

CHAPTER 12

COMPLEX NUMBERS

12.1 INTRODUCTION

If a, b are natural numbers such that $a > b$, then the equation $x + a = b$ is not solvable in N , the set of natural numbers i.e. there is no natural number satisfying the equation $x + a = b$. So, the set of natural numbers is extended to form the set I of integers in which every equation of the form $x + a = b$; $a, b \in N$ is solvable. But, equations of the form $xa = b$, where $a, b \in I$, $a \neq 0$ are not solvable in I also. Therefore, the set I of integers is extended to obtain the set Q of all rational numbers in which every equation of the form $xa = b$, $a \neq 0$, $a, b \in I$ is uniquely solvable. The equations of the form $x^2 = 2$, $x^2 = 3$ etc. are not solvable in Q because there is no rational number whose square is 2. Such numbers are known as irrational numbers. The set Q of all rational numbers is extended to obtain the set R which includes both rational and irrational numbers. This set is known as the set of real numbers. The equations of the form $x^2 + 1 = 0$, $x^2 + 4 = 0$ etc. are not solvable in R i.e. there is no real number whose square is a negative real number. Euler was the first mathematician to introduce the symbol i (iota) for the square root of -1 i.e. a solution of $x^2 + 1 = 0$ with the property $i^2 = -1$. He also called this symbol as the imaginary unit.

12.2 INTEGRAL POWERS OF IOTA (i)

Positive integral powers of i : We have, $i = \sqrt{-1}$

$$\therefore i^2 = -1, i^3 = i^2 \times i = -i, i^4 = (i^2)^2 = (-1)^2 = 1$$

In order to compute i^n for $n > 4$, we divide n by 4 and obtain the remainder r . Let m be the quotient when n is divided by 4. Then,

$$n = 4m + r, \text{ where } 0 \leq r < 4 \Rightarrow i^n = i^{4m+r} = (i^4)^m i^r = i^r$$

Thus, the value of i^n for $n > 4$ is i^r , where r is the remainder when n is divided by 4.

Negative integral powers of i : By the law of indices, we have

$$i^{-1} = \frac{1}{i} = \frac{i^3}{i^4} = i^3 = -i, i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1, i^{-3} = \frac{1}{i^3} = \frac{i}{i^4} = i, i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$$

If $n > 4$, then $i^{-n} = \frac{1}{i^n} = \frac{1}{i^r}$, where r is the remainder when n is divided by 4

NOTE i^0 is defined as 1.

The above discussion suggests the following algorithm to find integral exponents of i .

ALGORITHM

To find the value of i^n for $n \in Z$, we may follow the following steps.

Step I If $n = 0$, then write $i^n = 1$.

Step II If $n > 0$, then

$$i^n = \begin{cases} i, & \text{if } n = 1 \\ -1, & \text{if } n = 2 \\ -i, & \text{if } n = 3 \\ 1, & \text{if } n = 4 \\ i^r, & \text{if } n > 4, \text{ where } r \text{ is the remainder when } n \text{ is divided by } 4 \end{cases}$$

Step III If $n < 0$, then $n = -m$, where $m > 0$.

$$\therefore i^n = \begin{cases} i^{-1} = \frac{1}{i} = -i, & \text{if } n = -1 \\ i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1, & \text{if } n = -2 \\ i^{-3} = \frac{1}{i^3} = \frac{i}{i^4} = i, & \text{if } n = -3 \\ i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1, & \text{if } n = -4 \\ i^{-m} = \frac{1}{i^m} = \frac{1}{i^r}, & \text{where } r \text{ is the remainder when } m \text{ is divided by } 4, \text{ if } n < -4. \end{cases}$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 Evaluate the following:

(i) i^{135}

(ii) i^{19}

(iii) i^{-999}

(iv) $(-\sqrt{-1})^{4n+3}, n \in \mathbb{N}$

SOLUTION (i) 135 leaves remainder as 3 when it is divided by 4. Therefore, $i^{135} = i^3 = -i$

(ii) The remainder is 3 when 19 is divided by 4. Therefore, $i^{19} = i^3 = -i$.

(iii) We have, $i^{-999} = 1/i^{999}$. On dividing 999 by 4, we obtain 3 as the remainder. Therefore, $i^{999} = i^3$. Hence, $i^{-999} = \frac{1}{i^3} = \frac{1}{i^3} = \frac{i}{i^4} = \frac{i}{1} = i$.

(iv) We have, $(-\sqrt{-1})^{4n+3} = (-i)^{4n+3} = (-i)^{4n}(-i)^3 = \{(-i)^4\}^n(-i)^3 = 1 \times -i^3 = i$

EXAMPLE 2 Show that:

(i) $\left\{i^{19} + \left(\frac{1}{i}\right)^{25}\right\}^2 = -4$

(ii) $\left\{i^{17} - \left(\frac{1}{i}\right)^{34}\right\}^2 = 2i$

(iii) $\left\{i^{18} + \left(\frac{1}{i}\right)^{24}\right\}^3 = 0$

(iv) $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$, for all $n \in \mathbb{N}$.

SOLUTION (i) $\left\{i^{19} + \left(\frac{1}{i}\right)^{25}\right\}^2 = \left\{i^{19} + \frac{1}{i^{25}}\right\}^2 = \left\{i^3 + \frac{1}{i}\right\}^2 = \left\{-i + \frac{i^3}{i^4}\right\}^2$

$$= [-i + i^3]^2 = (-i - i)^2 = 4i^2 = -4.$$

(ii) $\left\{i^{17} - \left(\frac{1}{i}\right)^{34}\right\}^2 = \left\{i^{17} - \frac{1}{i^{34}}\right\}^2 = \left\{i - \frac{1}{i^2}\right\}^2 = \left\{i - \frac{1}{(-1)}\right\}^2 = (i+1)^2$

$$= i^2 + 2i + 1 = -1 + 2i + 1 = 2i$$

$$(iii) \quad \left\{ i^{18} + \left(\frac{1}{i} \right)^{24} \right\}^3 = \left\{ i^{18} + \frac{1}{i^{24}} \right\}^3 = \left(i^2 + \frac{1}{1} \right)^3 = (-1 + 1)^3 = 0$$

$$(iv) \quad i^n + i^{n+1} + i^{n+2} + i^{n+3} = i^n + i^n \times i + i^n \times i^2 + i^n \times i^3 \\ = i^n (1 + i + i^2 + i^3) = i^n (1 + i - 1 - i) = i^n (0) = 0$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 3 Evaluate $\sum_{n=1}^{13} (i^n + i^{n+1})$, where $n \in N$.

[NCERT EXEMPLAR]

SOLUTION $\sum_{n=1}^{13} (i^n + i^{n+1}) = \sum_{n=1}^{13} (i+1) i^n = (i+1) \sum_{n=1}^{13} i^n = (i+1) (i + i^2 + i^3 + \dots + i^{13})$

$$= (i+1) \times i \left(\frac{i^{13} - 1}{i - 1} \right) = (i^2 + i) \left(\frac{i - 1}{i - 1} \right) = (-1 + i) \quad [\because i^{13} = i]$$

EXAMPLE 4 Evaluate $1 + i^2 + i^4 + i^6 + \dots + i^{2n}$.

SOLUTION Let $S = 1 + i^2 + i^4 + i^6 + \dots + i^{2n}$. Then

$$S = 1 + i^2 + (i^2)^2 + (i^2)^3 + \dots + (i^2)^n$$

$$\Rightarrow S = 1 \left\{ \frac{1 - (i^2)^{n+1}}{1 - i^2} \right\} = \frac{1 - (i^2)^{n+1}}{1 + 1} = \frac{1}{2} \left\{ 1 - (-1)^{n+1} \right\} = \begin{cases} \frac{1}{2} (1 - 1) = 0, & \text{if } n \text{ is odd} \\ \frac{1}{2} (1 + 1) = 1, & \text{if } n \text{ is even} \end{cases}$$

EXAMPLE 5 For a positive integer n , find the value of $(1-i)^n \left(1 - \frac{1}{i} \right)^n$.

[NCERT EXEMPLAR]

SOLUTION $(1-i)^n \left(1 - \frac{1}{i} \right)^n = (1-i)^n (1+i)^n$

$$\left[\because \frac{1}{i} = -i \right]$$

$$= \{(1-i)(1+i)\}^n = (1-i^2)^n = (1+1)^n = 2^n$$

EXERCISE 12.1

BASIC

1. Evaluate the following:

(i) i^{457}

(ii) i^{528}

(iii) $\frac{1}{i^{58}}$

(iv) $i^{37} + \frac{1}{i^{67}}$

(v) $\left(i^{41} + \frac{1}{i^{257}} \right)^9$

(vi) $(i^{77} + i^{70} + i^{87} + i^{414})^3$

(vii) $i^{30} + i^{40} + i^{60}$

(viii) $i^{49} + i^{68} + i^{89} + i^{110}$

2. Show that $1 + i^{10} + i^{20} + i^{30}$ is a real number.

3. Find the values of the following expressions:

(i) $i^{49} + i^{68} + i^{89} + i^{110}$

(ii) $i^{30} + i^{80} + i^{120}$

(iii) $i + i^2 + i^3 + i^4$

(iv) $i^5 + i^{10} + i^{15}$

(v) $\frac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}}$

(vi) $1 + i^2 + i^4 + i^6 + i^8 + \dots + i^{20}$

(vii) $(1+i)^6 + (1-i)^3$

[NCERT EXEMPLAR]

ANSWERS

1. (i) i (ii) 1 (iii) -1 (iv) $2i$ (v) 0 (vi) -8 (vii) 1 (viii) $2i$
 3. (i) $2i$ (ii) 1 (iii) 0 (iv) -1 (v) -1 (vi) 1 (vii) $-2-10i$

12.3 IMAGINARY QUANTITIES

The square root of a negative real number is called an imaginary quantity or an imaginary number.

For example, $\sqrt{-3}$, $\sqrt{-4}$, $\sqrt{-9/4}$ etc. are imaginary quantities.

THEOREM If a, b are positive real numbers, then $\sqrt{-a} \times \sqrt{-b} = -\sqrt{ab}$.

PROOF We have,

$$\sqrt{-a} = \sqrt{-1 \times a} = \sqrt{-1} \times \sqrt{a} = i\sqrt{a} \text{ and, } \sqrt{-b} = \sqrt{-1 \times b} = \sqrt{-1} \times \sqrt{b} = i\sqrt{b}$$

$$\therefore \sqrt{-a} \times \sqrt{-b} = (i\sqrt{a})(i\sqrt{b}) = i^2(\sqrt{a} \times \sqrt{b}) = -1(\sqrt{ab}) = -\sqrt{ab}$$

NOTE 1 For any two real numbers $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is true only when at least one of a and b is either positive or zero. In other words, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is not valid if a and b both are negative.

NOTE 2 For any positive real number a , we have $\sqrt{-a} = \sqrt{-1 \times a} = \sqrt{-1} \times \sqrt{a} = i\sqrt{a}$.

ILLUSTRATION 1 Compute the following:

$$(i) \sqrt{-144} \quad (ii) \sqrt{-4} \times \sqrt{\frac{-9}{4}} \quad (iii) \sqrt{-25} + 3\sqrt{-4} + 2\sqrt{-9}$$

SOLUTION (i) $\sqrt{-144} = \sqrt{-1 \times 144} = \sqrt{-1} \times \sqrt{144} = 12i$

(ii) $\sqrt{-4} \times \sqrt{\frac{-9}{4}} = (2i) \left(\frac{3i}{2} \right) = 3i^2 = -3$

(iii) $\sqrt{-25} + 3\sqrt{-4} + 2\sqrt{-9} = 5i + 6i + 6i = 17i$

ILLUSTRATION 2 A student writes the formula $\sqrt{ab} = \sqrt{a} \sqrt{b}$. Then he substitutes $a = -1$ and $b = -1$ and finds $1 = -1$. Explain where is he wrong?

SOLUTION Since a and b both are negative. Therefore, $\sqrt{ab}$ cannot be written as $\sqrt{a} \sqrt{b}$. In fact, for a and b both negative, we have $\sqrt{a} \sqrt{b} = -\sqrt{ab}$.

ILLUSTRATION 3 Is the following computation correct? If not give the correct computation:

$$"\sqrt{(-2)} \cdot \sqrt{(-3)} = \sqrt{(-2) \cdot (-3)} = \sqrt{6}"$$

SOLUTION The said computation is not correct, because -2 and -3 both are negative and $\sqrt{ab} = \sqrt{a} \sqrt{b}$ is true when at least one of a and b is positive or zero. The correct computation is

$$(\sqrt{-2})(\sqrt{-3}) = (i\sqrt{2})(i\sqrt{3}) = i^2 \sqrt{6} = -\sqrt{6}$$

12.4 COMPLEX NUMBERS

COMPLEX NUMBER If a, b are two real numbers, then a number of the form $a + ib$ is called a complex number.

For example, $7 + 2i$, $-1 + i$, $3 - 2i$, $0 + 2i$, $1 + 0i$ etc. are complex numbers.

Real and imaginary parts of a complex number: If $z = a + ib$ is a complex number, then 'a' is called the real part of z and 'b' is known as the imaginary part of z . The real part of z is denoted by $\text{Re}(z)$ and the imaginary part by $\text{Im}(z)$.

If $z = 3 - 4i$, then $\text{Re}(z) = 3$ and $\text{Im}(z) = -4$.

Purely real and purely imaginary complex numbers: A complex number z is purely real if its imaginary part is zero i.e. $\text{Im}(z) = 0$ and purely imaginary if its real part is zero i.e. $\text{Re}(z) = 0$.

Set of complex numbers: The set of all complex numbers is denoted by C i.e. $C = \{a + ib : a, b \in R\}$.

Since a real number 'a' can be written as $a + 0i$. Therefore, every real number is a complex number. Hence, $R \subset C$, where R is the set of all real numbers.

12.5 EQUALITY OF COMPLEX NUMBERS

DEFINITION Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal if $a_1 = a_2$ and $b_1 = b_2$.

i.e. $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.

Thus, $z_1 = z_2 \Leftrightarrow \text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.

ILLUSTRATION 1 If $z_1 = 2 - iy$ and $z_2 = x + 3i$ are equal, find x and y .

SOLUTION We have,

$$z_1 = z_2 \Rightarrow 2 - iy = x + 3i \Rightarrow 2 = x \text{ and } -y = 3 \Rightarrow x = 2 \text{ and } y = -3.$$

ILLUSTRATION 2 If $(a + b) - i(3a + 2b) = 5 + 2i$, find a and b .

SOLUTION We have,

$$(a + b) - i(3a + 2b) = 5 + 2i \Rightarrow a + b = 5 \text{ and } -(3a + 2b) = 2 \Rightarrow a = -12, b = 17$$

12.6 ADDITION OF COMPLEX NUMBERS

DEFINITION Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers. Then their sum $z_1 + z_2$ is defined as the complex number $(a_1 + a_2) + i(b_1 + b_2)$.

It follows from this definition that the sum $z_1 + z_2$ is a complex number such that

$$\text{Re}(z_1 + z_2) = \text{Re}(z_1) + \text{Re}(z_2) \text{ and } \text{Im}(z_1 + z_2) = \text{Im}(z_1) + \text{Im}(z_2)$$

For example, If $z_1 = 2 + 3i$ and $z_2 = 3 - 2i$, then $z_1 + z_2 = (2 + 3) + (3 - 2)i = 5 + i$

12.6.1 PROPERTIES OF ADDITION OF COMPLEX NUMBERS

(i) **Addition is Commutative:** For any two complex numbers z_1 and z_2

$$z_1 + z_2 = z_2 + z_1$$

PROOF Let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$, where a_1, a_2 and b_1, b_2 are real numbers. Then,

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2) \quad [\text{By definition of addition}]$$

$$= (a_2 + a_1) + i(b_2 + b_1) \quad [\text{By commutativity of addition of real numbers}]$$

$$= z_2 + z_1 \quad [\text{By definition of addition}]$$

Thus, $z_1 + z_2 = z_2 + z_1$ for all $z_1, z_2 \in C$.

Hence, addition of complex number is commutative.

(ii) **Addition is Associative:** For any three complex numbers z_1, z_2, z_3

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

PROOF Let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$ and $z_3 = a_3 + ib_3$, where a_1, a_2, a_3 and b_1, b_2, b_3 are real numbers. Then,

$$(z_1 + z_2) + z_3 = [(a_1 + a_2) + i(b_1 + b_2)] + (a_3 + ib_3) \quad [\text{By definition of addition}]$$

$$= [(a_1 + a_2) + a_3] + i[(b_1 + b_2) + b_3] \quad [\text{By definition of addition}]$$

$$= [(a_1 + (a_2 + a_3))] + i[b_1 + (b_2 + b_3)] \quad [\text{By associativity of addition on } R]$$

$$\begin{aligned}
 &= (a_1 + i b_1) + [(a_2 + a_3) + i (b_2 + b_3)] && \text{[By definition of addition]} \\
 &= z_1 + (z_2 + z_3) && \text{[By definition of addition]}
 \end{aligned}$$

Thus, $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ for all $z_1, z_2, z_3 \in \mathbb{C}$.

Hence, addition of complex numbers is associative.

(iii) *Existence of Additive Identity:* The complex number $0 = 0 + i 0$ is the identity element for addition i.e. $z + 0 = z = 0 + z$ for all $z \in \mathbb{C}$.

PROOF Let $z = a + i b$ be an arbitrary complex number. Then,

$$z + 0 = (a + i b) + (0 + i 0) = (a + 0) + i (b + 0) = a + i b = z$$

$$\text{and, } 0 + z = (0 + i 0) + (a + i b) = (0 + a) + i (0 + b) = a + i b = z$$

Thus, $z + 0 = z = 0 + z$ for all $z \in \mathbb{C}$

Hence, the complex number $0 = 0 + i 0$ is the identity element for addition.

(iv) *Existence of Additive Inverse:* For any complex number $z = a + i b$, there exists $-z = (-a) + i(-b)$ such that $z + (-z) = 0 = (-z) + z$.

PROOF Let $z = a + i b$ be an arbitrary complex number. Then, $-z = (-a) + i(-b)$ is also a complex number such that

$$z + (-z) = (a + i b) + \{(-a) + i(-b)\} = \{a + (-a)\} + i \{b + (-b)\} = 0 + i 0 = 0$$

$$\text{and } (-z) + z = \{(-a) + i(-b)\} + (a + i b) = \{(-a) + a\} + i \{(-b) + b\} = 0 + i 0 = 0.$$

Thus, for each complex number $z = a + i b$, there exists a complex number $-z = (-a) + i(-b)$ such that $z + (-z) = 0 = (-z) + z$.

The complex number $-z$ is called the *additive inverse* of z .

12.7 SUBTRACTION OF COMPLEX NUMBERS

DEFINITION Let $z_1 = a_1 + i b_1$ and $z_2 = a_2 + i b_2$ be two complex numbers. Then the subtraction of z_2 from z_1 is denoted by $z_1 - z_2$ and is defined as the addition of z_1 and $-z_2$.

$$\text{Thus, } z_1 - z_2 = z_1 + (-z_2) = (a_1 + i b_1) + (-a_2 - i b_2) = (a_1 - a_2) + i (b_1 - b_2)$$

For example, If $z_1 = -2 + 3i$ and $z_2 = 4 + 5i$, then

$$z_1 - z_2 = (-2 + 3i) + (-4 - 5i) = (-2 - 4) + i (3 - 5) = -6 - 2i$$

12.8 MULTIPLICATION OF COMPLEX NUMBERS

Let $z_1 = a_1 + i b_1$ and $z_2 = a_2 + i b_2$ be two complex numbers. Then the multiplication of z_1 with z_2 is denoted by $z_1 z_2$ and is defined as the complex number $(a_1 a_2 - b_1 b_2) + i (a_1 b_2 + a_2 b_1)$.

Thus, $z_1 z_2 = (a_1 + i b_1) (a_2 + i b_2)$

$$= (a_1 a_2 - b_1 b_2) + i (a_1 b_2 + a_2 b_1)$$

$$= [\text{Re}(z_1) \text{Re}(z_2) - \text{Im}(z_1) \text{Im}(z_2)] + i [\text{Re}(z_1) \text{Im}(z_2) + \text{Re}(z_2) \text{Im}(z_1)]$$

For example, If $z_1 = 3 + 2i$ and $z_2 = 2 - 3i$, then

$$z_1 z_2 = (3 + 2i) (2 - 3i) = (3 \times 2 - 2 \times (-3)) + i (3 \times -3 + 2 \times 2) = 12 - 5i$$

NOTE The product $z_1 z_2$ can also be obtained if we actually carry out the multiplication $(a_1 + i b_1) (a_2 + i b_2)$ as given below:

$$(a_1 + i b_1) (a_2 + i b_2) = a_1 a_2 + i a_1 b_2 + i a_2 b_1 + i^2 b_1 b_2$$

$$= (a_1 a_2 - b_1 b_2) + i (a_1 b_2 + a_2 b_1) \quad [\because i^2 = -1]$$

12.8.1 PROPERTIES OF MULTIPLICATION

(i) *Multiplication is commutative:* For any two complex numbers z_1 and z_2

$$z_1 z_2 = z_2 z_1$$

PROOF Let $z_1 = a_1 + i b_1$ and $z_2 = a_2 + i b_2$, where a_1, a_2 , and b_1, b_2 are real numbers. Then,

$$z_1 z_2 = (a_1 + i b_1)(a_2 + i b_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

$$\text{and, } z_2 z_1 = (a_2 + i b_2)(a_1 + i b_1) = (a_2 a_1 - b_2 b_1) + i(b_2 a_1 + b_1 a_2) \\ = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1) \quad [\text{By commutativity of mult. of real numbers}]$$

$$\therefore z_1 z_2 = z_2 z_1$$

Thus, $z_1 z_2 = z_2 z_1$ for all $z_1, z_2 \in \mathbb{C}$.

Hence, the multiplication of complex numbers is commutative on $\mathbb{C}$.

(ii) *Multiplication is associative: For any three complex numbers z_1, z_2, z_3*

$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

PROOF Let $z_1 = a_1 + i b_1, z_2 = a_2 + i b_2$ and $z_3 = a_3 + i b_3$ be any three complex numbers. Then,

$$\begin{aligned} (z_1 z_2) z_3 &= \{(a_1 + i b_1)(a_2 + i b_2)\}(a_3 + i b_3) \\ &= \{(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)\}(a_3 + i b_3) \\ &= \{(a_1 a_2 - b_1 b_2) a_3 - (a_1 b_2 + a_2 b_1) b_3\} + i\{(a_1 a_2 - b_1 b_2) b_3 + (a_1 b_2 + a_2 b_1) a_3\} \\ &= \{a_1(a_2 a_3 - b_2 b_3) - b_1(a_2 b_3 + a_3 b_2)\} + i\{b_1(a_2 a_3 - b_2 b_3) + a_1(a_3 b_2 + a_2 b_3)\} \\ &= (a_1 + i b_1)\{(a_2 a_3 - b_2 b_3) + i(a_2 b_3 + a_3 b_2)\} \\ &= z_1 (z_2 z_3) \end{aligned}$$

Thus, $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ for all $z_1, z_2, z_3 \in \mathbb{C}$.

Hence, multiplication is associative on $\mathbb{C}$.

(iii) *Existence of identity element for multiplication: The complex number $1 = 1 + i0$ is the identity element for multiplication i.e. for every complex number z , $z \cdot 1 = z = 1 \cdot z$.*

PROOF Let $z = a + i b$. Then,

$$z \cdot 1 = (a + i b)(1 + i0) = (a \times 1 - b \times 0) + i(a \times 0 + 1 \times b) = a + i b.$$

Similarly, we obtain $1 \cdot z = z$

Thus, $z \cdot 1 = z = 1 \cdot z$, for all $z \in \mathbb{C}$.

Hence, $1 = 1 + 0i$ is the multiplicative identity in $\mathbb{C}$.

(iv) *Existence of multiplicative inverse: Corresponding to every non-zero complex number $z = a + i b$ there exists a complex number $z_1 = x + i y$ such that $z \cdot z_1 = 1 = z_1 \cdot z$.*

PROOF Clearly,

$$z \cdot z_1 = 1$$

$$\Rightarrow (a + i b)(x + i y) = 1 + i0 \Rightarrow (ax - by) + i(ay + bx) = 1 + i0 \Rightarrow ax - by = 1 \text{ and } ay + bx = 0.$$

$$\text{Solving these two equations, we obtain: } x = \frac{a}{a^2 + b^2}, \quad y = -\frac{b}{a^2 + b^2} \quad [\because a \neq 0, b \neq 0]$$

Thus, every non-zero complex number $z = a + i b$ possesses multiplicative inverse given by

$$\left\{ \frac{a}{a^2 + b^2} \right\} + i \left\{ \frac{-b}{a^2 + b^2} \right\}$$

NOTE The multiplicative inverse of z is denoted by z^{-1} or, $\frac{1}{z}$

ILLUSTRATION Find the multiplicative inverse of $z = 3 - 2i$.

SOLUTION Using the above formula, we obtain

$$z^{-1} = \frac{3}{3^2 + (-2)^2} + \frac{i(-(-2))}{3^2 + (-2)^2} = \frac{3}{13} + \frac{2}{13}i$$

(v) *Multiplication of complex numbers is distributive over addition of complex numbers : For any three complex numbers z_1, z_2, z_3*

$$(i) \quad z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3 \quad \text{(Left distributivity)}$$

$$(ii) \quad (z_2 + z_3)z_1 = z_2 z_1 + z_3 z_1 \quad \text{(Right distributivity)}$$

PROOF Let $z_1 = a_1 + i b_1, z_2 = a_2 + i b_2$ and $z_3 = a_3 + i b_3$. Then,

$$\begin{aligned} z_1(z_2 + z_3) &= (a_1 + i b_1) \{(a_2 + a_3) + i(b_2 + b_3)\} \\ &= \{a_1(a_2 + a_3) - b_1(b_2 + b_3)\} + i\{a_1(b_2 + b_3) + b_1(a_2 + a_3)\} \\ &= [(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)] + [(a_1 a_3 - b_1 b_3) + i(a_1 b_3 + a_3 b_1)] \\ &= z_1 z_2 + z_1 z_3 \end{aligned}$$

Similarly, it can be established that $(z_2 + z_3)z_1 = z_2 z_1 + z_3 z_1$.

