CHAPTER 10

TRIGONOMETRIC EQUATIONS

10.1 SOME DEFINITIONS

TRIGONOMETRIC EQUATIONS *The equations containing trigonometric functions of unknown angles are known as trigonometric equations.*

$\cos x = \frac{1}{2}$, $\sin x = 0$, $\tan x = \sqrt{3}$ etc. are trigonometric equations.

SOLUTION OF A TRIGONOMETRIC EQUATION *A solution of a trigonometric equation is the value of the unknown angle that satisfies the equation.*

Consider the equation $\sin x = \frac{1}{2}$. This equation is clearly satisfied by $x = \frac{\pi}{6}, \frac{5\pi}{6}$ etc. So, these are its solutions.

Solving an equation means to find the set of all values of the unknown angle which satisfy the given equation.

Consider the equation $2\cos x + 1 = 0$ or $\cos x = -1/2$. This equation is clearly satisfied by $x = \frac{2\pi}{3}, \frac{4\pi}{3}$ etc.

Since the trigonometric functions are periodic. Therefore, if a trigonometric equation has a solution, it will have infinitely many solutions. For example, $x = \frac{2\pi}{3}$, $2\pi \pm \frac{2\pi}{3}$, $4\pi \pm \frac{2\pi}{3}$, ... are solutions of $2\cos x + 1 = 0$. These solutions can be put together in compact form as $2n\pi \pm \frac{2\pi}{3}$, where n is an integer. This solution is known as the general solution.

Thus, a solution generalised by means of periodicity is known as the general solution.

It also follows from the above discussion that solving an equation means to find its general solution.

10.2 GENERAL SOLUTIONS OF TRIGONOMETRIC EQUATIONS

In this section, we shall obtain the general solutions of the trigonometric equations $\sin x = 0$, $\cos x = 0$, $\tan x = 0$ and $\cot x = 0$.

THEOREM 1 *Prove that the general solution of* $\sin x = 0$ *is given by* $x = n\pi, n \in Z$.

PROOF In ΔOMP, we obtain

$$\sin x = \frac{PM}{OP}$$

$\therefore \quad \sin x = 0$

$\Rightarrow \quad \frac{PM}{OP} = 0$

$\Rightarrow \quad PM = 0$

$\Rightarrow \quad OP$ coincides with OX or, OX'

$\Rightarrow \quad x = 0, \pi, 2\pi, \ldots, -\pi, -2\pi, -3\pi, \ldots$

$\Rightarrow \quad x = n\pi, n \in Z.$

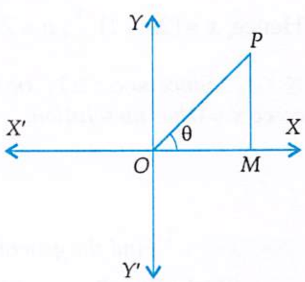

Fig. 10.1

Hence, $x = n\pi, n \in Z$ is the general solution of $\sin x = 0$. **Q.E.D.**

ALITER In chapter 6, we have learnt that the curve $y = \sin x$ cuts x-axis at points $(0, 0)$, $(\pm \pi, 0)$, $(\pm 2\pi, 0)$ etc. Thus, $\sin x = 0$ at $x = n\pi$, $n \in Z$.

THEOREM 2 *Prove that the general solution of* $\tan x = 0$ *is* $x = n\pi$, $n \in Z$.

PROOF By definition,

$$\tan x = \frac{PM}{OM} \qquad \text{[See Fig. 10.1]}$$

$$\therefore \quad \tan x = 0$$

$$\Rightarrow \quad \frac{PM}{OM} = 0$$

$$\Rightarrow \quad PM = 0$$

$\Rightarrow$ OP coincides with OX or, $OX' \Rightarrow x = 0, \pi, 2\pi, \ldots, -\pi, -2\pi, \ldots \Rightarrow x = n\pi, n \in Z$.

Hence, $x = n\pi$, $n \in Z$ is general solution of $\tan x = 0$. **Q.E.D.**

THEOREM 3 *Prove that the general solution of* $\cos x = 0$ *is* $x = (2n+1)\frac{\pi}{2}$, $n \in Z$.

PROOF By definition

$$\cos x = \frac{OM}{OP} \qquad \text{[See Fig. 10.1]}$$

$$\therefore \quad \cos x = 0$$

$$\Rightarrow \quad \frac{OM}{OP} = 0$$

$$\Rightarrow \quad OM = 0$$

$\Rightarrow$ OP coincides with OY or, $OY' \Rightarrow x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots \Rightarrow x = (2n+1)\frac{\pi}{2}, n \in Z$.

Hence, the general solution of $\cos x = 0$ is $x = (2n+1)\frac{\pi}{2}$, $n \in Z$. **Q.E.D.**

THEOREM 4 *Prove that the general solution of* $\cot x = 0$ *is* $x = (2n+1)\frac{\pi}{2}, n \in Z$.

PROOF By definition,

$$\cot x = \frac{OM}{PM} \qquad \text{[See Fig. 10.1]}$$

$$\therefore \quad \cot x = 0$$

$$\Rightarrow \quad \frac{OM}{PM} = 0$$

$$\Rightarrow \quad OM = 0$$

$\Rightarrow$ OP coincides with OY or, $OY' \Rightarrow x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots \Rightarrow x = (2n+1)\frac{\pi}{2}, n \in Z$.

Hence, $x = (2n+1)\frac{\pi}{2}, n \in Z$ is the general solution of $\cot x = 0$. **Q.E.D.**

NOTE *Since* $\sec x \geq 1$, or $\sec x \leq -1$, *therefore* $\sec x = 0$ *does not have any solution. Similarly,* $\operatorname{cosec} x = 0$ *has no solution.*

ILLUSTRATIVE EXAMPLES

BASIC

EXAMPLE 1 *Find the general solutions of the following equations:*

(i) $\sin 2x = 0$ (ii) $\sin \frac{3x}{2} = 0$ (iii) $\sin^2 2x = 0$

SOLUTION (i) We have,

$$\sin 2x = 0$$

$\Rightarrow \quad 2x = n\pi, \text{ where } n \in Z \qquad [\because \sin x = 0 \Rightarrow x = n\pi]$

$\Rightarrow \quad x = \dfrac{n\pi}{2}, \ n \in Z.$

(ii) $\sin \dfrac{3x}{2} = 0$

$\Rightarrow \quad \dfrac{3x}{2} = n\pi, \ n \in Z \qquad [\sin x = 0 \Rightarrow x = n\pi]$

$\Rightarrow \quad x = \dfrac{2n\pi}{3}, n \in Z.$

(iii) $\sin^2 2x = 0 \Rightarrow \sin 2x = 0 \Rightarrow 2x = n\pi \Rightarrow x = \dfrac{n\pi}{2}, \ n \in Z.$

EXAMPLE 2 *Find the general solutions of the following equations:*

(i) $\cos 3x = 0$ (ii) $\cos \dfrac{3x}{2} = 0$ (iii) $\cos^2 3x = 0$

SOLUTION We know that the general solution of the equation $\cos x = 0$ is $x = (2n+1)\dfrac{\pi}{2}, n \in Z.$

Therefore,

(i) $\cos 3x = 0 \Rightarrow 3x = (2n+1)\dfrac{\pi}{2} \Rightarrow x = (2n+1)\dfrac{\pi}{6}, \ n \in Z$

(ii) $\cos \dfrac{3x}{2} = 0 \Rightarrow \dfrac{3x}{2} = (2n+1)\dfrac{\pi}{2} \Rightarrow x = (2n+1)\dfrac{\pi}{3}, \ n \in Z$

(iii) $\cos^2 3x = 0 \Rightarrow \cos 3x = 0 \Rightarrow 3x = (2n+1)\dfrac{\pi}{2} \Rightarrow x = (2n+1)\dfrac{\pi}{6}, \ n \in Z.$

EXAMPLE 3 *Find the general solutions of the following equations:*

(i) $\tan 2x = 0$ (ii) $\tan \dfrac{x}{2} = 0$ (iii) $\tan \dfrac{3x}{4} = 0$

SOLUTION We know that the general solution of the equation $\tan x = 0$ is $x = n\pi, n \in Z.$
Therefore,

(i) $\tan 2x = 0 \Rightarrow 2x = n\pi \Rightarrow x = \dfrac{n\pi}{2}, \ n \in Z$

(ii) $\tan \dfrac{x}{2} = 0 \Rightarrow \dfrac{x}{2} = n\pi \Rightarrow x = 2n\pi, \ n \in Z$

(iii) $\tan \dfrac{3x}{4} = 0 \Rightarrow \dfrac{3x}{4} = n\pi \Rightarrow x = \dfrac{4n\pi}{3}, n \in Z$

THEOREM 5 *Prove that the general solution of* $\sin x = \sin \alpha$ *is given by:* $x = n\pi + (-1)^n \alpha, n \in Z.$

PROOF We have,

$\sin x = \sin \alpha$

$\Leftrightarrow \quad \sin x - \sin \alpha = 0$

$\Leftrightarrow \quad 2 \sin\left(\dfrac{x-\alpha}{2}\right) \cos\left(\dfrac{x+\alpha}{2}\right) = 0$

$\Leftrightarrow \quad \sin\left(\dfrac{x-\alpha}{2}\right) = 0 \ \text{ or, } \ \cos\left(\dfrac{x+\alpha}{2}\right) = 0$

$\Leftrightarrow \quad \dfrac{x-\alpha}{2} = m\pi, \ \text{or, } \ \dfrac{x+\alpha}{2} = (2m+1)\dfrac{\pi}{2}, m \in Z$

$\Leftrightarrow \quad x = 2m\pi + \alpha \ \text{ or, } \ x = (2m+1)\pi - \alpha, m \in Z$

$\Leftrightarrow \quad x = (\text{Any even multiple of } \pi) + \alpha \ \text{ or, } \ x = (\text{Any odd multiple of } \pi) - \alpha$

$\Leftrightarrow \quad x = n\pi + (-1)^n \alpha, \ \text{ where } n \in Z.$

Q.E.D.

REMARK 1 *The equation* cosec x = cosec α *is equivalent to* $\sin x = \sin \alpha$. *Thus,* cosec x = cosec α *and* $\sin x = \sin \alpha$ *have the same general solution.*

THEOREM 6 *Prove that the general solution of* $\cos x = \cos \alpha$ *is given by:* $x = 2n\pi \pm \alpha$, where $n \in Z$.

PROOF We have,

$$\cos x = \cos \alpha$$

$$\Leftrightarrow \quad \cos x - \cos \alpha = 0$$

$$\Leftrightarrow \quad -2 \sin\left(\frac{x+\alpha}{2}\right) \sin\left(\frac{x-\alpha}{2}\right) = 0$$

$$\Leftrightarrow \quad \sin\left(\frac{x+\alpha}{2}\right) = 0 \text{ or, } \sin\left(\frac{x-\alpha}{2}\right) = 0$$

$$\Leftrightarrow \quad \frac{x+\alpha}{2} = n\pi, \text{ or } \frac{x-\alpha}{2} = n\pi, \ n \in Z$$

$$\Leftrightarrow \quad x = 2n\pi - \alpha \text{ or, } x = 2n\pi + \alpha, n \in Z \Leftrightarrow x = 2n\pi \pm \alpha, n \in Z.$$

Q.E.D.

REMARK 2 *Since* $\sec x = \sec \alpha \Leftrightarrow \cos x = \cos \alpha$. *So, the general solutions of* $\cos x = \cos \alpha$ *and* $\sec x = \sec \alpha$ *are same.*

THEOREM 7 *Prove that the general solution of* $\tan x = \tan \alpha$ *is given by:* $x = n\pi + \alpha, n \in Z$.

