congruent? Give reasons to justify your answer.
6. In Fig. 8, $\angle AOC \cong \angle PYR$ and $\angle BOC \cong \angle QYR$. Name the angle which is congruent to $\angle AOB$.

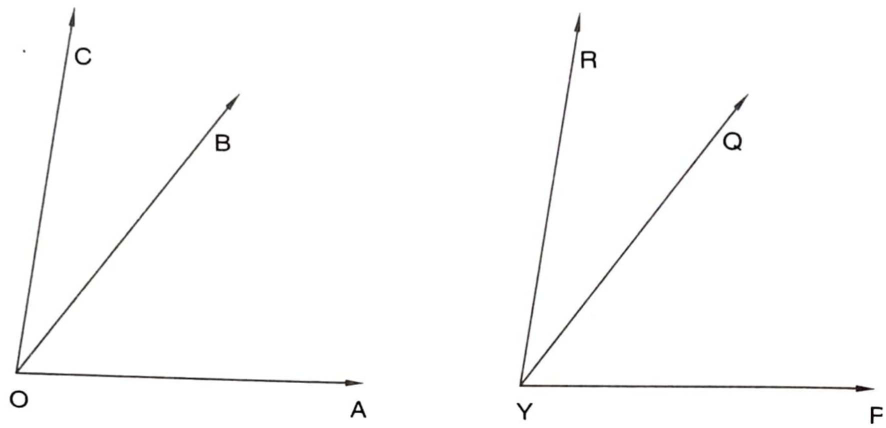


Fig. 8

7. Which of the following statements are true and which are false ;
   (i) All squares are congruent.
   (ii) If two squres have equal areas, they are congruent.
   (iii) If two rectangles have equal area, they are congruent.
   (iv) If two triangles are equal in area, they are congruent.

ANSWERS

2. (i) they are of equal lengths (ii) their measures are equal (iii) they have the same side length
   (iv) their dimensions are same (v) they have the same radii

3. Yes 4. $\angle BOD$ 5. Yes 6. $\angle PYQ$

7. (i) F (ii) T (iii) F (iv) F

16.3 CONGRUENCE OF TWO TRIANGLES

Two triangles are congruent if and only if one of them can be made to superpose on the other, so as to cover it exactly.

Let ΔABC and ΔDEF be two congruent triangles. Then, we can superpose $\Delta\, ABC$ on $\Delta\, DEF$, so as to cover it exactly. In such a superposition, the vertices of $\Delta\, DEF$, in some order. Let us assume that the vertex A falls on vertex D, vertex B on vertex E and vertex C on vertex F.

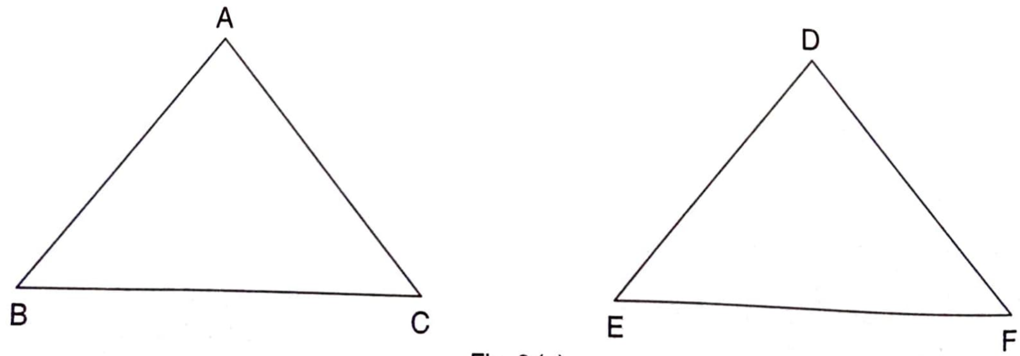


Fig. 9 (a)

Then, side AB falls on DE, BC on EF and CA on FD. Also $\angle A$ superposes on the corresponding angle $\angle D$, $\angle B$ on $\angle E$ and $\angle C$ on $\angle F$. Thus, the order in which the vertices match, automatically determines a correspondence between the sides and angles of the two triangles. And, if the superposition is exact, i.e. the triangles are congruent, the corresponding sides and angles are congruent. Consequently, we get six equalities; three of the corresponding sides and three of the corresponding angles.

It follows from the above discussion that, if ΔABC superposes on ΔDEF exactly such that the vertices of ΔABC fall on the vertices of ΔDEF in the following order

$$A \leftrightarrow D, B \leftrightarrow E, C \leftrightarrow F$$

Then, we have the following six equalities

$AB = DE, BC = EF, CA = FD$ (i.e. Corresponding sides are congruent)

$\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$ (i.e. Corresponding angles are congruent)

In the above discussion we have considered one correspondence between the vertices of triangles ABC and DEF viz. $A \rightarrow D$, $B \rightarrow E$ and $C \rightarrow F$. But there can be many other matchings between the vertices of two triangles as discussed below.

In two triangles ABC and DEF, we have the following six matchings or correspondences between their vertices:

$A \leftrightarrow D, B \leftrightarrow E$ and $C \leftrightarrow F$ written as $ABC \leftrightarrow DEF$

$A \leftrightarrow E, B \leftrightarrow F$ and $C \leftrightarrow D$ written as $ABC \leftrightarrow EFD$

$A \leftrightarrow F, B \leftrightarrow D$ and $C \leftrightarrow E$ written as $ABC \leftrightarrow FDE$

$A \leftrightarrow D, B \leftrightarrow F$ and $C \leftrightarrow E$ written as $ABC \leftrightarrow DFE$

$A \leftrightarrow E, B \leftrightarrow D$ and $C \leftrightarrow F$ written as $ABC \leftrightarrow EDF$

$A \leftrightarrow F, B \leftrightarrow E$ and $C \leftrightarrow D$ written as $ABC \leftrightarrow FED$

If ΔABC is congruent to ΔDEF, then in one of these six matchings ΔABC superposes on ΔDEF exactly and in that particular matching corresponding sides and angles will be congruent. Consequently, we will have three equalities of corresponding sides and three equali-ties of the measures of corresponding angles.

If ΔABC is not congruent to ΔDEF, then ΔABC will not superpose exactly ΔDEF in none of the above six possible matchings. Infact, in each matching at least one part (a side or an angle) of ΔABC will not be equal to the corresponding part of ΔDEF.

From the above discussion we obtain the following general condition for the congruence of two triangles:

Two triangles are congruent if and only if there exists a correspondence between their vertices such that the corresponding sides and the corresponding angles of the two triangles are equal or congruent.

If ΔABC is congruent to ΔDEF and the correspondence $ABC \leftrightarrow DEF$ makes the six pairs of corresponding parts of the two triangles congruent, then we write

$$\Delta ABC \cong \Delta DEF$$

Thus, $\Delta ABC \cong \Delta DEF$ if and only if $AB = DE$, $BC = EF$, $CA = FD$, $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$.

NOTE 1: *In the subsequent discussion, the order of the letters in the names of two triangles will indicate the correspondence between the vertices of two triangles. For example, $\Delta ABC \cong \Delta DEF$ will indicate the correspondence $ABC \leftrightarrow DEF$ and $\Delta ABC \cong \Delta DFE$ will indicate the correspondence $ABC \leftrightarrow DFE$. Thus, we can easily infer the six equalities between the corresponding parts of two triangles from the notation $\Delta ABC \cong \Delta DEF$.*

NOTE 2: *Note that $\Delta PQR \cong \Delta UVW$ will mean that $\angle P = \angle U$, $\angle Q = \angle V$, $\angle R = \angle W$, $PQ = UV$, $QR = VW$ and $PR = UW$.*

16.4 SUFFICIENT CONDITIONS FOR CONGRUENCE OF TWO TRIANGLES

In the previous section, we have learnt that two triangles are congruent if and only if there exists a correspondence between their vertices such that the corresponding sides and corresponding angles of two triangles are congruent or equal, that is, six equalities (three of the corresponding sides and three of the corresponding angles) hold good. Now the question arises: What are the least possible conditions to ensure the congruence of two triangles? Do we need all the six conditions or a less number of conditions will ensure the congruence of two triangles? In this section, we shall see that if three properly chosen conditions out of the six conditions, mentioned in the previous section, are satisfied, then the other three conditions are automatically satisfied. There are four such cases. In each case we have a different combination of three matching parts. Let us now discuss those combinations of three matching parts.

16.4.1 THE SIDE–SIDE–SIDE (SSS) CONGRUENCE CONDITION

Two triangles are congruent, if the three sides of one triangle are respectively equal to the three sides of the other triangle.

In order to verify the above property, let us perform the following experiment.

<u>*Experiment:*</u> Draw a ΔABC with $AB = 5.5$ cm, $BC = 6.5$ cm and $CA = 4.3$ cm. Draw another ΔPQR with $PQ = 5.5$ cm, $QR = 6.5$ cm and $RP = 4.3$ cm.

Thus, we have

$$AB = PQ, BC = QR \text{ and } CA = RP.$$

Now, make a trace copy of ΔPQR and try to make it cover ΔABC with P on A, Q on B and R on C. You will find that the two triangles cover each other exactly.