12.9 DIVISION OF COMPLEX NUMBERS

The division of a complex number z_1 by a non-zero complex number z_2 is defined as the multiplication of z_1 by the multiplicative inverse of z_2 and is denoted by $\frac{z_1}{z_2}$.

$$\text{Thus, } \frac{z_1}{z_2} = z_1 z_2^{-1} = z_1 \left\{ \frac{1}{z_2} \right\}$$

Let $z_1 = a_1 + i b_1$ and $z_2 = a_2 + i b_2$. Then,

$$\begin{aligned} \frac{z_1}{z_2} &= (a_1 + i b_1) \left\{ \frac{a_2}{a_2^2 + b_2^2} + i \frac{(-b_2)}{a_2^2 + b_2^2} \right\} \quad \left[\because z = a + i b \Rightarrow \frac{1}{z} = \frac{a}{a^2 + b^2} + \frac{i(-b)}{a^2 + b^2} \right] \\ \Rightarrow \frac{z_1}{z_2} &= \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} \right) + i \left(\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2} \right) \quad \text{[By definition of multiplication]} \end{aligned}$$

For example, If $z_1 = 2 + 3i$ and $z_2 = 1 + 2i$, then

$$\frac{z_1}{z_2} = z_1 \times \frac{1}{z_2} = (2 + 3i) \times \frac{1}{1 + 2i} = (2 + 3i) \frac{1}{5} - \frac{2}{5}i = \left(\frac{2}{5} + \frac{6}{5} \right) + i \left(-\frac{4}{5} + \frac{3}{5} \right) = \frac{8}{5} - \frac{1}{5}i$$

12.10 CONJUGATE OF A COMPLEX NUMBER

DEFINITION Let $z = a + i b$ be a complex number. Then the conjugate of z is denoted by $\bar{z}$ and is equal to $a - i b$.

$$\text{Thus, } z = a + i b \Rightarrow \bar{z} = a - i b$$

It follows from this definition that the conjugate of a complex number is obtained by replacing i by $-i$. For example, if $z = 3 + 4i$, then $\bar{z} = 3 - 4i$.

12.10.1 PROPERTIES OF CONJUGATE

THEOREM If z, z_1, z_2 are complex numbers, then

- | | | |
|--|--|---|
| (i) $(\bar{\bar{z}}) = z$ | (ii) $z + \bar{z} = 2 \operatorname{Re}(z)$ | (iii) $z - \bar{z} = 2i \operatorname{Im}(z)$ |
| (iv) $z = \bar{z} \Leftrightarrow z$ is purely real | (v) $z + \bar{z} = 0 \Rightarrow z$ is purely imaginary | |
| (vi) $z \bar{z} = [\operatorname{Re}(z)]^2 + [\operatorname{Im}(z)]^2$ | (vii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ | (viii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ |
| (ix) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ | (x) $\left(\frac{z_1}{z_2} \right) = \frac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$ | |

PROOF Let $z = a + ib$, $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$.

- (i) $z = a + ib \Rightarrow \bar{z} = a - ib \Rightarrow (\bar{\bar{z}}) = \overline{(a - ib)} = a + ib \Rightarrow (\bar{\bar{z}}) = z$.
- (ii) $z + \bar{z} = (a + ib) + (a - ib) = 2a = 2 \operatorname{Re}(z)$
- (iii) $z - \bar{z} = (a + ib) - (a - ib) = 2ib = 2i \operatorname{Im}(z)$
- (iv) $z = \bar{z} \Leftrightarrow a + ib = a - ib \Leftrightarrow 2ib = 0 \Leftrightarrow b = 0 \Leftrightarrow \operatorname{Im}(z) = 0 \Rightarrow z$ is purely real
- (v) $z + \bar{z} = 0 \Leftrightarrow (a + ib) + (a - ib) = 0 \Leftrightarrow 2a = 0 \Leftrightarrow a = 0 \Leftrightarrow \operatorname{Re}(z) = 0 \Leftrightarrow z$ is purely imaginary
- (vi) $z\bar{z} = (a + ib)(a - ib) = a^2 + b^2 = [\operatorname{Re}(z)]^2 + [\operatorname{Im}(z)]^2$
- (vii) We have, $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$
- $\therefore \overline{z_1 + z_2} = \overline{(a_1 + a_2) + i(b_1 + b_2)} = (a_1 + a_2) - i(b_1 + b_2) = (a_1 - ib_1) + (a_2 - ib_2) = \overline{(a_1 + ib_1)} + \overline{(a_2 + ib_2)} = \bar{z}_1 + \bar{z}_2$.
- (viii) We have, $z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$
- $\therefore \overline{z_1 - z_2} = \overline{(a_1 - a_2) + i(b_1 - b_2)} = (a_1 - a_2) - i(b_1 - b_2) = (a_1 - ib_1) - (a_2 - ib_2) = \overline{(a_1 + ib_1)} - \overline{(a_2 + ib_2)} = \bar{z}_1 - \bar{z}_2$
- (ix) We have, $z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$
- $\therefore \overline{z_1 z_2} = \overline{(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)} = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + a_2 b_1) = \overline{(a_1 - ib_1)(a_2 - ib_2)} = \bar{z}_1 \bar{z}_2$
- (x) We have, $\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} = \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} \right) + i \left(\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2} \right)$
- $\therefore \overline{\left(\frac{z_1}{z_2} \right)} = \overline{\left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} \right) + i \left(\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2} \right)} = \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} \right) - i \left(\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2} \right) \quad \dots(i)$

Now, $\frac{\bar{z}_1}{\bar{z}_2} = \bar{z}_1 \times \frac{1}{\bar{z}_2} = (a_1 - ib_1) \left(\frac{a_2}{a_2^2 + b_2^2} + i \frac{b_2}{a_2^2 + b_2^2} \right) = \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} \right) - i \left(\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2} \right) \quad \dots(ii)$

From (i) and (ii), we obtain: $\overline{\left(\frac{z_1}{z_2} \right)} = \frac{\bar{z}_1}{\bar{z}_2}$.

12.11 MODULUS OF A COMPLEX NUMBER

DEFINITION The modulus of a complex number $z = a + ib$ is denoted by $|z|$ and is defined as

$$|z| = \sqrt{a^2 + b^2} = \sqrt{[\operatorname{Re}(z)]^2 + [\operatorname{Im}(z)]^2}$$

Clearly, $|z| \geq 0$ for all $z \in C$.

For example, If $z_1 = 3 - 4i$, $z_2 = -5 + 2i$ and $z_3 = 1 + \sqrt{-3}$, then

$$|z_1| = \sqrt{3^2 + (-4)^2} = 5, |z_2| = \sqrt{(-5)^2 + 2^2} = \sqrt{29} \text{ and } |z_3| = |1 + i\sqrt{3}| = \sqrt{1^2 + (\sqrt{3})^2} = 2.$$

REMARK In the set C of all complex numbers, the order relation is not defined. As such $z_1 > z_2$ or, $z_1 < z_2$ has no meaning but $|z_1| > |z_2|$ or, $|z_1| < |z_2|$ has got its meaning as $|z_1|$ and $|z_2|$ are real numbers.

12.11.1 PROPERTIES OF MODULUS

THEOREM If $z, z_1, z_2 \in C$, then

- (i) $|z| = 0 \Leftrightarrow z = 0$ i.e. $\operatorname{Re}(z) = \operatorname{Im}(z) = 0$
- (ii) $|z| = |\bar{z}| = |-z|$
- (iii) $-|z| \leq \operatorname{Re}(z) \leq |z|$; $-|z| \leq \operatorname{Im}(z) \leq |z|$
- (iv) $z\bar{z} = |z|^2$
- (v) $|z_1 z_2| = |z_1| |z_2|$
- (vi) $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$; $z_2 \neq 0$
- (vii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2 \operatorname{Re}(z_1 \bar{z}_2)$
- (viii) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2 \operatorname{Re}(z_1 \bar{z}_2)$

$$(ix) |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

$$(x) |az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2), \text{ where } a, b \in \mathbb{R}.$$

[NCERT EXEMPLAR]

PROOF Let $z = a + ib$. Then,

$$(i) |z| = 0 \Leftrightarrow \sqrt{a^2 + b^2} = 0 \Leftrightarrow a^2 + b^2 = 0 \Leftrightarrow a = 0 \text{ and } b = 0 \Leftrightarrow \operatorname{Re}(z) = \operatorname{Im}(z) = 0$$

$$(ii) \text{ Let } z = a + ib. \text{ Then, } \bar{z} = a - ib \text{ and } -z = -a - ib.$$

$$\therefore |z| = \sqrt{a^2 + b^2}, |\bar{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} \text{ and, } |-z| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2}$$

$$\text{Clearly, } |z| = |\bar{z}| = |-z|$$

$$(iii) \text{ Let } z = a + ib. \text{ Then, } |z| = \sqrt{a^2 + b^2}.$$

$$\text{Clearly, } -\sqrt{a^2 + b^2} \leq a \leq \sqrt{a^2 + b^2} \text{ and } -\sqrt{a^2 + b^2} \leq b \leq \sqrt{a^2 + b^2}$$

$$\Rightarrow -|z| \leq \operatorname{Re}(z) \leq |z| \text{ and } -|z| \leq \operatorname{Im}(z) \leq |z|$$

$$(iv) \text{ Let } z = a + ib. \text{ Then, } \bar{z} = a - ib.$$

$$\therefore z\bar{z} = (a + ib)(a - ib) = a^2 - i^2 b^2 = a^2 + b^2 = \left\{ \sqrt{a^2 + b^2} \right\}^2 = |z|^2$$

$$(v) \text{ Let } z_1 = a_1 + ib_1 \text{ and } z_2 = a_2 + ib_2, \text{ where } a_1, a_2 \text{ and } b_1, b_2 \text{ are real numbers. Then,}$$

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

$$\begin{aligned} \Rightarrow |z_1 z_2| &= \sqrt{(a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + a_2 b_1)^2} = \sqrt{a_1^2 a_2^2 + b_1^2 b_2^2 + a_1^2 b_2^2 + a_2^2 b_1^2} \\ &= \sqrt{a_1^2 (a_2^2 + b_2^2) + b_1^2 (a_2^2 + b_2^2)} = \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} = \sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2} \\ &= |z_1| |z_2| \end{aligned}$$

$$(vi) \text{ Let } z_1 = a_1 + ib_1 \text{ and } z_2 = a_2 + ib_2, \text{ where } a_1, a_2 \text{ and } b_1, b_2 \text{ are real numbers. Then,}$$

$$\frac{z_1}{z_2} = z_1 \times \frac{1}{z_2} = (a_1 + ib_1) \left(\frac{a_2}{a_2^2 + b_2^2} + i \frac{(-b_2)}{a_2^2 + b_2^2} \right) = \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} \right) + i \left(\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2} \right)$$

$$\begin{aligned} \Rightarrow \left| \frac{z_1}{z_2} \right| &= \sqrt{\left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} \right)^2 + \left(\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2} \right)^2} = \sqrt{\frac{(a_1 a_2 + b_1 b_2)^2 + (a_2 b_1 - a_1 b_2)^2}{(a_2^2 + b_2^2)^2}} \\ &= \sqrt{\frac{a_1^2 a_2^2 + b_1^2 b_2^2 + a_2^2 b_1^2 + a_1^2 b_2^2}{(a_2^2 + b_2^2)^2}} = \sqrt{\frac{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}{(a_2^2 + b_2^2)^2}} = \sqrt{\frac{a_1^2 + b_1^2}{a_2^2 + b_2^2}} \\ &= \frac{\sqrt{a_1^2 + b_1^2}}{\sqrt{a_2^2 + b_2^2}} = \frac{|z_1|}{|z_2|} \end{aligned}$$

(vii) Clearly,

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1 + z_2}) & [\because z\bar{z} = |z|^2] \\ &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) & [\because \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2] \\ &= z_1 \bar{z}_1 + z_2 \bar{z}_2 + z_1 \bar{z}_2 + z_2 \bar{z}_1 & [\text{By distributivity of multiplication}] \\ &= |z_1|^2 + |z_2|^2 + z_1 \bar{z}_2 + (\overline{z_1 \bar{z}_2}) & [\because \overline{(z_1 \bar{z}_2)} = \bar{z}_1 (\overline{\bar{z}_2}) = \bar{z}_1 z_2 = z_2 \bar{z}_1] \\ &= |z_1|^2 + |z_2|^2 + 2 \operatorname{Re}(z_1 \bar{z}_2) & [\because z + \bar{z} = 2 \operatorname{Re}(z)] \end{aligned}$$

(viii) Clearly,

$$\begin{aligned}
 |z_1 - z_2|^2 &= (z_1 - z_2)(\overline{z_1 - z_2}) & [\because z\bar{z} = |z|^2] \\
 &= (z_1 - z_2)(\overline{z_1} - \overline{z_2}) & [\because \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}] \\
 &= z_1\overline{z_1} + z_2\overline{z_2} - z_1\overline{z_2} - z_2\overline{z_1} & [\text{By distributivity of multiplication}] \\
 &= |z_1|^2 + |z_2|^2 - z_1\overline{z_2} - \overline{z_1}z_2 & [\because \overline{(z_1\overline{z_2})} = \overline{z_1}(\overline{\overline{z_2}}) = \overline{z_1}z_2] \\
 &= |z_1|^2 + |z_2|^2 - 2\operatorname{Re}(z_1\overline{z_2})
 \end{aligned}$$

(ix) Using (vii) and (viii), we get

$$\begin{aligned}
 |z_1 + z_2|^2 + |z_1 - z_2|^2 &= |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1\overline{z_2}) + |z_1|^2 + |z_2|^2 - 2\operatorname{Re}(z_1\overline{z_2}) \\
 &= 2(|z_1|^2 + |z_2|^2)
 \end{aligned}$$

$$\begin{aligned}
 \text{(x)} \quad |az_1 - bz_2|^2 &= (az_1 - bz_2)(\overline{az_1 - bz_2}) \\
 &= (az_1 - bz_2)(a\overline{z_1} - b\overline{z_2}) \\
 &= a^2 z_1\overline{z_1} - (az_1)(b\overline{z_2}) - (bz_2)(a\overline{z_1}) + b^2 z_2\overline{z_2} \\
 &= a^2 |z_1|^2 - ab(z_1\overline{z_2} + \overline{z_1}z_2) + b^2 |z_2|^2 \\
 &= a^2 |z_1|^2 - ab(z_1\overline{z_2} + \overline{z_1}z_2) + b^2 |z_2|^2 \\
 &= a^2 |z_1|^2 - ab\{2\operatorname{Re}(z_1\overline{z_2})\} + b^2 |z_2|^2 & [\because z_1\overline{z_2} + \overline{z_1}z_2 = 2\operatorname{Re}(z_1\overline{z_2})] \\
 &= a^2 |z_1|^2 - 2ab\operatorname{Re}(z_1\overline{z_2}) + b^2 |z_2|^2
 \end{aligned}$$

Similarly, we obtain

$$|bz_1 + az_2|^2 = b^2 |z_1|^2 + a^2 |z_2|^2 + 2ab\operatorname{Re}(z_1\overline{z_2})$$

$$\begin{aligned}
 \therefore |az_1 - bz_2|^2 + |bz_1 + az_2|^2 &= a^2 |z_1|^2 - 2ab\operatorname{Re}(z_1\overline{z_2}) + b^2 |z_2|^2 + b^2 |z_1|^2 + a^2 |z_2|^2 + 2ab\operatorname{Re}(z_1\overline{z_2}) \\
 &= |z_1|^2 (a^2 + b^2) + |z_2|^2 (b^2 + a^2) \\
 &= (a^2 + b^2)(|z_1|^2 + |z_2|^2)
 \end{aligned}$$

12.12 RECIPROCAL OF A COMPLEX NUMBER

Let $z = a + ib$ be a non-zero complex number. Then,

$$\frac{1}{z} = \frac{1}{a + ib} = \frac{1}{a + ib} \times \frac{a - ib}{a - ib} \quad \left[\begin{array}{l} \text{Multiplying numerator and denominator} \\ \text{by conjugate of denominator} \end{array} \right]$$

$$\Rightarrow \frac{1}{z} = \frac{a - ib}{a^2 - i^2 b^2} = \frac{a - ib}{a^2 + b^2} = \frac{a}{a^2 + b^2} + \frac{i(-b)}{a^2 + b^2}$$

Clearly, $\frac{1}{z}$ is equal to the multiplicative inverse of z . Also, $\frac{1}{z} = \frac{a - ib}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$

Thus, the multiplicative inverse of a non-zero complex number z is same as its reciprocal and is given by

$$\frac{\operatorname{Re}(z)}{|z|^2} + i \frac{(-\operatorname{Im}(z))}{|z|^2} = \frac{\bar{z}}{|z|^2}$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I EXPRESSING A COMPLEX NUMBER IN THE STANDARD FORM $a + ib$

In order to express a complex number in the standard form, we may follow the following algorithm.

ALGORITHM

Step I Write the complex number in the form $\frac{a + ib}{c + id}$ by using fundamental operations of addition, subtraction and multiplication.

Step II Multiply the numerator and denominator by the conjugate of the denominator.

EXAMPLE 1 Express the following in the form $a + ib$:

$$(i) (-5i)\left(\frac{1}{8}i\right) \quad \text{[NCERT]} \quad (ii) (-i)(2i)\left(-\frac{1}{8}i\right)^3 \quad \text{[NCERT]}$$

$$(iii) (5i)\left(-\frac{3}{5}i\right) \quad \text{[NCERT]} \quad (iv) i^9 + i^{19}$$

$$(v) i^{-39} \quad \text{[NCERT]} \quad (vi) (1-i)^4 \quad \text{[NCERT]}$$

SOLUTION (i) $(-5i)\left(\frac{1}{8}i\right) = -\frac{5}{8}i^2 = -\frac{5}{8} \times -1 = \frac{5}{8} = \frac{5}{8} + 0i$

$$(ii) (-i)(2i)\left(-\frac{1}{8}i\right)^3 = -2i^2 \times -\frac{1}{512}i^3 = \frac{1}{256} \times i^2 \times i^3 = \frac{1}{256}i^5 = \frac{i}{256} = 0 + \frac{1}{256}i$$

$$(iii) (5i)\left(-\frac{3}{5}i\right) = -3i^2 = -3 \times -1 = 3 = 3 + 0i$$

$$(iv) i^9 + i^{19} = (i^4)^2 i + (i^4)^4 i^3 = i + i^3 = i - i = 0 = 0 + 0i$$

$$(v) i^{-39} = (i^4)^{-10} i = i = 0 + 1i$$

$$(vi) (1-i)^4 = \left\{(1-i)^2\right\}^2 = (1-2i+i^2)^2 = (1-2i-1)^2 = (-2i)^2 = 4i^2 = -4 = -4 + 0i$$

EXAMPLE 2 Express each of the following in the form $a + ib$:

$$(i) 3(7+7i) + i(7+7i) \quad \text{[NCERT]} \quad (ii) (1-i) - (-1+6i) \quad \text{[NCERT]}$$

$$(iii) \left(\frac{1}{5} + \frac{2}{5}i\right) - \left(4 + \frac{5}{2}i\right) \quad \text{[NCERT]} \quad (iv) \left\{\left(\frac{1}{3} + \frac{7}{3}i\right) + \left(4 + \frac{1}{3}i\right)\right\} - \left(-\frac{4}{3} + i\right) \quad \text{[NCERT]}$$

SOLUTION (i) $3(7+7i) + i(7+7i) = 21 + 21i + 7i + 7i^2 = 21 + 21i + 7i - 7 = 14 + 28i$

$$(ii) (1-i) - (-1+6i) = 1-i+1-6i = 2-7i$$

$$(iii) \left(\frac{1}{5} + \frac{2}{5}i\right) - \left(4 + \frac{5}{2}i\right) = \left(\frac{1}{5} - 4\right) + \frac{2i}{5} - \frac{5i}{2} = -\frac{19}{5} - \frac{21}{10}i$$

$$\begin{aligned} (iv) \left\{\left(\frac{1}{3} + \frac{7}{3}i\right) + \left(4 + \frac{1}{3}i\right)\right\} - \left(-\frac{4}{3} + i\right) &= \left\{\left(\frac{1}{3} + 4\right) + i\left(\frac{7}{3} + \frac{1}{3}\right)\right\} - \left(-\frac{4}{3} + i\right) \\ &= \left(\frac{13}{3} + \frac{8}{3}i\right) + \frac{4}{3} - i \\ &= \left(\frac{13}{3} + \frac{4}{3}\right) + \left(\frac{8}{3} - 1\right)i = \frac{17}{3} + \frac{5}{3}i \end{aligned}$$

EXAMPLE 3 Express each of the following in the form $a + ib$:

(i) $\left(\frac{1}{3} + 3i\right)^3$ [NCERT]

(ii) $\left(-2 - \frac{1}{3}i\right)^3$ [NCERT]

(iii) $(5 - 3i)^3$ [NCERT]

(iv) $(-\sqrt{3} + \sqrt{-2})(2\sqrt{3} - i)$

SOLUTION (i) $\left(\frac{1}{3} + 3i\right)^3 = \left(\frac{1}{3}\right)^3 + (3i)^3 + 3 \times \frac{1}{3} \times 3i \left(\frac{1}{3} + 3i\right) = \frac{1}{27} + 27i^3 + 3i \left(\frac{1}{3} + 3i\right)$
 $= \frac{1}{27} + 27i^3 + i + 9i^2 = \frac{1}{27} - 27i + i - 9 = -\frac{242}{27} - 26i$

(ii) $\left(-2 - \frac{1}{3}i\right)^3 = (-2)^3 + \left(-\frac{1}{3}i\right)^3 + 3 \times -2 \times -\frac{1}{3}i \left(-2 - \frac{1}{3}i\right) = -8 - \frac{1}{27}i^3 + 2i \left(-2 - \frac{1}{3}i\right)$
 $= -8 + \frac{1}{27}i - 4i - \frac{2}{3}i^2 = -8 + \frac{1}{27}i - 4i + \frac{2}{3} = -\frac{22}{3} - \frac{107}{27}i$

(iii) $(5 - 3i)^3 = 5^3 + (-3i)^3 + 3 \times 25 \times -3i + 3 \times 5 \times (-3i)^2 = 125 + 27i - 225i - 135 = -10 - 198i$

(iv) $(-\sqrt{3} + \sqrt{-2})(2\sqrt{3} - i) = (-\sqrt{3} + i\sqrt{2})(2\sqrt{3} - i) = -6 + \sqrt{3}i + 2\sqrt{6}i - \sqrt{2}i^2$
 $= -6 + (\sqrt{3} + 2\sqrt{6})i + \sqrt{2} = (\sqrt{2} - 6) + (\sqrt{3} + 2\sqrt{6})i$

EXAMPLE 4 Express each one of the following in the standard form $a + ib$.

(i) $\frac{1}{3 - 4i}$

(ii) $\frac{5 + 4i}{4 + 5i}$

(iii) $\frac{(1 + i)^2}{3 - i}$

(iv) $\frac{(3 - 2i)(2 + 3i)}{(1 + 2i)(2 - i)}$

(v) $\frac{1}{-2 + \sqrt{-3}}$

(vi) $\left(\frac{1}{1 - 2i} + \frac{3}{1 + i}\right) \left(\frac{3 + 4i}{2 - 4i}\right)$

(vii) $\frac{1}{1 - \cos \theta + 2i \sin \theta}$

(viii) $\frac{(3 + i\sqrt{5})(3 - i\sqrt{5})}{(\sqrt{3} + \sqrt{2}i) - (\sqrt{3} - i\sqrt{2})}$ [NCERT]

SOLUTION (i) $\frac{1}{3 - 4i} = \frac{1}{3 - 4i} \times \frac{3 + 4i}{3 + 4i} = \frac{3 + 4i}{9 - 16i^2} = \frac{3 + 4i}{9 + 16} = \frac{3}{25} + \frac{4}{25}i$

(ii) $\frac{5 + 4i}{4 + 5i} = \frac{5 + 4i}{4 + 5i} \times \frac{4 - 5i}{4 - 5i} = \frac{(20 + 20) + i(16 - 25)}{16 - 25i^2} = \frac{40 - 9i}{41} = \frac{40}{41} - \frac{9}{41}i$

(iii) $\frac{(1 + i)^2}{3 - i} = \frac{1 + 2i + i^2}{3 - i} = \frac{2i}{3 - i} = \frac{2i}{3 - i} \cdot \frac{3 + i}{3 + i} = \frac{6i + 2i^2}{9 - i^2} = \frac{-2 + 6i}{10} = -\frac{1}{5} + \frac{3}{5}i$

(iv) $\frac{(3 - 2i)(2 + 3i)}{(1 + 2i)(2 - i)} = \frac{(6 + 6) + i(-4 + 9)}{(2 + 2) + i(4 - 1)} = \frac{12 + 5i}{4 + 3i} = \frac{12 + 5i}{4 + 3i} \times \frac{4 - 3i}{4 - 3i}$
 $= \frac{(48 + 15) + i(-36 + 20)}{16 - 9i^2} = \frac{63 - 16i}{25} = \frac{63}{25} - \frac{16}{25}i$

(v) $\frac{1}{-2 + \sqrt{-3}} = \frac{1}{-2 + i\sqrt{3}} = \frac{1}{-2 + i\sqrt{3}} \times \frac{-2 - i\sqrt{3}}{-2 - i\sqrt{3}} = \frac{-2 - i\sqrt{3}}{4 - 3i^2} = -\frac{2}{7} - \frac{\sqrt{3}}{7}i$

(vi) $\left(\frac{1}{1 - 2i} + \frac{3}{1 + i}\right) \left(\frac{3 + 4i}{2 - 4i}\right) = \frac{1 + i + 3 - 6i}{(1 + 2) + i(-2 + 1)} \times \frac{3 + 4i}{2 - 4i} = \frac{4 - 5i}{3 - i} \times \frac{3 + 4i}{2 - 4i} = \frac{(12 + 20) + i(16 - 15)}{(6 - 4) + i(-2 - 12)}$
 $= \frac{32 + i}{2 - 4i} = \frac{32 + i}{2 - 4i} \times \frac{2 + 4i}{2 + 4i} = \frac{(64 - 14) + i(2 + 448)}{4 - 196i^2} = \frac{50 + 450i}{200} = \frac{1}{4} + \frac{9}{4}i$

$$\begin{aligned}
 \text{(vii)} \quad \frac{1}{1 - \cos \theta + 2i \sin \theta} &= \frac{1}{1 - \cos \theta + 2i \sin \theta} \times \frac{1 - \cos \theta - 2i \sin \theta}{1 - \cos \theta - 2i \sin \theta} \\
 &= \frac{1 - \cos \theta - 2i \sin \theta}{(1 - \cos \theta)^2 - 4i^2 \sin^2 \theta} = \frac{1 - \cos \theta - 2i \sin \theta}{(1 - \cos \theta)^2 + 4 \sin^2 \theta} \\
 &= \frac{1 - \cos \theta - 2i \sin \theta}{1 - 2 \cos \theta + \cos^2 \theta + 4 \sin^2 \theta} = \frac{1 - \cos \theta - 2i \sin \theta}{2 - 2 \cos \theta + 3 \sin^2 \theta} \\
 &= \left(\frac{1 - \cos \theta}{2 - 2 \cos \theta + 3 \sin^2 \theta} \right) + i \left(\frac{-2 \sin \theta}{2 - 2 \cos \theta + 3 \sin^2 \theta} \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{(viii)} \quad \frac{(3 + i\sqrt{5})(3 - i\sqrt{5})}{(\sqrt{3} + \sqrt{2}i) - (\sqrt{3} - i\sqrt{2})} &= \frac{(9 - \sqrt{5} \times -\sqrt{5}) + i(3 \times -\sqrt{5} + 3\sqrt{5})}{\sqrt{3} + \sqrt{2}i - \sqrt{3} + \sqrt{2}i} \\
 &= \frac{(9 + 5) + i \times 0}{2\sqrt{2}i} = \frac{14}{2\sqrt{2}i} = \frac{7}{\sqrt{2}i} = \frac{-7}{\sqrt{2}}i = 0 - \frac{7}{\sqrt{2}}i
 \end{aligned}$$

EXAMPLE 5 Prove that the following complex numbers are purely real:

$$\text{(i)} \left(\frac{2 + 3i}{3 + 4i} \right) \left(\frac{2 - 3i}{3 - 4i} \right) \qquad \text{(ii)} \left(\frac{3 + 2i}{2 - 3i} \right) + \left(\frac{3 - 2i}{2 + 3i} \right)$$

SOLUTION (i) $\left(\frac{2 + 3i}{3 + 4i} \right) \left(\frac{2 - 3i}{3 - 4i} \right) = \frac{(2 + 3i)(2 - 3i)}{(3 + 4i)(3 - 4i)} = \frac{4 - 9i^2}{9 - 16i^2} = \frac{13}{25}$, which is purely real.