PROOF We have,

$$\tan x = \tan \alpha \Leftrightarrow \frac{\sin x}{\cos x} = \frac{\sin \alpha}{\cos \alpha} \Leftrightarrow \sin x \cos \alpha - \cos x \sin \alpha = 0$$

$$\Leftrightarrow \quad \sin(x - \alpha) = 0 \Leftrightarrow x - \alpha = n\pi, n \in Z \Leftrightarrow x = n\pi + \alpha, n \in Z$$

Q.E.D.

REMARK 3 *Since* $\tan x = \tan \alpha \Leftrightarrow \cot x = \cot \alpha$. *So, general solutions of* $\cot x = \cot \alpha$ *and* $\tan x = \tan \alpha$ *are same.*

In order to find the general solutions of trigonometrical equations of the form $\sin x = \sin \alpha$, $\cos x = \cos \alpha$ and $\tan x = \tan \alpha$, we may use the following algorithm.

ALGORITHM

Step II *Find a value of x, preferably between 0 and* 2π or between $-\pi$ and π, *satisfying the given equation and call it* α.

Step II *If the equation is* $\sin x = \sin \alpha$, write $x = n\pi + (-1)^n \alpha, n \in Z$ *as the general solution.*

For the equation $\cos x = \cos \alpha$, write $x = 2n\pi \pm \alpha, n \in Z$ *as the general solution.*

For the equation $\tan x = \tan \alpha$, write $x = n\pi + \alpha, n \in Z$ *as the general solution.*

Following examples illustrate the algorithm.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I **ON FINDING THE GENERAL SOLUTIONS OF THE EQUATIONS OF THE FORM** $\sin x = \sin \alpha$, $\cos x = \cos \alpha$, $\tan x = \tan \alpha$

EXAMPLE 1 *Find the general solutions of the following equations:*

(i) $\sin x = \frac{\sqrt{3}}{2}$ (ii) $2 \sin x + 1 = 0$ (iii) cosec $x = 2$

SOLUTION (i) A value of x satisfying $\sin x = \frac{\sqrt{3}}{2}$ is $\frac{\pi}{3}$.

$$\therefore \quad \sin x = \frac{\sqrt{3}}{2} \Rightarrow \sin x = \sin\frac{\pi}{3} \Rightarrow x = n\pi + (-1)^n \frac{\pi}{3}, n \in Z$$

(ii) We have, $2 \sin x + 1 = 0 \Rightarrow \sin x = -\frac{1}{2}$. A value of x satisfying this equation is $-\pi/6$.

$\therefore \quad \sin x = -\frac{1}{2} \Rightarrow \sin x = \sin\left(-\frac{\pi}{6}\right) \Rightarrow x = n\pi + (-1)^n\left(-\frac{\pi}{6}\right) \Rightarrow x = n\pi + (-1)^{n+1}\frac{\pi}{6}, n \in Z.$

(iii) We have, $\operatorname{cosec} x = 2 \Rightarrow \sin x = \frac{1}{2} \Rightarrow \sin x = \sin\frac{\pi}{6} \Rightarrow x = n\pi + (-1)^n\frac{\pi}{6}, n \in Z.$

EXAMPLE 2 *Find the general solutions of the following equations:*

(i) $\cos x = \frac{1}{2}$ (ii) $\cos 3x = -\frac{1}{2}$ (iii) $\sqrt{3} \sec 2x = 2$

(iv) $\sec x \cos 5x + 1 = 0, \ 0 < x \leq \frac{\pi}{2}$ [NCERT EXEMPLAR]

SOLUTION (i) $\cos x = \frac{1}{2} \Rightarrow \cos x = \cos\frac{\pi}{3} \Rightarrow x = 2n\pi \pm \frac{\pi}{3}, \ n \in Z$

(ii) $\cos 3x = -\frac{1}{2} \Rightarrow \cos 3x = \cos\frac{2\pi}{3} \Rightarrow 3x = 2n\pi \pm \frac{2\pi}{3} \Rightarrow x = \frac{2n\pi}{3} \pm \frac{2\pi}{9}, n \in Z$

(iii) $\sqrt{3} \sec 2x = 2 \Rightarrow \cos 2x = \frac{\sqrt{3}}{2} \Rightarrow \cos 2x = \cos\frac{\pi}{6} \Rightarrow 2x = 2n\pi \pm \frac{\pi}{6} \Rightarrow x = n\pi \pm \frac{\pi}{12}, \ n \in Z$

(iv) We have, $\sec x \cos 5x + 1 = 0, \ 0 < x \leq \frac{\pi}{2}$

$\Rightarrow \quad \frac{\cos 5x}{\cos x} + 1 = 0$

$\Rightarrow \quad \cos 5x + \cos x = 0$

$\Rightarrow \quad 2 \cos\left(\frac{5x + x}{2}\right) \cos\left(\frac{5x - x}{2}\right) = 0$

$\Rightarrow \quad 2 \cos 3x \cos 2x = 0$

$\Rightarrow \quad \cos 3x = 0$ or, $\cos 2x = 0$

$\Rightarrow \quad 3x = \frac{\pi}{2}, \frac{3\pi}{2}$ or, $2x = \frac{\pi}{2}$ $\quad \left[\because 0 < x \leq \frac{\pi}{2} \therefore 0 < 3x \leq \frac{3\pi}{2} \text{ and } 0 < 2x \leq \pi\right]$

$\Rightarrow \quad x = \frac{\pi}{6}, \frac{\pi}{2}$ or, $x = \frac{\pi}{4} \Rightarrow x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{\pi}{4}$

EXAMPLE 3 *Solve the following trigonometric equations:*

(i) $\tan x = \frac{1}{\sqrt{3}}$ (ii) $\tan 2x = \sqrt{3}$ (iii) $\tan 3x = -1$

(iv) $2 \tan^2 x \sec^2 x + 1 = 2, \ 0 \leq x \leq 2\pi$ [NCERT EXEMPLAR]

SOLUTION (i) $\tan x = \frac{1}{\sqrt{3}} \Rightarrow \tan x = \tan\frac{\pi}{6} \Rightarrow x = n\pi + \frac{\pi}{6}, n \in Z$

(ii) $\tan 2x = \sqrt{3} \Rightarrow \tan 2x = \tan\frac{\pi}{3} \Rightarrow 2x = n\pi + \frac{\pi}{3} \Rightarrow x = \frac{n\pi}{2} + \frac{\pi}{6}, n \in Z$

(iii) $\tan 3x = -1 \Rightarrow \tan 3x = \tan\left(-\frac{\pi}{4}\right) \Rightarrow 3x = n\pi + \left(-\frac{\pi}{4}\right) \Rightarrow x = \frac{n\pi}{3} - \frac{\pi}{12}, n \in Z.$

(iv) We have, $\cot x + \tan x = 2 \operatorname{cosec} x$

$\Rightarrow \quad \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} = \frac{2}{\sin x}$ and $x \neq n\pi, n \in Z$

$$\Rightarrow \quad \frac{\cos^2 x + \sin^2 x}{\sin x \cos x} = \frac{2}{\sin x}$$

$$\Rightarrow \quad \frac{1}{\sin x \cos x} = \frac{2}{\sin x} \Rightarrow \frac{1}{\cos x} = 2 \qquad [\because x \neq n\pi \therefore \sin x \neq 0]$$

$$\Rightarrow \quad \cos x = \frac{1}{2} \Rightarrow \cos x = \cos\frac{\pi}{3} \Rightarrow x = 2n\pi \pm \frac{\pi}{3}, \quad n \in Z.$$

EXAMPLE 4 *Solve the following trigonometric equations:*

(i) $\sin\frac{x}{2} = -1$ (ii) $\cos\frac{3x}{2} = \frac{1}{2}$ (iii) $\tan\left(\frac{2}{3}x\right) = \sqrt{3}$

(iv) $\cot x + \tan x = 2 \operatorname{cosec} x$ **[NCERT EXEMPLAR]**

SOLUTION (i) $\sin\frac{x}{2} = -1$

$$\Rightarrow \quad \sin\frac{x}{2} = \sin\left(-\frac{\pi}{2}\right) \Rightarrow \frac{x}{2} = n\pi + (-1)^n\left(-\frac{\pi}{2}\right) \Rightarrow x = 2n\pi + (-1)^{n+1}\pi, n \in Z$$

(ii) $$\cos\frac{3x}{2} = \frac{1}{2} \Rightarrow \cos\frac{3x}{2} = \cos\frac{\pi}{3} \Rightarrow \frac{3x}{2} = 2n\pi \pm \frac{\pi}{3} \Rightarrow x = \frac{4n\pi}{3} \pm \frac{2\pi}{9}, n \in Z$$

(iii) $$\tan\left(\frac{2x}{3}\right) = \sqrt{3} \Rightarrow \tan\left(\frac{2x}{3}\right) = \tan\frac{\pi}{3} \Rightarrow \frac{2x}{3} = n\pi + \frac{\pi}{3} \Rightarrow x = \frac{3n\pi}{2} + \frac{\pi}{2}, n \in Z.$$

(iv) We have, $2\tan^2 x + \sec^2 x = 2, \ 0 \leq x \leq 2\pi$

$$\Rightarrow \quad 2\tan^2 x + 1 + \tan^2 x = 2 \Rightarrow 3\tan^2 x = 1 \Rightarrow \tan^2 x = \frac{1}{3} \Rightarrow \tan x = \pm\frac{1}{\sqrt{3}}$$

Case I *When* $\tan x = \frac{1}{\sqrt{3}}$: In this case, we obtain $x = \frac{\pi}{6}, \frac{7\pi}{6}$

Case II *When* $\tan x = -\frac{1}{\sqrt{3}}$: In this case, we obtain $x = \frac{5\pi}{6}, \frac{11\pi}{6}$

Hence, the possible solutions of the given equation are: $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

Type II ON FINDING THE GENERAL SOLUTION OF THE EQUATIONS REDUCIBLE TO THE FORMS $\sin x = \sin\alpha$, $\cos x = \cos\alpha$, $\tan x = \tan\alpha$

EXAMPLE 5 *Solve the equation:* $\sin x + \sin 3x + \sin 5x = 0$. **[NCERT EXEMPLAR]**

SOLUTION We have,

$$\sin x + \sin 3x + \sin 5x = 0 \Rightarrow (\sin 5x + \sin x) + \sin 3x = 0$$

$$\Rightarrow \quad 2\sin 3x \cos 2x + \sin 3x = 0$$

$$\Rightarrow \quad \sin 3x(2\cos 2x + 1) = 0 \Rightarrow \sin 3x = 0 \text{ or } 2\cos 2x + 1 = 0 \Rightarrow \sin 3x = 0 \text{ or, } \cos 2x = -\frac{1}{2}$$

Now, $\sin 3x = 0 \Rightarrow 3x = n\pi \Rightarrow x = \frac{n\pi}{3}, n \in Z$

And, $\cos 2x = -\frac{1}{2} \Rightarrow \cos 2x = \cos\frac{2\pi}{3} \Rightarrow 2x = 2m\pi \pm \frac{2\pi}{3} \Rightarrow x = m\pi \pm \frac{\pi}{3}, m \in Z.$

$$\Rightarrow \quad x = (3m \pm 1)\frac{\pi}{3}, \quad m \in Z.$$

These values of x are contained in $x=\frac{n\pi}{3}$, $n\in Z$. Hence, the general solution of the given equation is : $x = \frac{n\pi}{3}, n\in Z$.

EXAMPLE 6 *Solve the equation:* $\cos x+\cos 3x-2\cos 2x = 0$

SOLUTION We have,

$$\cos x+\cos 3x-2\cos 2x=0$$

$$\Leftrightarrow \quad 2\cos 2x\cos x-2\cos 2x=0 \Leftrightarrow 2\cos 2x(\cos x-1)=0 \Rightarrow \cos 2x=0 \text{ or, } \cos x-1=0$$

Now, $\cos 2x=0 \Rightarrow 2x=(2n+1)\frac{\pi}{2} \Rightarrow x=(2n+1)\frac{\pi}{4}, n\in Z$

And, $\cos x-1=0 \Rightarrow \cos x = 1 \Rightarrow \cos x=\cos 0 \Rightarrow x=2m\pi\pm 0 \Rightarrow x=2m\pi, m\in Z$

Hence, $x = (2n+1)\frac{\pi}{4}$ or, $x=2m\pi$, where $m, n\in Z$.