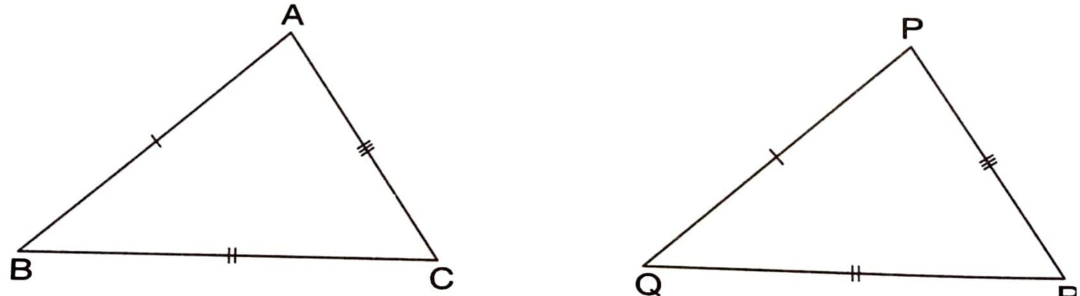


Fig. 9

$\therefore \quad \Delta ABC \cong \Delta PQR$

<u>Remark</u>: *If two triangles are congruent by SSS condition of congruence, then they are identical. So, the corresponding angles (angles opposite to equal sides) will also be equal.*

ILLUSTRATIVE EXAMPLES

Example 1 Without drawing the triangles, state the correspondence between the sides and the angles of the following pairs of congruent triangles:

(i) $\Delta ABC \cong \Delta PQR$ (ii) $\Delta ABC \cong \Delta QRP$

Solution (i) We have,

$\Delta ABC \cong \Delta PQR$

$\Rightarrow \quad A \leftrightarrow P, B \leftrightarrow Q, C \leftrightarrow R$

$\Rightarrow \quad AB = PQ, BC = QR, AC = PR, \angle A = \angle P, \angle B = \angle Q$ and $\angle C = \angle R$

(ii) We have,

$\Delta ABC \cong \Delta QRP$

$\Rightarrow \quad A \leftrightarrow Q, B \leftrightarrow R$ and $C \leftrightarrow P$

$\Rightarrow \quad AB = QR, BC = RP, AC = QP, \ \angle A = \angle Q, \angle B = \angle R$ and $\angle C = \angle P$

Example 2 In the following pairs of triangles (Fig. 10), by applying SSS condition, state which are congruent. State the result in symbolic form:

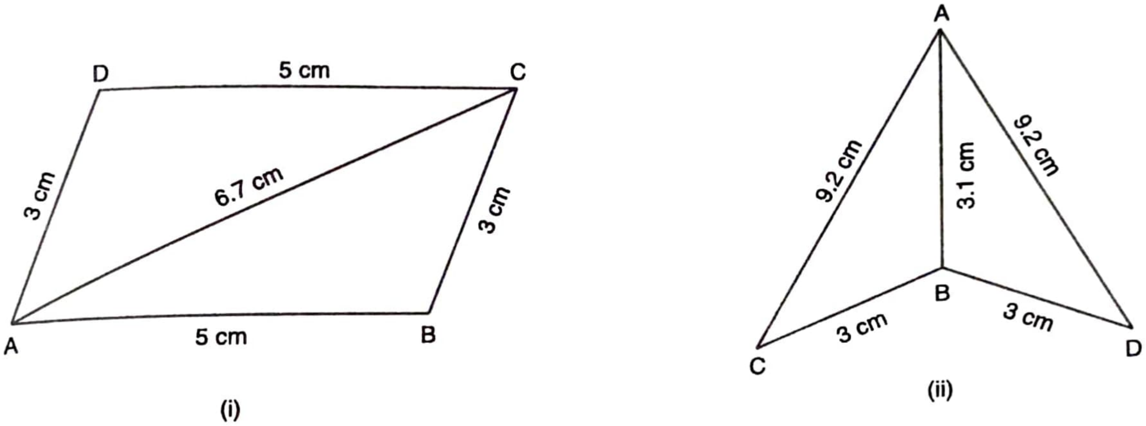


Fig. 10

Solution (i) In triangles ABC and CDA, we have

$AB = CD = 5$ cm, $BC = DA = 3$ cm, and, $AC = AC = 6.7$ cm. (Common)

$\therefore$ By SSS condition of congruence, we have

$$\Delta ABC \cong \Delta CDA$$

(ii) In triangles ABC and ABD, we have

$AC = AD = 9.2$ cm, $BC = BD = 3$ cm and, $AB = AB = 3.1$ cm (Common)

So, by SSS condition of congruence, we have

$$\Delta ABC \cong \Delta ABD$$

Example 3 Which of the following pairs of triangles are congruent? If they are congruent, write out the pairs of equal angles.

(i) $\Delta\ ABC$: $AB = 3$ cm, $BC = 4$ cm, $CA = 2$ cm

$\Delta\ DEF$: $DE = 2$ cm, $EF = 3$ cm and $FD = 4$ cm

(ii) $\Delta\ PQR$: $PQ = 17$ cm, $QR = 15$ cm, $PR = 18$ cm

$\Delta\ DEF$: $DE = 18$ cm, $EF = 17$ cm, $DF = 15$ cm.

Solution (i) In triangles ABC and DEF, we have

$AB = EF = 3$ cm, $BC = FD = 4$ cm and $CA = DE = 2$ cm

So, by SSS condition of congruence, we have

$$\Delta ABC \cong \Delta EFD$$

Since $AB = EF$, therefore angles opposite to these sides are equal.

$\Rightarrow \quad \angle C = \angle D$

Similarly, $BC = FD \quad \Rightarrow \quad \angle A = \angle E$

and, $\quad CA = DE \quad \Rightarrow \quad \angle B = \angle F$

(ii) In triangles PQR and DEF, we have

$PQ = EF = 17$ cm, $QR = FD = 15$ cm and $RP = DE = 18$ cm

So, by SSS condition of congruence, we have

$$\Delta PQR \cong \Delta EFD$$

$\Rightarrow \quad \angle P = \angle E, \angle Q = \angle F$ and $\angle R = \angle D$

Example 4 In Fig. 11, it is given that $AB = CD$ and $AD = BC$. Prove that $\Delta ADC \cong \Delta CBA$.

Solution In ΔADC and ΔCBA, we have

$AB = CD$ [Given]

$AD = BC$ [Given]

$AC = AC$ [Common side]

So, by SSS criterion of congruence, we have

$\Delta ADC \cong \Delta CBA$

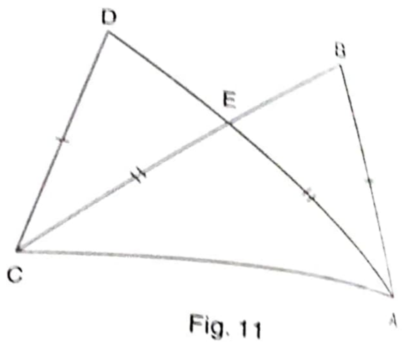


Fig. 11

EXERCISE 16.2

1. In the following pairs of triangles (Fig. 12 to 15), the lengths of the sides are indicated along sides. By applying SSS condition, determine which are congruent. State the result in symbolic form.

(i)

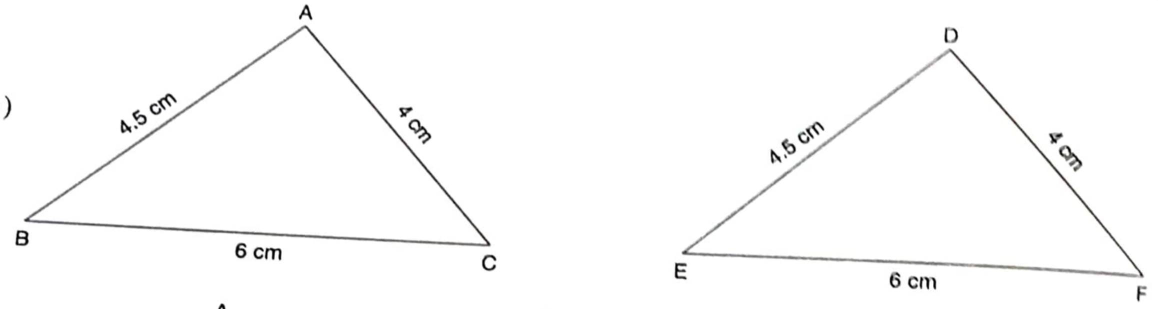


Fig. 12

(ii)

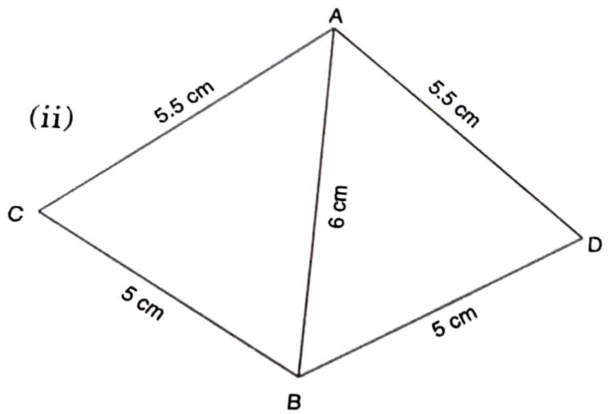


Fig. 13

(iii)

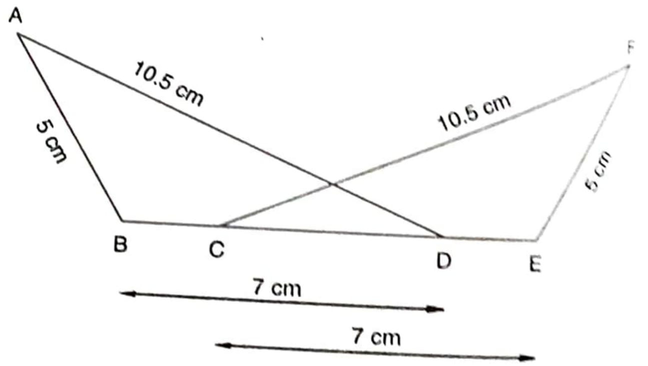


Fig. 14

(iv)