(ii) $\left(\frac{3 + 2i}{2 - 3i} \right) + \left(\frac{3 - 2i}{2 + 3i} \right) = \frac{3 + 2i}{2 - 3i} \times \frac{2 + 3i}{2 + 3i} + \frac{3 - 2i}{2 + 3i} \times \frac{2 - 3i}{2 - 3i} = \frac{(3 + 2i)(2 + 3i)}{4 - 9i^2} + \frac{(3 - 2i)(2 - 3i)}{4 - 9i^2}$
 $= \frac{13i}{13} - \frac{13i}{13} = 0$, which is purely real.

EXAMPLE 6 Express $(1 - 2i)^{-3}$ in the standard form $a + ib$.

SOLUTION $(1 - 2i)^{-3} = \frac{1}{(1 - 2i)^3} = \frac{1}{1 - 8i^3 - 6i + 12i^2} = \frac{1}{1 + 8i - 6i - 12} = \frac{1}{-11 + 2i}$
 $= \frac{1}{-11 + 2i} \times \frac{-11 - 2i}{-11 - 2i} = \frac{-11 - 2i}{(-11)^2 - (2i)^2} = \frac{-11 - 2i}{125} = \frac{-11}{125} - \frac{2}{125}i$

EXAMPLE 7 Perform the suitable operations to express the result in the form $a + ib$.

$$\text{(i)} \frac{2 - \sqrt{-25}}{1 - \sqrt{-16}} \qquad \text{(ii)} \frac{3 - \sqrt{-16}}{1 - \sqrt{-9}}$$

SOLUTION We have,

(i) $\frac{2 - \sqrt{-25}}{1 - \sqrt{-16}} = \frac{2 - 5i}{1 - 4i} = \frac{2 - 5i}{1 - 4i} \times \frac{1 + 4i}{1 + 4i} = \frac{(2 + 20) + i(8 - 5)}{1 - 16i^2} = \frac{22 + 3i}{17} = \frac{22}{17} + \frac{3}{17}i$

(ii) $\frac{3 - \sqrt{-16}}{1 - \sqrt{-9}} = \frac{3 - 4i}{1 - 3i} = \frac{3 - 4i}{1 - 3i} \times \frac{1 + 3i}{1 + 3i} = \frac{(3 + 12) + i(-4 + 9)}{1 - 9i^2} = \frac{15 + 5i}{10} = \frac{3}{2} + \frac{1}{2}i$

EXAMPLE 8 If z_1, z_2 are $1 - i, -2 + 4i$, respectively, find $\text{Im} \left(\frac{z_1 z_2}{\bar{z}_1} \right)$.

SOLUTION $\frac{z_1 z_2}{\bar{z}_1} = \frac{(1 - i)(-2 + 4i)}{(1 - i)} = \frac{(-2 + 4) + i(2 + 4)}{1 + i} = \frac{2 + 6i}{1 + i}$

$$= \frac{2+6i}{1+i} \times \frac{1-i}{1-i} = \frac{(2+6)+i(6-2)}{1+1} = 4+2i$$

$$\therefore \operatorname{Im}\left(\frac{z_1 z_2}{\bar{z}_1}\right) = 2$$

Type II ON EQUALITY OF COMPLEX NUMBERS

Recall that two complex numbers z_1 and z_2 are equal iff $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$.

EXAMPLE 9 Find the real values of x and y , if

$$\begin{aligned} \text{(i)} \quad (3x-7) + 2iy &= -5y + (5+x)i & \text{(ii)} \quad (1-i)x + (1+i)y &= 1-3i \\ \text{(iii)} \quad (x+iy)(2-3i) &= 4+i & \text{(iv)} \quad \frac{x-1}{3+i} + \frac{y-1}{3-i} &= i \end{aligned}$$

SOLUTION (i) We have

$$(3x-7) + 2iy = -5y + (5+x)i$$

$$\Rightarrow 3x-7 = -5y \text{ and } 2y = 5+x \Rightarrow 3x+5y=7 \text{ and } x-2y=-5 \Rightarrow x=-1, y=2.$$

(ii) We have,

$$(1-i)x + (1+i)y = 1-3i$$

$$\Rightarrow (x+y) + i(-x+y) = 1-3i \Rightarrow x+y=1 \text{ and } -x+y=-3 \Rightarrow x=2, y=-1$$

(iii) We have,

$$(x+iy)(2-3i) = 4+i$$

$$\Rightarrow (2x+3y) + i(-3x+2y) = 4+i \Rightarrow 2x+3y=4 \text{ and } -3x+2y=1 \Rightarrow x = \frac{5}{13}, y = \frac{14}{13}$$

(iv) We have,

$$\frac{x-1}{3+i} + \frac{y-1}{3-i} = i$$

$$\Rightarrow \frac{(x-1)(3-i) + (y-1)(3+i)}{(3+i)(3-i)} = i \Rightarrow \frac{(3x+3y-6) + i(y-x)}{9-i^2} = i$$

$$\Rightarrow \left(\frac{3x+3y-6}{10}\right) + i\left(\frac{y-x}{10}\right) = 0+i \Rightarrow \frac{3x+3y-6}{10} = 0 \text{ and } \frac{y-x}{10} = 1$$

$$\Rightarrow x+y-2=0 \text{ and } y-x=10 \Rightarrow x=-4, y=6.$$

EXAMPLE 10 Find real values of x and y for which the following equalities hold:

$$\text{(i)} \quad (1+i)y^2 + (6+i) = (2+i)x \quad \text{(ii)} \quad (x^4 + 2xi) - (3x^2 + iy) = (3-5i) + (1+2iy)$$

SOLUTION (i) We have,

$$(1+i)y^2 + (6+i) = (2+i)x \Rightarrow (y^2+6) + i(y^2+1) = 2x+ix$$

$$\Rightarrow y^2+6=2x \quad \dots\text{(i)} \quad \text{and,} \quad y^2+1=x \quad \dots\text{(ii)}$$

From (i) and (ii), we get

$$y^2+6=2(y^2+1) \Rightarrow y^2=4 \Rightarrow y=\pm 2$$

Substituting $y=\pm 2$ in (ii), we get $x=5$. Thus, $x=5$ and $y=2$ or, $x=5$ and $y=-2$

(ii) We have,

$$(x^4 + 2xi) - (3x^2 + iy) = (3-5i) + (1+2iy)$$

$$\Rightarrow (x^4 - 3x^2) + i(2x - y) = 4 + i(2y - 5)$$

$$\Rightarrow x^4 - 3x^2 = 4 \text{ and, } 2x - y = 2y - 5 \quad [\text{On equating real and imaginary parts}]$$

$$\Rightarrow x^4 - 3x^2 - 4 = 0, \quad 2x - 3y + 5 = 0$$

$$\text{Now, } x^4 - 3x^2 - 4 = 0 \Rightarrow (x^2 - 4)(x^2 + 1) = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x = \pm 2$$

Putting $x=\pm 2$ in $2x-3y+5=0$, we get

$y = 3$ when $x = 2$ and $y = 1/3$ when $x = -2$

Thus, $x = 2$ and $y = 3$ or, $x = -2$ and $y = 1/3$.

EXAMPLE 11 If $a + ib = \frac{c+i}{c-i}$, where c is real, prove that: $a^2 + b^2 = 1$ and $\frac{b}{a} = \frac{2c}{c^2 - 1}$.

SOLUTION We have,

$$a + ib = \frac{c+i}{c-i} \Rightarrow a + ib = \frac{(c+i)(c+i)}{(c-i)(c+i)} \Rightarrow a + ib = \frac{(c+i)^2}{c^2 - i^2} \Rightarrow a + ib = \frac{c^2 + 2ic + i^2}{c^2 - i^2}$$

$$\Rightarrow a + ib = \frac{c^2 - 1}{c^2 + 1} + \frac{i 2c}{c^2 + 1} \Rightarrow a = \frac{c^2 - 1}{c^2 + 1} \text{ and } b = \frac{2c}{c^2 + 1}$$

$$\Rightarrow a^2 + b^2 = \left(\frac{c^2 - 1}{c^2 + 1} \right)^2 + \frac{4c^2}{(c^2 + 1)^2} \text{ and, } \frac{b}{a} = \left(\frac{2c}{c^2 + 1} \right) \div \left(\frac{c^2 - 1}{c^2 + 1} \right)$$

$$\Rightarrow a^2 + b^2 = \frac{(c^2 + 1)^2}{(c^2 + 1)^2} = 1 \text{ and, } \frac{b}{a} = \frac{2c}{c^2 - 1}$$

EXAMPLE 12 If $(x + iy)^{1/3} = a + ib$, $x, y, a, b \in \mathbb{R}$. Show that

$$(i) \frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2) \quad (ii) \frac{x}{a} - \frac{y}{b} = -2(a^2 + b^2)$$

[NCERT EXEMPLAR]

SOLUTION We have,

$$(x + iy)^{1/3} = a + ib$$

$$\Rightarrow (x + iy) = (a + ib)^3$$

[On cubing both sides]

$$\Rightarrow x + iy = a^3 + 3a^2 ib + 3a i^2 b^2 + i^3 b^3$$

$$\Rightarrow x + iy = (a^3 - 3ab^2) + i(3a^2 b - b^3)$$

$$\Rightarrow x = a^3 - 3ab^2 \text{ and } y = 3a^2 b - b^3$$

$$\Rightarrow \frac{x}{a} = a^2 - 3b^2 \text{ and } \frac{y}{b} = 3a^2 - b^2$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = a^2 - 3b^2 + 3a^2 - b^2 = 4(a^2 - b^2) \text{ and } \frac{x}{a} - \frac{y}{b} = (a^2 - 3b^2) - (3a^2 - b^2) = -2(a^2 + b^2)$$

Type III ON CONJUGATE OF A COMPLEX NUMBER

EXAMPLE 13 Multiply $3 - 2i$ by its conjugate.

SOLUTION The conjugate of $3 - 2i$ is $3 + 2i$.

$$\therefore \text{ Required product} = (3 - 2i)(3 + 2i) = 9 - 4i^2 = 9 + 4 = 13$$

ALITER Let $z = 3 - 2i$. Then, $\bar{z} = 3 + 2i$. Therefore, $z\bar{z} = |z|^2 \Rightarrow z\bar{z} = 3^2 + (-2)^2 = 13$

EXAMPLE 14 Find the conjugate of $\frac{1}{3 + 4i}$.

$$\text{SOLUTION Let } z = \frac{1}{3 + 4i}. \text{ Then, } z = \frac{1}{3 + 4i} = \frac{1}{3 + 4i} \times \frac{3 - 4i}{3 - 4i} = \frac{3 - 4i}{9 + 16} = \frac{3}{25} - \frac{4}{25}i$$

$$\therefore \bar{z} = \frac{3}{25} + \frac{4}{25}i.$$

EXAMPLE 15 Express the following complex numbers in the standard form. Also, find their conjugate:

(i) $\frac{1-i}{1+i}$

(ii) $\frac{(1+i)^2}{3-i}$

(iii) $\frac{(2+3i)^2}{2-i}$

(iv) $\frac{\sqrt{5+12i} + \sqrt{5-12i}}{\sqrt{5+12i} - \sqrt{5-12i}}$ [NCERT EXEMPLAR]

SOLUTION (i) We have,

$$z = \frac{1-i}{1+i} = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{(1-i)^2}{1^2 - i^2} = \frac{1-2i+i^2}{1-(-1)} = \frac{1-2i-1}{1+1} = 0-i$$

$\therefore \bar{z} = 0+i$

(ii) We have,

$$z = \frac{(1+i)^2}{3-i} = \frac{1+2i+i^2}{3-i} \times \frac{3+i}{3+i} = \frac{2i}{3-i} \times \frac{3+i}{3+i} = \frac{6i+2i^2}{9-i^2} = \frac{6i-2}{10} = -\frac{1}{5} + \frac{3}{5}i$$

$\therefore \bar{z} = -\frac{1}{5} - \frac{3}{5}i$

(iii) We have,

$$z = \frac{(2+3i)^2}{2-i} = \frac{4+12i+9i^2}{2-i} = \frac{4+12i-9}{2-i} \times \frac{2+i}{2+i} = \frac{-5+12i}{2-i} \times \frac{2+i}{2+i} = \frac{-22+19i}{4-i^2} = -\frac{22}{5} + \frac{19}{5}i$$

$\therefore \bar{z} = -\frac{22}{5} - \frac{19}{5}i$

(iv) Let $\frac{\sqrt{5+12i} + \sqrt{5-12i}}{\sqrt{5+12i} - \sqrt{5-12i}}$. Then,

$$z = \frac{\sqrt{5+12i} + \sqrt{5-12i}}{\sqrt{5+12i} - \sqrt{5-12i}} \times \frac{\sqrt{5+12i} + \sqrt{5-12i}}{\sqrt{5+12i} + \sqrt{5-12i}} = \frac{(\sqrt{5+12i} + \sqrt{5-12i})^2}{(5+12i) - (5-12i)}$$

$$\Rightarrow z = \frac{5+12i+5-12i+2\sqrt{5+12i}\sqrt{5-12i}}{5+12i-5+12i} = \frac{10+2\sqrt{25+144}}{24i} = \frac{3}{2i} = -\frac{3}{2}i = 0 - \frac{3}{2}i$$

$\therefore \bar{z} = 0 + \frac{3}{2}i$

EXAMPLE 16 Find real values of x and y for which the complex numbers $-3 + ix^2y$ and $x^2 + y + 4i$ are conjugate of each other.

SOLUTION Since $-3 + ix^2y$ and $x^2 + y + 4i$ are complex conjugates.

$\therefore -3 + ix^2y = \overline{x^2 + y + 4i}$

$\Rightarrow -3 + ix^2y = x^2 + y - 4i$

$\Rightarrow -3 = x^2 + y \quad \dots(i) \quad \text{and,} \quad x^2y = -4 \quad \dots(ii)$

$\Rightarrow -3 = x^2 - \frac{4}{x^2} \quad [\text{Putting } y = -4/x^2 \text{ from (ii) in (i)}]$

$\Rightarrow x^4 + 3x^2 - 4 = 0$

$\Rightarrow (x^2 + 4)(x^2 - 1) = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1 \quad [\because x^2 + 4 \neq 0 \text{ for any real } x]$

From (ii), $y = -4$, when $x = \pm 1$. Hence, $x = 1, y = -4$ or, $x = -1, y = -4$

EXAMPLE 17 Find the real numbers x and y , if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

[NCERT]

SOLUTION We have, $(x - iy)(3 + 5i) = (3x + 5y) + i(5x - 3y)$

It is given that $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

$$\therefore (x - iy)(3 + 5i) = -6 - 24i$$

$$\Rightarrow (3x + 5y) + i(5x - 3y) = -6 + 24i \Rightarrow 3x + 5y = -6 \text{ and } 5x - 3y = 24$$

Solving these equations, we get $x = 3, y = -3$.

Type IV ON FINDING THE MULTIPLICATIVE INVERSE OR RECIPROCAL OF A NON-ZERO COMPLEX NUMBER

EXAMPLE 18 Find the multiplicative inverse of the following complex numbers:

(i) $3 + 2i$ **[NCERT]**

(ii) $(2 + \sqrt{3}i)^2$

SOLUTION (i) Let $z = 3 + 2i$. Then,

$$\frac{1}{z} = \frac{1}{3 + 2i} = \frac{3 - 2i}{(3 + 2i)(3 - 2i)} = \frac{3 - 2i}{9 - 4i^2} = \frac{3 - 2i}{13} = \frac{3}{13} - \frac{2}{13}i$$

ALITER Let $z = 3 + 2i$. Then, $\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{3 - 2i}{9 + 4} = \frac{3}{13} - \frac{2i}{13}$.

(ii) Let $z = (2 + \sqrt{3}i)^2$. Then,

$$z = 4 + 3i^2 + 4\sqrt{3}i = 4 - 3 + 4\sqrt{3}i = 1 + 4\sqrt{3}i$$

$$\therefore \frac{1}{z} = \frac{1}{1 + 4\sqrt{3}i} = \frac{1 - 4\sqrt{3}i}{(1 + 4\sqrt{3}i)(1 - 4\sqrt{3}i)} = \frac{1 - 4\sqrt{3}i}{1 + 48} = \frac{1}{49} - \frac{4\sqrt{3}i}{49}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

Type V PROBLEMS BASED UPON CONJUGATE AND MODULUS OF A COMPLEX NUMBER

EXAMPLE 19 If $\frac{a + ib}{c + id} = x + iy$, prove that $\frac{a - ib}{c - id} = x - iy$ and $\frac{a^2 + b^2}{c^2 + d^2} = x^2 + y^2$.

SOLUTION We have,

$$\frac{a + ib}{c + id} = x + iy$$

$$\Rightarrow \overline{\left(\frac{a + ib}{c + id}\right)} = \overline{x + iy}$$

[Taking Conjugate of both sides]

$$\Rightarrow \frac{\overline{a + ib}}{\overline{c + id}} = \overline{x + iy}$$

$$\Rightarrow \frac{a - ib}{c - id} = x - iy$$

$$\left[\because \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \right]$$

Thus, we have $\frac{a + ib}{c + id} = x + iy$ and $\frac{a - ib}{c - id} = x - iy$

$$\Rightarrow \frac{a + ib}{c + id} \times \frac{a - ib}{c - id} = (x + iy)(x - iy) \Rightarrow \frac{(a + ib)(a - ib)}{(c + id)(c - id)} = (x + iy)(x - iy)$$

$$\Rightarrow \frac{a^2 + b^2}{c^2 + d^2} = x^2 + y^2$$

[Using: $z\bar{z} = |z|^2$]

EXAMPLE 20 If $\frac{(a + i)^2}{(2a - i)} = p + iq$, show that: $p^2 + q^2 = \frac{(a^2 + 1)^2}{(4a^2 + 1)}$.

SOLUTION We have,

$$\frac{(a+i)^2}{(2a-i)} = (p+iq) \quad \dots(i)$$

$$\Rightarrow \left\{ \frac{(a+i)^2}{(2a-i)} \right\} = \overline{(p+iq)} \quad [\text{Taking conjugate of both sides}]$$

$$\Rightarrow \frac{\overline{(a+i)^2}}{(2a-i)} = \overline{(p+iq)} \Rightarrow \frac{(a-i)^2}{(2a+i)} = p-iq \quad \dots(ii)$$

Multiplying (i) and (ii), we obtain:

$$\frac{(a+i)^2}{(2a-i)} \times \frac{(a-i)^2}{(2a+i)} = (p+iq)(p-iq)$$

$$\Rightarrow \frac{\{(a+i)(a-i)\}^2}{(2a-i)(2a+i)} = (p+iq)(p-iq) \Rightarrow \frac{(a^2+1)^2}{4a^2+1} = p^2+q^2 \quad [\text{Using : } z\bar{z}=|z|^2]$$

EXAMPLE 21 If $a+ib = \frac{(x+i)^2}{2x^2+1}$, prove that $a^2+b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$. [NCERT]

SOLUTION We have,

$$a+ib = \frac{(x+i)^2}{2x^2+1} \quad \dots(i)$$

$$\Rightarrow \overline{a+ib} = \overline{\left\{ \frac{(x+i)^2}{2x^2+1} \right\}} \quad [\text{Taking conjugate of both sides}]$$

$$\Rightarrow \overline{a+ib} = \frac{\overline{(x+i)^2}}{(2x^2+1)} \Rightarrow a-ib = \frac{(x-i)^2}{2x^2+1} \quad \dots(ii)$$

Multiplying (i) and (ii), we get

$$(a+ib)(a-ib) = \frac{(x+i)^2(x-i)^2}{(2x^2+1)(2x^2+1)}$$

$$\Rightarrow a^2+b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2} \quad \left[\because (x+i)(x-i) = x^2 - i^2 = x^2 + 1 \right]$$

EXAMPLE 22 If $x+iy = \sqrt{\frac{a+ib}{c+id}}$, prove that: $(x^2+y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$. [NCERT]

SOLUTION We have,

$$x+iy = \sqrt{\frac{a+ib}{c+id}} \Rightarrow x-iy = \sqrt{\frac{a-ib}{c-id}} \quad [\text{Taking conjugate of both sides}]$$

$$\therefore (x+iy)(x-iy) = \sqrt{\frac{a+ib}{c+id}} \times \sqrt{\frac{a-ib}{c-id}} = \sqrt{\frac{a+ib}{c+id}} \times \frac{a-ib}{c-id}$$

$$\Rightarrow x^2+y^2 = \sqrt{\frac{a^2+b^2}{c^2+d^2}} \Rightarrow (x^2+y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$$

EXAMPLE 23 Find the least positive value of n , if $\left(\frac{1+i}{1-i}\right)^n = 1$. [NCERT]

SOLUTION We have,

$$\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{(1+i)^2}{1-i^2} = \frac{1+2i+i^2}{1-(-1)} = \frac{1+2i-1}{1+1} = i$$

$$\therefore \left(\frac{1+i}{1-i} \right)^n = 1 \Rightarrow i^n = 1 \Rightarrow n \text{ is a multiple of } 4 \Rightarrow \text{The smallest positive value of } n \text{ is } 4.$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 24 Find real θ such that $\frac{3+2i \sin \theta}{1-2i \sin \theta}$ is purely real.

[NCERT]

SOLUTION Clearly,

$$\begin{aligned} \frac{3+2i \sin \theta}{1-2i \sin \theta} &= \frac{(3+2i \sin \theta)(1+2i \sin \theta)}{(1-2i \sin \theta)(1+2i \sin \theta)} \\ &= \frac{(3-4 \sin^2 \theta) + i(6 \sin \theta + 2 \sin \theta)}{1+4 \sin^2 \theta} \\ &= \frac{(3-4 \sin^2 \theta) + i(6 \sin \theta + 2 \sin \theta)}{1+4 \sin^2 \theta} = \frac{3-4 \sin^2 \theta}{1+4 \sin^2 \theta} + \frac{i 8 \sin \theta}{1+4 \sin^2 \theta} \end{aligned}$$

It is given that $\frac{3+2i \sin \theta}{1-2i \sin \theta}$ is purely real. Therefore, its imaginary part is zero.

$$\text{i.e.} \quad \frac{8 \sin \theta}{1+4 \sin^2 \theta} = 0 \Rightarrow \sin \theta = 0 \Rightarrow \theta = n\pi, n \in \mathbb{Z}$$

EXAMPLE 25 The sum and product of two complex numbers are real if and only if they are conjugate of each other.

SOLUTION First, let the two complex numbers be conjugate of each other. Let complex numbers be $z_1 = a + ib$ and $z_2 = a - ib$. Then,

$$z_1 + z_2 = (a + ib) + (a - ib) = 2a, \text{ which is real.}$$

$$\text{And, } z_1 z_2 = (a + ib)(a - ib) = a^2 - i^2 b^2 = a^2 + b^2, \text{ which is also real.}$$

Thus, if z_1 and z_2 are conjugate of each other. Then, Their sum $z_1 + z_2$ and product $z_1 z_2$ both are real.

Conversely, let z_1 and z_2 be two complex numbers such that their sum $z_1 + z_2$ and product $z_1 z_2$ both are real. Then, we have to prove that z_1 and z_2 are conjugate of each other.