EXAMPLE 7 *Solve the equation:* $\sin mx+\sin nx = 0$.

SOLUTION We have,

$$\sin mx+\sin nx = 0$$

$$\Rightarrow \quad 2\sin\left(\frac{m+n}{2}\right)x\cos\left(\frac{m-n}{2}\right)x = 0 \Rightarrow \sin\left(\frac{m+n}{2}\right)x = 0 \text{ or, } \cos\left(\frac{m-n}{2}\right)x = 0$$

Now, $\sin\left(\frac{m+n}{2}\right)x = 0 \Rightarrow \left(\frac{m+n}{2}\right)x = r\pi \Rightarrow x = \frac{2r\pi}{m+n}, \ r\in Z$

And, $\cos\left(\frac{m-n}{2}\right)x=0 \Rightarrow \left(\frac{m-n}{2}\right)x=(2s+1)\frac{\pi}{2} \Rightarrow x = \frac{(2s+1)\pi}{m-n}, s\in Z$

Hence, $x = \frac{2r\pi}{m+n}$ or, $x = \frac{(2s+1)\pi}{m-n}$, where $r, s\in Z$.

EXAMPLE 8 *Solve the following equations:*

(i) $\sin 2x+\cos x=0$ [NCERT] (ii) $\sin 3x+\cos 2x=0$ (iii) $\sin 2x+\sin 4x+\sin 6x=0$

SOLUTION (i) $\sin 2x+\cos x=0$

$$\Rightarrow \quad \cos x = -\sin 2x \Rightarrow \cos x = \cos\left(\frac{\pi}{2}+2x\right) \Rightarrow x = 2n\pi\pm\left(\frac{\pi}{2}+2x\right), n\in Z$$

Taking positive sign, we obtain

$$x=2n\pi+\frac{\pi}{2}+2x \Rightarrow -x=2n\pi+\frac{\pi}{2} \Rightarrow x=-2n\pi-\frac{\pi}{2} \Rightarrow x=2m\pi-\frac{\pi}{2}, \text{ where } m=-n\in Z.$$

Taking negative sign, we obtain

$$x = 2n\pi-\left(\frac{\pi}{2}+2x\right) \Rightarrow 3x = 2n\pi-\frac{\pi}{2} \Rightarrow x = \frac{2n\pi}{3}-\frac{\pi}{6}, n\in Z.$$

Hence, $x = 2m\pi-\frac{\pi}{2}$, or, $x=\frac{2n\pi}{3}-\frac{\pi}{6}$, where $m, n\in Z$.

(ii) $\sin 3x+\cos 2x = 0$

$$\Rightarrow \cos 2x = -\sin 3x \Rightarrow \cos 2x = \cos\left(\frac{\pi}{2}+3x\right) \Rightarrow 2x = 2n\pi\pm\left(\frac{\pi}{2}+3x\right), n\in Z$$

Taking positive sign, we obtain

$$2x = 2n\pi+\frac{\pi}{2}+3x \Rightarrow -x = 2n\pi+\frac{\pi}{2} \Rightarrow x = -2n\pi-\frac{\pi}{2} \Rightarrow x = 2m\pi-\frac{\pi}{2}, \text{ where } -n=m.$$

Taking negative sign, we obtain

$$2x = 2n\pi - \frac{\pi}{2} - 3x \Rightarrow 5x = 2n\pi - \frac{\pi}{2} \Rightarrow x = \frac{2n\pi}{5} - \frac{\pi}{10},\ n \in Z$$

Hence, $x = \frac{2n\pi}{5} - \frac{\pi}{10}$ or, $x = 2m\pi - \frac{\pi}{2}$, where $m, n \in Z$.

(iii) We have,

$$\sin 2x + \sin 4x + \sin 6x = 0$$

$\Rightarrow \quad \sin 4x + (\sin 2x + \sin 6x) = 0$

$\Rightarrow \quad \sin 4x + 2\sin 4x \cos 2x = 0$

$\Rightarrow \quad \sin 4x(1 + 2\cos 2x) = 0$

$\Rightarrow \quad \sin 4x = 0$ or, $1 + 2\cos 2x = 0 \Rightarrow \sin 4x = 0$ or, $\cos 2x = -\frac{1}{2}$

Now, $\quad \sin 4x = 0 \Rightarrow 4x = n\pi \Rightarrow x = \frac{n\pi}{4},\ n \in Z$

And, $\quad \cos 2x = -\frac{1}{2}$

$\Rightarrow \quad \cos 2x = \cos\frac{2\pi}{3} \Rightarrow 2x = 2m\pi \pm \frac{2\pi}{3} \Rightarrow x = m\pi \pm \frac{\pi}{3},\ m \in Z$

Hence, $x = \frac{n\pi}{4}$ or, $x = m\pi \pm \frac{\pi}{3}$, where $m, n \in Z$.

EXAMPLE 9 *Find the sum of all solutions of* $\cos x \cos\left(x + \frac{\pi}{3}\right)\cos\left(\frac{\pi}{3} - x\right) = \frac{1}{4},\ x \in [0, 6\pi]$.

SOLUTION We have,

$$\cos x \cos\left(\frac{\pi}{3} + x\right)\cos\left(\frac{\pi}{3} - x\right) = \frac{1}{4}$$

$\Rightarrow \quad \frac{1}{4}\cos 3x = \frac{1}{4} \qquad \left[\because \cos x \cos\left(\frac{\pi}{3} + x\right)\cos\left(\frac{\pi}{3} - x\right) = \frac{1}{4}\cos 3x\right]$

$\Rightarrow \quad \cos 3x = 1 \Rightarrow \cos 3x = \cos 0 \Rightarrow 3x = 2n\pi \Rightarrow x = \frac{2n\pi}{3},\ n \in Z$

But, $x \in [0, 6\pi]$. Therefore, $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{6\pi}{3}, \frac{8\pi}{3}, \dots, \frac{18\pi}{3}$.

Sum of all these solutions $= 0 + \frac{2\pi}{3} + \frac{4\pi}{3} + \dots + \frac{18\pi}{3} = \frac{2\pi}{3}(1 + 2 + 3 + \dots + 9) = \frac{2\pi}{3} \times \frac{9 \times 10}{2} = 30\pi$

Hence, required sum $= 30\pi$.

EXAMPLE 10 *Solve :* $\sin x - 2\sin 2x + \sin 3x = \cos x - 2\cos 2x + \cos 3x$

SOLUTION We have,

$$\sin x - 2\sin 2x + \sin 3x = \cos x - 2\cos 2x + \cos 3x$$

$\Rightarrow \quad (\sin 3x + \sin x) - 2\sin 2x = (\cos 3x + \cos x) - 2\cos 2x$

$\Rightarrow \quad 2\sin 2x \cos x - 2\sin 2x = 2\cos 2x \cos x - 2\cos 2x$

$\Rightarrow \quad 2\sin 2x(\cos x - 1) = 2\cos 2x(\cos x - 1)$

$\Rightarrow \quad 2(\cos x - 1)(\sin 2x - \cos 2x) = 0$

$\Rightarrow \quad 2(\cos x - 1) = 0$ or, $\sin 2x - \cos 2x = 0$

$\Rightarrow \quad \cos x = 1$ or, $\sin 2x = \cos 2x$

$\Rightarrow \quad \cos x = \cos 0$ or, $\tan 2x = 1 \Rightarrow \cos x = \cos 0$ or, $\tan 2x = \tan\frac{\pi}{4}$

$\Rightarrow \quad x = 2n\pi$ or, $2x = n\pi + \dfrac{\pi}{4} \Rightarrow x = 2n\pi$ or, $x = n\pi + \dfrac{\pi}{8}, n \in Z$

EXAMPLE 11 *Find the general solution of the equation*

$\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$ **[NCERT EXEMPLAR]**

SOLUTION We have,

$$\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$$

$$\Rightarrow \quad (\sin x + \sin 3x) - 3\sin 2x = (\cos x + \cos 3x) - 3\cos 2x$$

$$\Rightarrow \quad 2\sin 2x\cos x - 3\sin 2x = 2\cos 2x\cos x - 3\cos 2x$$

$$\Rightarrow \quad \sin 2x(2\cos x - 3) = \cos 2x(2\cos x - 3)$$

$$\Rightarrow \quad \sin 2x = \cos 2x \qquad [\because 2\cos x - 3 \neq 0]$$

$$\Rightarrow \quad \tan 2x = 1 \Rightarrow \tan 2x = \tan\frac{\pi}{4} \Rightarrow 2x = n\pi + \frac{\pi}{4} \Rightarrow x = \frac{n\pi}{2} + \frac{\pi}{8},\ n \in Z.$$

EXAMPLE 12 *Solve the following equations:*

(i) $2\cos^2 x + 3\sin x = 0$ **[NCERT]** (ii) $2\sin^2 x = 3\cos x, 0 \leq x \leq 2\pi$ **[NCERT EXEMPLAR]**

(iii) $\cot^2 x + \dfrac{3}{\sin x} + 3 = 0$ (iv) $2\tan x - \cot x = -1$

(v) $4\cos x - 3\sec x = \tan x$ (vi) $\tan^2 x + (1 - \sqrt{3})\tan x - \sqrt{3} = 0$

(vii) $\sec^2 2x = 1 - \tan 2x$ **[NCERT]**

SOLUTION (i) $2\cos^2 x + 3\sin x = 0$

$$\Rightarrow \quad 2(1 - \sin^2 x) + 3\sin x = 0$$

$$\Rightarrow \quad 2\sin^2 x - 3\sin\theta - 2 = 0$$

$$\Rightarrow \quad 2\sin^2 x - 4\sin x + \sin x - 2 = 0$$

$$\Rightarrow \quad 2\sin x(\sin x - 2) + 1(\sin x - 2) = 0$$

$$\Rightarrow \quad (\sin x - 2)(2\sin x + 1) = 0$$

$$\Rightarrow \quad 2\sin x + 1 = 0 \qquad [\because \sin x \neq 2 \quad \therefore \sin x - 2 \neq 0]$$

$$\Rightarrow \quad \sin x = -\frac{1}{2}$$

$$\Rightarrow \quad \sin x = \sin\left(-\frac{\pi}{6}\right) \Rightarrow x = n\pi + (-1)^n\left(-\frac{\pi}{6}\right) \Rightarrow x = n\pi + (-1)^{n+1}\frac{\pi}{6},\ n \in Z.$$

(ii) We have, $2\sin^2 x = 3\cos x,\ 0 \leq x \leq 2\pi$

$$\Rightarrow \quad 2(1 - \cos^2 x) = 3\cos x$$

$$\Rightarrow \quad 2\cos^2 x + 3\cos x - 2 = 0$$

$$\Rightarrow \quad 2\cos^2 x + 4\cos x - \cos x - 2 = 0$$

$$\Rightarrow \quad 2\cos x(\cos x + 2) - (\cos x + 2) = 0$$

$$\Rightarrow \quad (\cos x + 2)(2\cos x - 1) = 0$$

$$\Rightarrow \quad 2\cos x - 1 = 0 \qquad [\because \cos x + 2 \neq 0]$$

$$\Rightarrow \quad \cos x = \frac{1}{2}$$

$\Rightarrow \quad x = \frac{\pi}{3}, \frac{5\pi}{3}$ $\quad [\because\ 0 \le x \le 2\pi]$

(iii) $\cot^2 x + \frac{3}{\sin x} + 3 = 0$

$\Rightarrow \quad \operatorname{cosec}^2 x - 1 + 3 \operatorname{cosec} x + 3 = 0$

$\Rightarrow \quad \operatorname{cosec}^2 x + 3 \operatorname{cosec} x + 2 = 0$

$\Rightarrow \quad (\operatorname{cosec} x + 2)(\operatorname{cosec} x + 1) = 0 \Rightarrow \operatorname{cosec} x + 2 = 0$ or, $\operatorname{cosec} x + 1 = 0$