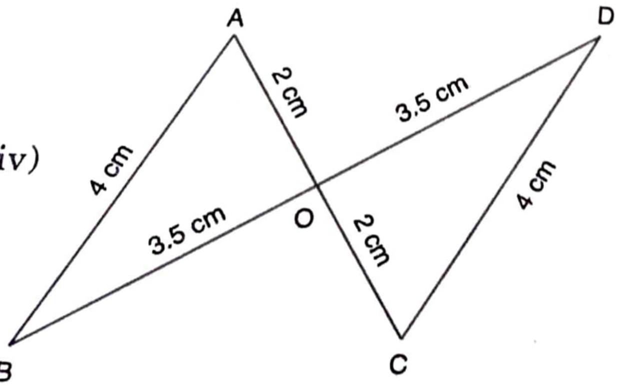


Fig. 15

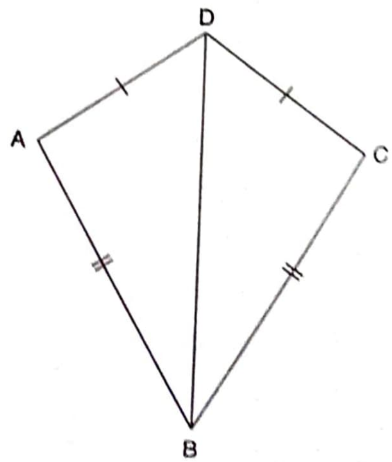


Fig. 16

2. In Fig. 16, $AD = DC$ and $AB = BC$.

(i) Is $\Delta ABD \cong \Delta CBD$?

(ii) State the three parts of matching pairs you have used to answer (i).

3. In Fig. 17, $AB = DC$ and $BC = AD$.

(i) Is $\Delta ABC \cong \Delta CDA$?

(ii) What congruence condition have you used?

(iii) You have used some fact, not given in the question, what is that?

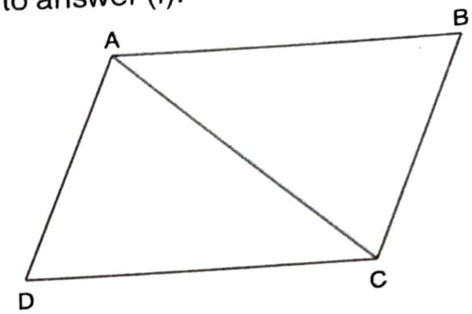


Fig. 17

4. If $\Delta PQR \cong \Delta EFD$,

(i) Which side of ΔPQR equals ED? (ii) Which angle of ΔPQR equals $\angle E$?

5. Triangles ABC and PQR are both isosceles with $AB = AC$ and $PQ = PR$ respectively. If also, $AB = PQ$ and $BC = QR$, are the two triangles congruent? Which condition do you use? If $\angle B = 50°$, what is the measure of $\angle R$?

6. ABC and DBC are both isosceles triangles on a common base BC such that A and D lie on the same side of BC. Are triangles ADB and ADC congruent? Which condition do you use? If $\angle BAC = 40°$ and $\angle BDC = 100°$; then find $\angle ADB$.

7. ΔABC and ΔABD are on a common base AB, and $AC = BD$ and $BC = AD$ as shown in Fig. 18. Which of the following statements is true?

(i) $\Delta ABC \cong \Delta ABD$

(ii) $\Delta ABC \cong \Delta ADB$

(iii) $\Delta ABC \cong \Delta BAD$

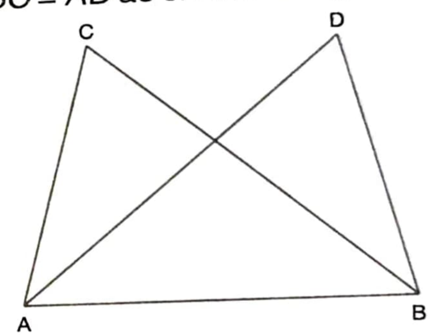


Fig. 18

8. In Fig. 19, ΔABC is isosceles with $AB = AC$, D is the mid-point of base BC.

(i) Is $\Delta ADB \cong \Delta ADC$?

(ii) State the three pairs of matching parts you use to arrive at your answer.

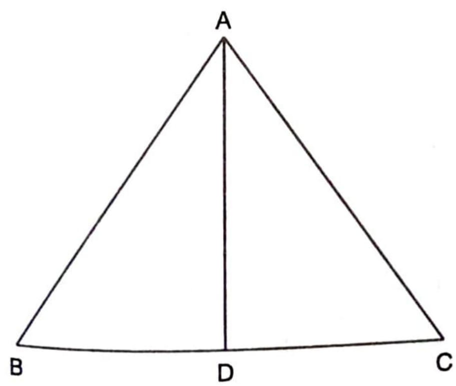


Fig. 19

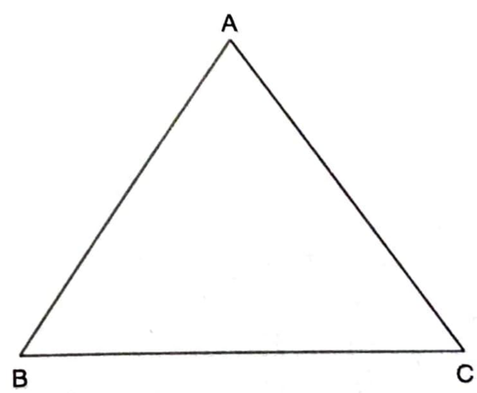


Fig. 20

9. In Fig. 20, ΔABC is isosceles with $AB = AC$. State if $\Delta ABC \cong \Delta ACB$. If yes, state three relations that you use to arrive at your answer.

10. Triangles ABC and DBC have side BC common, $AB = BD$ and $AC = CD$. Are the two triangles congruent? State in symbolic form. Which congruence condition do you use? Does $\angle ABD$ equal $\angle ACD$? Why or why not?

ANSWERS

1. (i) $ABC \cong DEF$ (ii) $ABC \cong ABD$ (iii) $\Delta ABD \cong \Delta FEC$ (iv) $\Delta OAB \cong \Delta OCD$
2. Yes; AB, CB ; AD, CD ; BD, BD
3. (i) Yes (ii) SSS (iii) AC = CA
4. (i) PR (ii) $\angle P$
5. Yes, SSS, 50°
6. Yes, SSS, 130°
7. (iii)
8. (i) Yes (ii) AB, AC ; AD, AD ; BD, DC
9. Yes, $AB = AC$, $BC = CB$ and $AC = AB$
10. Yes, $\Delta ABC \cong \Delta$ DBC, SSS, Yes

16.4.2 SAS CONGRUENCE CONDITION

Two triangles are congruent if two sides and the included angle of the one are respectively equal to the two sides and the included angle of the other.

In order to verify the above fact, we perform the following experiment.

Experiment: Draw a ΔABC with $AB = 6$ cm, $AC = 4$ cm and $\angle A = 60°$. Also, draw a ΔPQR with $PQ = 6$ cm, $PR = 4$ cm and $\angle P = 60°$.

Thus, we have

$$AB = PQ, AC = PR \text{ and } \angle A = \angle P$$

Make a trace copy of ΔABC and try to cover ΔPQR with A on P, B on Q and C on R.

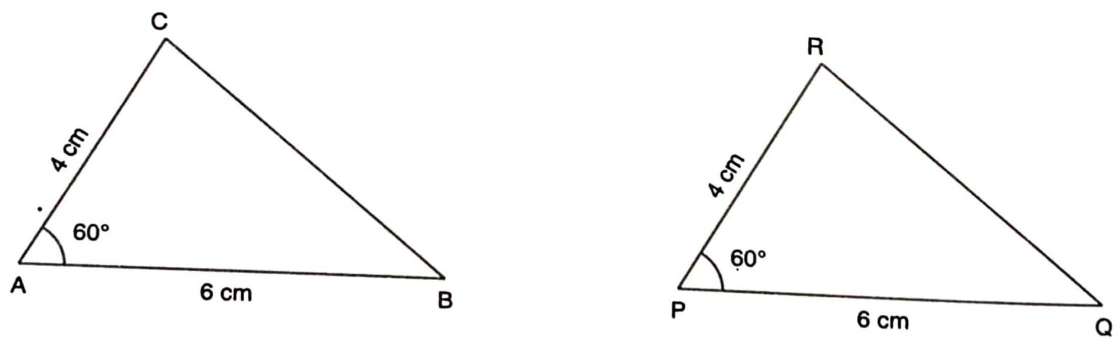


Fig. 21

You will find that the two triangles will cover each other exactly.

Hence, $\Delta ABC \cong \Delta PQR$.

ILLUSTRATIVE EXAMPLES

Example 1 In each of the following pairs of triangles the measures of some parts are indicated along side. By the application of SAS congruence condition, state which are congruent. State the result in symbolic form.