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$. Then,

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2) \text{ and } z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

Now, $z_1 + z_2$ and $z_1 z_2$ are real

$$\Rightarrow (a_1 + a_2) + i(b_1 + b_2) \text{ and } (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1) \text{ are real}$$

$$\Rightarrow b_1 + b_2 = 0 \text{ and } a_1 b_2 + a_2 b_1 = 0 \quad [\because z \text{ is real} \Leftrightarrow \text{Im}(z) = 0]$$

$$\Rightarrow b_2 = -b_1 \text{ and } a_1 b_2 + a_2 b_1 = 0$$

$$\Rightarrow b_2 = -b_1 \text{ and } -a_1 b_1 + a_2 b_1 = 0$$

$$\Rightarrow b_2 = -b_1 \text{ and } (a_2 - a_1) b_1 = 0 \Rightarrow b_2 = -b_1 \text{ and } a_2 - a_1 = 0 \Rightarrow b_2 = -b_1 \text{ and } a_2 = a_1$$

$$\therefore z_2 = a_2 + ib_2 = a_1 - ib_1 \Rightarrow z_2 = \bar{z}_1 \Rightarrow z_1 \text{ and } z_2 \text{ are conjugate of each other}$$

EXAMPLE 26 If $(1+i)(1+2i)(1+3i) \dots (1+ni) = (x+iy)$, show that: $2.5.10 \dots (1+n^2) = x^2 + y^2$.

SOLUTION We have,

$$(1+i)(1+2i)(1+3i) \dots (1+ni) = x+iy$$

$$\Rightarrow |(1+i)(1+2i) \dots (1+ni)| = |x+iy| \quad [\text{Taking modulus of both sides}]$$

$$\begin{aligned} \Rightarrow & |1+i| |1+2i| \dots |1+ni| = |x+iy| & [\because |z_1 z_2 \dots z_n| = |z_1| |z_2| \dots |z_n|] \\ \Rightarrow & \sqrt{1+1} \sqrt{1+4} \dots \sqrt{1+n^2} = \sqrt{x^2+y^2} \\ \Rightarrow & 25.10 \dots (1+n^2) = (x^2+y^2) & [\text{On squaring both sides}] \end{aligned}$$

EXAMPLE 27 If $(a+ib)(c+id)(e+if)(g+ih) = A+iB$, prove that $(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2+B^2$

SOLUTION We have,

$$\begin{aligned} & (a+ib)(c+id)(e+if)(g+ih) = A+iB \\ \Rightarrow & |(a+ib)(c+id)(e+if)(g+ih)| = |A+iB| & [\text{Taking modulus of both sides}] \\ \Rightarrow & |a+ib| |c+id| |e+if| |g+ih| = |A+iB| & [\text{Using: } |z_1 z_2 \dots z_n| = |z_1| |z_2| \dots |z_n|] \\ \Rightarrow & \sqrt{a^2+b^2} \sqrt{c^2+d^2} \sqrt{e^2+f^2} \sqrt{g^2+h^2} = \sqrt{A^2+B^2} \\ \Rightarrow & (a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2+B^2 & [\text{On squaring both sides}] \end{aligned}$$

EXAMPLE 28 If z_1, z_2 are complex numbers such that $\frac{2z_1}{3z_2}$ is purely imaginary number, then

find $\left| \frac{z_1 - z_2}{z_1 + z_2} \right|$.

SOLUTION It is given that $\frac{2z_1}{3z_2}$ is purely imaginary. Therefore,

$$\frac{2z_1}{3z_2} = \lambda i \text{ for some } \lambda \in \mathbb{R} \Rightarrow \frac{z_1}{z_2} = \frac{3\lambda}{2} i$$

$$\text{Now, } \left| \frac{z_1 - z_2}{z_1 + z_2} \right| = \left| \frac{\frac{z_1}{z_2} - 1}{\frac{z_1}{z_2} + 1} \right| = \left| \frac{\frac{3}{2} \lambda i - 1}{\frac{3}{2} \lambda i + 1} \right| = \frac{|-2 + 3\lambda i|}{|2 + 3\lambda i|} = \frac{\sqrt{4 + 9\lambda^2}}{\sqrt{4 + 9\lambda^2}} = 1 \quad \left[\because \frac{z_1}{z_2} = \frac{3\lambda}{2} i \right]$$

Type VI ON FINDING THE VALUE OF A POLYNOMIAL FOR A GIVEN COMPLEX VALUE OF THE VARIABLE

EXAMPLE 29 If $x = -5 + 2\sqrt{-4}$, find the value of $x^4 + 9x^3 + 35x^2 - x + 4$.

SOLUTION We have, $x = -5 + 2\sqrt{-4}$

$$\begin{aligned} \Rightarrow & x+5 = 2i \Rightarrow (x+5)^2 = 4i^2 \Rightarrow x^2 + 10x + 25 = -4 \Rightarrow x^2 + 10x + 29 = 0 \\ \therefore & x^4 + 9x^3 + 35x^2 - x + 4 = x^2(x^2 + 10x + 29) - x(x^2 + 10x + 29) + 4(x^2 + 10x + 29) - 160 \\ & = x^2(0) - x(0) + 4(0) - 160 = -160 \quad [\because x^2 + 10x + 29 = 0] \end{aligned}$$

Thus, the value of the given polynomial for $x = -5 + 2\sqrt{-4}$ is -160 .

EXAMPLE 30 Find the value of $x^3 + 7x^2 - x + 16$, when $x = 1 + 2i$.

SOLUTION We have,

$$\begin{aligned} x = 1 + 2i & \Rightarrow x - 1 = 2i \Rightarrow (x - 1)^2 = 4i^2 \Rightarrow x^2 - 2x + 1 = -4 \Rightarrow x^2 - 2x + 5 = 0 \\ \therefore & x^3 + 7x^2 - x + 16 = x(x^2 - 2x + 5) + 9(x^2 - 2x + 5) + (12x - 29) \\ & = x(0) + 9(0) + 12x - 29 & [\because x^2 - 2x + 5 = 0] \\ & = 12(1 + 2i) - 29 = -17 + 24i & [\because x = 1 + 2i] \end{aligned}$$

Hence, the value of the given polynomial when $x = 1 + 2i$ is $-17 + 24i$.

EXAMPLE 31 Find the value of $2x^4 + 5x^3 + 7x^2 - x + 41$, when $x = -2 - \sqrt{3}i$. [NCERT EXEMPLAR]

SOLUTION We have, $x = -2 - \sqrt{3}i$

$$\Rightarrow x + 2 = -\sqrt{3}i \Rightarrow (x + 2)^2 = 3i^2 \Rightarrow x^2 + 4x + 4 = -3 \Rightarrow x^2 + 4x + 7 = 0.$$

$$\begin{aligned} \therefore 2x^4 + 5x^3 + 7x^2 - x + 41 &= 2x^2(x^2 + 4x + 7) - 3x(x^2 + 4x + 7) + 5(x^2 + 4x + 7) + 6 \\ &= 2x^2 \times 0 - 3x \times 0 + 5 \times 0 + 6 = 6 \end{aligned} \quad [\because x^2 + 4x + 7 = 0]$$

EXAMPLE 32 If $z = 2 - 3i$, show that $z^2 - 4z + 13 = 0$ and hence find the value of $4z^3 - 3z^2 + 169$.

SOLUTION We have,

$$z = 2 - 3i \Rightarrow z - 2 = -3i \Rightarrow (z - 2)^2 = (-3i)^2 \Rightarrow z^2 - 4z + 4 = 9i^2 \Rightarrow z^2 - 4z + 13 = 0$$

$$\therefore 4z^3 - 3z^2 + 169 = 4z(z^2 - 4z + 13) + 13(z^2 - 4z + 13) = 4z(0) + 13(0) = 0 \quad [\because z^2 - 4z + 13 = 0]$$

Type VII MISCELLANEOUS PROBLEMS

EXAMPLE 33 Prove that: $x^4 + 4 = (x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - i)$.

SOLUTION We have, $(x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - i)$

$$\begin{aligned} &= \{(x + 1)^2 - i^2\} \{(x - 1)^2 - i^2\} = \{(x + 1)^2 + 1\} \{(x - 1)^2 + 1\} = \{x^2 + 2x + 2\} \{x^2 - 2x + 2\} \\ &= \{x^2 + 2 + 2x\} \{x^2 + 2 - 2x\} = (x^2 + 2)^2 - (2x)^2 = x^4 + 4x^2 + 4 - 4x^2 = x^4 + 4 \end{aligned}$$

EXAMPLE 34 If $z (\neq 1)$ is a complex number such that $|z| = 1$, prove that $\frac{z-1}{z+1}$ is purely imaginary.

What will be your conclusion if $z = 1$?

SOLUTION Let $z = x + iy$. Then, $|z| = 1 \Rightarrow \sqrt{x^2 + y^2} = 1 \Rightarrow x^2 + y^2 = 1$

$$\begin{aligned} \therefore \frac{z-1}{z+1} &= \frac{x + iy - 1}{x + iy + 1} = \frac{(x-1) + iy}{(x+1) + iy} = \frac{(x-1) + iy}{(x+1) + iy} \times \frac{(x+1) - iy}{(x+1) - iy} \\ &= \frac{(x^2 - 1 + y^2) + i(xy + y - xy + y)}{(x+1)^2 + y^2} = \frac{(x^2 + y^2 - 1) + 2iy}{(x+1)^2 + y^2} \\ &= \frac{2iy}{(x+1)^2 + y^2}, \text{ which is purely imaginary} \end{aligned} \quad [\because x^2 + y^2 = 1]$$

Now, $z = 1 \Rightarrow x + iy = 1 + i \cdot 0 \Rightarrow x = 1$ and $y = 0$.

$$\therefore \frac{z-1}{z+1} = \frac{x + iy - 1}{x + iy + 1} = \frac{1 + i \cdot 0 - 1}{1 + i \cdot 0 + 1} = 0, \text{ which is purely real}$$

EXAMPLE 35 If $z = x + iy$ and $w = \frac{1-iz}{z-i}$, show that $|w| = 1 \Rightarrow z$ is purely real.

SOLUTION We have, $|w| = 1 \Rightarrow \left| \frac{1-iz}{z-i} \right| = 1 \Rightarrow \frac{|1-iz|}{|z-i|} = 1 \Rightarrow |1-iz| = |z-i|$

$$\Rightarrow |1 - i(x + iy)| = |x + iy - i|, \text{ where } z = x + iy$$

$$\Rightarrow |1 + y - ix| = |x + i(y-1)|$$

$$\Rightarrow \sqrt{(1+y)^2 + (-x)^2} = \sqrt{x^2 + (y-1)^2}$$

$$\Rightarrow (1+y)^2 + x^2 = x^2 + (y-1)^2 \Rightarrow y = 0 \Rightarrow z = x + i \cdot 0 = x, \text{ which is purely real}$$

EXAMPLE 36 Show that a real value of x will satisfy the equation $\frac{1-ix}{1+ix} = a-ib$ if $a^2 + b^2 = 1$, where a, b are real.

SOLUTION We have, $\frac{1-ix}{1+ix} = \frac{a-ib}{1}$

Applying componendo and dividendo, we obtain

$$\begin{aligned} \frac{(1-ix) + (1+ix)}{(1-ix) - (1+ix)} &= \frac{a-ib+1}{a-ib-1} \Rightarrow \frac{2}{-2ix} = \frac{1+a-ib}{-(1-a+ib)} \Rightarrow ix = \frac{1-a+ib}{1+a-ib} \\ \Rightarrow ix &= \frac{(1-a+ib)}{(1+a-ib)} \times \frac{(1+a+ib)}{(1+a+ib)} = \frac{1-a^2-b^2+2ib}{(1+a)^2-i^2b^2} = \frac{1-a^2-b^2+2ib}{(1+a)^2+b^2} \\ \Rightarrow ix &= \frac{2ib}{(1+a)^2+b^2}, \text{ if } a^2+b^2=1 \Rightarrow x = \frac{2b}{(1+a)^2+b^2}, \text{ which is real} \end{aligned}$$

EXAMPLE 37 If α and β are different complex numbers with $|\beta|=1$, find $\left| \frac{\beta-\alpha}{1-\bar{\alpha}\beta} \right|$. [NCERT]

SOLUTION Clearly,

$$\begin{aligned} \left| \frac{\beta-\alpha}{1-\bar{\alpha}\beta} \right|^2 &= \left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta} \right) \overline{\left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta} \right)} = \left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta} \right) \left(\frac{\bar{\beta}-\bar{\alpha}}{1-\alpha\bar{\beta}} \right) = \frac{(\beta-\alpha)(\bar{\beta}-\bar{\alpha})}{(1-\bar{\alpha}\beta)(1-\alpha\bar{\beta})} \\ &= \frac{\beta\bar{\beta}-\beta\bar{\alpha}-\alpha\bar{\beta}+\alpha\bar{\alpha}}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+\bar{\alpha}\beta\alpha\bar{\beta}} = \frac{|\beta|^2-\alpha\bar{\beta}-\bar{\alpha}\beta+|\alpha|^2}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+(\alpha\bar{\alpha})(\beta\bar{\beta})} \\ &= \frac{|\alpha|^2-\alpha\bar{\beta}-\bar{\alpha}\beta+|\beta|^2}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+|\alpha|^2|\beta|^2} = \frac{|\alpha|^2-\alpha\bar{\beta}-\bar{\alpha}\beta+1}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+|\alpha|^2} = 1 \quad [\because |\beta|=1] \end{aligned}$$

$$\therefore \left| \frac{\beta-\alpha}{1-\bar{\alpha}\beta} \right| = 1.$$

EXAMPLE 38 If $|z_1|=|z_2|=...=|z_n|=1$, prove that $|z_1+z_2+z_3+...+z_n| = \left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + ... + \frac{1}{z_n} \right|$. [NCERT EXEMPLAR]

SOLUTION We find that

$$\begin{aligned} |z_1+z_2+z_3+...+z_n| &= \left| \frac{z_1\bar{z}_1}{\bar{z}_1} + \frac{z_2\bar{z}_2}{\bar{z}_2} + \frac{z_3\bar{z}_3}{\bar{z}_3} + ... + \frac{z_n\bar{z}_n}{\bar{z}_n} \right| = \left| \frac{|z_1|^2}{\bar{z}_1} + \frac{|z_2|^2}{\bar{z}_2} + \frac{|z_3|^2}{\bar{z}_3} + ... + \frac{|z_n|^2}{\bar{z}_n} \right| \\ &= \left| \frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3} + ... + \frac{1}{\bar{z}_n} \right| \quad [\because |z_1|=|z_2|=...=|z_n|=1] \\ &= \left| \left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + ... + \frac{1}{z_n} \right) \right| = \left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + ... + \frac{1}{z_n} \right| \quad [\because |z| = |\bar{z}|] \end{aligned}$$

EXAMPLE 39 Find non-zero integral solutions of $|1-i|^x = 2^x$. [NCERT]

SOLUTION We have, $|1-i|^x = 2^x$

$$\Rightarrow (\sqrt{2})^x = 2^x \Rightarrow 2^{x/2} = 2^x \Rightarrow 2^{x/2} = 1 \Rightarrow 2^{x/2} = 2^0 \Rightarrow \frac{x}{2} = 0 \Rightarrow x = 0.$$

Hence, the given equation has no non-zero integral solution.

EXAMPLE 40 Find all non-zero complex numbers z satisfying $\bar{z} = iz^2$.

SOLUTION Let $z = x + iy$. Then,

$$\bar{z} = iz^2$$

$$\Rightarrow x - iy = i(x^2 - y^2 + 2ixy) \Rightarrow x - iy = i(x^2 - y^2) - 2xy \Rightarrow (x + 2xy) - i(x^2 - y^2 + y) = 0$$

$$\Rightarrow x + 2xy = 0 \quad \dots(i) \quad \text{and,} \quad x^2 - y^2 + y = 0 \quad \dots(ii)$$

Now,

$$x + 2xy = 0 \Rightarrow x(1 + 2y) = 0 \Rightarrow x = 0 \text{ or, } 1 + 2y = 0 \Rightarrow x = 0 \text{ or, } y = -\frac{1}{2}$$

Case I When $x = 0$: Putting $x = 0$ in (ii), we obtain

$$-y^2 + y = 0 \Rightarrow y(y - 1) = 0 \Rightarrow y = 0, y = 1$$

Thus, the pairs of values of x and y are: $x = 0, y = 0$; $x = 0, y = 1$

$$\therefore z = 0 + i0 = 0, z = 0 + 1i = i$$

Case II When $y = -\frac{1}{2}$: Putting $y = -\frac{1}{2}$ in (ii), we get

$$x^2 - y^2 + y = 0 \Rightarrow x^2 - \frac{1}{4} - \frac{1}{2} = 0 \Rightarrow x^2 - \frac{3}{4} = 0 \Rightarrow x = \pm \frac{\sqrt{3}}{2}$$

Thus, the pairs of values of x and y are: $x = \frac{\sqrt{3}}{2}, y = -\frac{1}{2}$ and, $x = -\frac{\sqrt{3}}{2}, y = -\frac{1}{2}$

$$\therefore z = \frac{\sqrt{3}}{2} - \frac{1}{2}i, z = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

$$\text{Hence, } z = 0, i, \frac{\sqrt{3}}{2} - \frac{1}{2}i, -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

EXAMPLE 41 If $iz^3 + z^2 - z + i = 0$, then show that $|z| = 1$.

SOLUTION We have, $iz^3 + z^2 - z + i = 0$

Dividing both sides by i , we obtain

$$z^3 - iz^2 + iz + 1 = 0 \Rightarrow z^2(z - i) + i(z - i) = 0 \Rightarrow (z - i)(z^2 + i) = 0 \Rightarrow z = i \text{ or, } z^2 = -i$$

Now, $z = i \Rightarrow |z| = |i| = 1$ and, $z^2 = -i \Rightarrow |z^2| = |-i| = 1 \Rightarrow |z|^2 = 1 \Rightarrow |z| = 1$.

Hence, in either case, we obtain: $|z| = 1$.

EXAMPLE 42 Solve the equation $z^2 + |z| = 0$, where z is a complex number.

SOLUTION Let $z = x + iy$. Then,

$$z^2 + |z| = 0 \Rightarrow (x + iy)^2 + \sqrt{x^2 + y^2} = 0 \Rightarrow (x^2 - y^2) + \sqrt{x^2 + y^2} + 2ixy = 0$$

$$\Rightarrow x^2 - y^2 + \sqrt{x^2 + y^2} = 0 \quad \dots(i) \quad \text{and,} \quad 2xy = 0 \quad \dots(ii)$$

Now, $2xy = 0 \Rightarrow xy = 0 \Rightarrow x = 0$ or, $y = 0$

Case I When $y = 0$: Putting $y = 0$ in (i), we get

$$x^2 + \sqrt{x^2} = 0 \Rightarrow x^2 + |x| = 0$$

Clearly, $x^2 + |x| > 0$ for all $x > 0$. So, let $x < 0$. In this case, we have

$$x^2 + |x| = 0$$

$$\Rightarrow x^2 - x = 0$$

$$[\because x < 0 \therefore |x| = -x]$$

$$\Rightarrow x(x - 1) = 0 \Rightarrow x = 0, x = 1$$

But, $x < 0$. So, the equation $x^2 + |x| = 0$ has no solution for $x < 0$.

Clearly, $x = 0$ satisfies the equation $x^2 + |x| = 0$. Thus, we have $x = 0$, $y = 0$. Therefore, $z = 0$

Case I When $x = 0$: Putting $x = 0$ in (i), we get

$$-y^2 + \sqrt{y^2} = 0 \Rightarrow -y^2 + |y| = 0$$

If $y > 0$, then $|y| = y$.

$$\therefore -y^2 + |y| = 0 \Rightarrow -y^2 + y = 0 \Rightarrow y = 0, y = 1 \Rightarrow y = 1 \quad [\because y > 0]$$

If $y < 0$, then $|y| = -y$.

$$-y^2 + |y| = 0 \Rightarrow -y^2 - y = 0 \Rightarrow y = 0, -1 \Rightarrow y = -1 \quad [\because y < 0]$$

Thus, we obtain $x = 0$, $y = 1$ or, $x = 0$, $y = -1$. Therefore, $z = 0 + i$ or, $z = 0 - i$.

Hence, $z = 0, i$ and $-i$ are solutions of $z^2 + |z| = 0$.

EXAMPLE 43 Solve the equation $z^2 = \bar{z}$.

[NCERT EXEMPLAR]

SOLUTION Let $z = x + iy$. Then,

$$z^2 = \bar{z}$$

$$\Rightarrow (x + iy)^2 = x - iy \Rightarrow x^2 + 2ixy + (iy)^2 = x - iy \Rightarrow (x^2 - y^2) + 2ixy = x - iy$$

$$\Rightarrow x^2 - y^2 = x \quad \dots(i) \quad \text{and,} \quad 2xy = -y \quad \dots(ii)$$

$$\text{Now, } 2xy = -y \Rightarrow (2x + 1)y = 0 \Rightarrow 2x + 1 = 0 \text{ or } y = 0 \Rightarrow x = -\frac{1}{2} \text{ or } y = 0$$

Following cases arise :

Case I When $y = 0$: Putting $y = 0$ in (i), we obtain

$$x^2 = x \Rightarrow x(x - 1) = 0 \Rightarrow x = 0 \text{ or, } x = 1$$

Thus, we obtain $(x = 0 \text{ and } y = 0)$ or $(x = 1 \text{ and } y = 0)$. Therefore, $z = 0 + i0 = 0$ or, $z = 1 + i0$.

Case II When $x = -\frac{1}{2}$: Putting $x = -\frac{1}{2}$ in (i), we obtain

$$\frac{1}{4} - y^2 = -\frac{1}{2} \Rightarrow y^2 = \frac{3}{4} \Rightarrow y = \pm \frac{\sqrt{3}}{2}$$

$$\text{Thus, we obtain } \left(x = -\frac{1}{2} \text{ and } y = \frac{\sqrt{3}}{2} \right) \text{ or } \left(x = -\frac{1}{2} \text{ and } y = -\frac{\sqrt{3}}{2} \right).$$

$$\therefore z = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \text{ or } z = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Hence the values of z satisfying the given equation are

$$z = 0 + i0, z = 1 + i0, z = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \text{ and } z = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

EXAMPLE 44 Solve the equation $|z + 1| = z + 2(1 + i)$.

[NCERT EXEMPLAR]

SOLUTION Let $z = x + iy$. Then, $z + 1 = (x + 1) + iy$ and, $|z + 1| = \sqrt{(x + 1)^2 + y^2}$

Given that:

$$|z + 1| = z + 2(1 + i)$$

$$\Rightarrow \sqrt{(x + 1)^2 + y^2} = (x + iy) + 2(1 + i) \Rightarrow \sqrt{(x + 1)^2 + y^2} + 0i = (x + 2) + (y + 2)i$$

$$\Rightarrow \sqrt{(x + 1)^2 + y^2} = x + 2 \text{ and } y + 2 = 0 \Rightarrow (x + 1)^2 + y^2 = (x + 2)^2 \text{ and } y = -2$$

$$\Rightarrow y^2 = 2x + 3 \text{ and } y = -2 \Rightarrow 4 = 2x + 3 \text{ and } y = -2 \Rightarrow x = \frac{1}{2} \text{ and } y = -2$$

$$\text{Hence, } z = \frac{1}{2} - 2i$$

EXAMPLE 45 If $|z^2 - 1| = |z|^2 + 1$, then show that z lies on the imaginary axis.

[NCERT EXEMPLAR]

SOLUTION Let $z = x + iy$. Then, $z^2 = x^2 - y^2 + 2ixy$ and $|z|^2 = x^2 + y^2$.

$$\therefore |z^2 - 1| = |z|^2 + 1$$

$$\Rightarrow |(x^2 - y^2) + 2ixy| = x^2 + y^2 + 1$$

$$\Rightarrow \sqrt{(x^2 - y^2 - 1)^2 + 4x^2y^2} = x^2 + y^2 + 1 \Rightarrow (x^2 - y^2 - 1)^2 + 4x^2y^2 = (x^2 + y^2 + 1)^2$$

$$\Rightarrow x^4 + y^4 + 1 - 2x^2 + 2y^2 - 2x^2y^2 + 4x^2y^2 = x^4 + y^4 + 1 + 2x^2y^2 + 2x^2 + 2y^2$$

$$\Rightarrow 4x^2 = 0 \Rightarrow x = 0$$

$$\therefore z = x + iy = 0 + iy, \text{ which is purely imaginary.}$$

Thus, z is purely imaginary and hence it lies on y -axis.

EXAMPLE 46 If the imaginary part of $\frac{2z+1}{iz+1}$ is -2 , then show that the locus of the point representing z in the argand plane is a straight line.

[NCERT EXEMPLAR]

SOLUTION Let $z = x + iy$. Then,

$$\begin{aligned} \frac{2z+1}{iz+1} &= \frac{2(x+iy)+1}{i(x+iy)+1} = \frac{(2x+1)+i2y}{(1-y)+ix} = \frac{(2x+1)+i2y}{(1-y)+ix} \times \frac{(1-y)-ix}{(1-y)-ix} \\ &= \frac{(2x+1-y)+i(2y-2y^2-2x^2-x)}{(1-y)^2+x^2} = \left\{ \frac{2x+1-y}{x^2+(1-y)^2} \right\} + i \left\{ \frac{2y-2y^2-2x^2-x}{x^2+(1-y)^2} \right\} \end{aligned}$$

$$\therefore \operatorname{Im} \left(\frac{2z+1}{iz+1} \right) = \frac{2y-2y^2-2x^2-x}{x^2+(1-y)^2}. \text{ But, it is given that } \operatorname{Im} \left(\frac{2z+1}{iz+1} \right) = -2.$$

$$\therefore \frac{2y-2y^2-2x^2-x}{x^2+(1-y)^2} = -2 \Rightarrow 2y-2y^2-2x^2-x = -2x^2-2(1-y)^2 \Rightarrow x+2y-2=0,$$

which is a straight line. Hence, the locus of z is a straight line.

EXAMPLE 47 If the real part of $\frac{\bar{z}+2}{z-1}$ is 4, then show that the locus of the point representing z in the complex plane is a circle.