Now, $\operatorname{cosec} x + 2 = 0$

$\Rightarrow \quad \frac{1}{\sin x} + 2 = 0$

$\Rightarrow \quad \sin x = -\frac{1}{2}$

$\Rightarrow \quad \sin x = \sin\left(-\frac{\pi}{6}\right) \Rightarrow x = n\pi + (-1)^n\left(-\frac{\pi}{6}\right), \Rightarrow x = n\pi + (-1)^{n+1}\frac{\pi}{6}, \ n \in Z$

And, $\operatorname{cosec} x + 1 = 0$

$\Rightarrow \frac{1}{\sin x} + 1 = 0$

$\Rightarrow \sin x = -1$

$\Rightarrow \sin x = \sin\left(-\frac{\pi}{2}\right) \Rightarrow x = m\pi + (-1)^m\left(-\frac{\pi}{2}\right) \Rightarrow x = m\pi + (-1)^{m+1}\frac{\pi}{2}, m \in Z$

Hence, $x = n\pi + (-1)^{n+1}\frac{\pi}{6}$ or, $x = m\pi + (-1)^{m+1}\frac{\pi}{2}, m, n \in Z$

(iv) $2\tan x - \cot x = -1$

$\Rightarrow 2\tan x - \frac{1}{\tan x} = -1$

$\Rightarrow 2\tan^2 x + \tan x - 1 = 0$

$\Rightarrow 2\tan^2 x + 2\tan x - \tan x - 1 = 0$

$\Rightarrow 2\tan x(\tan x + 1) - (\tan x + 1) = 0 \Rightarrow (\tan x + 1)(2\tan x - 1) = 0 \Rightarrow \tan x = -1$ or, $\tan x = \frac{1}{2}$

Now,

$\tan x = -1 \Rightarrow \tan x = \tan\left(-\frac{\pi}{4}\right) \Rightarrow x = n\pi + \left(-\frac{\pi}{4}\right), \Rightarrow x = n\pi - \frac{\pi}{4}, \ n \in Z$

And, $\tan x = \frac{1}{2}$

$\Rightarrow \tan x = \tan\alpha$, where $\tan\alpha = \frac{1}{2} \Rightarrow x = m\pi + \alpha$, where $\tan\alpha = \frac{1}{2}$ and $m \in Z$

Hence, $x = n\pi - \frac{\pi}{4}$ or, $x = m\pi + \alpha$, where $m, n \in Z$ and $\tan\alpha = \frac{1}{2}$

(v) $4\cos x - 3\sec x = \tan x$

$\Rightarrow 4\cos x - \frac{3}{\cos x} = \frac{\sin x}{\cos x}$

$\Rightarrow 4\cos^2 x - 3 = \sin x$

$\Rightarrow 4(1 - \sin^2 x) - 3 = \sin x$

$\Rightarrow 4\sin^2 x + \sin x - 1 = 0$

$$\Rightarrow \sin x = \frac{-1 \pm \sqrt{1+16}}{8} \Rightarrow \sin x = \frac{-1 \pm \sqrt{17}}{8} \Rightarrow \sin x = \frac{-1+\sqrt{17}}{8} \text{ or, } \sin x = \frac{-1-\sqrt{17}}{8}$$

Now, $\sin x = \frac{-1+\sqrt{17}}{8}$

$\Rightarrow \sin x = \sin \alpha$, where $\sin \alpha = \frac{-1+\sqrt{17}}{8}$

$\Rightarrow x = n\pi + (-1)^n \alpha$, where $\sin \alpha = \frac{-1+\sqrt{17}}{8}$ and $n \in Z$

And, $\sin x = \frac{-1-\sqrt{17}}{8}$

$\Rightarrow \sin x = \sin \beta$, where $\sin \beta = \frac{-1-\sqrt{17}}{8}$

$\Rightarrow x = n\pi + (-1)^n \beta$, where $\sin \beta = \frac{-1-\sqrt{17}}{8}$

(vi) $\tan^2 x + (1-\sqrt{3}) \tan x - \sqrt{3} = 0$

$\Rightarrow \tan^2 x + \tan x - \sqrt{3} \tan x - \sqrt{3} = 0$

$\Rightarrow \tan (\tan x + 1) - \sqrt{3} (\tan x + 1) = 0$

$\Rightarrow (\tan x + 1)(\tan x - \sqrt{3}) = 0$

$\Rightarrow \tan x + 1 = 0$ or $\tan x - \sqrt{3} = 0 \Rightarrow \tan x = -1$ or, $\tan x = \sqrt{3}$

Now, $\tan x = -1 \Rightarrow \tan x = \tan\left(-\frac{\pi}{4}\right) \Rightarrow x = n\pi - \frac{\pi}{4}$, $n \in Z$

And, $\tan x = \sqrt{3} \Rightarrow \tan x = \tan \frac{\pi}{3} \Rightarrow x = m\pi + \frac{\pi}{3}$, $m \in Z$

Hence, $x = n\pi - \frac{\pi}{4}$ or, $x = m\pi + \frac{\pi}{3}$, where $m, n \in Z$.

(vii) $\sec^2 2x = 1 - \tan 2x$

$\Rightarrow 1 + \tan^2 2x = 1 - \tan 2x$

$\Rightarrow \tan^2 2x + \tan 2x = 0$

$\Rightarrow \tan 2x (\tan 2x + 1) = 0$

$\Rightarrow \tan 2x = 0$ or, $\tan 2x + 1 = 0$

$\Rightarrow \tan 2x = 0$ or, $\tan 2x = -1 \Rightarrow \tan 2x = 0$ or, $\tan 2x = \tan \frac{3\pi}{4}$

$\Rightarrow 2x = n\pi$ or, $2x = n\pi + \frac{3\pi}{4} \Rightarrow x = \frac{n\pi}{2}$ or, $x = \frac{n\pi}{2} + \frac{3\pi}{8}$, $n \in Z$

EXAMPLE 13 *Solve :* $\tan\left(x + \frac{\pi}{12}\right) = 3 \tan\left(x - \frac{\pi}{12}\right)$

SOLUTION We have,

$$\tan\left(x+\frac{\pi}{12}\right) = 3\tan\left(x-\frac{\pi}{12}\right) \Rightarrow \frac{\tan\left(x+\frac{\pi}{12}\right)}{\tan\left(x-\frac{\pi}{12}\right)} = \frac{3}{1}$$

$$\Rightarrow \quad \frac{\tan\left(x+\frac{\pi}{12}\right)+\tan\left(x-\frac{\pi}{12}\right)}{\tan\left(x+\frac{\pi}{12}\right)-\tan\left(x-\frac{\pi}{12}\right)}=\frac{3+1}{3-1} \qquad \text{[Applying Componendo and dividendo]}$$

$$\Rightarrow \quad \frac{\sin\left(x+\frac{\pi}{12}+x-\frac{\pi}{12}\right)}{\sin\left(x+\frac{\pi}{12}-x+\frac{\pi}{12}\right)}=2 \qquad \left[\because \frac{\tan A+\tan B}{\tan A-\tan B}=\frac{\sin(A+B)}{\sin(A-B)}\right]$$

$$\Rightarrow \quad \frac{\sin 2x}{\sin\frac{\pi}{6}}=2 \Rightarrow \sin 2x=1 \Rightarrow \sin 2x=\sin\frac{\pi}{2} \Rightarrow 2x=n\pi+(-1)^n\frac{\pi}{2} \Rightarrow x=\frac{n\pi}{2}+(-1)^n\frac{\pi}{4}, n\in Z$$

EXAMPLE 14 *Solve the following equations:*

(i) $\tan x+\tan 2x+\tan x\tan 2x=1$ (ii) $\tan x+\tan 2x+\tan 3x = \tan x\tan 2x\tan 3x$

(iii) $\tan x+\tan 2x+\sqrt{3}\tan x\tan 2x=\sqrt{3}$ (iv) $\tan x+\tan\left(x+\frac{\pi}{3}\right)+\tan\left(x+\frac{2\pi}{3}\right)=3$

SOLUTION (i) $\tan x+\tan 2x+\tan x\tan 2x = 1$

$$\Rightarrow \quad \tan x+\tan 2x = 1-\tan x\tan 2x$$

$$\Rightarrow \quad \frac{\tan x+\tan 2x}{1-\tan x\tan 2x} = 1$$

$$\Rightarrow \quad \tan 3x = 1 \Rightarrow \tan 3x = \tan\frac{\pi}{4} \Rightarrow 3x = n\pi+\frac{\pi}{4} \Rightarrow x = \frac{n\pi}{3}+\frac{\pi}{12}, \; n\in Z$$

(ii) $\tan x+\tan 2x+\tan 3x = \tan x\tan 2x\tan 3x$

$$\Rightarrow \quad \tan x+\tan 2x = -\tan 3x+\tan x\tan 2x\tan 3x$$

$$\Rightarrow \quad \tan x+\tan 2x = -\tan 3x(1-\tan x\tan 2x)$$

$$\Rightarrow \quad \frac{\tan x+\tan 2x}{1-\tan x\tan 2x} = -\tan 3x$$

$$\Rightarrow \quad \tan(x+2x) = -\tan 3x \Rightarrow \tan 3x = -\tan 3x \Rightarrow 2\tan 3x = 0 \Rightarrow \tan 3x = 0$$

$$\Rightarrow \quad 3x = n\pi \Rightarrow x = \frac{n\pi}{3}, \quad n\in Z.$$

(iii) $\tan x+\tan 2x+\sqrt{3}\tan x\tan 2x = \sqrt{3}$

$$\Rightarrow \quad \tan x+\tan 2x = \sqrt{3}(1-\tan x\tan 2x)$$

$$\Rightarrow \quad \frac{\tan x+\tan 2x}{1-\tan x\tan 2x} = \sqrt{3}$$

$$\Rightarrow \quad \tan 3x = \sqrt{3} \Rightarrow \tan 3x = \tan\frac{\pi}{3} \Rightarrow 3x = n\pi+\frac{\pi}{3} \Rightarrow x = \frac{n\pi}{3}+\frac{\pi}{9}, \; n\in Z$$

(iv) $\tan x+\tan\left(x+\frac{\pi}{3}\right)+\tan\left(x+\frac{2\pi}{3}\right) = 3$

$$\Rightarrow \quad \tan x+\frac{\tan x+\tan\frac{\pi}{3}}{1-\tan x\tan\frac{\pi}{3}}+\frac{\tan x+\tan\frac{2\pi}{3}}{1-\tan x\tan\frac{2\pi}{3}} = 3$$

$$\Rightarrow \quad \tan x+\frac{\tan x+\sqrt{3}}{1-\sqrt{3}\tan x}+\frac{\tan x-\sqrt{3}}{1+\sqrt{3}\tan x} = 3$$

$$\Rightarrow \quad \tan x + \frac{8\tan x}{1-3\tan^2 x} = 3 \Rightarrow \frac{\tan x - 3\tan^3 x + 8\tan x}{1-3\tan^2 x} = 3 \Rightarrow \frac{3(3\tan x - \tan^3 x)}{1-3\tan^2 x} = 3$$

$$\Rightarrow \quad 3\tan 3x = 3 \Rightarrow \tan 3x = 1 \Rightarrow \tan 3x = \tan\frac{\pi}{4} \Rightarrow 3x = n\pi + \frac{\pi}{4} \Rightarrow x = \frac{n\pi}{3} + \frac{\pi}{12}, n \in Z.$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 15 *Solve :* $\tan^2 x \tan^2 3x \tan 4x = \tan^2 x - \tan^2 3x + \tan 4x.$