Solution (i) In triangles ABC and PQR, we have

$$AB = PQ, AC = PR \text{ and } \angle A = \angle P = 40°$$

So, by SAS-congruence condition, we have

$$\Delta ABC \cong \Delta PQR$$

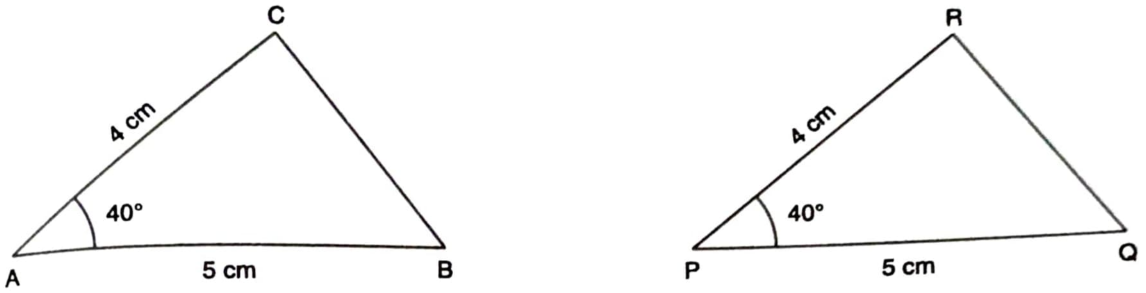


Fig. 22 (i)

(ii) In triangles ABC and PQR, we have

$CB = PQ = 5.5$ cm, $CA = PR = 4$ cm and $\angle C = \angle P = 50°$

So, by SAS congruence condition, we have

$$\Delta CBA \cong \Delta PQR$$

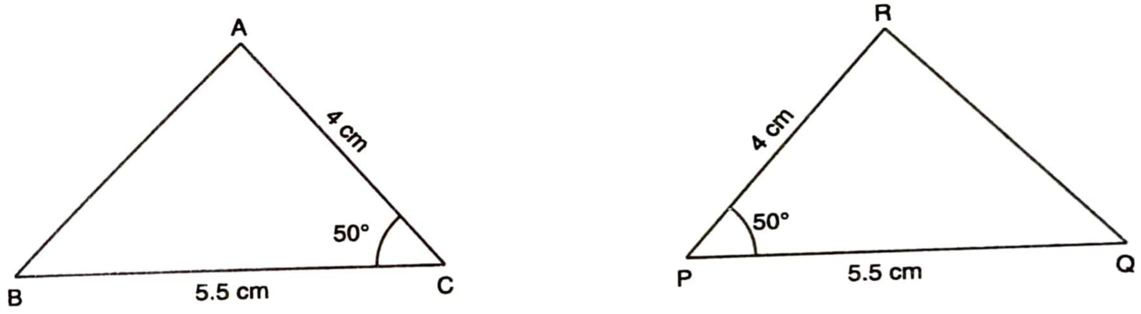


Fig. 22 (ii)

(iii) In triangles ABE and CBD, we have

$AB = CB = 5.2$ cm, $AE = CD = 5$ cm and $\angle A = \angle C = 40°$

So, by SAS congruence condition, we have

$$\Delta EAB \cong \Delta DCB$$

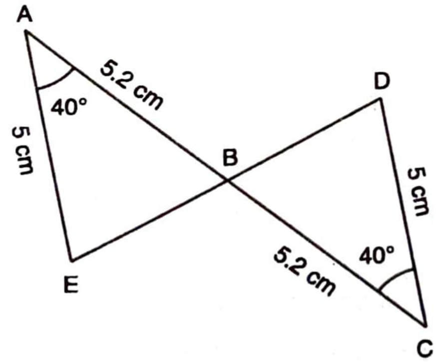


Fig. 22 (iii)

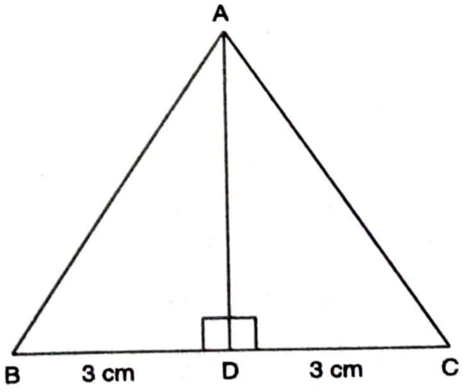


Fig. 22 (iv)

(iv) In triangles ABD and ACD [Fig. 22 (iv)], we have

$BD = CD = 3$ cm, $AD = AD$ and $\angle ADB = \angle ADC = 90°$.

So, by SAS congruence condition, we have

$$\Delta ADB \cong \Delta ADC$$

Example 2 Which of the following pairs of triangles are congruent?

(i) ΔABC: $AB = 2$ cm, $AC = 4$ cm, $\angle A = 40°$; ΔXYZ : $XZ = 2$ cm, $YZ = 4$ cm, $\angle Z = 40°$

(ii) ΔPQR: $PQ = 5$ cm, $PR = 6$ cm, $\angle P = 55°$; ΔDEF: $DE = 6$ cm, $EF = 5$ cm, $\angle D = 55°$

Solution (i) In ΔABC and ΔXYZ, we have

$AB = XZ = 2$ cm, $AC = YZ = 4$ cm and $\angle A = \angle Z = 40°$

Thus, in ΔABC and XYZ, the two sides and included angle of one triangle are equal to two sides and the corresponding included angle of the other.

So, by SAS congruence condition, we have

$\Delta ABC \cong \Delta ZXY$

(ii) In ΔPQR, the included angle between PQ and PR is $\angle P$

In ΔDEF, the included angle between DE and EF is $\angle E$

We have,

$$PQ = EF = 5 \text{ cm and } PR = DE = 6 \text{ cm but } \angle P \neq \angle E$$

So, the given triangles are not congruent.

Example 3 Show that in an isosceles triangle, angles opposite to equal sides are equal.

Solution Let ΔABC be an isosceles triangle such that $AB = AC$.

Then, we have to prove that $\angle B = \angle C$.

Draw the bisector AD of $\angle A$ meeting BC in D.

Now, in Δ s ABD and ACD, we have

$AB = AC$ [Given]

$\angle BAD = \angle CAD$ [$\because$ AD is the bisector of $\angle A$]

and, $AD = AD$ [Common side]

Therefore, by SAS congruence condition, we have

$\Delta ABD \cong \Delta ACD$

$\Rightarrow \angle B = \angle C$ [Corresponding parts of congruent triangles are equal]

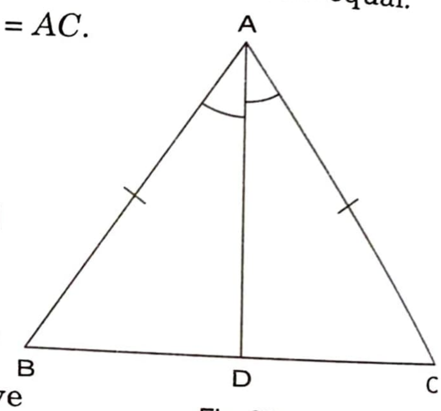


Fig. 23

Example 4 Show that the bisector of vertical angle of an isosceles triangle bisects the base at right angles.

Solution Let ΔABC be an isosceles triangle such that $AB = AC$.

Let AD be the bisector of vertical angle $\angle A$ meeting BC in D.

Now, in Δs ABD and ACD, we have

$AB = AC$ [Given]

$\angle BAD = \angle CAD$ [$\because$ AD is the bisector of $\angle A$]

and, $AD = AD$ [Common side]

$\Rightarrow BD = CD$ and $\angle ADB = \angle ADC$

Therefore, by SAS congruence condition, we have

$\Delta ABD \cong \Delta ACD$

But, $\angle ADB + \angle ADC = 180°$ [Linear pair property]

$\therefore \angle ADB = \angle ADC = 90°$

Hence, AD bisects BC at right angles.

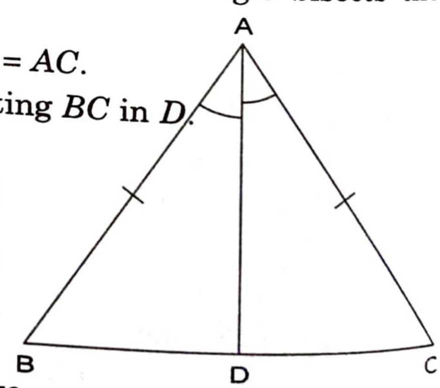


Fig. 24

Example 5 In ΔABC, $\angle A = 100°$ and $AB = AC$. Find $\angle B$ and $\angle C$.

Solution We have,

$AB = AC$ [Given]

$\Rightarrow \angle B = \angle C$ [$\because$ Angles opp. to equal sides are equal]

Now,

$\angle A + \angle B + \angle C = 180°$ [$\because$ Sum of angles of a triangle = 180°]

$\Rightarrow \angle A + \angle B + \angle B = 180°$ [$\because \angle B = \angle C$]

$\Rightarrow 100° + 2\angle B = 180°$

$\Rightarrow 2\angle B = 80°$

$\Rightarrow \angle B = 40°$

Hence, $\angle B = \angle C = 40°$

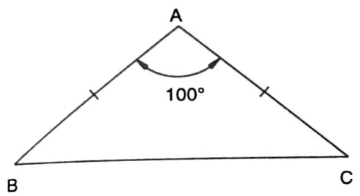


Fig. 25

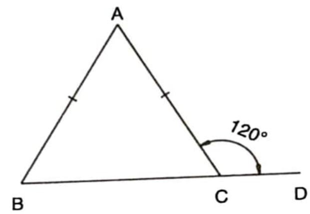


Fig. 26

Example 6 In Fig. 26, $AB = AC$ and $\angle ACD = 120°$. Find $\angle A$.