[NCERT EXEMPLAR]

SOLUTION Let $z = x + iy$. Then, $\bar{z} = x - iy$

$$\begin{aligned} \therefore \frac{\bar{z}+2}{z-1} &= \frac{x-iy+2}{x-iy-1} = \frac{(x+2)-iy}{(x-1)-iy} = \frac{(x+2)-iy}{(x-1)-iy} \times \frac{(x-1)+iy}{(x-1)+iy} \\ &= \frac{(x^2+y^2+x-2)+3iy}{(x-1)^2+y^2} = \left\{ \frac{x^2+y^2+x-2}{(x-1)^2+y^2} \right\} + i \left\{ \frac{3y}{(x-1)^2+y^2} \right\} \end{aligned}$$

It is given that the real part of $\frac{\bar{z}+2}{z-1}$ is 4.

$$\therefore \frac{x^2+y^2+x-2}{(x-1)^2+y^2} = 4 \Rightarrow 3x^2+3y^2-9x+6=0 \Rightarrow x^2+y^2-3x+2=0, \text{ which represents a circle.}$$

EXAMPLE 48 If $z = x + iy$, then show that $\bar{z}z + 2(z + \bar{z}) + a = 0$, where $a \in \mathbb{R}$, represents a circle.

[NCERT EXEMPLAR]

SOLUTION We have, $z = x + iy \Rightarrow \bar{z} = x - iy$

$$\therefore \bar{z}z + 2(z + \bar{z}) + a = 0$$

$$\Rightarrow (x + iy)(x - iy) + 2(x + iy + x - iy) + a = 0$$

$$\Rightarrow x^2 + y^2 + 4x + a = 0 \Rightarrow (x + 2)^2 + (y - 0)^2 = (\sqrt{4 - a})^2, \text{ which represents a circle for all } a \leq 4.$$

EXAMPLE 49 Show that $\left| \frac{z-2}{z-3} \right| = 2$ represents a circle. Find its centre and radius.

[NCERT EXEMPLAR]

SOLUTION Let $z = x + iy$. Then,

$$\left| \frac{z-2}{z-3} \right| = 2 \Rightarrow \left| \frac{(x-2) + iy}{(x-3) + iy} \right| = 2 \Rightarrow |(x-2) + iy| = 2|(x-3) + iy|$$

$$\Rightarrow \sqrt{(x-2)^2 + y^2} = 2\sqrt{(x-3)^2 + y^2}$$

$$\Rightarrow (x-2)^2 + y^2 = 4\{(x-3)^2 + y^2\}$$

$$\Rightarrow 3x^2 + 3y^2 - 20x + 32 = 0$$

$$\Rightarrow x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0$$

$$\Rightarrow \left(x - \frac{10}{3}\right)^2 + (y - 0)^2 = \left(\frac{2}{3}\right)^2, \text{ which represents a circle with centre at } \left(\frac{10}{3}, 0\right) \text{ and radius } \frac{2}{3}$$

EXAMPLE 50 Find a complex number z satisfying the equation $z + \sqrt{2}|z+1| + i = 0$.

[NCERT EXEMPLAR]

SOLUTION Let $z = x + iy$. Then,

$$z + \sqrt{2}|z+1| + i = 0$$

$$\Rightarrow x + iy + \sqrt{2}|(x+1) + iy| + i = 0 \Rightarrow x + \sqrt{2}\sqrt{(x+1)^2 + y^2} + (y+1)i = 0$$

$$\Rightarrow x + \sqrt{2(x+1)^2 + 2y^2} = 0 \text{ and } (y+1) = 0 \Rightarrow x + \sqrt{2(x+1)^2 + 2y^2} = 0 \text{ and } y = -1$$

$$\Rightarrow x + \sqrt{2(x+1)^2 + 2} = 0 \text{ and } y = -1 \Rightarrow \sqrt{2(x+1)^2 + 2} = -x \text{ and } y = -1$$

$$\Rightarrow 2(x+1)^2 + 2 = x^2 \text{ and } y = -1 \Rightarrow x^2 + 4x + 4 = 0 \text{ and } y = -1$$

$$\Rightarrow (x+2)^2 = 0 \text{ and } y = -1 \Rightarrow x = -2 \text{ and } y = -1$$

Hence, $z = x + iy = -2 - i$.

EXAMPLE 51 Let z_1 and z_2 be two complex numbers such that

$$|1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 = k(1 - |z_1|^2)(1 - |z_2|^2). \text{ Find the value of } k. \text{ [NCERT EXEMPLAR]}$$

SOLUTION We find that

$$\begin{aligned} |1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 &= (1 - \bar{z}_1 z_2)(1 - \overline{\bar{z}_1 z_2}) - (z_1 - z_2)(\overline{z_1 - z_2}) \\ &= (1 - \bar{z}_1 z_2)(1 - z_1 \bar{z}_2) - (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \\ &= (1 - \bar{z}_1 z_2 - z_1 \bar{z}_2 + \bar{z}_1 z_2 z_1 \bar{z}_2) - (z_1 \bar{z}_1 - z_1 \bar{z}_2 - \bar{z}_1 z_2 + \bar{z}_2 z_2) \\ &= 1 - \bar{z}_1 z_2 - z_1 \bar{z}_2 + (z_1 \bar{z}_1)(z_2 \bar{z}_2) - (|z_1|^2 - z_1 \bar{z}_2 - \bar{z}_1 z_2 + |z_2|^2) \end{aligned}$$

$$\begin{aligned}
 &= 1 - \bar{z}_1 z_2 - z_1 \bar{z}_2 + |z_1|^2 |z_2|^2 - |z_1|^2 + z_1 \bar{z}_2 + \bar{z}_1 z_2 - |z_2|^2 \\
 &= 1 - |z_1|^2 - |z_2|^2 + |z_1|^2 |z_2|^2 = (1 - |z_1|^2)(1 - |z_2|^2)
 \end{aligned}$$

$$\therefore |1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 = k(1 - |z_1|^2)(1 - |z_2|^2)$$

$$\Rightarrow \left(1 - |z_1|^2\right) \left(1 - |z_2|^2\right) = k \left(1 - |z_1|^2\right) \left(1 - |z_2|^2\right) \Rightarrow k = 1.$$

EXERCISE 12.2

BASIC

1. Express the following complex numbers in the standard form $a + ib$:

(i) $(1 + i)(1 + 2i)$

(ii) $\frac{3 + 2i}{-2 + i}$

(iii) $\frac{1}{(2 + i)^2}$

(iv) $\frac{1 - i}{1 + i}$

(v) $\frac{(2 + i)^3}{2 + 3i}$

(vi) $\frac{(1 + i)(1 + \sqrt{3}i)}{1 - i}$

(vii) $\frac{2 + 3i}{4 + 5i}$

(viii) $\frac{(1 - i)^3}{1 - i^3}$

(ix) $(1 + 2i)^{-3}$

(x) $\frac{3 - 4i}{(4 - 2i)(1 + i)}$

(xi) $\left(\frac{1}{1 - 4i} - \frac{2}{1 + i}\right) \left(\frac{3 - 4i}{5 + i}\right)$

[NCERT]

(xii) $\frac{5 + \sqrt{2}i}{1 - \sqrt{2}i}$

[NCERT]

2. Find the real values of x and y , if

(i) $(x + iy)(2 - 3i) = 4 + i$

(ii) $(3x - 2iy)(2 + i)^2 = 10(1 + i)$

(iii) $\frac{(1 + i)x - 2i}{3 + i} + \frac{(2 - 3i)y + i}{3 - i} = i$

(iv) $(1 + i)(x + iy) = 2 - 5i$

3. Find the conjugates of the following complex numbers:

(i) $4 - 5i$

(ii) $\frac{1}{3 + 5i}$

(iii) $\frac{1}{1 + i}$

(iv) $\frac{(3 - i)^2}{2 + i}$

(v) $\frac{(1 + i)(2 + i)}{3 + i}$

(vi) $\frac{(3 - 2i)(2 + 3i)}{(1 + 2i)(2 - i)}$

[NCERT]

4. Find the multiplicative inverse of the following complex numbers:

(i) $1 - i$

(ii) $(1 + i\sqrt{3})^2$

(iii) $4 - 3i$

(iv) $\sqrt{5} + 3i$

5. If $z_1 = 2 - i$, $z_2 = 1 + i$, find $\left| \frac{z_1 + z_2 + 1}{z_1 - z_2 + i} \right|$.

[NCERT]

6. If $z_1 = 2 - i$, $z_2 = -2 + i$, find (i) $\operatorname{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right)$ (ii) $\operatorname{Im}\left(\frac{1}{z_1 \bar{z}_1}\right)$

[NCERT]

7. Find the modulus of the complex number $z = \frac{1 + i}{1 - i} - \frac{1 - i}{1 + i}$

[NCERT]

BASED ON LOTS

8. If $x + iy = \frac{a + ib}{a - ib}$, prove that $x^2 + y^2 = 1$

[NCERT]

9. Find the least positive integral value of n for which $\left(\frac{1+i}{1-i}\right)^n$ is real.
10. Find the real values of θ for which the complex number $\frac{1+i\cos\theta}{1-2i\cos\theta}$ is purely real.
11. Find the smallest positive integer value of n for which $\frac{(1+i)^n}{(1-i)^{n-2}}$ is a real number.
12. If $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x+iy$, find (x, y) [NCERT EXEMPLAR]
13. If $\frac{(1+i)^2}{2-i} = x+iy$, find $x+y$. [NCERT EXEMPLAR]
14. If $\left(\frac{1-i}{1+i}\right)^{100} = a+ib$, find (a, b) . [NCERT EXEMPLAR]
15. If $a = \cos\theta + i\sin\theta$, find the value of $\frac{1+a}{1-a}$. [NCERT EXEMPLAR]

BASED ON HOTS

16. Evaluate the following:
 (i) $2x^3 + 2x^2 - 7x + 72$, when $x = \frac{3-5i}{2}$ (ii) $x^4 - 4x^3 + 4x^2 + 8x + 44$, when $x = 3 + 2i$
 (iii) $x^4 + 4x^3 + 6x^2 + 4x + 9$, when $x = -1 + i\sqrt{2}$ (iv) $x^6 + x^4 + x^2 + 1$, when $x = \frac{1+i}{\sqrt{2}}$.
17. Find the number of solutions of $z^2 + |z|^2 = 0$. [NCERT EXEMPLAR]
18. If $(1+i)z = (1-i)\bar{z}$, then show that $z = -i\bar{z}$. [NCERT EXEMPLAR]
19. Solve the system of equations $\operatorname{Re}(z^2) = 0, |z| = 2$. [NCERT EXEMPLAR]
20. If $\frac{z-1}{z+1}$ is purely imaginary number ($z \neq -1$), find the value of $|z|$. [NCERT EXEMPLAR]
21. If z_1 is a complex number other than -1 such that $|z_1| = 1$ and $z_2 = \frac{z_1-1}{z_1+1}$, then show that the real part of z_2 is zero. [NCERT EXEMPLAR]
22. If $|z+1| = z+2(1+i)$, find z . [NCERT EXEMPLAR]
23. Solve the equation $|z| = z+1+2i$. [NCERT EXEMPLAR]
24. What is the smallest positive integer n for which $(1+i)^{2n} = (1-i)^{2n}$?
25. If z_1, z_2, z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$, then find the value of $|z_1 + z_2 + z_3|$. [NCERT EXEMPLAR]

ANSWERS

1. (i) $-1 + 3i$ (ii) $-\frac{4}{5} - \frac{7}{5}i$ (iii) $\frac{3}{25} - \frac{4}{25}i$ (iv) $-i$
 (v) $\frac{37}{13} + \frac{16}{13}i$ (vi) $-\sqrt{3} + i$ (vii) $\frac{23}{41} + \frac{2}{41}i$ (viii) $-2 + 0i$
 (ix) $\frac{-11}{125} + \frac{2i}{125}$ (x) $\frac{1}{4} - \frac{3}{4}i$ (xi) $\frac{307}{442} + i\frac{599}{442}$ (xii) $1 + 2\sqrt{2}i$

2. (i) $x = \frac{5}{13}, y = \frac{14}{13}$ (ii) $x = \frac{14}{15}, y = \frac{1}{5}$ (iii) $x = 3, y = -1$ (iv) $x = -\frac{3}{2}, y = -\frac{7}{2}$
3. (i) $4 + 5i$ (ii) $\frac{1}{34}(3 + 5i)$ (iii) $\frac{1}{2} + \frac{1}{2}i$ (iv) $2 + 4i$
- (v) $\frac{3}{5} - \frac{4}{5}i$ (vi) $\frac{63}{25} + \frac{16}{25}i$
4. (i) $\frac{1}{2} + \frac{1}{2}i$ (ii) $-\frac{1}{8} - i\frac{\sqrt{3}}{8}$ (iii) $\frac{4}{25} + \frac{3}{25}i$ (iv) $\frac{\sqrt{5}}{14} - \frac{3i}{14}$
5. $\frac{4}{\sqrt{2}}$ 6. (i) $-\frac{2}{5}$ (ii) 0 7. 2 9. $n = 2$
10. $\theta = 2n\pi \pm \frac{\pi}{2}, n \in \mathbb{Z}$ 11. 1 12. $(0, -2)$ 13. $\frac{2}{5}$ 14. $(1, 0)$
15. $i \cot \frac{\theta}{2}$ 16. (i) 4 (ii) 5 (iii) 12 (iv) 0
17. Infinitely many solutions of the form $z = 0 + iy, y \in \mathbb{R}$. 19. $\sqrt{2}(1 \pm i), \sqrt{2}(-1 \pm i)$
20. 1 22. $\frac{1}{2} - 2i$ 23. $\frac{3}{2} - 2i$ 24. $n = 2$ 25. 1

HINTS TO SELECTED PROBLEMS

12. $\frac{1+i}{1-i} = \frac{(1+i)^2}{(1-i)(1+i)} = \frac{1+2i-1}{2} = i$ and $\frac{1-i}{1+i} = \frac{(1-i)^2}{(1-i)(1+i)} = \frac{1-2i-1}{2} = -i$
- $\therefore \left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + iy$
- $\Rightarrow i^3 - (-i)^3 = x + iy \Rightarrow i^3 + i^3 = x + iy \Rightarrow 2i^3 = x + iy \Rightarrow 0 - 2i = x + iy \Rightarrow x = 0, y = -2$
13. $\frac{(1+i)^2}{2-i} = x + iy \Rightarrow \frac{1+2i+i^2}{2-i} = x + iy \Rightarrow \frac{2i}{2-i} = x + iy$
- $\Rightarrow \frac{2i(2+i)}{(2+i)(2-i)} = x + iy \Rightarrow \frac{4i+2i^2}{4-i^2} = x + iy \Rightarrow \frac{-2+4i}{5} = x + iy \Rightarrow -\frac{2}{5} + \frac{4}{5}i = x + iy$
- $\Rightarrow x = -\frac{2}{5}, y = \frac{4}{5}$
14. $\left(\frac{1-i}{1+i}\right)^{100} = a + ib \Rightarrow (-i)^{100} = a + ib \Rightarrow 1 + 0i = a + ib \Rightarrow a = 1, b = 0$
15. We have, $a = \cos \theta + i \sin \theta$
- $\therefore \frac{1+a}{1-a} = \frac{1+\cos \theta + i \sin \theta}{1-\cos \theta - i \sin \theta} = \left(\frac{1+\cos \theta + i \sin \theta}{1-\cos \theta - i \sin \theta}\right) \times \left(\frac{1-\cos \theta + i \sin \theta}{1-\cos \theta + i \sin \theta}\right)$
- $\Rightarrow \frac{1+a}{1-a} = \frac{(1-\cos^2 \theta - \sin^2 \theta) + 2i \sin \theta}{(1-\cos \theta)^2 + \sin^2 \theta} = \frac{i \sin \theta}{1-\cos \theta} = \frac{2i \sin \theta / 2 \cos \theta / 2}{2 \sin^2 \theta / 2} = i \cot \frac{\theta}{2}$
17. Let $z = x + iy$. Then, $z^2 = x^2 - y^2 + 2ixy$ and $|z|^2 = x^2 + y^2$.
- $\therefore z^2 + |z|^2 = 0$
- $\Rightarrow x^2 - y^2 + 2ixy + x^2 + y^2 = 0 \Rightarrow 2x^2 + 2ixy = 0 \Rightarrow 2x^2 = 0$ and $2xy = 0 \Rightarrow x = 0$ and $y \in \mathbb{R}$
- $\therefore z = 0 + iy$, where $y \in \mathbb{R}$.
18. We have, $(1+i)z = (1-i)\bar{z}$

$$\Rightarrow \frac{z}{\bar{z}} = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{1-2i+i^2}{1+i^2} = -i \Rightarrow z = -i\bar{z}$$

19. Let $z = x + iy$. Then, $z^2 = x^2 - y^2 + 2ixy$ and $|z| = \sqrt{x^2 + y^2}$.

$$\therefore \operatorname{Re}(z^2) = 0 \text{ and } |z| = 2 \Rightarrow x^2 - y^2 = 0 \text{ and } x^2 + y^2 = 4 \Rightarrow x^2 = y^2 = 2 \Rightarrow x = \pm\sqrt{2}, y = \pm\sqrt{2}$$

$$\therefore z = \pm\sqrt{2} \pm \sqrt{2}i$$

20. Let $z = x + iy$. Then,

$$\frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} = \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy} = \frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2}$$

If $\frac{z-1}{z+1}$ is purely imaginary, then

$$\operatorname{Re}\left(\frac{z-1}{z+1}\right) = 0 \Rightarrow \frac{x^2+y^2-1}{(x+1)^2+y^2} = 0 \Rightarrow x^2+y^2 = 1 \Rightarrow |z| = 1$$

21. Let $z_1 = x + iy$. Then,

$$z_2 = \frac{z_1-1}{z_1+1} = \frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2} \Rightarrow \operatorname{Re}(z_2) = \frac{x^2+y^2-1}{(x+1)^2+y^2} = 0 \quad [\because |z_1| = 1 \Rightarrow x^2+y^2 = 1]$$

22. Let $z = x + iy$. Then, $z+1 = (x+1)+iy$ and $z+2(1+i) = (x+2)+i(y+2)$

$$\therefore |z+1| = |z+2(1+i)|$$

$$\Rightarrow \sqrt{(x+1)^2+y^2} = \sqrt{(x+2)^2+(y+2)^2} \Rightarrow \sqrt{(x+1)^2+y^2} = x+2 \text{ and } y+2 = 0$$

$$\Rightarrow (x+1)^2+y^2 = (x+2)^2 \text{ and } y = -2 \Rightarrow y^2 = 2x+3 \text{ and } y = -2$$

$$\Rightarrow 4 = 2x+3 \text{ and } y = -2 \Rightarrow x = \frac{1}{2} \text{ and } y = -2 \Rightarrow z = x+iy = \frac{1}{2}-2i$$

23. Let $z = x + iy$. Then, $|z| = \sqrt{x^2+y^2}$ and $z+1+2i = (x+1)+i(y+2)$.

$$\therefore |z| = |z+1+2i| \Rightarrow \sqrt{x^2+y^2} = \sqrt{(x+1)^2+(y+2)^2} \Rightarrow \sqrt{x^2+y^2} = x+1 \text{ and } y+2 = 0$$

$$\Rightarrow x^2+y^2 = (x+1)^2 \text{ and } y = -2 \Rightarrow x^2+4 = (x+1)^2 \text{ and } y = -2 \Rightarrow x = \frac{3}{2} \text{ and } y = -2$$

$$\text{Hence, } z = x+iy = \frac{3}{2}-2i$$

24. $(1+i)^{2n} = (1-i)^{2n} \Rightarrow \{(1+i)^2\}^n = \{(1-i)^2\}^n \Rightarrow (1+2i+i^2)^n = (1-2i+i^2)^n$

$$\Rightarrow (2i)^n = (-2i)^n \Rightarrow i^n = (-1)^n i^n \Rightarrow (-1)^n = 1 \Rightarrow n \text{ is a multiple of } 2.$$

12.13 SQUARE ROOTS OF A COMPLEX NUMBER

Let $a + ib$ be a complex number such that $\sqrt{a+ib} = x + iy$, where x and y are real numbers. Then,

$$\sqrt{a+ib} = x + iy \Rightarrow (a+ib) = (x+iy)^2 \Rightarrow a+ib = (x^2-y^2) + 2ixy$$

On equating real and imaginary parts, we obtain

$$x^2 - y^2 = a \quad \dots(i) \quad \text{and} \quad 2xy = b \quad \dots(ii)$$

Now, $(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$$\Rightarrow (x^2+y^2)^2 = a^2 + b^2 \Rightarrow (x^2+y^2) = \sqrt{a^2+b^2} \quad [\because x^2+y^2 > 0] \quad \dots(iii)$$

Solving equations (i) and (ii), we get

$$x^2 = \frac{1}{2} \left\{ \sqrt{a^2 + b^2 + a} \right\} \text{ and } y^2 = \frac{1}{2} \left\{ \sqrt{a^2 + b^2 - a} \right\}$$

$$\Rightarrow x = \pm \sqrt{\frac{1}{2} \left\{ \sqrt{a^2 + b^2 + a} \right\}} \text{ and } y = \pm \sqrt{\frac{1}{2} \left\{ \sqrt{a^2 + b^2 - a} \right\}}$$

If b is positive, then from equation (ii), we find that x and y are of the same sign.

$$\therefore \sqrt{a + ib} = \pm \left[\sqrt{\frac{1}{2} \left\{ \sqrt{a^2 + b^2 + a} \right\}} + i \sqrt{\frac{1}{2} \left\{ \sqrt{a^2 + b^2 - a} \right\}} \right]$$

If b is negative, then from equation (ii), we find that x and y are of different signs.

$$\therefore \sqrt{a + ib} = \pm \left[\sqrt{\frac{1}{2} \left\{ \sqrt{a^2 + b^2 + a} \right\}} - i \sqrt{\frac{1}{2} \left\{ \sqrt{a^2 + b^2 - a} \right\}} \right]$$

REMARK It is evident from the above discussion that for any complex number z , we have

$$(i) \sqrt{z} = \pm \left\{ \sqrt{\frac{|z| + \operatorname{Re}(z)}{2}} + i \sqrt{\frac{|z| - \operatorname{Re}(z)}{2}} \right\}, \text{ if } \operatorname{Im}(z) > 0$$

$$(ii) \sqrt{z} = \pm \left\{ \sqrt{\frac{|z| + \operatorname{Re}(z)}{2}} - i \sqrt{\frac{|z| - \operatorname{Re}(z)}{2}} \right\}, \text{ if } \operatorname{Im}(z) < 0$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 Find the square roots of the following complex numbers: (i) $7 - 24i$ (ii) $5 + 12i$

SOLUTION (i) Let $\sqrt{7 - 24i} = x + iy$. Then,

$$\sqrt{7 - 24i} = x + iy \Rightarrow 7 - 24i = (x + iy)^2 \Rightarrow 7 - 24i = (x^2 - y^2) + 2ixy$$

$$\Rightarrow x^2 - y^2 = 7 \quad \dots(i) \quad \text{and, } 2xy = -24 \quad \dots(ii)$$

$$\therefore (x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$$

$$\Rightarrow (x^2 + y^2)^2 = 49 + 576 = 625 \Rightarrow x^2 + y^2 = 25 \quad [\because x^2 + y^2 > 0] \quad \dots(iii)$$

On solving (i) and (iii), we obtain:

$$x^2 = 16 \text{ and } y^2 = 9 \Rightarrow x = \pm 4 \text{ and } y = \pm 3$$

From (ii) we observe that $2xy$ is negative. So, x and y are of opposite signs.

$$\therefore (x = 4 \text{ and } y = -3) \text{ or, } (x = -4 \text{ and } y = 3)$$

$$\text{Hence, } \sqrt{7 - 24i} = x + iy = \pm(4 - 3i)$$

ALITER Let $z = 7 - 24i$. Then, $\operatorname{Re}(z) = 7$ and $|z| = \sqrt{49 + 576} = 25$. We find that $\operatorname{Im}(z) < 0$.

$$\therefore \sqrt{7 - 24i} = \pm \left\{ \sqrt{\frac{|z| + \operatorname{Re}(z)}{2}} - i \sqrt{\frac{|z| - \operatorname{Re}(z)}{2}} \right\} = \pm \left\{ \sqrt{\frac{25 + 7}{2}} - i \sqrt{\frac{25 - 7}{2}} \right\} = \pm(4 - 3i)$$

(ii) Let $\sqrt{5 + 12i} = x + iy$. Then,

$$\sqrt{5 + 12i} = x + iy \Rightarrow 5 + 12i = (x + iy)^2 \Rightarrow 5 + 12i = (x^2 - y^2) + 2ixy$$

$$\Rightarrow x^2 - y^2 = 5 \quad \dots(i) \quad \text{and, } 2xy = 12 \quad \dots(ii)$$

$$\therefore (x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$$

$$\Rightarrow (x^2 + y^2)^2 = 25 + 144 = 169 \Rightarrow x^2 + y^2 = 13 \quad [\because x^2 + y^2 > 0] \quad \dots(iii)$$

On solving (i) and (iii), we get

$$x^2 = 9 \text{ and } y^2 = 4 \Rightarrow x = \pm 3 \text{ and } y = \pm 2$$

From (ii) we observe that $2xy$ is positive. So, x and y are of the same sign.

$$\therefore (x = 3 \text{ and } y = 2) \text{ or, } (x = -3 \text{ and } y = -2)$$

$$\text{Hence, } \sqrt{5+12i} = x+iy = \pm(3+2i).$$

ALITER Let $z = 5 + 12i$. Then, $\operatorname{Re}(z) = 5$, and $|z| = \sqrt{25 + 144} = 13$. We find that $\operatorname{Im}(z) > 0$.

$$\therefore \sqrt{5+12i} = \pm \left\{ \sqrt{\frac{|z| + \operatorname{Re}(z)}{2}} + i \sqrt{\frac{|z| - \operatorname{Re}(z)}{2}} \right\} = \pm \left\{ \sqrt{\frac{13+5}{2}} + i \sqrt{\frac{13-5}{2}} \right\} = \pm(3+2i)$$

EXAMPLE 2 Find the square roots of $-15 - 8i$.