SOLUTION We have,

$$\tan^2 x \tan^2 3x \tan 4x = \tan^2 x - \tan^2 3x + \tan 4x$$

$$\Rightarrow \quad \tan^2 3x - \tan^2 x = \tan 4x - \tan^2 x \tan^2 3x \tan 4x$$

$$\Rightarrow \quad \tan^2 3x - \tan^2 x = \tan 4x\,(1 - \tan^2 x \tan^2 3x)$$

$$\Rightarrow \quad \tan 4x = \frac{\tan^2 3x - \tan^2 x}{1 - \tan^2 x \tan^2 3x}$$

$$\Rightarrow \quad \tan 4x = \frac{\tan 3x + \tan x}{1 - \tan 3x \tan x} \times \frac{\tan 3x - \tan x}{1 + \tan 3x \tan x}$$

$$\Rightarrow \quad \tan 4x = \tan(3x + x)\tan(3x - x) \Rightarrow \tan 4x = \tan 4x \tan 2x \Rightarrow \tan 4x(\tan 2x - 1) = 0$$

$$\Rightarrow \quad \tan 4x = 0 \text{ or, } \tan 2x - 1 = 0 \Rightarrow \tan 4x = 0 \text{ or, } \tan 2x = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad 4x = n\pi \text{ or, } 2x = n\pi + \frac{\pi}{4} \Rightarrow x = \frac{n\pi}{4} \text{ or, } x = \frac{n\pi}{2} + \frac{\pi}{8}, n \in Z$$

EXAMPLE 16 *Solve :* $\sec x - 1 = (\sqrt{2} - 1)\tan x,\ x \neq (2n-1)\frac{\pi}{2}, n \in Z$

SOLUTION We have,

$$\sec x - 1 = (\sqrt{2} - 1)\tan x$$

$$\Rightarrow \quad \frac{1}{\cos x} - 1 = \tan\frac{\pi}{8}\tan x \qquad \left[\because \tan\frac{\pi}{8} = \sqrt{2} - 1\right]$$

$$\Rightarrow \quad \frac{1 - \cos x}{\cos x} = \frac{\sin x \sin\frac{\pi}{8}}{\cos x \cos\frac{\pi}{8}} \qquad \left[\because \cos x \neq 0 \text{ as } x \neq (2n-1)\frac{\pi}{2}\right]$$

$$\Rightarrow \quad \cos\frac{\pi}{8} - \cos x \cos\frac{\pi}{8} = \sin x \sin\frac{\pi}{8} \Rightarrow \sin x \sin\frac{\pi}{8} + \cos x \cos\frac{\pi}{8} = \cos\frac{\pi}{8}$$

$$\Rightarrow \quad \cos\left(x - \frac{\pi}{8}\right) = \cos\frac{\pi}{8} \Rightarrow x - \frac{\pi}{8} = 2n\pi \pm \frac{\pi}{8} \Rightarrow x = 2n\pi + \frac{\pi}{4} \text{ or, } x = 2n\pi, n \in Z$$

EXAMPLE 17 *Find the value of* $x \in (-\pi, \pi)$ *satisfying the equation*

$$8^{1 + |\cos x| + \cos^2 x + |\cos^3 x| + \ldots to\ \infty} = 4^3.$$

SOLUTION We have,

$$8^{1 + |\cos x| + \cos^2 x + |\cos^3 x| + \cos^4 x \ldots \text{to } \infty} = 4^3$$

$$\Rightarrow \quad 8^{1 + |\cos x| + \cos^2 x + |\cos^3 x| + \cos^4 x \ldots \text{to } \infty} = 64$$

$$\Rightarrow \quad 8^{\frac{1}{1 - |\cos x|}} = 8^2$$

$$\Rightarrow \quad \frac{1}{1-|\cos x|}=2 \Rightarrow 1-|\cos x|=\frac{1}{2} \Rightarrow |\cos x|=\frac{1}{2} \Rightarrow \cos x=\pm\frac{1}{2} \Rightarrow x=\pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$$

10.3 GENERAL SOLUTIONS OF TRIGONOMETRICAL EQUATIONS OF THE FORM $\sin^2 x = \sin^2 \alpha$, $\cos^2 x = \cos^2 \alpha$, $\tan^2 x = \tan^2 \alpha$

THEOREM *Prove that:*

(i) $\sin^2 x = \sin^2 \alpha \Rightarrow x = n\pi \pm \alpha, n \in Z$ (ii) $\cos^2 x = \cos^2 \alpha \Rightarrow x = n\pi \pm \alpha, n \in Z$

(iii) $\tan^2 x = \tan^2 \alpha \Rightarrow x = n\pi \pm \alpha, n \in Z$

PROOF (i) $\sin^2 x = \sin^2 \alpha$

$\Rightarrow \quad 2\sin^2 x = 2\sin^2 \alpha$

$\Rightarrow \quad 1-\cos 2x = 1-\cos 2\alpha \Rightarrow \cos 2x = \cos 2\alpha \Rightarrow 2x = 2n\pi \pm 2\alpha \Rightarrow x = n\pi \pm \alpha,\ n \in Z$

(ii) $\cos^2 x = \cos^2 \alpha$

$\Rightarrow \quad 2\cos^2 x = 2\cos^2 \alpha$

$\Rightarrow \quad 1+\cos 2x = 1+\cos 2\alpha \Rightarrow \cos 2x = \cos 2\alpha \Rightarrow 2x = 2n\pi \pm 2\alpha \Rightarrow x = n\pi \pm \alpha, n \in Z$

(iii) $\tan^2 x = \tan^2 \alpha$

$$\Rightarrow \quad \frac{1-\tan^2 x}{1+\tan^2 x} = \frac{1-\tan^2 \alpha}{1+\tan^2 \alpha} \Rightarrow \cos 2x = \cos 2\alpha \Rightarrow 2x = 2n\pi \pm 2\alpha \Rightarrow x = n\pi \pm \alpha,\ n \in Z$$

Q.E.D.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Solve:* $7\cos^2 x + 3\sin^2 x = 4$

SOLUTION We have,

$$7\cos^2 x + 3\sin^2 x = 4 \Rightarrow 7(1-\sin^2 x) + 3\sin^2 x = 4 \Rightarrow 4\sin^2 x = 3$$

$$\Rightarrow \quad 4\sin^2 x = 3 \Rightarrow \sin^2 x = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2 \Rightarrow \sin^2 x = \sin^2 \frac{\pi}{3} \Rightarrow x = n\pi \pm \frac{\pi}{3},\ n \in Z$$

EXAMPLE 2 *Solve:* $5\cos^2 x + 7\sin^2 x - 6 = 0$ [NCERT EXEMPLAR]

SOLUTION We have, $5\cos^2 x + 7\sin^2 x - 6 = 0$

$$\Rightarrow \quad 5(1-\sin^2 x) + 7\sin^2 x - 6 = 0 \Rightarrow 2\sin^2 x - 1 = 0 \Rightarrow \sin^2 x = \frac{1}{2} = \left(\frac{1}{\sqrt{2}}\right)^2$$

$$\Rightarrow \quad \sin^2 x = \sin^2 \frac{\pi}{4} \Rightarrow x = n\pi \pm \frac{\pi}{4}, \quad n \in Z.$$

EXAMPLE 3 *Solve:* $2\sin^2 x + \sin^2 2x = 2$

SOLUTION We have,

$$2\sin^2 x + \sin^2 2x = 2$$

$$\Rightarrow \quad 2\sin^2 x + (2\sin x \cos x)^2 = 2$$

$$\Rightarrow \quad 4\sin^2 x\cos^2 x + 2\sin^2 x = 2$$

$$\Rightarrow \quad 2\sin^2 x\cos^2 x + \sin^2 x = 1$$

$$\Rightarrow \quad 2\sin^2 x\cos^2 x - (1-\sin^2 x) = 0$$

$$\Rightarrow \quad 2\sin^2 x\cos^2 x - \cos^2 x = 0$$

$$\Rightarrow \quad \cos^2 x(2\sin^2 x - 1) = 0$$

$$\Rightarrow \quad \cos^2 x = 0 \text{ or, } 2\sin^2 x - 1 = 0 \Rightarrow \cos^2 x = 0 \text{ or, } \sin^2 x = \frac{1}{2}$$

Now, $\cos^2 x = 0 \Rightarrow \cos^2 x = \cos^2\frac{\pi}{2} \Rightarrow x = n\pi \pm \frac{\pi}{2},\ n \in Z$

And, $\sin^2 x = \frac{1}{2} \Rightarrow \sin^2 x = \sin^2\frac{\pi}{4} \Rightarrow x = m\pi \pm \frac{\pi}{4},\ m \in Z$

Hence, $x = n\pi \pm \frac{\pi}{2}$ or $x = m\pi \pm \frac{\pi}{4}$, where $m,\ n \in Z$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *Solve:* $\sin 3\alpha = 4\sin\alpha\sin(x+\alpha)\sin(x-\alpha)$, where $\alpha \neq n\pi, n \in Z$

SOLUTION We have,

$$\sin 3\alpha = 4\sin\alpha\sin(x+\alpha)\sin(x-\alpha)$$

$$\Rightarrow \sin 3\alpha = 4\sin\alpha(\sin^2 x - \sin^2\alpha)$$

$$\Rightarrow 3\sin\alpha - 4\sin^3\alpha = 4\sin^2 x\sin\alpha - 4\sin^3\alpha$$

$$\Rightarrow 3\sin\alpha = 4\sin^2 x\sin\alpha \Rightarrow \sin^2 x = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2 \Rightarrow \sin^2 x = \sin^2\frac{\pi}{3} \Rightarrow x = n\pi \pm \frac{\pi}{3},\ n \in Z$$

EXAMPLE 5 *Solve:* $4\sin x\sin 2x\sin 4x = \sin 3x$

SOLUTION We have,

$$4\sin x\sin 2x\sin 4x = \sin 3x$$

$$\Rightarrow 4\sin x\sin(3x-x)\,.\,\sin(3x+x) = \sin 3x$$

$$\Rightarrow 4[\sin x(\sin^2 3x - \sin^2 x)] = 3\sin x - 4\sin^3 x$$

$$\Rightarrow 4\sin x\sin^2 3x - 4\sin^3 x = 3\sin x - 4\sin^3 x$$

$$\Rightarrow 4\sin x\sin^2 3x = 3\sin x$$

$$\Rightarrow \sin x(4\sin^2 3x - 3) = 0 \Rightarrow \sin x = 0 \text{ or}, 4\sin^2 3x - 3 = 0 \Rightarrow \sin x = 0 \text{ or}, \sin^2 3x = \frac{3}{4}$$

Now, $\sin x = 0 \Rightarrow x = n\pi,\ n \in Z$

And, $\sin^2 3x = \frac{3}{4}$

$$\Rightarrow \quad \sin^2 3x = \left(\frac{\sqrt{3}}{2}\right)^2 \Rightarrow \sin^2 3x = \sin^2\frac{\pi}{3} \Rightarrow 3x = m\pi \pm \frac{\pi}{3} \Rightarrow x = \frac{m\pi}{3} \pm \frac{\pi}{9}, m \in Z$$

Hence, $x = n\pi$ or, $x = \frac{m\pi}{3} \pm \frac{\pi}{9}$, where $m,\ n \in Z$.

EXAMPLE 6 *Solve:* $81^{\sin^2 x} + 81^{\cos^2 x} = 30,\ 0 \leq x \leq \pi$.