Solution We have,

$AB = AC$ [Given]

$\Rightarrow \angle B = \angle C$ [$\because$ Angles opposite to equal sides are equal]

Now,

$\angle ACB + \angle ACD = 180°$ [Angles of a linear pair]

$\Rightarrow \angle C + 120° = 180°$ [$\because$ Sum of angles of a triangle is 180°]

$\Rightarrow \angle C = 60°$

$\therefore \angle B = 60°$ [$\because \angle B = \angle C$]

Now,

$\angle A + \angle B + \angle C = 180°$ [$\because$ Sum of angles of a triangle is 180°]

$\Rightarrow \angle A + 60° + 60° = 180°$

$\Rightarrow \angle A = 60°$

Example 7 Prove that measure of each angle of an equilateral triangle is 60°

Solution Let ΔABC be an equilateral triangle. Then, $AB = BC = CA$

Since angles opposite to equal sides of a triangle are equal.

$\therefore AB = BC$

$\Rightarrow \angle C = \angle A$ ...(i)

and, $BC = CA$

$\Rightarrow \angle A = \angle B$ ...(ii)

From (i) and (ii), we get

$\angle A = \angle B = \angle C$

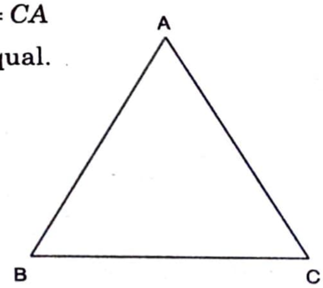


Fig. 27

But, $\angle A + \angle B + \angle C = 180°$ [$\because$ Sum angle property of a triangle]

$\therefore \quad \angle A + \angle A + \angle A = 180°$

$\Rightarrow \quad 3\angle A = 180°$

$\Rightarrow \quad \angle A = 60°$

Hence, $\angle A = \angle B = \angle C = 60°$.

EXERCISE 16.3

1. By applying SAS congruence condition, state which of the following pairs (Fig. 28) of triangles are congruent. State the result in symbolic form

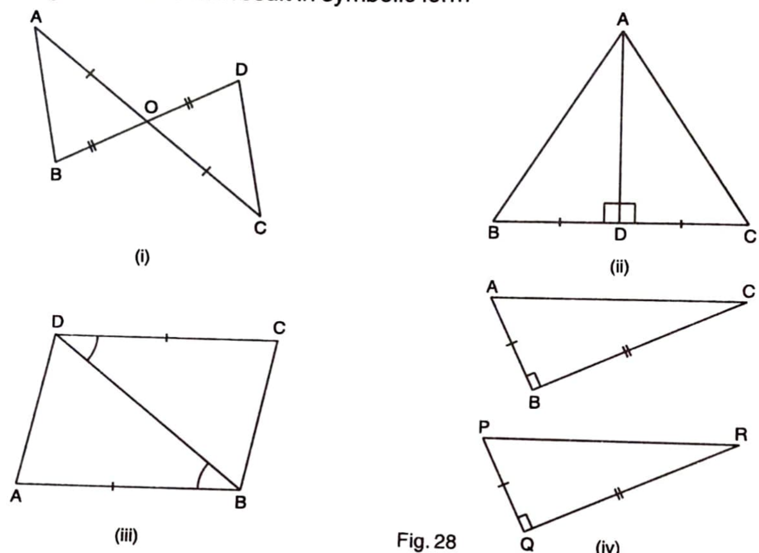


Fig. 28

2. State the condition by which the following pairs of triangles are congruent.

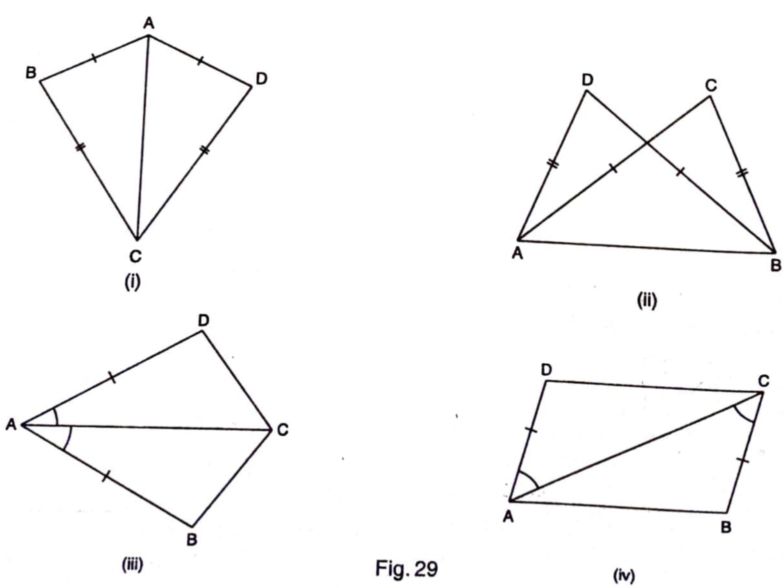


Fig. 29

3. In Fig. 30, line segments AB and CD bisect each other at O. Which of the following statements is true?
   (i) $\Delta AOC \cong \Delta DOB$ (ii) $\Delta AOC \cong \Delta BOD$
   (iii) $\Delta AOC \cong \Delta ODB$.
   State the three pairs of matching parts, you have used to arive at the answer.

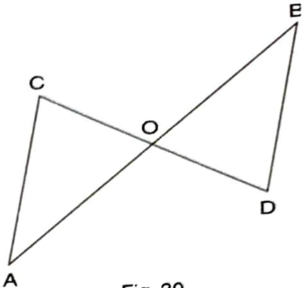


Fig. 30

4. Line-segments AB and CD bisect each other at O. AC and BD are joined forming triangles AOC and BOD. State the three equality relations between the parts of the two triangles, that are given or otherwise known. Are the two triangles congruent? State in symbolic form. Which congruence condition do you use?
5. ΔABC is isosceles with $AB = AC$. Line segment AD bisects $\angle A$ and meets the base BC in D.
   (i) Is $\Delta ADB \cong \Delta ADC$?
   (ii) State the three pairs of matching parts used to answer (i).
   (iii) Is it true to say that $BD = DC$?
6. In Fig. 31, $AB = AD$ and $\angle BAC = \angle DAC$.
   (i) State in symbolic form the congruence of two triangles ABC and ADC that is true.
   (ii) Complete each of the following, so as to make it true:
   (a) $\angle ABC =$ ....
   (b) $\angle ACD =$ ....
   (c) Line segment AC bisects ..... and .....

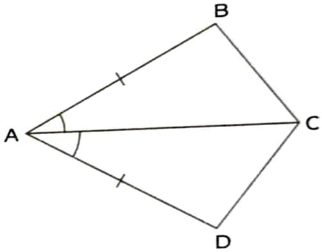


Fig. 31

7. In Fig. 32, $AB \parallel DC$ and $AB = DC$.
   (i) Is $\Delta ACD \cong \Delta CAB$?
   (ii) State the three pairs of matching parts used to answer (i).
   (iii) Which angle is equal to $\angle CAD$?
   (iv) Does it follow from (iii) that $AD \parallel BC$?

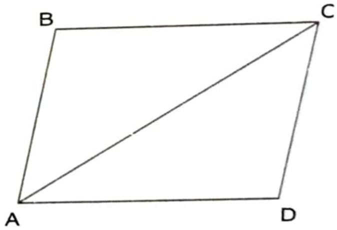


Fig. 32

ANSWERS

1. (i) $\Delta AOB \cong \Delta COD$ (ii) $\Delta ADC \cong \Delta ADB$ (iii) $\Delta ABD \cong \Delta CDB$
   (iv) $\Delta ABC \cong \Delta PQR$
2. (i) SSS (ii) SSS (iii) SAS (iv) SAS
3. (ii) AO, BO; CO, DO; $\angle AOC, \angle BOD$
4. $OA = OB$, $OC = OD$, $\angle AOC = \angle BOD$; Yes; $\Delta AOC \cong \Delta BOD$; SAS
5. (i) Yes (ii) AB, AC; AD, AD; $\angle BAD = \angle CAD$ (iii) Yes
6. (i) $\Delta ABC \cong \Delta ADC$ (a) $\angle ADC$ (b) $\angle ACB$ (c) $\angle A, \angle C$
7. (i) Yes (ii) AC, CA; DC, BA; $\angle DCA, \angle BAC$ (iii) $\angle ACB$ (iv) Yes

16.4.3 ASA CONGRUENCE CONDITION

Two triangles are congruent if two angles and the included side of the one are respectively equal to the two angles and the included side of the other.

In order to verify the above statement, let us perform the following experiment.

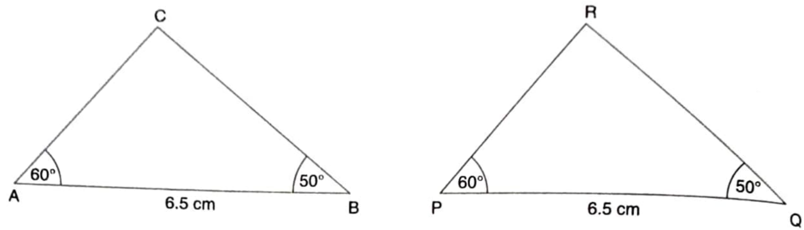


Fig. 32a

Experiment: Draw a ΔABC with $AB = 6.5$ cm, $\angle A = 60°$ and $\angle B = 50°$. Also, draw a ΔPQR with $PQ = 6.5$ cm, $\angle P = 60°$ and $\angle Q = 50°$. Thus, we have two triangles ABC and PQR such that $AB = PQ$, $\angle A = \angle P$ and $\angle B = \angle Q$. Make a trace copy of ΔABC and try to cover ΔPQR with A on P, B on Q and C on R.

You will find that the two triangles will cover each other exactly.

Hence, $\Delta ABC \cong \Delta PQR$.