SOLUTION Let $\sqrt{-15-8i} = x + iy$. Then,

$$\sqrt{-15-8i} = x + iy \Rightarrow -15 - 8i = (x + iy)^2 \Rightarrow -15 - 8i = (x^2 - y^2) + 2i xy$$

$$\Rightarrow -15 = x^2 - y^2 \quad \dots(i) \quad \text{and,} \quad 2xy = -8 \quad \dots(ii)$$

$$\therefore (x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2 y^2$$

$$\Rightarrow (x^2 + y^2)^2 = (-15)^2 + 64 = 289 \Rightarrow x^2 + y^2 = 17 \quad \dots(iii)$$

On solving (i) and (iii), we get

$$x^2 = 1 \text{ and } y^2 = 16 \Rightarrow x = \pm 1 \text{ and } y = \pm 4$$

From (ii), we observe that $2xy$ is negative. So, x and y are of opposite signs.

$$\therefore (x = 1 \text{ and } y = -4) \text{ or, } (x = -1 \text{ and } y = 4)$$

$$\text{Hence, } \sqrt{-15-8i} = x + iy = \pm(1 - 4i)$$

EXAMPLE 3 Find the square root of i .

SOLUTION Let $\sqrt{i} = x + iy$. Then,

$$\sqrt{i} = x + iy \Rightarrow i = (x + iy)^2 \Rightarrow (x^2 - y^2) + 2i xy = 0 + i$$

$$\Rightarrow x^2 - y^2 = 0 \quad \dots(i) \quad \text{and,} \quad 2xy = 1 \quad \dots(ii)$$

$$\therefore (x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2 y^2$$

$$\Rightarrow (x^2 + y^2)^2 = 0 + 1 = 1 \Rightarrow x^2 + y^2 = 1 \quad [\because x^2 + y^2 > 0] \quad \dots(iii)$$

Solving (i) and (iii), we get

$$x^2 = 1/2 \text{ and } y^2 = 1/2 \Rightarrow x = \pm 1/\sqrt{2} \text{ and } y = \pm 1/\sqrt{2}$$

From equation (ii) we find that $2xy$ is positive. So, x and y are of same sign.

$$\therefore \left(x = \frac{1}{\sqrt{2}} \text{ and } y = \frac{1}{\sqrt{2}} \right) \text{ or, } \left(x = -\frac{1}{\sqrt{2}} \text{ and } y = -\frac{1}{\sqrt{2}} \right)$$

$$\text{Hence, } \sqrt{i} = \pm \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \right) = \pm \frac{1}{\sqrt{2}} (1 + i)$$

ALITER Let $z = i$. Then, $\operatorname{Re}(z) = 0$ and $|z| = 1$. We find that $\operatorname{Im}(z) > 0$.

$$\therefore \sqrt{i} = \pm \left\{ \sqrt{\frac{|z| + \operatorname{Re}(z)}{2}} + i \sqrt{\frac{|z| - \operatorname{Re}(z)}{2}} \right\} = \pm \left\{ \sqrt{\frac{1+0}{2}} + i \sqrt{\frac{1-0}{2}} \right\} = \pm \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) = \pm \frac{1}{\sqrt{2}} (1 + i)$$

EXERCISE 12.3

BASIC

1. Find the square root of the following complex numbers:

- (i) $-5 + 12i$ (ii) $-7 - 24i$ (iii) $1 - i$ (iv) $-8 - 6i$
 (v) $8 - 15i$ (vi) $-11 - 60\sqrt{-1}$ (vii) $1 + 4\sqrt{-3}$ (viii) $-i$

ANSWERS

1. (i) $\pm(2 + 3i)$ (ii) $\pm(3 - 4i)$ (iii) $\pm \left\{ \left(\frac{\sqrt{2} + 1}{2} \right) - \left(\frac{\sqrt{2} - 1}{2} \right) i \right\}$ (iv) $\pm(1 - 3i)$
 (v) $\pm \frac{1}{\sqrt{2}}(5 - 3i)$ (vi) $\pm(5 - 6i)$ (vii) $\pm(2 + \sqrt{3}i)$ (viii) $\pm \frac{1}{\sqrt{2}}(1 - i)$

12.14 REPRESENTATIONS OF A COMPLEX NUMBER

A complex number can be represented in the following forms:

- (i) Geometrical form (ii) Vectorial form (iii) Trigonometrical form or, Polar form

In this section, we shall learn about these three representations of a complex number.

12.14.1 GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER

A complex number $z = x + iy$ can be represented by a point (x, y) on the plane which is known as the Argand plane. To represent $z = x + iy$ geometrically we take two mutually perpendicular straight lines $X'OX$ and $Y'OY$. Now plot a point whose x and y coordinates are respectively the real and imaginary parts of z . This point $P(x, y)$ represents the complex number $z = x + iy$.

If a complex number is purely real, then its imaginary part is zero. Therefore, a purely real number is represented by a point on x -axis. A purely imaginary complex number is represented by a point on y -axis. That is why x -axis is known as the *real axis* and y -axis, as the *imaginary axis*.

Conversely, if $P(x, y)$ is a point in the plane, then the point $P(x, y)$ represents a complex number $z = x + iy$. The complex number $z = x + iy$ is known as the *affix* of the point P .

Thus, there exists a one-one correspondence between the points of the plane and the members (elements) of the set C of all complex numbers, i.e., for every complex number $z = x + iy$ there exists uniquely a point (x, y) on the plane and for every point (x, y) of the plane there exists uniquely a complex number $z = x + iy$.

The plane in which we represent a complex number geometrically is known as the **complex plane** or **Argand plane** or the **Gaussian plane**. The point P , plotted on the Argand plane, is called the Argand diagram.

The length of the line segment OP is called the *modulus* of z and is denoted by $|z|$.

From Fig. 12.1, we obtain

$$OP^2 = OM^2 + MP^2 \Rightarrow OP^2 = x^2 + y^2 \Rightarrow OP = \sqrt{x^2 + y^2}$$

$$\text{Thus, } |z| = \sqrt{x^2 + y^2} = \sqrt{[\operatorname{Re}(z)]^2 + [\operatorname{Im}(z)]^2}$$

The angle θ which OP makes with positive direction of x -axis in anticlockwise sense is called the *argument* or *amplitude* of z and is denoted by $\arg(z)$ or $\operatorname{amp}(z)$.

From Fig. 12.1, we have

$$\tan \theta = \frac{PM}{OM} = \frac{y}{x} = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} \Rightarrow \theta = \tan^{-1} \left(\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} \right)$$

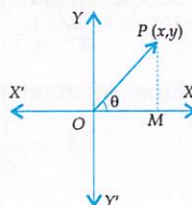


Fig. 12.1

This angle θ has infinitely many values differing by multiples of 2π . The unique value of θ such that $-\pi < \theta \leq \pi$ is called the *principal value of the amplitude* or *principal argument*. This formula for

determining the argument of $z = x + iy$ has severe drawback, because $z_1 = 1 + i\sqrt{3}$ and $z_2 = -1 - i\sqrt{3}$ are two distinct complex numbers represented by two distinct points in the Argand plane but their arguments seem to be $\tan^{-1} \sqrt{3} = \pi/3$ or $4\pi/3$ which is not correct. In fact the argument is the common solution of the simultaneous trigonometric equations

$$\cos \theta = \frac{x}{\sqrt{x^2 + y^2}} \text{ and, } \sin \theta = \frac{y}{\sqrt{x^2 + y^2}}$$

Since the above system of equations has infinitely many solutions. Therefore, there can be infinitely many arguments of $z = x + iy$. The argument θ which satisfies the inequality $-\pi < \theta \leq \pi$ is usually known as the *principal argument* of z . The argument of z depends upon the quadrant in which the point P lies as discussed below.

12.14.2 ARGUMENT OR AMPLITUDE OF A COMPLEX NUMBER FOR DIFFERENT SIGNS OF REAL AND IMAGINARY PARTS

(i) *Argument of $z = x + iy$ when $x > 0$ and $y > 0$:* Since x and y both are positive, therefore the point $P(x, y)$ representing $z = x + iy$ in the Argand plane lies in the first quadrant. Let θ be the argument of z and let α be the acute angle satisfying $\tan \alpha = |y/x|$. Then it is evident from Fig. 12.2 that $\theta = \alpha$.

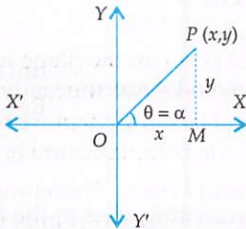


Fig.12.2

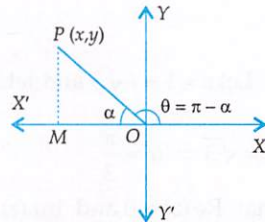


Fig.12.3

Thus, if x and y both are positive, then the argument of $z = x + iy$ is the acute angle given by $\tan \alpha = \frac{y}{x}$.

(ii) *Argument of $z = x + iy$ when $x < 0$ and $y > 0$:* In this case, the point $P(x, y)$ representing $z = x + iy$ in the Argand plane lies in the second quadrant. Let θ be the argument of z and let α be the acute angle satisfying $\tan \alpha = |y/x|$. Then it is evident from Fig. 12.3 that $\theta = \pi - \alpha$.

Thus, if $x < 0$ and $y > 0$, then the argument of $z = x + iy$ is $\pi - \alpha$, where α is the acute angle given by $\tan \alpha = \left| \frac{y}{x} \right|$.

(iii) *Argument of $z = x + iy$ when $x < 0$ and $y < 0$:* In this case, the point $P(x, y)$ representing $z = x + iy$ lies in the third quadrant. Let θ be the argument of z and α be the acute angle given by $\tan \alpha = |y/x|$. Then from Fig. 12.4, we obtain $\theta = -(\pi - \alpha) = \alpha - \pi$.

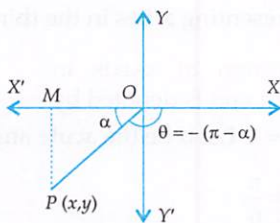


Fig.12.4

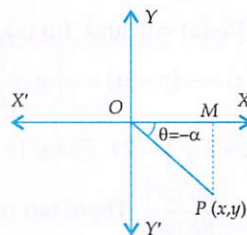


Fig.12.5

Thus, if $x < 0$ and $y < 0$ then the argument of $z = x + iy$ is $\alpha - \pi$ where α is the acute angle given by $\tan \alpha = |y/x|$.

(iv) *Argument of $z = x + iy$ when $x > 0$ and $y < 0$:* In this case, the point $P(x, y)$ representing $z = x + iy$ lies in the fourth quadrant. Let θ be the argument of z and let α be the acute angle given by $\tan \alpha = |y/x|$. Then from Fig. 12.5, we obtain $\theta = -\alpha$.

Thus, if $x > 0$ and $y < 0$, then the argument of $z = x + iy$ is $-\alpha$ where α is the acute angle given by $\tan \alpha = |y/x|$.

The above discussion suggests us the following algorithm for finding the argument of a complex number $z = x + iy$.

ALGORITHM

Step I Find the acute angle α given by $\tan \alpha = |y/x|$.

Step II Determine quadrant in which the point $P(x, y)$ lies.

If $P(x, y)$ belongs to the first quadrant, then $\arg(z) = \alpha$.

If $P(x, y)$ belongs to the second quadrant, then $\arg(z) = \pi - \alpha$.

If $P(x, y)$ belongs to the third quadrant, the $\arg(z) = -(\pi - \alpha)$ or $\pi + \alpha$.

If $P(x, y)$ belongs to the fourth quadrant, then $\arg(z) = -\alpha$ or $2\pi - \alpha$.

ILLUSTRATION 1 Find the modulus and argument of each of the following complex numbers:

(i) $1 + i\sqrt{3}$ [NCERT]

(ii) $-2 + 2i\sqrt{3}$

(iii) $-\sqrt{3} - i$

(iv) $2\sqrt{3} - 2i$

SOLUTION (i) Let $z = 1 + i\sqrt{3}$ and let α be the acute angle given by $\tan \alpha = \left| \frac{\text{Im}(z)}{\text{Re}(z)} \right|$. Then,

$$\tan \alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$$

We observe that $\text{Re}(z) > 0$ and $\text{Im}(z) > 0$. So, the point representing z lies in the first quadrant.

$$\therefore \arg(z) = \alpha = \frac{\pi}{3}. \text{ We find that } |z| = \sqrt{1^2 + (\sqrt{3})^2} = 2.$$

(ii) Let $z = -2 + 2\sqrt{3}i$. Then, $|z| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = 4$. Let α be the angle given by

$$\tan \alpha = \left| \frac{\text{Im}(z)}{\text{Re}(z)} \right|. \text{ Then, } \tan \alpha = \left| \frac{2\sqrt{3}}{-2} \right| = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}.$$

We find that $\text{Re}(z) < 0$ and $\text{Im}(z) > 0$. So, the point representing z lies in the second quadrant.

$$\therefore \arg(z) = \pi - \alpha = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

(iii) Let $z = -\sqrt{3} - i$. Then, $|z| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2$. Let α be the acute angle given by

$$\tan \alpha = \left| \frac{\text{Im}(z)}{\text{Re}(z)} \right|. \text{ Then, } \tan \alpha = \left| \frac{-1}{-\sqrt{3}} \right| = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}.$$

We find that $\text{Re}(z) < 0$ and $\text{Im}(z) < 0$. So, the point representing z lies in the third quadrant.

$$\therefore \arg(z) = -(\pi - \alpha) = -\left(\pi - \frac{\pi}{6}\right) = -\frac{5\pi}{6}$$

(iv) Let $z = 2\sqrt{3} - 2i$. Then, $|z| = \sqrt{(2\sqrt{3})^2 + (-2)^2} = 4$. Let α be the acute angle given by

$$\tan \alpha = \left| \frac{\text{Im}(z)}{\text{Re}(z)} \right|. \text{ Then, } \tan \alpha = \left| \frac{-2}{2\sqrt{3}} \right| = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}.$$

We observe that $\text{Re}(z) > 0$ and $\text{Im}(z) < 0$. So, the point representing z lies in the fourth quadrant.

$$\therefore \arg(z) = -\alpha = -\pi/6$$

ILLUSTRATION 2 Find the modulus and argument of the following complex numbers:

(i) $\frac{1+i}{1-i}$

[NCERT]

(ii) $\frac{1}{1+i}$

[NCERT]

SOLUTION (i) Let $z = \frac{1+i}{1-i}$. Then,

$$z = \frac{1+i}{1-i} = \frac{1+i}{1-i} \cdot \frac{(1+i)^2}{(1-i)^2} = \frac{1+2i+i^2}{1-i^2} = \frac{1+2i-1}{1+1} = i = 0+i \Rightarrow |z| = \sqrt{0^2+1^2} = 1$$

Let α be the acute angle given by $\tan \alpha = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then, $\tan \alpha = \frac{1}{0} = \infty \Rightarrow \alpha = \frac{\pi}{2}$.

We find that $\operatorname{Re}(z) = 0$ and $\operatorname{Im}(z) = 1 > 0$. So the point representing z lies on y -axis. Consequently, $\arg(z) = \alpha = \frac{\pi}{2}$. Hence, $|z| = 1$ and $\arg(z) = \frac{\pi}{2}$.

(ii) Let $z = \frac{1}{1+i}$. Then, $z = \frac{1}{1+i} = \frac{1}{1+i} \times \frac{1-i}{1-i} = \frac{1-i}{1-i^2} = \frac{1-i}{1+1} = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i$

$$\therefore |z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

Let α be the acute angle given by $\tan \alpha = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then, $\tan \alpha = \frac{-1/2}{1/2} = -1 \Rightarrow \alpha = \frac{\pi}{4}$.

We observe that $\operatorname{Re}(z) = \frac{1}{2} > 0$ and $\operatorname{Im}(z) = -\frac{1}{2} < 0$. So, the point representing z lies in the fourth quadrant. Therefore, $\arg(z) = -\alpha = -\frac{\pi}{4}$. Hence, $|z| = \frac{1}{\sqrt{2}}$ and $\arg(z) = -\frac{\pi}{4}$.

12.14.3 VECTORIAL REPRESENTATION OF A COMPLEX NUMBER

A complex number $z = x + iy$ can be represented by the position vector OP of point $P(x, y)$ in a two dimensional plane because a complex number depends on two things viz. (i) its modulus and (ii) its argument which are also the requirements of a vector on a plane.

In Fig. 12.6, the complex number $z = x + iy$ is represented by the vector $\vec{OP}$ and in such a case $|z|$ is the length OP and $\arg(z)$ is the angle which the directed line OP makes with the positive direction of x -axis.

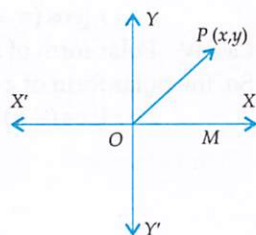


Fig.12.6

12.14.4 POLAR OR TRIGONOMETRICAL FORM OF A COMPLEX NUMBER

Let $z = x + iy$ be a complex number represented by a point $P(x, y)$ in the Argand plane. Then by the geometrical representation of $z = x + iy$, we obtain

$$OP = |z| \text{ and, } \angle POX = \theta = \arg(z)$$

In $\triangle POM$, we obtain

$$\cos \theta = \frac{OM}{OP} = \frac{x}{|z|} \Rightarrow x = |z| \cos \theta$$

$$\text{and, } \sin \theta = \frac{PM}{OP} = \frac{y}{|z|} \Rightarrow y = |z| \sin \theta$$

$$\therefore z = x + iy$$

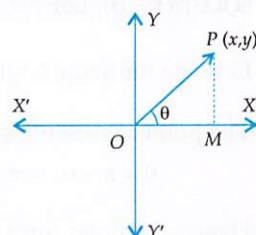


Fig.12.7

$\Rightarrow z = |z| \cos \theta + i |z| \sin \theta = |z| (\cos \theta + i \sin \theta) = r (\cos \theta + i \sin \theta)$, where $r = |z|$ and $\theta = \arg(z)$

This form of z is called a polar form of z . If we use the general value of the argument θ , then the polar form of z is given by

$$z = r \{ \cos (2n\pi + \theta) + i \sin (2n\pi + \theta) \}, \text{ where } r = |z|, \theta = \arg(z) \text{ and } n \text{ is an integer.}$$

12.14.5 MULTIPLICATION OF A COMPLEX NUMBER BY IOTA

Let $z = x + iy$ be a complex number represented by a point $P(x, y)$ in the argand plane. Let $r (\cos \theta + i \sin \theta)$ be the polar form of z . Then, $r = |z|$ and $\arg(z) = \theta$.

Now, $z = r (\cos \theta + i \sin \theta)$

$$\Rightarrow iz = ir (\cos \theta + i \sin \theta) = r (-\sin \theta + i \cos \theta) = r \{ \cos (\pi/2 + \theta) + i \sin (\pi/2 + \theta) \}$$

This means that iz is a complex number such that $|iz| = r = |z|$ and $\arg(iz) = \frac{\pi}{2} + \theta = \frac{\pi}{2} + \arg(z)$.

Thus, multiplication of a complex number by i results in rotating the vector joining the origin to point representing z through a right angle.

12.14.6 POLAR FORM OF A COMPLEX NUMBER FOR DIFFERENT SIGNS OF REAL AND IMAGINARY PARTS

Let $|z| = r$ and α be the acute angle given by $\tan \alpha = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Let θ be the argument of z .

Case I Polar form of $z = x + iy$ when $x > 0$ and $y > 0$: In this case, we have $\theta = \alpha$.

So, the polar form of $z = x + iy$ is $z = r (\cos \alpha + i \sin \alpha)$

Case II Polar form of $z = x + iy$ when $x < 0$ and $y > 0$: In this case, we have $\theta = \pi - \alpha$.

So, the polar form of $z = x + iy$ is

$$z = r [\cos (\pi - \alpha) + i \sin (\pi - \alpha)] = r (-\cos \alpha + i \sin \alpha)$$

Case III Polar form of $z = x + iy$ when $x < 0$ and $y < 0$: In this case, we have $\theta = -(\pi - \alpha)$.

So, the polar form of z is given by

$$z = r [\cos (\pi - \alpha) + i \sin (-\pi + \alpha)] = r (-\cos \alpha - i \sin \alpha)$$

Case IV Polar form of $z = x + iy$ when $x > 0$ and $y < 0$: In this case, we have $\theta = -\alpha$.

So, the polar form of z is

$$z = r [\cos (-\alpha) + i \sin (-\alpha)] = r (\cos \alpha - i \sin \alpha)$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 Write the following complex numbers in the polar form:

(i) $-3\sqrt{2} + 3\sqrt{2}i$

(ii) $1 + i$

(iii) $-1 - i$

[NCERT]

(iv) $1 - i$

[NCERT EXEMPLAR]

SOLUTION (i) Let $z = -3\sqrt{2} + 3\sqrt{2}i$. Then, $r = |z| = \sqrt{(-3\sqrt{2})^2 + (3\sqrt{2})^2} = 6$.

Let α be the acute angle given by $\tan \alpha = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then, $\tan \alpha = 1 \Rightarrow \alpha = \frac{\pi}{4}$.

The point representing z lies in the second quadrant. So, the argument θ of z is given by

$$\theta = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Hence, the polar form of $z = -3\sqrt{2} + 3\sqrt{2}i$ is

$$z = r (\cos \theta + i \sin \theta) = 6 \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$$

(ii) Let $z = 1 + i$. Then, $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Let α be the acute angle given by $\tan \alpha = \left| \frac{\text{Im}(z)}{\text{Re}(z)} \right|$. Then, $\tan \alpha = \frac{1}{1} = 1 \Rightarrow \alpha = \frac{\pi}{4}$.

We find that the point $(1, 1)$ representing z lies in first quadrant. Therefore, the argument of z is given by $\theta = \alpha = \frac{\pi}{4}$. Hence, the polar form of $z = 1 + i$ is

$$z = r (\cos \theta + i \sin \theta) = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

(iii) Let $z = -1 - i$. Then, $r = |z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$. Let α be the acute angle given by $\tan \alpha = \left| \frac{\text{Im}(z)}{\text{Re}(z)} \right|$. Then, $\tan \alpha = \left| \frac{-1}{-1} \right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$.

Clearly, the point $(-1, -1)$ representing z lies in the third quadrant. Therefore, the argument of z is given by

$$\theta = -(\pi - \alpha) = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4}.$$

Hence, the polar form of $z = -1 - i$ is

$$z = r (\cos \theta + i \sin \theta) = \sqrt{2} \left\{ \cos \left(-\frac{3\pi}{4} \right) + i \sin \left(-\frac{3\pi}{4} \right) \right\} = \sqrt{2} \left(\cos \frac{3\pi}{4} - i \sin \frac{3\pi}{4} \right)$$

(iv) Let $z = 1 - i$. Then, $|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$. Let α be the acute angle given by $\tan \alpha = \left| \frac{\text{Im}(z)}{\text{Re}(z)} \right|$.

$$\text{Then, } \tan \alpha = \left| \frac{-1}{1} \right| = 1 \Rightarrow \alpha = \frac{\pi}{4}.$$

We find that the point $(1, -1)$ representing z lies in the fourth quadrant. Therefore, the argument of z is given by $\theta = -\alpha = -\frac{\pi}{4}$. Hence, the polar form of $z = 1 - i$ is

$$r (\cos \theta + i \sin \theta) = \sqrt{2} \left\{ \cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right\} = \sqrt{2} \left(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right)$$

EXAMPLE 2 Find the modulus and principal argument of $(1 + i)$ and hence express it in the polar form. [NCERT]

SOLUTION Let $z = 1 + i$. Then, $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Let α be the acute angle given by $\tan \alpha = \left| \frac{\text{Im}(z)}{\text{Re}(z)} \right|$. Then, $\tan \alpha = \left| \frac{1}{1} \right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$. Clearly, the point $(1, 1)$ representing $z = 1 + i$ lies

in first quadrant. Therefore, $\theta = \arg(z) = \frac{\pi}{4}$.

Hence, the polar form of $z = 1 + i$ is $z = |z| (\cos \theta + i \sin \theta) = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$.

EXAMPLE 3 Find the modulus and principal argument of $-2i$.

SOLUTION Let $z = -2i = 0 + (-2)i$. Then, $|z| = \sqrt{0^2 + (-2)^2} = 2$. We find that the point $(0, -2)$ representing $z = -2i$ lies on the negative side of imaginary axis. Therefore, principal argument of z is $-\frac{\pi}{2}$.

EXAMPLE 4 Find the modulus and principal argument of -4 .

[NCERT]

SOLUTION Let $z = -4 + 0i$. Then, $|z| = \sqrt{(-4)^2 + 0} = 4$. We find that the point $(-4, 0)$ representing $z = -4 + 0i$ lies on the negative side of real axis. Therefore, principal argument of z is π .

EXAMPLE 5 Express the following complex numbers in the polar form:

(i) $\frac{1+i}{1-i}$

(ii) $\frac{2+6\sqrt{3}i}{5+\sqrt{3}i}$

SOLUTION (i) Let $z = \frac{1+i}{1-i}$. and, let $r(\cos \theta + i \sin \theta)$ be the polar form of z . Then, $r = |z|$ and $\theta = \arg(z)$.

$$\text{Now, } z = \frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{1-2i+i^2}{1-i^2} = \frac{1+2i-1}{1+1} = i = 0+1i.$$

$$\therefore r = |z| = \sqrt{0^2 + 1^2} = 1.$$

Clearly, the point $(0, 1)$ representing $z = 0 + i$ lies on positive direction of imaginary axis. Therefore, $\arg(z) = \frac{\pi}{2}$. Hence, the polar form of z is

$$z = 1 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$$

(ii) Let $z = \frac{2+6\sqrt{3}i}{5+\sqrt{3}i}$ and, let $r(\cos \theta + i \sin \theta)$ be the polar form of z . Then, $r = |z|$ and $\theta = \arg(z)$.

$$\text{Now, } z = \frac{2+6\sqrt{3}i}{5+\sqrt{3}i} = \frac{2+6\sqrt{3}i}{5+\sqrt{3}i} \cdot \frac{(5-\sqrt{3}i)}{(5-\sqrt{3}i)} = \frac{28+28\sqrt{3}i}{28} = 1+i\sqrt{3}$$

$$\therefore r = |z| = \sqrt{1^2 + 3^2} = 2.$$

$$\text{Let } \alpha \text{ be acute angle given by } \tan \alpha = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|} = \frac{\sqrt{3}}{1} = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}.$$

Clearly, the point $(1, \sqrt{3})$ representing z lies in first quadrant. Therefore, $\theta = \arg(z) = \alpha = \frac{\pi}{3}$.