SOLUTION We have,

$$81^{\sin^2 x} + 81^{\cos^2 x} = 30 \Rightarrow 81^{\sin^2 x} + 81^{1-\sin^2 x} = 30 \Rightarrow 81^{\sin^2 x} + \frac{81}{81^{\sin^2 x}} = 30$$

$$\Rightarrow\ y+\frac{81}{y}=30, \text{ where } y=81^{\sin^2 x}$$

$$\Rightarrow\ y^2-30y+81=0 \Rightarrow (y-27)(y-3)=0 \Rightarrow y=27 \text{ or, } y=3 \Rightarrow 81^{\sin^2 x}=27 \text{ or, } 81^{\sin^2 x}=3$$

Now, $\quad 81^{\sin^2 x}=27$

$\Rightarrow\quad (3^4)^{\sin^2 x}=3^3$

$\Rightarrow\quad 3^{4\sin^2 x}=3^3$

$\Rightarrow\quad 4\sin^2 x=3 \Rightarrow \sin^2 x=\frac{3}{4}=\left(\frac{\sqrt{3}}{2}\right)^2 \Rightarrow \sin^2 x=\sin^2\frac{\pi}{3} \Rightarrow x=n\pi\pm\frac{\pi}{3}, n\in Z$

and,

$\quad 81^{\sin^2 x}=3$

$\Rightarrow\quad (3^4)^{\sin^2 x}=3$

$\Rightarrow\quad 3^{4\sin^2 x}=3^1$

$\Rightarrow\quad 4\sin^2 x=1 \Rightarrow \sin^2 x=\frac{1}{4}=\left(\frac{1}{2}\right)^2 \Rightarrow \sin^2 x=\sin^2\frac{\pi}{6} \Rightarrow x=n\pi\pm\frac{\pi}{6}, n\in Z$

Hence, $x=n\pi\pm\frac{\pi}{3}$ or $x=n\pi\pm\frac{\pi}{6}, n\in Z$

10.4 TRIGONOMETRIC EQUATIONS OF THE FORM

$a\cos x+b\sin x=c$, where $a, b, c\in R$ such that $|c|\le\sqrt{a^2+b^2}$

To solve this type of equations, we first reduce them in the form $\cos x=\cos\alpha$, or $\sin x=\sin\alpha$.

The following algorithm provides the method of solution.

ALGORITHM

Step I *Obtain the equation* $a\cos x+b\sin x=c$.

Step II *Put* $a=r\cos\alpha$ *and* $b=r\sin\alpha$, *where* $r=\sqrt{a^2+b^2}$ *and* $\tan\alpha=b/a$ i.e. $\alpha=\tan^{-1}(b/a)$.

Step III *Using the substitution in step II, the equation reduces to*

$r\cos(x-\alpha)=c \Rightarrow \cos(x-\alpha)=\frac{c}{r}=\cos\beta$ (say).

Step IV *Solve the equation obtained in step III by using the formulas discussed earlier.*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Solve:* $\sqrt{3}\cos x+\sin x=\sqrt{2}$ **[NCERT EXEMPLAR]**

SOLUTION We have, $\sqrt{3}\cos x+\sin x=\sqrt{2}$ ...(i)

This is of the form $a\cos x+b\sin x=c$, where $a=\sqrt{3}, b=1$ and $c=\sqrt{2}$. Let $a=r\cos\alpha$ and $b=r\sin\alpha$. Then,

$$\sqrt{3}=r\cos\alpha \text{ and } 1=r\sin\alpha \Rightarrow r=\sqrt{a^2+b^2}=\sqrt{(\sqrt{3})^2+1^2}=2 \text{ and}$$

$$\tan\alpha=\frac{1}{\sqrt{3}} \Rightarrow \alpha=\frac{\pi}{6}$$

Substituting $a = \sqrt{3} = r\cos\alpha$ and $b = 1 = r\sin\alpha$ in the equation (i) it reduces to

$$r\cos\alpha\cos x + r\sin\alpha\sin x = \sqrt{2}$$

$$\Rightarrow \quad r\cos(x-\alpha) = \sqrt{2}$$

$$\Rightarrow \quad 2\cos\left(x-\frac{\pi}{6}\right) = \sqrt{2}$$

$$\Rightarrow \quad \cos\left(x-\frac{\pi}{6}\right) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos\left(x-\frac{\pi}{6}\right) = \cos\frac{\pi}{4}$$

$$\Rightarrow \quad x-\frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4}, n \in Z$$

$$\Rightarrow \quad x = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{6}, \ n \in Z$$

$$\Rightarrow \quad x = 2n\pi + \frac{\pi}{4} + \frac{\pi}{6} \text{ or, } x = 2n\pi - \frac{\pi}{4} + \frac{\pi}{6} \Rightarrow x = 2n\pi + \frac{5\pi}{12} \text{ or, } x = 2n\pi - \frac{\pi}{12}$$

Hence, $x = 2n\pi + \frac{5\pi}{12}$ or, $x = 2n\pi - \frac{\pi}{12}$, where $n \in Z$

ALITER We have, $\sqrt{3}\cos x + \sin x = \sqrt{2}$. Dividing both sides by $\sqrt{(\sqrt{3})^2 + 1^2} = 2$, we obtain

$$\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \quad \cos x\cos\frac{\pi}{6} + \sin x\sin\frac{\pi}{6} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos\left(x-\frac{\pi}{6}\right) = \cos\frac{\pi}{4}$$

$$\Rightarrow \quad x-\frac{\pi}{6} = 2n\pi + \frac{\pi}{4}, \text{ where } n \in Z$$

$$\Rightarrow \quad x-\frac{\pi}{6} = 2n\pi + \frac{\pi}{4} \text{ or, } x - \frac{\pi}{6} = 2n\pi - \frac{\pi}{4}, \ n \in Z$$

$$\Rightarrow \quad x = 2n\pi + \frac{5\pi}{12} \text{ or, } x = 2n\pi - \frac{\pi}{12}, \ n \in Z$$

EXAMPLE 2 *If $\sin x + \cos x = 1$, then find the general value of x.* **[NCERT EXEMPLAR]**

SOLUTION We have, $\sin x + \cos x = 1$. Dividing throughout by $\sqrt{1^2+1^2} = \sqrt{2}$, we obtain

$$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \sin x\sin\frac{\pi}{4} + \cos x\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos\left(x-\frac{\pi}{4}\right) = \cos\frac{\pi}{4}$$

$$\Rightarrow \quad x-\frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4} \Rightarrow x = \left(2n\pi \pm \frac{\pi}{4}\right) + \frac{\pi}{4} \Rightarrow x = 2n\pi + \frac{\pi}{2} \text{ or, } x = 2n\pi, \quad n \in Z$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 3 *Solve:* $\sqrt{2}\sec x + \tan x = 1$

SOLUTION We have, $\sqrt{2}\sec x + \tan x = 1$

$$\Rightarrow \quad \frac{\sqrt{2}}{\cos x} + \frac{\sin x}{\cos x} = 1 \Rightarrow \sqrt{2} + \sin x = \cos x \Rightarrow \cos x - \sin x = \sqrt{2} \quad \text{...(i)}$$

This is of the form, $a\cos x - b\sin x = c$, where $a = 1$, $b = 1$ and $c = \sqrt{2}$

Let $a = r\cos\alpha$, and $b = r\sin\alpha$. Then,

$$1 = r\cos\alpha \text{ and } 1 = r\sin\alpha$$

$$\Rightarrow \quad r = \sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \sqrt{2} \text{ and, } \tan\alpha = \frac{r\sin\alpha}{r\cos\alpha} = 1 \Rightarrow r = \sqrt{2} \text{ and, } \alpha = \frac{\pi}{4}$$

Substituting $a = 1 = r\cos\alpha$ and $b = 1 = r\sin\alpha$ in (i), we get

$$r\cos x\cos\alpha - r\sin x\sin\alpha = \sqrt{2}$$

$$\Rightarrow \quad r\cos(x+\alpha) = \sqrt{2}$$

$$\Rightarrow \quad \cos\left(x+\frac{\pi}{4}\right) = 1$$

$$\Rightarrow \quad \cos\left(x+\frac{\pi}{4}\right) = \cos 0° \Rightarrow x + \frac{\pi}{4} = 2n\pi \pm 0 \Rightarrow x = 2n\pi - \frac{\pi}{4},\ n \in Z$$

ALITER The given equation reduces to $\cos x - \sin x = \sqrt{2}$. Dividing throughout by $\sqrt{(1)^2+(-1)^2} = \sqrt{2}$, we obtain

$$\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x = 1$$

$$\Rightarrow \cos x\cos\frac{\pi}{4} - \sin x\sin\frac{\pi}{4} = 1 \Rightarrow \cos\left(x+\frac{\pi}{4}\right) = \cos 0 \Rightarrow x + \frac{\pi}{4} = 2n\pi \pm 0 \Rightarrow x = 2n\pi - \frac{\pi}{4},\ n \in Z.$$

EXAMPLE 4 *Solve:* $\cot x + \operatorname{cosec} x = \sqrt{3}$.

SOLUTION We observe that the LHS of the given equation is meaningful for all $x \neq n\pi$, $n \in Z$.

Now, $\cot x + \operatorname{cosec} x = \sqrt{3}$

$$\Rightarrow \quad \frac{\cos x}{\sin x} + \frac{1}{\sin x} = \sqrt{3} \Rightarrow \cos x + 1 = \sqrt{3}\sin x \Rightarrow \sqrt{3}\sin x - \cos x = 1 \quad \text{...(i)}$$

This is of the form $a\sin x + b\cos x = c$, where $a = \sqrt{3}$, $b = -1$ and $c = 1$.

Let $\sqrt{3} = r\sin\alpha$ and $1 = r\cos\alpha$. Then,

$$r = \sqrt{a^2+b^2} = \sqrt{3+1} = 2 \text{ and, } \tan\alpha = \frac{\sqrt{3}}{1} = \sqrt{3} \Rightarrow r = 2 \text{ and } \alpha = \pi/3$$

Substituting $a = \sqrt{3} = r\sin\alpha$ and $b = 1 = r\cos\alpha$ in (i), we get

$$r\sin\alpha\sin x - r\cos\alpha\cos x = 1$$

$$\Rightarrow \quad -r\cos(x+\alpha) = 1 \Rightarrow -2\cos\left(x+\frac{\pi}{3}\right) = 1 \Rightarrow \cos\left(x+\frac{\pi}{3}\right) = -\frac{1}{2} \Rightarrow \cos\left(x+\frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$$

$$\Rightarrow \quad x + \frac{\pi}{3} = 2n\pi \pm \frac{2\pi}{3},\ n \in Z$$

$$\Rightarrow \quad x = 2n\pi \pm \frac{2\pi}{3} - \frac{\pi}{3} \Rightarrow x = 2n\pi + \frac{\pi}{3} \text{ or, } x = 2n\pi - \pi = (2n-1)\pi,\ n \in Z$$

But, x cannot be equal to $(2n-1)\pi$ as it makes $\sin x = 0$. Hence, $x = 2n\pi + \frac{\pi}{3}$, $n \in Z$

ALITER The given equation reduces to the equation $\sqrt{3}\sin x - \cos x = 1$.