Remark: *By angle sum property of a triangle, given two angles of a triangle, the third can be determined. Thus, whenever one side and any two angles of a triangle are given, we can change this to the type, two angles and included side, and the condition ASA is applied.*

ILLUSTRATIVE EXAMPLES

Example 1 In each of the following pairs of triangles given in Fig. 33, the measures of some parts are indicated along side. By applying ASA congruence condition, state which are congruent. State the answer in symbolic form.

Solution (i) In Δ's ABC and PQR, we have

$BC = QR = 6$ cm, $\angle ABC = \angle PQR = 50°$ and $\angle ACB = \angle PRQ = 40°$

$\therefore$ By ASA congruence condition, we have

$$\Delta ABC \cong \Delta PQR$$

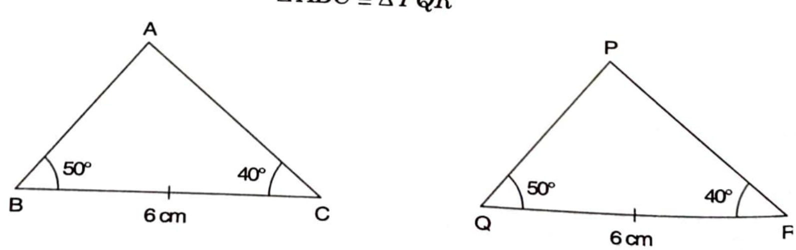


Fig. 33 (i)

(ii) In Δ's PQR and XYZ, we have

$QR = YZ = 5.5$ cm

$\angle PQR = \angle XYZ = 90°$

and, $\angle PRQ = \angle XZY = 40°$

So, by ASA congruence condition, we have

$$\Delta PQR \cong \Delta XYZ$$

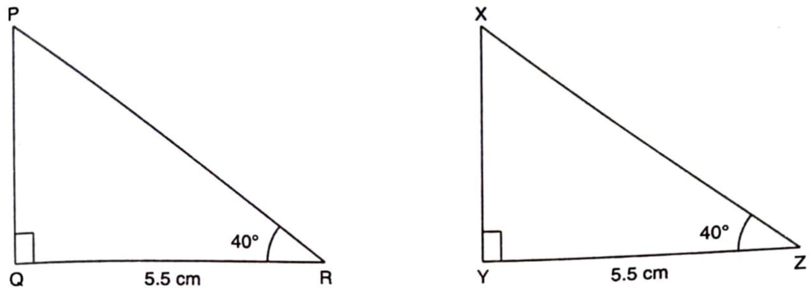


Fig. 33 (ii)

(iii) In Δ's AOD and BOC, we have

$\angle ADO = \angle CBO = 100°$

$\angle AOD = \angle COB$ [Vertically opposite angles]

and, $OB = OD = 2$ cm

So, by ASA congruence condition, we have

$\therefore\ \Delta AOD \cong \Delta COB$

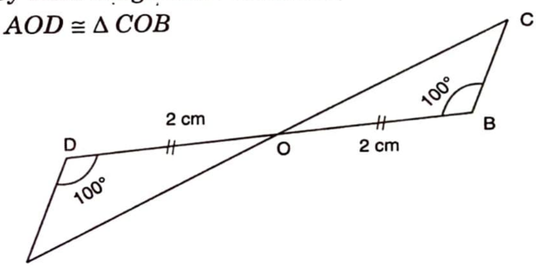


Fig. 33 (iii)

(iv) In Δ's ABC and BAD, we have

$\angle CAB = \angle DBA = 30°$

$\angle ABC = \angle BAD = 55°$

and, $AB = BA$

$\therefore$ By ASA congruence condition, we have

$$\Delta ABC \cong \Delta BAD$$

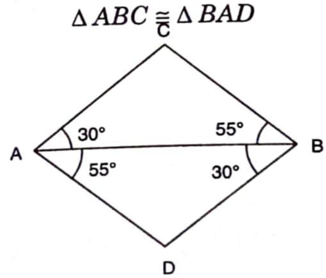


Fig. 33 (iv)

Example 2 Which of the following pairs of triangles are congruent?

(i) ΔABC: $AB = 10$ cm, $\angle A = 40°$, $\angle B = 55°$
ΔXYZ: $XY = 10$ cm, $\angle Y = 40°$, $\angle Z = 85°$

(ii) ΔPQR: $PR = 5$ cm, $\angle P = 37°$, $\angle R = 64°$
ΔDEF: $DE = 5$ cm, $\angle D = 37°$, $\angle F = 64°$

Solution (i) In ΔXYZ, we have

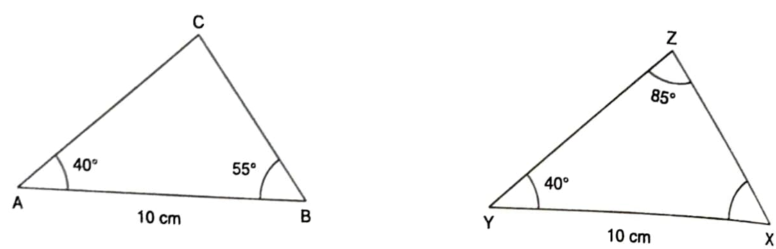


Fig. 34

$\angle Y = 40°$ and $\angle Z = 85°$

$\therefore$ $\angle X = 180° - (\angle Y + \angle Z) = 180° - (40° + 85°) = 55°$

Thus, in Δ s ABC and XYZ, we have

$AB = XY = 10$ cm

$\angle A = \angle Y = 40°$

and, $\angle B = \angle X = 55°$

$\therefore$ By ASA congruence condition, we have

$\Delta ABC \cong \Delta YXZ$

(ii) In Δ's PQR and DEF, we have

$PR = DE = 5$ cm

$\angle P = \angle D = 37°$

and, $\angle R = \angle E = 64°$

$\therefore$ By ASA congruence condition, we have

$\Delta PQR \cong \Delta DFE$ or $\Delta PRQ \cong \Delta DEF$

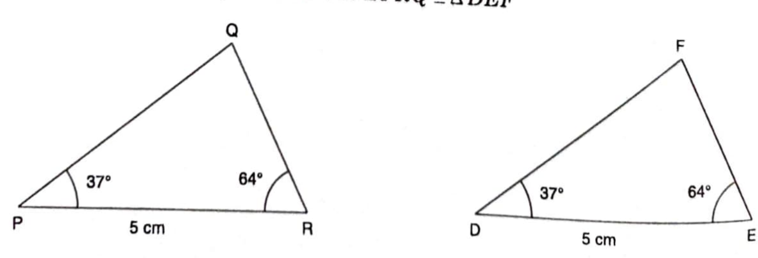


Fig. 34

EXERCISE 16.4

1. Which of the following pairs of triangles are congruent by ASA condition?

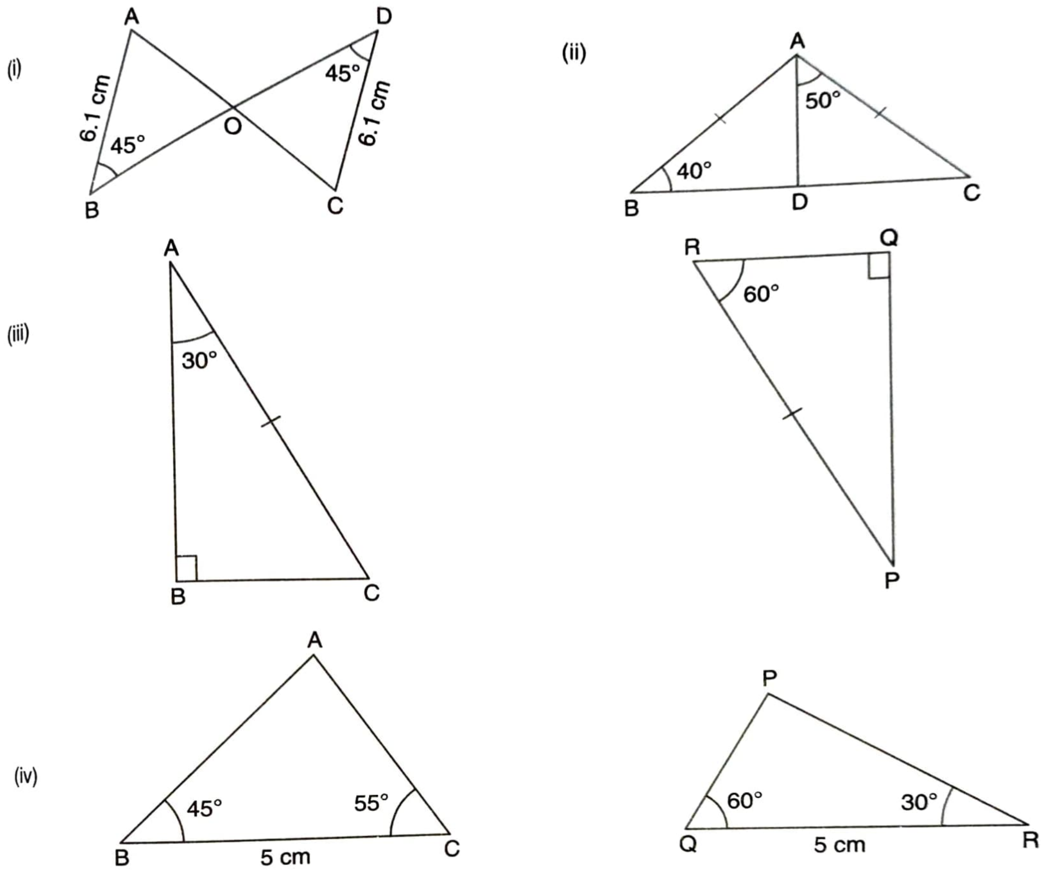


Fig. 36

2. In Fig. 37, AD bisects $\angle A$ and $AD \perp BC$.
   (i) Is $\Delta ADB \cong \Delta ADC$?
   (ii) State the three pairs of matching parts you have used in (i).
   (iii) Is it true to say that $BD = DC$?