Hence, the polar form of z is $2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 6 Put the complex number $\frac{1+7i}{(2-i)^2}$ in the form $r(\cos \theta + i \sin \theta)$, where r is a positive real number and $-\pi < \theta \leq \pi$.

[NCERT]

SOLUTION Let $z = \frac{1+7i}{(2-i)^2}$. Then,

$$z = \frac{1+7i}{4-4i+i^2} = \frac{1+7i}{3-4i} = \frac{1+7i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{-25+25i}{25} = -1+i$$

$$\therefore r = |z| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

$$\text{Let } \alpha \text{ be the acute angle given by } \tan \alpha = \left| \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} \right|. \text{ Then, } \tan \alpha = \left| -\frac{1}{1} \right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$$

Clearly, the point $(-1, 1)$ representing z lies in the second quadrant. Therefore,

$$\therefore \theta = \arg(z) = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

Hence, z in the polar form is given by $z = \sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$

EXAMPLE 7 Find the modulus and argument of the following complex numbers and convert them in polar form:

$$(i) \frac{1+2i}{1-3i} \quad [\text{NCERT}] \quad (ii) \frac{i-1}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}} \quad [\text{NCERT}] \quad (iii) \frac{1+3i}{1-2i} \quad [\text{NCERT}]$$

SOLUTION (i) Let $z = \frac{1+2i}{1-3i}$. Then, $z = \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i} = \frac{(1-6) + i(2+3)}{1+9} = -\frac{1}{2} + \frac{1}{2}i$

$$\therefore r = |z| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$$

Let α be the acute angle given by $\tan \alpha = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then, $\tan \alpha = \frac{|-1/2|}{|1/2|} = 1 \Rightarrow \alpha = \frac{\pi}{4}$

We find that $\operatorname{Re}(z) = -\frac{1}{2} < 0$ and $\operatorname{Im}(z) = \frac{1}{2} > 0$. So, the point representing z lies in the second quadrant. Therefore, $\theta = \arg(z) = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$. Hence, the polar form of z is

$$r(\cos \theta + i \sin \theta) = \frac{1}{\sqrt{2}} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right).$$

(ii) Let $z = \frac{i-1}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}}$. Then,

$$z = \frac{i-1}{\frac{1}{2} + i \frac{\sqrt{3}}{2}} = \frac{2(-1+i)}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}} = \frac{2\{(-1+\sqrt{3}) + i(1+\sqrt{3})\}}{1+3} = \left(\frac{\sqrt{3}-1}{2} \right) + i \left(\frac{\sqrt{3}+1}{2} \right)$$

$$\therefore |z| = \sqrt{\left(\frac{\sqrt{3}-1}{2}\right)^2 + \left(\frac{\sqrt{3}+1}{2}\right)^2} = \sqrt{\frac{(\sqrt{3}-1)^2 + (\sqrt{3}+1)^2}{4}} = \sqrt{\frac{2(3+1)}{4}} = \sqrt{2}$$

Let α be the acute angle given by $\tan \alpha = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then,

$$\tan \alpha = \frac{\left| \frac{\sqrt{3}+1}{2} \right|}{\left| \frac{\sqrt{3}-1}{2} \right|} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = \frac{\tan \frac{\pi}{4} + \tan \frac{\pi}{6}}{1 - \tan \frac{\pi}{4} \tan \frac{\pi}{6}} = \tan \left(\frac{\pi}{4} + \frac{\pi}{6} \right) = \tan \frac{5\pi}{12} \Rightarrow \alpha = \frac{5\pi}{12}$$

We find that $\operatorname{Re}(z) = \frac{\sqrt{3}-1}{2} > 0$ and, $\operatorname{Im}(z) = \frac{\sqrt{3}+1}{2} > 0$. So, the point representing z lies in the first quadrant. Therefore, $\theta = \arg(z) = \alpha = \frac{5\pi}{12}$. Hence, the polar form of z is

$$r(\cos \theta + i \sin \theta) = \sqrt{2} \left(\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12} \right).$$

(iii) Let $z = \frac{1+3i}{1-2i}$. Then, $z = \frac{1+3i}{1-2i} \times \frac{1+2i}{1+2i} = \frac{(1-6) + i(3+2)}{1+4} = -1 + i$

$$\therefore r = |z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}.$$

Let α be the acute angle given by $\tan \alpha = \left| \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} \right|$. Then, $\tan \alpha = \left| \frac{1}{-1} \right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$.

We find that $\operatorname{Re}(z) < 0$ and $\operatorname{Im}(z) > 0$. So, the point representing z lies in the second quadrant.

$$\therefore \theta = \arg(z) = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

Hence, the polar form of z is $r(\cos \theta + i \sin \theta) = \sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 8 For any complex number z , prove that $|\operatorname{Re}(z)| + |\operatorname{Im}(z)| \leq \sqrt{2}|z|$.

SOLUTION Let $z = r(\cos \theta + i \sin \theta)$. Then, $|z| = r$ and $\arg(z) = \theta$.

Now,

$$|\operatorname{Re}(z)| + |\operatorname{Im}(z)| = |r \cos \theta| + |r \sin \theta| = r \left\{ |\cos \theta| + |\sin \theta| \right\} \quad [\because r = |z| > 0]$$

$$\begin{aligned} \therefore \left\{ |\operatorname{Re}(z)| + |\operatorname{Im}(z)| \right\}^2 &= r^2 \left\{ |\cos \theta| + |\sin \theta| \right\}^2 = r^2 \left\{ 1 + |2 \sin \theta \cos \theta| \right\} \\ &= r^2 \left\{ 1 + |\sin 2\theta| \right\} \leq r^2 (1 + 1) \quad [\because |\sin 2\theta| \leq 1] \end{aligned}$$

$$\Rightarrow |\operatorname{Re}(z)| + |\operatorname{Im}(z)| \leq \sqrt{2}r \Rightarrow |\operatorname{Re}(z)| + |\operatorname{Im}(z)| \leq \sqrt{2}|z|$$

EXAMPLE 9 If z and w are two complex number such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$, then show that $\bar{z}w = -i$. [NCERT EXEMPLAR]

SOLUTION Let $|z| = r$ and $\arg(z) = \theta$. Then, $z = r(\cos \theta + i \sin \theta)$.

$$\text{Now, } |zw| = 1 \text{ and } \arg(z) - \arg(w) = \frac{\pi}{2}$$

$$\Rightarrow |z||w| = 1 \text{ and } \arg(w) = \arg(z) - \frac{\pi}{2} \Rightarrow |w| = \frac{1}{r} \text{ and } \arg(w) = \theta - \frac{\pi}{2}$$

$$\therefore w = |w| \{ \cos(\arg w) + i \sin(\arg w) \}$$

$$\Rightarrow w = \frac{1}{r} \left\{ \cos \left(\theta - \frac{\pi}{2} \right) + i \sin \left(\theta - \frac{\pi}{2} \right) \right\} = \frac{1}{r} \{ \sin \theta - i \cos \theta \} = -\frac{i}{r} (\cos \theta + i \sin \theta)$$

$$\therefore \bar{z}w = r(\cos \theta - i \sin \theta) \times -\frac{i}{r} (\cos \theta + i \sin \theta) = -i(\cos^2 \theta + \sin^2 \theta) = -i.$$

EXAMPLE 10 What is the locus of z , if amplitude of $(z - 2 - 3i)$ is $\frac{\pi}{4}$?

SOLUTION Let $z = x + iy$. Then, $z - 2 - 3i = (x + iy) - 2 - 3i = (x - 2) + i(y - 3)$

Let θ be the amplitude of $(x - 2) + i(y - 3)$. Then, $\tan \theta = \frac{y - 3}{x - 2}$. It is given that the argument of $z - 2 - 3i$ is $\frac{\pi}{4}$.

$$\therefore \tan \frac{\pi}{4} = \frac{y - 3}{x - 2} \Rightarrow 1 = \frac{y - 3}{x - 2} \Rightarrow x - y + 1 = 0, \text{ which is a straight line.}$$

Hence, the locus of z is a straight line.

EXAMPLE 11 Show that the complex number z , satisfying $\arg \left(\frac{z-1}{z+1} \right) = \frac{\pi}{4}$ lies on a circle.

[NCERT EXEMPLAR]

SOLUTION Let $z = x + iy$. Then,

$$\begin{aligned}\frac{z-1}{z+1} &= \frac{(x-1)+iy}{(x+1)+iy} = \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy} = \frac{(x^2-1+y^2)+2iy}{(x+1)^2+y^2} \\ &= \left\{ \frac{x^2+y^2-1}{(x+1)^2+y^2} \right\} + i \left\{ \frac{2y}{(x+1)^2+y^2} \right\}\end{aligned}$$

Let θ be the argument of $\frac{z-1}{z+1}$. Then, $\tan \theta = \frac{2y/(x+1)^2+y^2}{(x^2+y^2-1)/(x+1)^2+y^2} = \frac{2y}{x^2+y^2-1}$

But, it is given that $\arg\left(\frac{z-1}{z+1}\right)$ is $\frac{\pi}{4}$ i.e. $\theta = \frac{\pi}{4}$.

$$\therefore \tan \frac{\pi}{4} = \frac{2y}{x^2+y^2-1} \Rightarrow x^2+y^2-1=2y \Rightarrow x^2+y^2-2y-1=0 \Rightarrow (x-0)^2+(y-1)^2=(\sqrt{2})^2,$$

which represents a circle.

EXAMPLE 12 If $\arg(z-1) = \arg(z+3i)$, then find $(x-1):y$, where $z = x + iy$. [NCERT EXEMPLAR]

SOLUTION We have, $z = x + iy$.

$$\therefore z-1 = (x-1)+iy \text{ and } z+3i = x+i(y+3)$$

Let θ_1 and θ_2 be the arguments of $z-1$ and $z+3i$. Then, $\tan \theta_1 = \frac{y}{x-1}$ and $\tan \theta_2 = \frac{y+3}{x}$.

It is given that $\arg(z-1) = \arg(z+3i)$ i.e. $\theta_1 = \theta_2$.

$$\therefore \tan \theta_1 = \tan \theta_2$$

$$\Rightarrow \frac{y}{x-1} = \frac{y+3}{x} \Rightarrow 3x-y-3=0 \Rightarrow 3(x-1)=y \Rightarrow \frac{x-1}{y} = \frac{1}{3} \Rightarrow (x-1):y=1:3$$

EXAMPLE 13 If for complex numbers z_1 and z_2 , $\arg(z_1) - \arg(z_2) = 0$, then show that $|z_1 - z_2| = ||z_1| - |z_2||$. [NCERT EXEMPLAR]

SOLUTION Let $|z| = r_1$ and $|z_2| = r_2$. It is given that $\arg(z_1) - \arg(z_2) = 0$

i.e. $\arg(z_1) = \arg(z_2) = \theta$ (say)

$$\therefore z_1 = r_1(\cos \theta + i \sin \theta) \text{ and } z_2 = r_2(\cos \theta + i \sin \theta)$$

$$\Rightarrow z_1 - z_2 = (r_1 - r_2) \cos \theta + i(r_1 - r_2) \sin \theta$$

$$\Rightarrow |z_1 - z_2|^2 = (r_1 - r_2)^2 \cos^2 \theta + (r_1 - r_2)^2 \sin^2 \theta = (r_1 - r_2)^2 (\cos^2 \theta + \sin^2 \theta) = (r_1 - r_2)^2$$

$$\Rightarrow |z_1 - z_2| = |r_1 - r_2| \Rightarrow |z_1 - z_2| = ||z_1| - |z_2||$$

EXAMPLE 14 If z, z_1 and z_2 are complex numbers, prove that:

$$(i) \arg(\bar{z}) = -\arg(z). \text{ In general, } \arg(\bar{z}) = 2n\pi - \arg(z) \quad (ii) \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

$$(iii) \arg(z_1 \bar{z}_2) = \arg(z_1) - \arg(z_2) \quad (iv) \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

SOLUTION (i) Let $z = r(\cos \theta + i \sin \theta)$ be the polar form of z . Then, $|z| = r$ and $\arg(\bar{z}) = \theta$.

Now, $z = r(\cos \theta + i \sin \theta)$

$$\Rightarrow \bar{z} = r(\cos \theta - i \sin \theta) = r\{\cos(-\theta) + i \sin(-\theta)\} \Rightarrow |\bar{z}| = r \text{ and } \arg(\bar{z}) = -\theta$$

Since $\cos \theta$ and $\sin \theta$ are periodic functions with period 2π . Therefore, in general

$$\arg(\bar{z}) = 2n\pi - \arg(z)$$

(ii) Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ be two complex numbers in their polar forms. Then,

$$|z_1| = r_1, |z_2| = r_2, \arg(z_1) = \theta_1 \text{ and } \arg(z_2) = \theta_2$$

$$\therefore z_1 z_2 = r_1 (\cos \theta_1 + i \sin \theta_1) \times r_2 (\cos \theta_2 + i \sin \theta_2)$$

$$\Rightarrow z_1 z_2 = r_1 r_2 \{(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)\}$$

$$\Rightarrow z_1 z_2 = r_1 r_2 \{\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)\}$$

$$\Rightarrow |z_1 z_2| = r_1 r_2 \text{ and } \arg(z_1 z_2) = \theta_1 + \theta_2$$

$$\Rightarrow |z_1 z_2| = |z_1| |z_2| \text{ and } \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

REMARK It follows from the above result that

$$|z_1 z_2 \dots z_n| = |z_1| |z_2| \dots |z_n| \text{ and } \arg(z_1 z_2 \dots z_n) = \arg(z_1) + \arg(z_2) + \dots + \arg(z_n)$$

Replacing $z_1, z_2, z_3, \dots, z_n$ by z , we obtain: $|z^n| = |z|^n$ and $\arg(z^n) = n \arg(z)$

(iii) Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$. Then,

$$\overline{z_2} = r_2 (\cos \theta_2 - i \sin \theta_2) = r_2 \{\cos(-\theta_2) + i \sin(-\theta_2)\}$$

$$\therefore z_1 \overline{z_2} = r_1 r_2 [\cos\{\theta_1 + (-\theta_2)\} + i \sin\{\theta_1 + (-\theta_2)\}] \quad [\text{Using (ii)}]$$

$$\Rightarrow z_1 \overline{z_2} = r_1 r_2 \{\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)\}$$

$$\Rightarrow \arg(z_1 \overline{z_2}) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$$

(iv) Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$. Then,

$$|z_1| = r_1, |z_2| = r_2, \arg(z_1) = \theta_1 \text{ and } \arg(z_2) = \theta_2$$

$$\therefore \frac{z_1}{z_2} = \frac{r_1 (\cos \theta_1 + i \sin \theta_1)}{r_2 (\cos \theta_2 + i \sin \theta_2)} = \frac{r_1}{r_2} \frac{(\cos \theta_1 + i \sin \theta_1)}{(\cos \theta_2 + i \sin \theta_2)} \times \frac{(\cos \theta_2 - i \sin \theta_2)}{(\cos \theta_2 - i \sin \theta_2)}$$

$$\Rightarrow \frac{z_1}{z_2} = \frac{r_1}{r_2} \left\{ \frac{(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2)}{\cos^2 \theta_2 + \sin^2 \theta_2} \right\}$$

$$\Rightarrow \frac{z_1}{z_2} = \frac{r_1}{r_2} \left\{ \cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2) \right\} \Rightarrow \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$$

EXAMPLE 15 Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$ be two complex numbers. Then, prove that

$$(i) \quad |z_1 + z_2|^2 = r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2)$$

$$\text{or, } |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \cos(\theta_1 - \theta_2)$$

$$(ii) \quad |z_1 - z_2|^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)$$

$$\text{or, } |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2| \cos(\theta_1 - \theta_2)$$

SOLUTION We have, $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and, $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$

$$\therefore |z_1| = r_1, |z_2| = r_2, \arg(z_1) = \theta_1 \text{ and } \arg(z_2) = \theta_2$$

$$(i) \quad z_1 + z_2 = (r_1 \cos \theta_1 + r_2 \cos \theta_2) + i(r_1 \sin \theta_1 + r_2 \sin \theta_2)$$

$$\therefore |z_1 + z_2|^2 = (r_1 \cos \theta_1 + r_2 \cos \theta_2)^2 + (r_1 \sin \theta_1 + r_2 \sin \theta_2)^2$$

$$\Rightarrow |z_1 + z_2|^2 = r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2)$$

$$\Rightarrow |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \cos(\theta_1 - \theta_2)$$

$$(ii) \quad z_1 - z_2 = (r_1 \cos \theta_1 - r_2 \cos \theta_2) + i(r_1 \sin \theta_1 - r_2 \sin \theta_2)$$

$$\therefore |z_1 - z_2|^2 = (r_1 \cos \theta_1 - r_2 \cos \theta_2)^2 + (r_1 \sin \theta_1 - r_2 \sin \theta_2)^2$$

$$\Rightarrow |z_1 - z_2|^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)$$

$$\Rightarrow |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$$

EXAMPLE 16 For any two complex numbers z_1 and z_2 , prove that :

$$(i) |z_1 + z_2| = |z_1 - z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \frac{\pi}{2} \Leftrightarrow \frac{z_1}{z_2} \text{ is purely imaginary.}$$

$$(ii) |z_1 + z_2| = |z_1||z_2| \Leftrightarrow \arg(z_1) = \arg(z_2) \Leftrightarrow \frac{z_1}{z_2} \text{ is purely real.}$$

$$(iii) |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2} \text{ is purely imaginary}$$

SOLUTION Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$, $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$. Then, $|z_1| = r_1$, $|z_2| = r_2$, $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$.

(i) We have,

$$\begin{aligned} & |z_1 + z_2| = |z_1 - z_2| \\ \Leftrightarrow & |z_1 + z_2|^2 = |z_1 - z_2|^2 \\ \Leftrightarrow & r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2) = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2) \\ \Leftrightarrow & 4r_1 r_2 \cos(\theta_1 - \theta_2) = 0 \\ \Leftrightarrow & \cos(\theta_1 - \theta_2) = 0 \\ \Leftrightarrow & \theta_1 - \theta_2 = \frac{\pi}{2} \text{ i.e. } \arg(z_1) - \arg(z_2) = \frac{\pi}{2} \\ \Leftrightarrow & \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} \quad \left[\because \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \right] \\ \Leftrightarrow & \frac{z_1}{z_2} \text{ is purely imaginary.} \end{aligned}$$

(ii) We have,

$$\begin{aligned} & |z_1 + z_2| = |z_1| + |z_2| \\ \Leftrightarrow & |z_1 + z_2|^2 = (r_1 + r_2)^2 \quad [\because |z_1| = r_1 \text{ and } |z_2| = r_2] \\ \Leftrightarrow & r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2) = r_1^2 + r_2^2 + 2r_1 r_2 \\ \Leftrightarrow & \cos(\theta_1 - \theta_2) = 1 \\ \Leftrightarrow & \theta_1 - \theta_2 = 0 \text{ i.e. } \arg(z_1) - \arg(z_2) = 0 \text{ or, } \arg(z_1) = \arg(z_2) \\ \Leftrightarrow & \arg\left(\frac{z_1}{z_2}\right) = 0 \quad \left[\because \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \right] \\ \Leftrightarrow & \frac{z_1}{z_2} \text{ is purely real} \end{aligned}$$

(iii) We have,

$$\begin{aligned} & |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \\ \Leftrightarrow & r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2) = r_1^2 + r_2^2 \\ \Leftrightarrow & 2r_1 r_2 \cos(\theta_1 - \theta_2) = 0 \\ \Leftrightarrow & \cos(\theta_1 - \theta_2) = 0 \\ \Leftrightarrow & \theta_1 - \theta_2 = \frac{\pi}{2} \\ \Leftrightarrow & \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} \quad \left[\because \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = \theta_1 - \theta_2 \right] \end{aligned}$$

$\Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary.

EXAMPLE 17 For any two complex numbers z_1 and z_2 , prove the following triangle inequalities:

(i) $|z_1 + z_2| \leq |z_1| + |z_2|$

(ii) $|z_1 - z_2| \leq |z_1| + |z_2|$

(iii) $|z_1 + z_2| \geq |z_1| - |z_2|$

(iv) $|z_1 - z_2| \geq |z_1| - |z_2|$

SOLUTION (i) We have,

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2), \text{ where } \theta_1 = \arg(z_1) \text{ and } \theta_2 = \arg(z_2).$$

$$\therefore \cos(\theta_1 - \theta_2) \leq 1$$

$$\Rightarrow 2|z_1||z_2|\cos(\theta_1 - \theta_2) \leq 2|z_1||z_2| \quad [\text{Multiplying both sides by } 2|z_1||z_2|]$$

$$\Rightarrow |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2) \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$

$$\Rightarrow |z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2 \Rightarrow |z_1 + z_2| \leq |z_1| + |z_2|$$

(ii) We have,

$$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$$

$$\therefore -1 \leq \cos(\theta_1 - \theta_2) \leq 1$$

$$\therefore -1 \leq -\cos(\theta_1 - \theta_2) \leq 1$$

$$\Rightarrow -\cos(\theta_1 - \theta_2) \leq 1$$

$$\Rightarrow -2|z_1||z_2|\cos(\theta_1 - \theta_2) \leq 2|z_1||z_2|$$

$$\Rightarrow |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2) \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$

$$\Rightarrow |z_1 - z_2|^2 \leq (|z_1| + |z_2|)^2 \Rightarrow |z_1 - z_2| \leq |z_1| + |z_2|$$

(iii) We have,

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$$

$$\therefore -1 \leq \cos(\theta_1 - \theta_2) \leq 1$$

$$\Rightarrow \cos(\theta_1 - \theta_2) \geq -1$$

$$\Rightarrow 2|z_1||z_2|\cos(\theta_1 - \theta_2) \geq -2|z_1||z_2|$$

$$\Rightarrow |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2) \geq |z_1|^2 + |z_2|^2 - 2|z_1||z_2|$$

$$\Rightarrow |z_1 + z_2|^2 \geq (|z_1| - |z_2|)^2 \Rightarrow |z_1 + z_2| \geq |z_1| - |z_2|$$

(iv) We have,

$$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$$

$$\therefore -1 \leq \cos(\theta_1 - \theta_2) \leq 1$$

$$\Rightarrow \cos(\theta_1 - \theta_2) \leq 1$$

$$\Rightarrow -\cos(\theta_1 - \theta_2) \geq -1$$

$$\Rightarrow -2|z_1||z_2|\cos(\theta_1 - \theta_2) \geq -2|z_1||z_2|$$

$$\Rightarrow |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2) \geq |z_1|^2 + |z_2|^2 - 2|z_1||z_2|$$

$$\Rightarrow |z_1 - z_2|^2 \geq (|z_1| - |z_2|)^2 \Rightarrow |z_1 - z_2| \geq |z_1| - |z_2|$$

EXAMPLE 18 If $z_r = \cos\left(\frac{\pi}{3^r}\right) + i \sin\left(\frac{\pi}{3^r}\right)$, $r = 1, 2, 3, \dots$, prove that $z_1 z_2 z_3 \dots z_\infty = i$.

SOLUTION We know that, if $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$, $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$, ... are complex numbers, then

$$z_1 z_2 z_3 \dots z_n = r_1 r_2 r_3 \dots r_n \{\cos (\theta_1 + \theta_2 + \dots + \theta_n) + i \sin (\theta_1 + \theta_2 + \dots + \theta_n)\}$$

Here, $z_r = \cos \frac{\pi}{3^r} + i \sin \frac{\pi}{3^r}$, $r = 1, 2, 3, \dots$

$$\Rightarrow |z_r| = \sqrt{\cos^2 \frac{\pi}{3^r} + \sin^2 \frac{\pi}{3^r}} = 1, r = 1, 2, 3, \dots \text{ and } \arg(z_r) = \frac{\pi}{3^r}, r = 1, 2, 3, \dots$$

$$\begin{aligned} \therefore z_1 z_2 z_3 \dots z_n &= \cos \left\{ \frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \dots + \frac{\pi}{3^n} \right\} + i \sin \left\{ \frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \dots + \frac{\pi}{3^n} \right\} \\ &= \cos \left\{ \frac{\pi \left(1 - \frac{1}{3^n} \right)}{\left(1 - \frac{1}{3} \right)} \right\} + i \sin \left\{ \frac{\pi \left(1 - \frac{1}{3^n} \right)}{\left(1 - \frac{1}{3} \right)} \right\} = \cos \left\{ \frac{\pi}{2} \left(1 - \frac{1}{3^n} \right) \right\} + i \sin \left\{ \frac{\pi}{2} \left(1 - \frac{1}{3^n} \right) \right\} \end{aligned}$$

$$\begin{aligned} \text{Hence, } z_1 z_2 z_3 \dots z_\infty &= \lim_{n \rightarrow \infty} (z_1 z_2 z_3 \dots z_n) = \lim_{n \rightarrow \infty} \left[\cos \left\{ \frac{\pi}{2} \left(1 - \frac{1}{3^n} \right) \right\} + i \sin \left\{ \frac{\pi}{2} \left(1 - \frac{1}{3^n} \right) \right\} \right] \\ &= \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i \quad \left[\because \lim_{n \rightarrow \infty} \frac{1}{3^n} = 0 \right] \end{aligned}$$

EXAMPLE 19 If $x_n = \cos \frac{\pi}{2^n} + i \sin \frac{\pi}{2^n}$, prove that $x_1 x_2 x_3 \dots x_\infty = -1$.