Dividing throughout by $\sqrt{(\sqrt{3})^2 + (-1)^2} = 2$, we obtain

$$\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x = \frac{1}{2} \Rightarrow \sin\frac{\pi}{3}\sin x - \cos\frac{\pi}{3}\cos x = \frac{1}{2} \Rightarrow \cos x\cos\frac{\pi}{3} - \sin x\sin\frac{\pi}{3} = -\frac{1}{2}$$

$$\Rightarrow \cos\left(x + \frac{\pi}{3}\right) = \cos\frac{2\pi}{3} \Rightarrow x + \frac{\pi}{3} = 2n\pi \pm \frac{2\pi}{3} \Rightarrow x = 2n\pi + \frac{\pi}{3} \text{ or } x = 2n\pi - \pi, n \in Z$$

$$\Rightarrow x = 2n\pi + \frac{\pi}{3}, n \in Z \qquad [\because x \neq (2n-1)\pi, n \in Z]$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 5 *Solve:* $(\sqrt{3}-1)\cos x + (\sqrt{3}+1)\sin x = 2$. **[NCERT EXEMPLAR]**

SOLUTION We have, $(\sqrt{3}-1)\cos x + (\sqrt{3}+1)\sin x = 2$. ...(i)

Let $\sqrt{3}-1 = r\sin\alpha$ and $\sqrt{3}+1 = r\cos\alpha$. Then, $r = \sqrt{(\sqrt{3}-1)^2 + (\sqrt{3}+1)^2} = 2\sqrt{2}$

and $$\tan\alpha = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\tan\frac{\pi}{4} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{4}\tan\frac{\pi}{6}} = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \tan\frac{\pi}{12}$$

$$\Rightarrow \quad r = 2\sqrt{2} \text{ and } \alpha = \frac{\pi}{12}$$

Putting $\sqrt{3}-1 = r\sin\alpha$ and $\sqrt{3}+1 = r\cos\alpha$ in (i), we obtain

$$r\sin\alpha\cos x + r\cos\alpha\sin x = 2$$

$$r\sin(x+\alpha) = 2 \Rightarrow 2\sqrt{2}\sin\left(x + \frac{\pi}{12}\right) = 2 \Rightarrow \sin\left(x + \frac{\pi}{12}\right) = \frac{1}{\sqrt{2}} \Rightarrow \sin\left(x + \frac{\pi}{12}\right) = \sin\frac{\pi}{4}$$

$$\Rightarrow \quad x + \frac{\pi}{12} = n\pi + (-1)^n\frac{\pi}{4} \Rightarrow x = n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{12}, n \in Z$$

$$\Rightarrow \quad x = \begin{cases} n\pi + \frac{\pi}{4} - \frac{\pi}{12}, & \text{if } n \text{ is even} \\ n\pi - \frac{\pi}{4} - \frac{\pi}{12}, & \text{if } n \text{ is odd} \end{cases} \Rightarrow x = \begin{cases} n\pi + \frac{\pi}{6}, & \text{if } n \text{ is even} \\ n\pi - \frac{\pi}{3}, & \text{if } n \text{ is odd} \end{cases}$$

ALITER We have,

$$(\sqrt{3}-1)\cos x + (\sqrt{3}+1)\sin x = 2$$

Dividing throuhout by $\sqrt{(\sqrt{3}-1)^2 + (\sqrt{3}+1)^2} = 2\sqrt{2}$, we obtain

$$\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\cos x + \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\sin x = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \left(\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}}\right)\cos x + \left(\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}}\right)\sin x = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos x\left(\cos\frac{\pi}{6}\cos\frac{\pi}{4} - \sin\frac{\pi}{6}\sin\frac{\pi}{4}\right) + \left(\sin\frac{\pi}{6}\cos\frac{\pi}{4} + \cos\frac{\pi}{6}\sin\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos x\cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right) + \sin x\sin\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos x\cos\frac{5\pi}{12} + \sin x\sin\frac{5\pi}{12} = \cos\frac{\pi}{4}$$

$$\Rightarrow \quad \cos\left(x-\frac{5\pi}{12}\right)=\cos\frac{\pi}{4} \Rightarrow x-\frac{5\pi}{12}=2n\pi\pm\frac{\pi}{4}, n\in Z \Rightarrow x=2n\pi+\frac{2\pi}{3} \text{ or, } x=2n\pi+\frac{\pi}{6}, n\in Z$$

EXERCISE 10.1

BASIC

1. Find the general solutions of the following equations:
   (i) $\sin x = \frac{1}{2}$ (ii) $\cos x = -\frac{\sqrt{3}}{2}$ (iii) $\operatorname{cosec} x = -\sqrt{2}$
   (iv) $\sec x = \sqrt{2}$ (v) $\tan x = -\frac{1}{\sqrt{3}}$ (vi) $\sqrt{3} \sec x = 2$
2. Find the general solutions of the following equations:
   (i) $\sin 2x = \frac{\sqrt{3}}{2}$ (ii) $\cos 3x = \frac{1}{2}$ (iii) $\sin 9x = \sin x$
   (iv) $\sin 2x = \cos 3x$ (v) $\tan x + \cot 2x = 0$ (vi) $\tan 3x = \cot x$
   (vii) $\tan 2x \tan x = 1$ (viii) $\tan mx + \cot nx = 0$ (ix) $\tan px = \cot qx$
   (x) $\sin 2x + \cos x = 0$ (xi) $\sin x = \tan x$ (xii) $\sin 3x + \cos 2x = 0$
3. Solve the following equations:
   (i) $\sin^2 x - \cos x = \frac{1}{4}$ (ii) $2\cos^2 x - 5\cos x + 2 = 0$
   (iii) $2\sin^2 x + \sqrt{3}\cos x + 1 = 0$ (iv) $4\sin^2 x - 8\cos x + 1 = 0$
   (v) $\tan^2 x + (1-\sqrt{3})\tan x - \sqrt{3} = 0$ (vi) $3\cos^2 x - 2\sqrt{3}\sin x \cos x - 3\sin^2 x = 0$
   (vii) $\cos 4x = \cos 2x$

BASED ON LOTS

4. Solve the following equations:
   (i) $\cos x + \cos 2x + \cos 3x = 0$ (ii) $\cos x + \cos 3x - \cos 2x = 0$ [NCERT]
   (iii) $\sin x + \sin 5x = \sin 3x$ (iv) $\cos x \cos 2x \cos 3x = \frac{1}{4}$
   (v) $\cos x + \sin x = \cos 2x + \sin 2x$ (vi) $\sin x + \sin 2x + \sin 3x = 0$
   (vii) $\sin x + \sin 2x + \sin 3x + \sin 4x = 0$ (viii) $\sin 3x - \sin x = 4\cos^2 x - 2$
   (ix) $\sin 2x - \sin 4x + \sin 6x = 0$ [NCERT]
5. Solve the following equations:
   (i) $\tan x + \tan 2x + \tan 3x = 0$ (ii) $\tan x + \tan 2x = \tan 3x$
   (iii) $\tan 3x + \tan x = 2\tan 2x$
6. Solve the following equations:
   (i) $\sin x + \cos x = \sqrt{2}$ (ii) $\sqrt{3}\cos x + \sin x = 1$
   (iii) $\sin x + \cos x = 1$ (iv) $\operatorname{cosec} x = 1 + \cot x$
7. Solve the following equations:
   (i) $4\sin x\cos x + 2\sin x + 2\cos x + 1 = 0$ (ii) $\cos x + \sin x = \cos 2x + \sin 2x$
   (iii) $\sin x \tan x - 1 = \tan x - \sin x$ (iv) $3\tan x + \cot x = 5\operatorname{cosec} x$

BASED ON HOTS

Solve the following equations: (8 – 10)

8. $3 - 2\cos x - 4\sin x - \cos 2x + \sin 2x = 0$
9. $3\sin^2 x - 5\sin x\cos x + 8\cos^2 x = 2$

10. $2^{\sin^2 x} + 2^{\cos^2 x} = 2\sqrt{2}$

11. Find the most general value of x satisfying the equations $\tan x = -1$ and $\cos x = \dfrac{1}{\sqrt{2}}$.

[NCERT EXEMPLAR]

ANSWERS

1. (i) $x = n\pi + (-1)^n \dfrac{\pi}{6},\ n \in Z$ (ii) $x = 2n\pi \pm \dfrac{7\pi}{6},\ n \in Z$

(iii) $x = n\pi + (-1)^{n+1} \dfrac{\pi}{4},\ n \in Z$ (iv) $x = 2n\pi \pm \dfrac{\pi}{4},\ n \in Z$

(v) $x = n\pi - \dfrac{\pi}{6},\ n \in Z$ (vi) $x = 2n\pi \pm \dfrac{\pi}{6},\ n \in Z$

2. (i) $x = \dfrac{n\pi}{2} + (-1)^n \dfrac{\pi}{6},\ n \in Z$ (ii) $x = \dfrac{2n\pi}{3} \pm \dfrac{\pi}{9},\ n \in Z$

(iii) $x = \dfrac{r\pi}{4}$ or $x = (2r+1)\dfrac{\pi}{10}$, where $r \in Z$

(iv) $x = (4n+1)\dfrac{\pi}{10}$ or $x = (4n-1)\dfrac{\pi}{2}$, where $n \in Z$

(v) $x = n\pi - \dfrac{\pi}{2},\ n \in Z$ (vi) $x = \dfrac{n\pi}{4} + \dfrac{\pi}{8},\ n \in Z$

(vii) $x = \dfrac{n\pi}{3} + \dfrac{\pi}{6},\ n \in Z$ (viii) $x = \dfrac{(2r+1)\pi}{m-n},\ r \in Z$

(ix) $x = \left(\dfrac{2n+1}{p+q}\right)\dfrac{\pi}{2},\ n \in Z$

(x) $x = (4n-1)\dfrac{\pi}{2}$ or $x = (4m-1)\dfrac{\pi}{6}$, where $m, n \in Z$

(xi) $x = m\pi$ or $x = 2n\pi$, where $m, n \in Z$

(xii) $x = (4n-1)\dfrac{\pi}{10}$ or $x = (4m-1)\dfrac{\pi}{2},\ m, n \in Z$

3. (i) $x = 2n\pi \pm \dfrac{\pi}{3},\ n \in Z$ (ii) $x = 2n\pi \pm \dfrac{\pi}{3},\ n \in Z$

(iii) $x = 2n\pi \pm \dfrac{5\pi}{6},\ n \in Z$ (iv) $x = 2n\pi \pm \dfrac{\pi}{3},\ n \in Z$

(v) $x = n\pi - \dfrac{\pi}{4}$ or $x = m\pi + \dfrac{\pi}{3}$, where $m, n \in Z$

(vi) $x = n\pi - \dfrac{\pi}{3}$ or $x = m\pi + \dfrac{\pi}{6}$, where $m, n \in Z$

(vii) $x = n\pi,\ x = \dfrac{n\pi}{3},\ n \in Z$

4. (i) $x = (2n+1)\dfrac{\pi}{4}$ or $x = 2m\pi \pm \dfrac{2\pi}{3},\ m, n \in Z$

(ii) $x = (2n+1)\dfrac{\pi}{4}$ or $x = 2m\pi \pm \dfrac{\pi}{3},\ m, n \in Z$

(iii) $x = \dfrac{n\pi}{3}$ or $x = m\pi \pm \dfrac{\pi}{6},\ m, n \in Z$

(iv) $x = 2n + 1\dfrac{\pi}{8}$ or $x = m\pi \pm \dfrac{\pi}{3},\ m, n \in Z$

(v) $x = \frac{(2n\pi)}{3} + \frac{\pi}{6}$ or $x = 2m\pi,\ m, n \in Z$

(vi) $x = \frac{n\pi}{2}$ or $x = 2n\pi \pm \frac{2\pi}{3},\ m, n \in Z$

(vii) $x = n\pi + \frac{\pi}{2},\ x = (2m+1)\pi,\ x = \frac{2r\pi}{5}, m, n\, r \in Z$

(viii) $x = n\pi + (-1)^n \frac{\pi}{2}$ or $x = (2m+1)\frac{\pi}{4},\ m, n \in Z$

(ix) $x = \frac{n\pi}{4},\ x = n\pi \pm \frac{\pi}{6},\ n \in Z$

5. (i) $x = \frac{m\pi}{3}$ or $x = n\pi \pm \alpha$, where $\alpha = \tan^{-1}\frac{1}{\sqrt{2}}$ and $m, n \in Z$

(ii) $x = m\pi$ or $x = \frac{n\pi}{3}$, where $m, n \in Z$ (iii) $x = n\pi$, where $n \in Z$

6. (i) $x = (8n +)\frac{\pi}{4},\ n \in Z$

(ii) $x = (4n+1)\frac{\pi}{2}$ or $x = (12m-1)\frac{\pi}{6}, m, n \in Z$

(iii) $x = 2n\pi$ or $x = 2m\pi + \frac{\pi}{2},\ m, n \in Z$ (iv) $x = 2m\pi + \frac{\pi}{2},\ m, n \in Z$

7. (i) $x = 2n\pi + \frac{2\pi}{3}, x = n\pi + (-1)^n \frac{7\pi}{6}$ (ii) $x = 2n\pi$ or, $x = \frac{21\pi}{3} + \frac{\pi}{6}, n \in Z$

(iii) $x = n\pi + x = \frac{3\pi}{4}, n \in Z$ (iv) $x = 2n\pi \pm \frac{\pi}{3}, n \in Z$

8. $x = n\pi + (-1)^n \frac{\pi}{2}, x = 2n\pi + \frac{\pi}{2}, n \in Z$

9. $x = n\pi + \alpha, x = n\pi + \beta$, where $\tan\alpha = 2$, $\tan\beta = 2, n \in Z$

10. $x = n\pi \pm \frac{\pi}{4}, n \in Z$

HINTS TO SELECTED PROBLEMS

11. A value of x satisfying $\tan x = -1$ and $\cos x = \frac{1}{\sqrt{2}}$ is $x = \frac{7\pi}{4}$. Hence, the most general value of x is $x = 2n\pi + \frac{7\pi}{4}, n \in Z$.