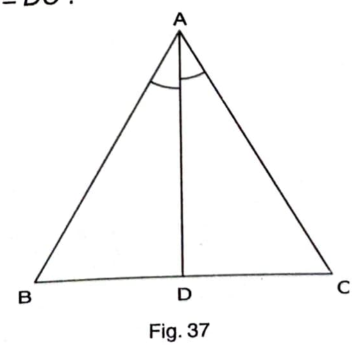


Fig. 37

3. Draw any triangle ABC. Use ASA condition to construct another triangle congruent to it.
4. In ΔABC, it is known that $\angle B = \angle C$. Imagine you have another copy of ΔABC
   (i) Is $\Delta ABC \cong \Delta ACB$?
   (ii) State the three pairs of matching parts you have used to answer (i).
   (iii) Is it true to say that $AB = AC$?
5. In Fig. 38, AX bisects $\angle BAC$ as well as $\angle BDC$. State the three facts needed to ensure that $\Delta ABD \cong \Delta ACD$.

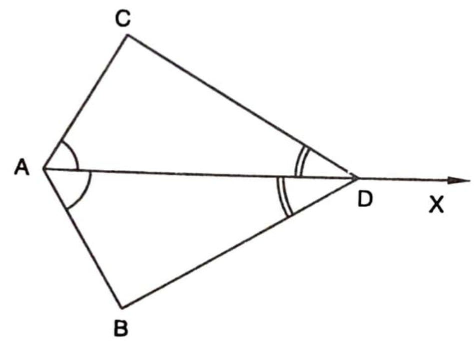


Fig. 38

6. In Fig. 39, $AO = OB$ and $\angle A = \angle B$.

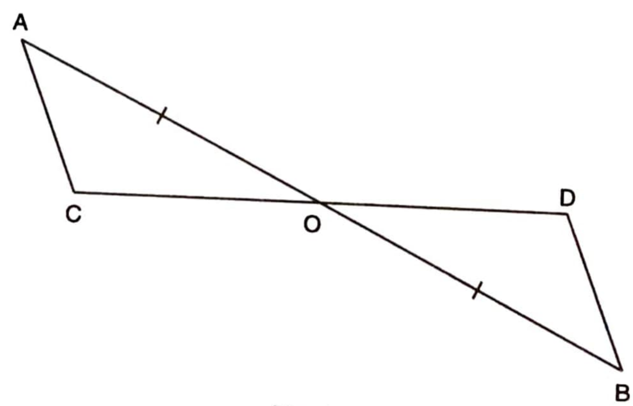


Fig. 39

   (i) Is $\Delta AOC \cong \Delta BOD$?
   (ii) State the matching pair you have used, which is not given in the question.
   (iii) Is it true to say that $\angle ACO = \angle BDO$?

ANSWERS

1. (i) $\Delta ABO \cong \Delta CDO$ (ii) $\Delta ADB \cong \Delta ADC$ (iii) $\Delta ABC \cong \Delta PQR$
   (iv) No 2. (i) Yes (ii) $\angle BAD, \angle CAD; AD, AD; \angle ADB, \angle ADC$ (iii) Yes
4. (i) Yes (ii) $\angle ABC, \angle ACB; \angle ACB, \angle ABC; BC, CD$ (iii) Yes
5. (i) $\angle CAD = \angle BAD$, $\angle CDA = \angle BDA$; $AD = AD$
6. (i) Yes (ii) $\angle AOC, \angle BOD$ (iii) Yes

16.4.4 RHS CONGRUENCE CONDITION

Two right triangles are congruent if the hypotenuse and one side of the one are respectively equal to the hypotenuse and one side of the other.

In order to verify the above fact, let us perform the following experiment.

Experiment: Draw a ΔABC such that $\angle C$ is a right angle, hypotenuse $AB = 10$ cm, and side $AC = 8$ cm. Also, draw a $\Delta\ PQR$ with $\angle R = 90°$; hypotenuse $PQ = 10$ cm and side $PR = 8$ cm.

Thus, we have two triangles ABC and PQR such that $\angle C = \angle R = 90°$, hypotenuse AB = hypotenuse PQ and side AC = side PR as shown in Fig. 40.

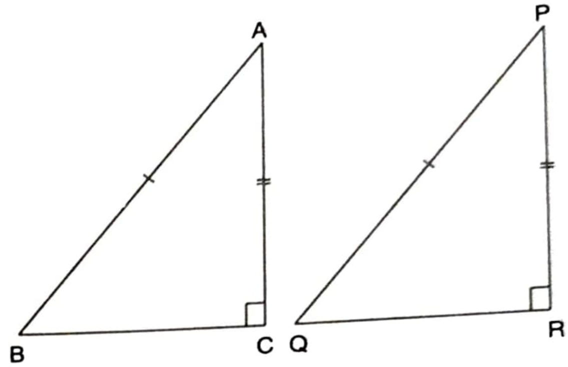


Fig. 40

Now, make a trace copy of $\Delta\ PQR$ and try to cover ΔABC with P on A, Q on B and R on C. You will find that the two triangles will cover each other exactly.

Hence, $\Delta ABC \cong \Delta PQR$.

ILLUSTRATIVE EXAMPLES

Example 1 In each of the following pairs of right triangles, the measures of some parts are indicated along side. State by the application of RHS congruence condition which are congruent. State each result in symbolic form.

Solution (i) In Δ's ADB and BCA, we have

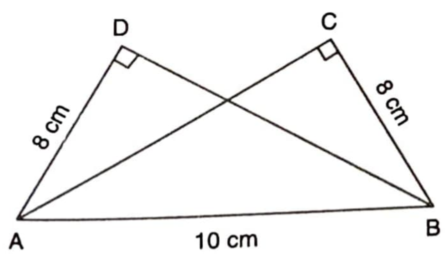
$\angle ADB = \angle BCA = 90°$

$AB = AB$ [Hypotenuse]

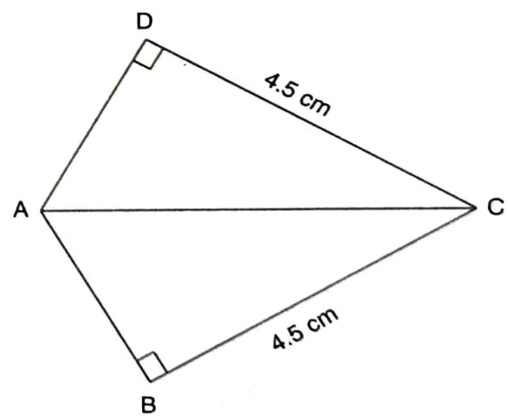
and, $AD = BC = 8$ cm

So, by RHS congruence condition, we have

$$\Delta ADB \cong \Delta BCA$$



(i)



(ii)

Fig. 41

(ii) In $\Delta\, s$ ABC and ADC, we have

$\angle ABC = \angle ADC = 90°$

$AC = AC$ [Hypotenuse]

and, $BC = DC = 4.5$ cm

So, by RHS congruence condition, we have

$$\Delta ABC \cong \Delta ADC$$

Example 2 In Fig. 42, $PL \perp OB$ and $PM \perp OA$ such that $PL = PM$. Is $\Delta\, PLO \cong \Delta\, PMO$? Give reasons in support of your answer.

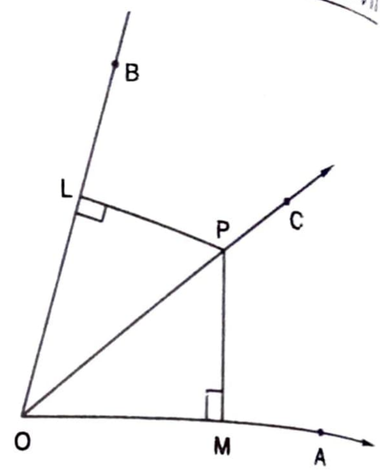


Fig. 42

Solution In Δs PLO and PMO, we have

$\angle PLO = \angle PMO = 90°$

$OP = OP$ [Hypotenuse]

and, $PL = PM$ [Given]

So, by RHS condition of congruence, we have

$$\Delta\, PLO \cong \Delta\, PMO.$$

Example 3 If $\Delta\, ABC$ is an isosceles triangle such that $AB = AC$, then altitude AD from A on BC bisects BC (Fig. 43).

Solution In right triangles ADB and ADC, we have

Hyp. AB = Hyp. AC [Given]

$AD = AD$ [Common side]

So, by RHS criterion of congruence,

$\Delta\, ABD \cong \Delta\, ACD$

$\Rightarrow$ $BD = DC$ [$\because$ Corresponding parts of above congruent triangles are equal]

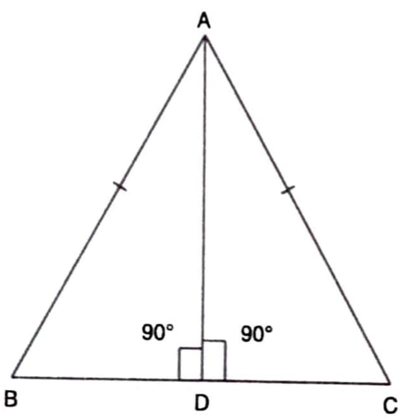


Fig. 43

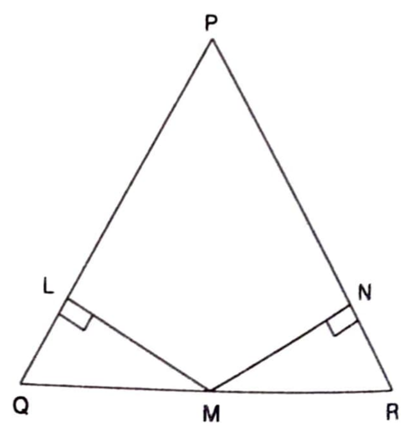


Fig. 44

Example 4 In Fig. 44, it is given that $LM = MN$, $QM = MR$, $ML \perp PQ$ and $MN \perp PR$. Prove that $PQ = PR$.