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. In a ΔABC, if $\frac{\sin(A-B)}{\sin(A+B)} = \frac{a^2 - b^2}{k}$, then $k =$ ....

2. In a ΔABC, if $c^2 + a^2 - b^2 = ac$, then the measure of angle B is ....

3. In a ΔABC, if $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ and $a = 2$, then area of ΔABC is equal to ....

4. In a triangle ABC, if $a = 2, b = 4$ and $A + B = \frac{2\pi}{3}$, then area of ΔABC is ....

5. The angles A, B, C of a ΔABC are in AP and the sides a, b, c are in G.P. If $a^2+c^2=\lambda b^2$, then $\lambda =$ ....

6. In a ΔABC, if $\angle C = 60°$, $a = 47$ cm and $b = 94$ cm, then $c^2 =$ ....

7. In a ΔABC, if $\angle C = \frac{\pi}{2}$, $\angle A = \frac{\pi}{6}$, $c = 20$, then $a =$ ....

8. In a ΔABC, if $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = k(a^2+b^2+c^2)$, then $k =$ ....

9. In a ΔABC, if $c^2 \sin A \sin B = ab$, then $A + B =$ ....

10. In a ΔABC, if $a = 8, b = 9$ and $3\cos C = 2$, then $C =$ ....

11. In a ΔABC, if $b = \sqrt{3}$, $c = 1$ and $B - C = \frac{\pi}{2}$, then $A =$ ....

12. If angles of a triangle are in A.P. and $b : c = \sqrt{3} : \sqrt{2}$, then $C =$ ....

13. If the sides of a ΔABC are $a, b, \sqrt{a^2+ab+b^2}$, then the measure of the largest angle is ....

14. In a ΔABC, if $a^4+b^4+c^4 = 2a^2b^2+2b^2c^2$, then $B =$ ....

15. In a ΔABC, if $a = 4, b = 3, A = \frac{\pi}{3.}$ Then side C is given by ....

ANSWERS

1. c^2	2. $\frac{\pi}{3}$	3. $\sqrt{3}$ sq. units	4. $\frac{\sqrt{3}}{2}$ sq. units	5. 2	6. 6627
7. 10	8. $\frac{1}{2abc}$	9. $\frac{\pi}{2}$	10. 7	11. $\frac{\pi}{6}$	12. $\frac{\pi}{4}$
13. $\frac{2\pi}{3}$	14. $\frac{\pi}{4}$ or $\frac{3\pi}{4}$	15. $c^2-3c-7=0$			

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Find the area of the triangle ΔABC in which $a = 1, b = 2$ and $\angle c = 60°$.
2. In a ΔABC, if $b = \sqrt{3}$, $c = 1$ and $\angle A = 30°$, find a.
3. In a ΔABC, if $\cos A = \frac{\sin B}{2\sin C}$, then show that $c = a$.
4. In a ΔABC, if $b = 20$, $c = 21$ and $\sin A = \frac{3}{5}$, find a.
5. In a ΔABC, if $\sin A$ and $\sin B$ are the roots of the equation $c^2x^2 - c(a+b)x + ab = 0$, then find $\angle C$.
6. In ΔABC, if $a = 8, b = 10, c = 12$ and $C = \lambda A$, find the value of λ.
7. If the sides of a triangle are proportional to $2, \sqrt{6}$ and $\sqrt{3}-1$, find the measure of its greatest angle.
8. If in a ΔABC, $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$, then find the measures of angles A, B, C.

9. In any triangle ABC, find the value of $a \sin(B-C) + b \sin(C-A) + c \sin(A-B)$.
10. In any ΔABC, find the value of $\Sigma\, a(\sin B - \sin C)$

ANSWERS

1. $\sqrt{3}$ sq. units 2. 1 4. 13 5. 90° 6. 2 7. 120°
8. $A = B = C = 60°$ 9. 0 10. 0

MULTIPLE CHOICES QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The smallest value of x satisfying the equation $\sqrt{3}(\cot x + \tan x) = 4$ is
   (a) $2\pi/3$ (b) $\pi/3$ (c) $\pi/6$ (d) $\pi/12$
2. If $\cos x + \sqrt{3} \sin x = 2$, then $x =$
   (a) $\pi/3$ (b) $2\pi/3$ (c) $4\pi/3$ (d) $5\pi/3$
3. If $\tan px - \tan qx = 0$, then the values of θ form a series in
   (a) AP (b) GP (c) HP (d) none of these
4. If a is any real number, the number of roots of $\cot x - \tan x = a$ in the first quadrant is (are).
   (a) 2 (b) 0 (c) 1 (d) none of these
5. The general solution of the equation $7\cos^2 x + 3\sin^2 x = 4$ is
   (a) $x = 2n\pi \pm \frac{\pi}{6}, n \in Z$ (b) $x = 2n\pi \pm \frac{2\pi}{3}, n \in Z$
   (c) $x = n\pi \pm \frac{\pi}{3}, n \in Z$ (d) none of these
6. A solution of the equation $\cos^2 x + \sin x + 1 = 0$, lies in the interval
   (a) $(-\pi/4, \pi/4)$ (b) $(\pi/4, 3\pi/4)$ (c) $(3\pi/4, 5\pi/4)$ (d) $(5\pi/4, 7\pi/4)$
7. The number of solution in $[0, \pi/2]$ of the equation $\cos 3x \tan 5x = \sin 7x$ is
   (a) 5 (b) 7 (c) 6 (d) none of these
8. The general value of x satisfying the equation $\sqrt{3} \sin x + \cos x = \sqrt{3}$ is given by
   (a) $x = n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{3}, n \in Z$ (b) $x = n\pi + (-1)^n \frac{\pi}{3} - \frac{\pi}{6}, n \in Z$
   (c) $x = n\pi \pm \frac{\pi}{6}, n \in Z$ (d) $x = n\pi \pm \frac{\pi}{3}, n \in Z$
9. The smallest positive angle which satisfies the equation $2\sin^2 x + \sqrt{3}\cos x + 1 = 0$ is
   (a) $\frac{5\pi}{6}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$
10. If $4\sin^2 x = 1$, then the values of x are
   (a) $2n\pi \pm \frac{\pi}{3}, n \in Z$ (b) $n\pi \pm \frac{\pi}{3}, n \in Z$ (c) $n\pi \pm \frac{\pi}{6}, n \in Z$ (d) $2n\pi \pm \frac{\pi}{6}, n \in Z$
11. If $\cot x - \tan x = \sec x$, then, x is equal to
   (a) $2n\pi + \frac{3\pi}{2}, n \in Z$ (b) $n\pi + (-1)^n \frac{\pi}{6}, n \in Z$
   (c) $n\pi + \frac{\pi}{2}, n \in Z$ (d) none of these

12. A value of x satisfying $\cos x + \sqrt{3}\sin x = 2$ is

(a) $\frac{5\pi}{3}$ (b) $\frac{4\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{3}$

13. In $(0, \pi)$, the number of solutions of the equation $\tan x + \tan 2x + \tan 3x = \tan x \tan 2x \tan 3x$ is

(a) 7 (b) 5 (c) 4 (d) 2

14. The number of values of x in $[0, 2\pi]$ that satisfy the equation $\sin^2 x - \cos x = \frac{1}{4}$

(a) 1 (b) 2 (c) 3 (d) 4

15. If $e^{\sin x} - e^{-\sin x} - 4 = 0$, then $x =$

(a) 0 (b) $\sin^{-1}\{\log_e(2-\sqrt{5})\}$

(c) 1 (d) none of these

16. The equation $3\cos x + 4\sin x = 6$ has .... solution

(a) finite (b) infinite (c) one (d) no

17. If $\sqrt{3}\cos x + \sin x = \sqrt{2}$, then general value of θ is

(a) $n\pi + (-1)^n \frac{\pi}{4}$, $n \in Z$ (b) $(-1)^n \frac{\pi}{4} - \frac{\pi}{3}$, $n \in Z$

(c) $n\pi + \frac{\pi}{4} - \frac{\pi}{3}$, $n \in Z$ (d) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{3}$, $n \in Z$

18. General solution of $\tan 5x = \cot 2x$ is

(a) $\frac{n\pi}{7} + \frac{\pi}{2}$, $n \in Z$ (b) $x = \frac{n\pi}{7} + \frac{\pi}{3}$, $n \in Z$

(c) $x = \frac{n\pi}{7} + \frac{\pi}{14}$, $n \in Z$ (d) $x = \frac{n\pi}{7} - \frac{\pi}{14}$, $n \in Z$

19. The solution of the equation $\cos^2 x + \sin x + 1 = 0$ lies in the interval

(a) $(-\pi/4, \pi/4)$ (b) $(\pi/4, 3\pi/4)$ (c) $(3\pi/4, 5\pi/4)$ (d) $(5\pi/4, 7\pi/4)$

20. If $\cos x = -\frac{1}{2}$ and $0 < x < 2\pi$, then the solutions are

(a) $x = \frac{\pi}{3}, \frac{4\pi}{3}$ (b) $x = \frac{2\pi}{3}, \frac{4\pi}{3}$ (c) $x = \frac{2\pi}{3}, \frac{7\pi}{6}$ (d) $\theta = \frac{2\pi}{3}, \frac{5\pi}{3}$

21. The number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3\sin^2 x - 7\sin x + 2 = 0$ is

(a) 0 (b) 5 (c) 6 (d) 10

22. Number of solutions of the equation $\tan x + \sec x = 2\cos x$ lying in the interval $[0, 2\pi]$ is

(a) 0 (b) 1 (c) 2 (d) 3

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1. (c) 2. (a) 3. (a) 4. (c) 5. (a) 6. (d) 7. (c) 8. (b) 9. (a)
10. (c) 11. (b) 12. (d) 13. (d) 14. (b) 15. (d) 16. (d) 17. (d) 18. (c)
19. (d) 20. (b) 21. (c) 22. (c)

SUMMARY

1. An equation containing trigonometric functions of unknown angles is known as a trigonometric equation.
2. A solution of a trigonometric equation is the value of the unknown angle that satisfies the equation.
3. Following are the general solutions of trigonometric equations in standar.' forms:

Trigonometric equation	*General solution*
(i) $\sin\theta = 0$	$\theta = n\pi, n \in Z$
(ii) $\cos\theta = 0$	$\theta = (2n+1)\frac{\pi}{2}, n \in Z$
(iii) $\tan\theta = 0$	$\theta = n\pi, n \in Z$
(iv) $\sin\theta = \sin\alpha$	$\theta = n\pi + (-1)^n\alpha, n \in Z$
(v) $\cos\theta = \cos\alpha$	$\theta = 2n\pi \pm \alpha, n \in Z$
(vi) $\tan\theta = \tan\alpha$	$\theta = n\pi + \alpha, n \in Z$
(vii) $\left.\begin{array}{l}\sin^2\theta = \sin^2\alpha \\ \cos^2\theta = \cos^2\alpha \\ \tan^2\theta = \tan^2\alpha\end{array}\right\}$	$\theta = n\pi \pm \alpha, n \in Z$

4. The equation $a\cos\theta + b\sin\theta = c$ is solvable for $|c| \le \sqrt{a^2+b^2}$.