Solution In right triangles QLM and MNR, we have

Hyp. QM = Hyp. MR [Given]

$LM = MN$ [Given]

So, by RHS criterion of congruence, we have

$\Delta\, QLM \cong \Delta\, MNR$

$\Rightarrow$ $\angle Q = \angle R$ [$\because$ Corresponding parts of congruent triangles are equal]

$\Rightarrow$ $PR = PQ$ [$\because$ Sides opposite to equal angles are equal]

Example 5 AD, BE and CF, the altitudes of ΔABC are equal. Prove that ΔABC is an equilateral triangle.

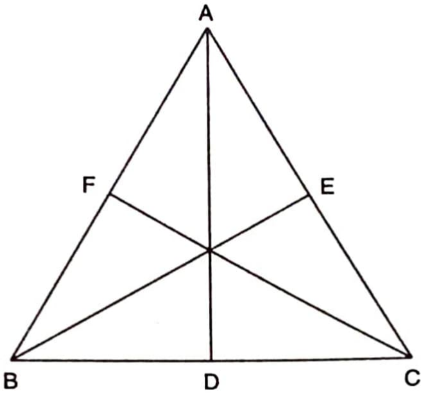


Fig. 45

Solution In right triangles BCE and BFC, we have

Hyp. BC = Hyp. BC

$BE = CF$ [Given]

So, by RHS criterion of congruence, we have

$\Delta BCE \cong \Delta BFC$

$\Rightarrow \angle B = \angle C$ [$\because$ Corresponding parts of congruent triangles are equal]

$\Rightarrow AC = AB$ ...(i) [$\because$ Sides opposite to equal angles are equal]

Similarly, $\Delta ABD \cong \Delta ABE$

$\Rightarrow \angle B = \angle A$ [Corresponding parts of congruent triangles are equal]

$\Rightarrow AC = BC$ ...(ii) [Sides opposite to equal angles are equal]

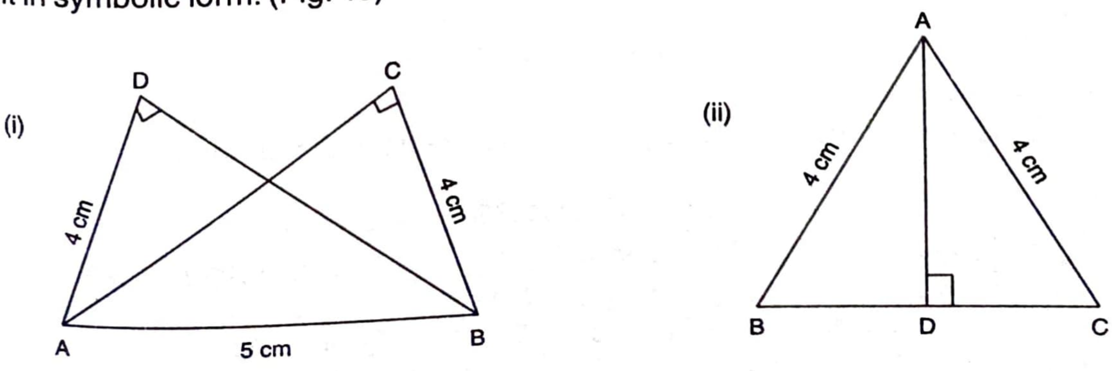
From (i) and (ii), we get

$AB = BC = AC$

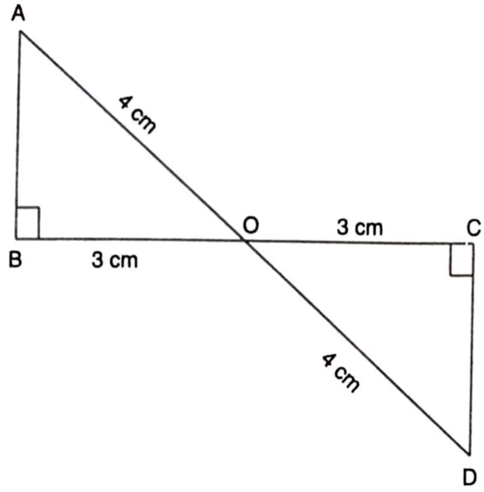
Hence, ΔABC is an equilateral triangle.

EXERCISE 16.5

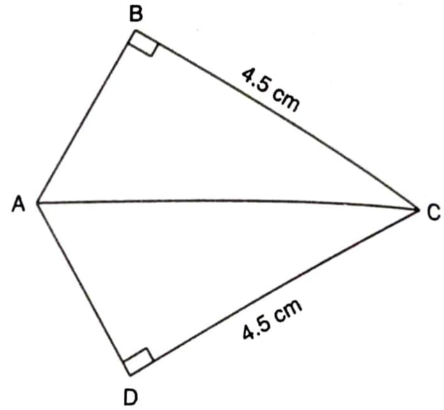
1. In each of the following pairs of right triangles, the measures of some parts are indicated along side. State by the application of RHS congruence condition which are congruent. State each result in symbolic form. (Fig. 46)



(iii)



(iv)



(v)

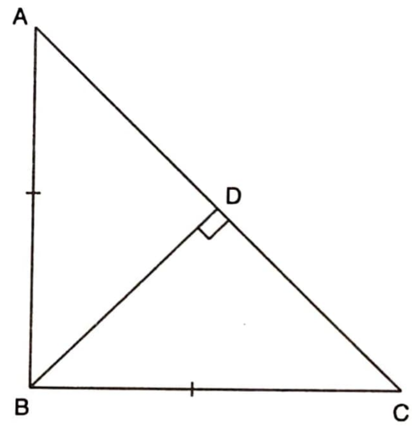


Fig. 46

2. ΔABC is isosceles with $AB = AC$. AD is the altitude from A on BC.
   (i) Is $\Delta ABD \cong \Delta ACD$?
   (ii) State the pairs of matching parts you have used to answer (i).
   (iii) Is it true to say that $BD = DC$?
3. ΔABC is isosceles with $AB = AC$. Also, $AD \perp BC$ meeting BC in D. Are the two triangles ABD and ACD congruent? State in symbolic form. Which congruence condition do you use? Which side of ΔADC equals BD? Which angle of ΔADC equals $\angle B$?
4. Draw a right triangle ABC. Use RHS condition to construct another triangle congruent to it.
5. In Fig. 47, BD and CE are altitudes of ΔABC and $BD = CE$.
   (i) Is $\Delta BCD \cong \Delta CBE$?
   (ii) State the three pairs of matching parts you have used to answer (i).

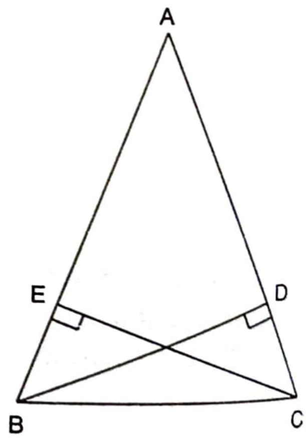


Fig. 47

ANSWERS

1. (i) $\Delta ADB \cong \Delta BCA$ (ii) $\Delta ADB \cong \Delta ADC$ (iii) $\Delta AOB \cong \Delta DOC$
   (iv) $\Delta ABC \cong \Delta ADC$ (v) $\Delta ABD \cong \Delta CBD$
2. (i) Yes (ii) AB, AC; AD, AD; $\angle ADB, \angle ADC$ (iii) Yes
3. Yes, $\Delta ABD \cong \Delta ACD$, RHS, CD, $\angle C$
5. (i) Yes (ii) BD, CE; CB; BC, $\angle BDC$, $\angle CEB$

THINGS TO REMEMBER

1. *Two figures are congruent, if they have exactly the same shape and size.*
2. *(i) Two line segments are congruent, if they have the same length.*
   (ii) Two angles are congruent, if they have the same measure.
   (iii) Two squares are congruent, if they have the same side length.
   (iv) Two rectangles are congruent, if they have the same length and breadth.
   (v) Two circles are congruent, if they have the same radius.
3. *Two triangles are congruent, if in matching of their vertices, the three sides and the three angles of one triangle are respectively equal to the corresponding parts of the other.*
4. SSS Congruence Condition: *Two triangles are congruent, if three sides of one triangle are respectively equal to the three sides of the other.*
5. SAS Congruence Condtion: *Two triangles are congruent, if two sides and the included angle of one are respectively equal to the two sides and the included angle of the other.*
6. ASA Congruence Condition: *Two triangles are congruent, if two angles and the included side of the one are respectively equal to the two angles and the included side of the other.*
7. RHS Congruence Condition: *Two right triangles are congruent, if the hypotenuse and one side of the one triangle are respectively equal to the hypotenuse and one side of the other.*
8. *In an isosceles triangle, the angles opposite to equal sides are equal.*
9. *The bisector of the vertical angle of an isosceles triangle bisects the base at right angles.*
10. *Two congruent figures are equal in area but two figures having the same area need not be congruent.*