SOLUTION We find that

$$\begin{aligned} x_1 x_2 \dots x_n &= \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) \left(\cos \frac{\pi}{2^2} + i \sin \frac{\pi}{2^2} \right) \left(\cos \frac{\pi}{2^3} + i \sin \frac{\pi}{2^3} \right) \dots \left(\cos \frac{\pi}{2^n} + i \sin \frac{\pi}{2^n} \right) \\ &= \cos \left\{ \frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \dots + \frac{\pi}{2^n} \right\} + i \sin \left\{ \frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \dots + \frac{\pi}{2^n} \right\} \\ &= \cos \left\{ \frac{\pi \left(1 - \frac{1}{2^n} \right)}{\left(1 - \frac{1}{2} \right)} \right\} + i \sin \left\{ \frac{\pi \left(1 - \frac{1}{2^n} \right)}{\left(1 - \frac{1}{2} \right)} \right\} = \cos \left\{ \pi \left(1 - \frac{1}{2^n} \right) \right\} + i \sin \left\{ \pi \left(1 - \frac{1}{2^n} \right) \right\} \end{aligned}$$

$$\begin{aligned} \therefore x_1 x_2 x_3 \dots x_\infty &= \lim_{n \rightarrow \infty} (x_1 x_2 x_3 \dots x_n) = \lim_{n \rightarrow \infty} \cos \left\{ \pi \left(1 - \frac{1}{2^n} \right) \right\} + i \sin \left\{ \pi \left(1 - \frac{1}{2^n} \right) \right\} \\ &= \cos \pi + i \sin \pi = -1 \end{aligned}$$

EXAMPLE 20 Let z_1 and z_2 be two complex numbers such that $\bar{z}_1 + i \bar{z}_2 = 0$ and $\arg(z_1 z_2) = \pi$. Then, find $\arg(z_1)$. [NCERT EXEMPLAR]

SOLUTION It is given that

$$\bar{z}_1 + i \bar{z}_2 = 0 \Rightarrow \bar{z}_1 = -i \bar{z}_2 \Rightarrow \overline{(\bar{z}_1)} = \overline{(-i \bar{z}_2)} \quad [\text{Taking conjugate of both sides}]$$

$$\Rightarrow z_1 = i z_2$$

$$\Rightarrow z_2 = -i z_1 \Rightarrow \arg(z_2) = \arg(-i z_1) \Rightarrow \arg(z_2) = \arg(-i) + \arg(z_1) \Rightarrow \arg(z_2) = -\frac{\pi}{2} + \arg(z_1)$$

...(i)

It is also given that

$$\arg(z_1 z_2) = \pi \Rightarrow \arg(z_1) + \arg(z_2) = \pi \Rightarrow \arg(z_1) - \frac{\pi}{2} + \arg(z_1) = \pi \quad [\text{Using (i)}]$$

$$\Rightarrow 2 \arg(z_1) = \frac{3\pi}{2} \Rightarrow \arg(z_1) = \frac{3\pi}{4}$$

EXAMPLE 21 If z_1 and z_2 both satisfy $z + \bar{z} = 2|z-1|$ and $\arg(z_1 - z_2) = \frac{\pi}{4}$, then find

$\text{Im}(z_1 + z_2)$.

[NCERT EXEMPLAR]

SOLUTION Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. It is given that z_1 and z_2 satisfy $z + \bar{z} = 2|z-1|$.

$$\therefore z_1 + \bar{z}_1 = 2|z_1 - 1| \quad \text{and} \quad z_2 + \bar{z}_2 = 2|z_2 - 1|$$

$$\Rightarrow 2x_1 = 2|(x_1 - 1) + iy_1| \quad \text{and} \quad 2x_2 = 2|(x_2 - 1) + iy_2|$$

$$\Rightarrow x_1 = \sqrt{(x_1 - 1)^2 + y_1^2} \quad \text{and} \quad x_2 = \sqrt{(x_2 - 1)^2 + y_2^2}$$

$$\Rightarrow x_1^2 = (x_1 - 1)^2 + y_1^2 \quad \text{and} \quad x_2^2 = (x_2 - 1)^2 + y_2^2$$

$$\Rightarrow 2x_1 = 1 + y_1^2 \quad \dots(i) \quad \text{and} \quad 2x_2 = 1 + y_2^2 \quad \dots(ii)$$

$$\Rightarrow 2(x_1 - x_2) = y_1^2 - y_2^2 \quad [\text{Subtracting (ii) from (i)}]$$

$$\Rightarrow 2 \left(\frac{x_1 - x_2}{y_1 - y_2} \right) = y_1 + y_2 \quad \dots(iii)$$

$$\text{Now, } z_1 = x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2 \Rightarrow z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

It is given that $\arg(z_1 - z_2) = \frac{\pi}{4}$. Therefore,

$$\tan \frac{\pi}{4} = \frac{y_1 - y_2}{x_1 - x_2} \Rightarrow 1 = \frac{y_1 - y_2}{x_1 - x_2} \Rightarrow \frac{x_1 - x_2}{y_1 - y_2} = 1 \quad \dots(iv)$$

From (iii) and (iv), we obtain: $2 = y_1 + y_2 \Rightarrow \text{Im}(z_1 + z_2) = 2$.

EXAMPLE 22 If a complex number z lies in the interior or on the boundary of a circle of radius 3 units and centre $(-4, 0)$, find the greatest and least values of $|z+1|$.

[NCERT EXEMPLAR]

SOLUTION Let $C(-4, 0)$ be the centre of a circle of radius 3 units and $P(z)$ be a point in the interior or on the boundary of the circle. Then,

$$CP \leq 3 \Rightarrow |z - (-4 + 0i)| \leq 3 \Rightarrow |z + 4| \leq 3 \quad \dots(i)$$

Now,

$$|z+1| = |z+4-3| \leq |z+4| + |-3| \quad [\because |z_1 + z_2| \leq |z_1| + |z_2|]$$

$$\Rightarrow |z+1| \leq 3 + 3 = 6 \quad [\text{Using (i)}]$$

So, the greatest value of $|z+1|$ is 6.

We know that the modulus of any complex number is greater than or equal to zero. Therefore,

$$|z+1| \geq 0 \quad \text{for all } z.$$

So, the least value of $|z+1|$ is 0.

EXAMPLE 23 Locate the points for which $3 < |z| < 4$.

[NCERT EXEMPLAR]

SOLUTION Let $z = x + iy$. Then, $|z| = \sqrt{x^2 + y^2}$.

Now

$$3 < |z| < 4$$

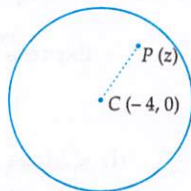


Fig. 12.8

$$\Leftrightarrow 9 < |z|^2 < 16 \Leftrightarrow 9 < x^2 + y^2 < 16 \Leftrightarrow 9 < x^2 + y^2 \text{ and } x^2 + y^2 < 16$$

Clearly, $x^2 + y^2 > 9$ represents the exterior of the circle $x^2 + y^2 = 9$ and $x^2 + y^2 < 16$ represents the interior of the circle $x^2 + y^2 = 16$. Hence, $9 < x^2 + y^2 < 16$ represents the shaded portion between the circles $x^2 + y^2 = 9$ and $x^2 + y^2 = 16$ i.e. the circular annulus.

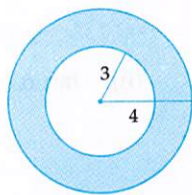


Fig. 12.9

EXERCISE 12.4

BASIC

1. Find the modulus and argument of the following complex numbers and hence express each of them in the polar form:

(i) $1+i$ (ii) $\sqrt{3}+i$ [NCERT] (iii) $1-i$ [NCERT]

(iv) $\frac{1-i}{1+i}$ (v) $\frac{1}{1+i}$ (vi) $\frac{1+2i}{1-3i}$

(vii) $\sin 120^\circ - i \cos 120^\circ$ (viii) $\frac{-16}{1+i\sqrt{3}}$ [NCERT]

2. Write $(i^{25})^3$ in polar form. [NCERT EXEMPLAR]

BASED ON HOTS

3. Express the following complex numbers in the form $r(\cos \theta + i \sin \theta)$:

(i) $1+i \tan \alpha$ (ii) $\tan \alpha - i$

(iii) $1 - \sin \alpha + i \cos \alpha$ (iv) $\frac{1-i}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}}$

[NCERT EXEMPLAR]

4. If z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$ and $\arg(z_1) + \arg(z_2) = \pi$, then show that $z_1 = -\bar{z}_2$. [NCERT EXEMPLAR]

5. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, prove that

$$\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right) = 0. \quad \text{[NCERT EXEMPLAR]}$$

6. Express $\sin \frac{\pi}{5} + i\left(1 - \cos \frac{\pi}{5}\right)$ in polar form. [NCERT EXEMPLAR]

ANSWERS

1. (i) $\sqrt{2}(\cos \pi/4 + i \sin \pi/4)$ (ii) $2(\cos \pi/6 + i \sin \pi/6)$
 (iii) $\sqrt{2}(\cos \pi/4 - i \sin \pi/4)$ (iv) $(\cos \pi/2 - i \sin \pi/2)$
 (v) $\frac{1}{\sqrt{2}}(\cos \pi/4 - i \sin \pi/4)$ (vi) $\frac{1}{\sqrt{2}}(\cos 3\pi/4 + i \sin 3\pi/4)$
 (vii) $\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}$ (viii) $8\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)$

2. $\cos \frac{\pi}{2} - i \sin \frac{\pi}{2}$

3. (i) $1+i \tan \alpha = \begin{cases} \sec \alpha (\cos \alpha + i \sin \alpha), & 0 \leq \alpha < \frac{\pi}{2} \\ -\sec \alpha (\cos(\alpha - \pi) + i \sin(\alpha - \pi)), & \frac{\pi}{2} < \alpha \leq \pi \end{cases}$

$$(ii) \tan \alpha - i = \begin{cases} \sec \alpha \left\{ \cos \left(\alpha - \frac{\pi}{2} \right) + i \sin \left(\alpha - \frac{\pi}{2} \right) \right\}, & 0 \leq \alpha < \frac{\pi}{2} \\ -\sec \alpha \left\{ \cos \left(\frac{\pi}{2} + \alpha \right) + i \sin \left(\frac{\pi}{2} + \alpha \right) \right\}, & \frac{\pi}{2} < \alpha \leq \pi \end{cases}$$

$$(iii) (1 - \sin \alpha) + i \cos \alpha = \begin{cases} \sqrt{2} \left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right) \left\{ \cos \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) + i \sin \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \right\}, & \text{if } 0 \leq \alpha < \frac{\pi}{2} \\ -\sqrt{2} \left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right) \left\{ \cos \left(\frac{\alpha}{2} - \frac{3\pi}{4} \right) + i \sin \left(\frac{\alpha}{2} - \frac{3\pi}{4} \right) \right\}, & \text{if } \frac{\pi}{2} < \alpha < \frac{3\pi}{2} \\ -\sqrt{2} \left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right) \left\{ \cos \left(\frac{\alpha}{2} - \frac{3\pi}{4} \right) + i \sin \left(\frac{\alpha}{2} - \frac{3\pi}{4} \right) \right\}, & \text{if } \frac{3\pi}{2} < \alpha < 2\pi \end{cases}$$

$$(iv) \frac{1-i}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}} = \sqrt{2} \left(\cos \frac{7\pi}{12} - i \sin \frac{7\pi}{12} \right)$$

$$6. 2 \sin \frac{\pi}{10} \left(\cos \frac{\pi}{10} + i \sin \frac{\pi}{10} \right)$$

HINTS TO SELECTED PROBLEMS

1. (ii) Let $z = \sqrt{3} + i$. Then, $|z| = \sqrt{(\sqrt{3})^2 + (1)^2} = 2$. Let θ be the argument of z and α be the acute angle given by $\tan \alpha = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then, $\tan \alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}$.

Clearly, z lies in the first quadrant. So, $\arg(z) = \alpha = \frac{\pi}{6}$.

- (iii) Let $z = 1 - i$. Then, $|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$. Let α be the acute angle given by $\tan \alpha = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then, $\tan \alpha = \frac{|-1|}{|1|} = 1 \Rightarrow \alpha = \frac{\pi}{4}$.

Clearly, z lies in the fourth quadrant. Therefore, $\arg(z) = -\alpha = -\frac{\pi}{4}$.

- (viii) Let $z = \frac{-16}{1+i\sqrt{3}} = \frac{-16(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} = \frac{-16(1-i\sqrt{3})}{1+3} = -4 + 4i\sqrt{3}$. Then,

$$|z| = \sqrt{(-4)^2 + (4\sqrt{3})^2} = \sqrt{16+48} = 8$$

Let α be the acute angle given by $\tan \alpha = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then, $\tan \alpha = \frac{|4\sqrt{3}|}{|-4|} = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$

Clearly, z lies in the second quadrant. Therefore, $\arg(z) = \pi - \alpha = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

3. (i) Let $z = 1 + i \tan \alpha$. Clearly, z is meaningful for $\alpha \neq (2n-1)\frac{\pi}{2}$, $n \in \mathbb{Z}$. Also, $\tan \alpha$ is a periodic function with period π . So, let us take α lying in the interval $[0, \pi/2) \cup (\pi/2, \pi]$.

Following cases arise:

Case I When $\alpha \in [0, \pi/2)$: We have, $z = 1 + i \tan \alpha$

$$\therefore |z| = \sqrt{1 + \tan^2 \alpha} = \sqrt{\sec^2 \alpha} = |\sec \alpha| = \sec \alpha \quad \left[\because \frac{\pi}{2} < \alpha < \pi \therefore \sec \alpha < 0 \right]$$

Let β be an acute angle given by $\tan \beta = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then, $\tan \beta = |\tan \alpha| = \tan \alpha \Rightarrow \beta = \alpha$.

As z is represented by a point lying in first quadrant. Therefore, $\arg(z) = \beta = \alpha$.

So, the polar form of z is $\sec \alpha (\cos \alpha + i \sin \alpha)$

Case II When $\alpha \in (\pi/2, \pi]$: We have, $z = 1 + i \tan \alpha$

$$\therefore |z| = \sqrt{1 + \tan^2 \alpha} = |\sec \alpha| = -\sec \alpha \quad \left[\because \frac{\pi}{2} < \alpha < \pi \therefore \sec \alpha < 0 \right]$$

Let β be an acute angle given by $\tan \beta = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then,

$$\tan \beta = |\tan \alpha| = -\tan \alpha \quad [\because \alpha \in (\pi/2, \pi)]$$

$$\Rightarrow \tan \beta = \tan(\pi - \alpha) \Rightarrow \beta = \pi - \alpha$$

We observe that z is represented by a point in fourth quadrant.

$$\therefore \arg(z) = -\beta = \alpha - \pi$$

Thus, z in polar form is $-\sec \alpha \{\cos(\alpha - \pi) + i \sin(\alpha - \pi)\}$.

(ii) Let $z = \tan \alpha - i$. Since $\tan \alpha$ is periodic with period π . So, let us take $\alpha \in [0, \pi/2) \cup (\pi/2, \pi]$.

Case I When $\alpha \in [0, \pi/2)$: We have, $z = \tan \alpha - i$

$$\therefore |z| = \sqrt{\tan^2 \alpha + 1} = |\sec \alpha| = \sec \alpha$$

Let β be the acute angle given by $\tan \beta = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then,

$$\tan \beta = \frac{1}{|\tan \alpha|} = |\cot \alpha| = \cot \alpha = \tan\left(\frac{\pi}{2} - \alpha\right) \Rightarrow \beta = \frac{\pi}{2} - \alpha$$

Clearly, $\operatorname{Re}(z) > 0$ and $\operatorname{Im}(z) < 0$. So, z lies in the fourth quadrant. Therefore, $\arg(z) = -\beta = \alpha - \frac{\pi}{2}$. Thus, z in polar form is given by $z = \sec \alpha \left\{ \cos\left(\alpha - \frac{\pi}{2}\right) + i \sin\left(\alpha - \frac{\pi}{2}\right) \right\}$

Case II When $\alpha \in (\pi/2, \pi]$: we have, $z = \tan \alpha - i \Rightarrow |z| = \sqrt{\tan^2 \alpha + 1} = |\sec \alpha| = -\sec \alpha$

Let β be the acute angle given by $\tan \beta = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then,

$$\tan \beta = \frac{1}{|\tan \alpha|} = |\cot \alpha| = -\cot \alpha = \tan\left(\alpha - \frac{\pi}{2}\right) \Rightarrow \beta = \alpha - \frac{\pi}{2}$$

Clearly, $\operatorname{Re}(z) < 0$ and $\operatorname{Im}(z) < 0$. So, z lies in third quadrant. Therefore, $\arg(z) = \pi + \beta = \frac{\pi}{2} + \alpha$

Thus, the polar form of z is $-\sec \alpha \left\{ \cos\left(\frac{\pi}{2} + \alpha\right) + i \sin\left(\frac{\pi}{2} + \alpha\right) \right\}$.

(iii) Let $z = (1 - \sin \alpha) + i \cos \alpha$. Since sine and cosine functions are periodic functions with period 2π . So, let us take α lying in the interval $[0, 2\pi]$.

Now, $z = 1 - \sin \alpha + i \cos \alpha$

$$\Rightarrow |z| = \sqrt{(1 - \sin \alpha)^2 + \cos^2 \alpha} = \sqrt{2 - 2 \sin \alpha} = \sqrt{2} \sqrt{1 - \sin \alpha} = \sqrt{2} \sqrt{\left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2}\right)^2}$$

$$\Rightarrow |z| = \sqrt{2} \left| \cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right|$$

Let β be the acute angle given by $\tan \beta = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then,

$$\begin{aligned}\tan \beta &= \frac{|\cos \alpha|}{|1 - \sin \alpha|} = \left| \frac{\cos \alpha}{1 - \sin \alpha} \right| = \left| \frac{\cos^2 \frac{\alpha}{2} - \sin^2 \frac{\alpha}{2}}{\left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right)^2} \right| = \left| \frac{\cos \frac{\alpha}{2} + \sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2}} \right| \\ \Rightarrow \tan \beta &= \left| \frac{1 + \tan \frac{\alpha}{2}}{1 - \tan \frac{\alpha}{2}} \right| = \left| \tan \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \right|\end{aligned}$$

Following cases arise:

Case I When $0 \leq \alpha < \frac{\pi}{2}$: In this case, we have $\cos \frac{\alpha}{2} > \sin \frac{\alpha}{2}$ and $\frac{\pi}{4} + \frac{\alpha}{2} \in \left[\frac{\pi}{4}, \frac{\pi}{2} \right)$

$$\therefore |z| = \sqrt{2} \left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right) \text{ and } \tan \beta = \left| \tan \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \right| = \tan \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \Rightarrow \beta = \frac{\pi}{4} + \frac{\alpha}{2}$$

Clearly, z lies in the first quadrant. Therefore, $\arg(z) = \frac{\pi}{4} + \frac{\alpha}{2}$.

Hence, the polar form of z is $\sqrt{2} \left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right) \left\{ \cos \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) + i \sin \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \right\}$.

Case II When $\frac{\pi}{2} < \alpha < \frac{3\pi}{2}$: In this case, we have $\cos \frac{\alpha}{2} < \sin \frac{\alpha}{2}$ and $\frac{\pi}{4} + \frac{\alpha}{2} \in \left(\frac{\pi}{2}, \pi \right)$

$$\therefore |z| = \sqrt{2} \left| \cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right| = -\sqrt{2} \left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right)$$

$$\text{and, } \tan \beta = \left| \tan \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \right| = -\tan \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) = \tan \left\{ \pi - \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \right\} = \tan \left(\frac{3\pi}{4} - \frac{\alpha}{2} \right) \Rightarrow \beta = \frac{3\pi}{4} - \frac{\alpha}{2}$$

Since $1 - \sin \alpha > 0$ and $\cos \alpha < 0$. So, z lies in fourth quadrant. Therefore, $\arg(z) = -\beta = \frac{\alpha}{2} - \frac{3\pi}{4}$.

Hence, the polar form of z is $-\sqrt{2} \left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right) \left\{ \cos \left(\frac{\alpha}{2} - \frac{3\pi}{4} \right) + i \sin \left(\frac{\alpha}{2} - \frac{3\pi}{4} \right) \right\}$.

Case III When $\frac{3\pi}{2} < \alpha < 2\pi$: In this case, we have

$$\cos \frac{\alpha}{2} < \sin \frac{\alpha}{2} \text{ and } \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \in \left(\pi, \frac{5\pi}{4} \right)$$

$$\therefore |z| = \sqrt{2} \left| \cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right| = -\sqrt{2} \left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right)$$

$$\text{and, } \tan \beta = \left| \tan \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \right| = \tan \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) = -\tan \left\{ \pi - \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \right\} = \tan \left(\frac{\alpha}{2} - \frac{3\pi}{4} \right) \Rightarrow \beta = \frac{\alpha}{2} - \frac{3\pi}{4}$$

Clearly, $\operatorname{Re}(z) < 0$ and $\operatorname{Im}(z) > 0$. So, z lies in the first quadrant. Therefore, $\arg(z) = \beta = \frac{\alpha}{2} - \frac{3\pi}{4}$.

Hence, the polar form of z is $-\sqrt{2} \left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right) \left\{ \cos \left(\frac{\alpha}{2} - \frac{3\pi}{4} \right) + i \sin \left(\frac{\alpha}{2} - \frac{3\pi}{4} \right) \right\}$.

$$(iv) \text{ Let } z = \frac{1-i}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}} = \frac{\sqrt{2} (\cos \pi/4 - i \sin \pi/4)}{(\cos \pi/3 + i \sin \pi/3)}$$

$$\Rightarrow z = \sqrt{2} \left\{ \frac{\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right)}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}} \right\} = \sqrt{2} \left\{ \cos \left(-\frac{\pi}{4} - \frac{\pi}{3} \right) + i \sin \left(-\frac{\pi}{4} - \frac{\pi}{3} \right) \right\}$$

$$\Rightarrow z = \sqrt{2} \left\{ \cos \left(-\frac{7\pi}{12} \right) + i \sin \left(-\frac{7\pi}{12} \right) \right\}$$

ALITER We have, $z = \frac{z_1}{z_2}$, where $z_1 = 1 - i$ and $z_2 = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$.

$$\text{Now, } z_1 = 1 - i = \sqrt{2} \left\{ \cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right\} \text{ and } z_2 = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$$

$$\Rightarrow |z_1| = \sqrt{2} \text{ and } \theta_1 = \arg(z_1) = -\frac{\pi}{4}, |z_2| = 1, \theta_2 = \arg(z_2) = \frac{\pi}{3}$$

$$\therefore \frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} \{ \cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2) \} = \sqrt{2} \left\{ \cos \left(-\frac{7\pi}{12} \right) + i \sin \left(-\frac{7\pi}{12} \right) \right\}$$

4. Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$. Then,

$$|z_1| = r_1, \arg(z_1) = \theta_1, |z_2| = r_2 \text{ and } \arg(z_2) = \theta_2$$

It is given that

$$|z_2| = |z_1| \text{ and } \arg(z_1) + \arg(z_2) = \pi \Rightarrow r_1 = r_2 \text{ and } \theta_1 + \theta_2 = \pi \Rightarrow r_1 = r_2 \text{ and } \theta_1 = \pi - \theta_2$$

$$\therefore z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$$

$$\Rightarrow z_1 = r_2 \{ \cos(\pi - \theta_2) + i \sin(\pi - \theta_2) \} = r_2 (-\cos \theta_2 + i \sin \theta_2) = -r_2 (\cos \theta_2 - i \sin \theta_2) = -\bar{z}_2$$

5. Let $\arg(z_1) = \theta_1$ and $\arg(z_3) = \theta_2$

It is given that $z_2 = \bar{z}_1$ and $z_4 = \bar{z}_3$. Therefore, $\arg(z_2) = -\theta_1$ and $\arg(z_4) = -\theta_2$

$$\text{Hence, } \arg \left(\frac{z_1}{z_4} \right) + \arg \left(\frac{z_2}{z_3} \right) = \arg(z_1) - \arg(z_4) + \arg(z_2) - \arg(z_3) = \theta_1 + \theta_2 - \theta_1 - \theta_2 = 0.$$

6. Let $z = \sin \frac{\pi}{5} + i \left(1 - \cos \frac{\pi}{5} \right) = 2 \sin \frac{\pi}{10} \cos \frac{\pi}{10} + 2i \sin^2 \frac{\pi}{10} = 2 \sin \frac{\pi}{10} \left(\cos \frac{\pi}{10} + i \sin \frac{\pi}{10} \right)$

Clearly, z is the polar form with $|z| = 2 \sin \frac{\pi}{10}$ and $\arg(z) = \frac{\pi}{10}$.

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

- The principal value of the argument of the complex number $1 - i$ is
- The polar form of $(i^{25})^3$ is
- The value of $\sqrt{-25} \times \sqrt{-9}$ is
- The complex number $\frac{(1-i)^3}{1-i^3}$ in polar form is
- The sum of the series $i + i^2 + i^3 + \dots$ upto 1000 terms is
- The multiplicative inverse of $(1 + i)$ is
- If $|z| = 4$ and $\arg(z) = \frac{5\pi}{6}$, then $z = \dots$

8. If z_1 and z_2 are two complex numbers such that $z_1 + z_2$ is a real number, then $z_2 = \dots\dots\dots$.
9. For any non-zero complex number z , $\arg(z) + \arg(\bar{z}) = \dots\dots\dots$.
10. If $|z + 4| \leq 3$, then the greatest and least values of $|z + 1|$ are $\dots\dots\dots$ and $\dots\dots\dots$.
11. The modulus and argument of $\sin \frac{\pi}{5} + i \left(1 - \cos \frac{\pi}{5}\right)$ are $\dots\dots\dots$ and $\dots\dots\dots$ respectively.
12. If $\left| \frac{z-2}{z+2} \right| = \frac{\pi}{6}$, then the locus of z is $\dots\dots\dots$.
13. If $|z + 2i| = |z - 2i|$, then the locus of z is $\dots\dots\dots$.
14. If $|z + 2| = |z - 2|$, then the locus of z is $\dots\dots\dots$.
15. If $z = -1 + \sqrt{-3}$, then $\arg(z) = \dots\dots\dots$.
16. If $x < 0$ is a real number, then $\arg(x) = \dots\dots\dots$.
17. The real value of ' a ' for which $3i^3 - 2ai^2 + (1-a)i + 5$ is real is $\dots\dots\dots$.
18. If $|z| = 2$ and $\arg(z) = \frac{\pi}{4}$, then $z = \dots\dots\dots$.
19. The value of $(-\sqrt{-1})^{4n-3}$, where $n \in N$, is $\dots\dots\dots$.
20. The locus of z satisfying $\arg(z) = \frac{\pi}{3}$ is $\dots\dots\dots$.
21. The conjugate of the complex number $\frac{1-i}{1+i}$ is $\dots\dots\dots$.
22. If $(2+i)(2+2i)(2+3i)\dots(2+ni) = x+iy$, then $5.8.13\dots(4+n^2) = \dots\dots\dots$.
23. If the point representing a complex number lies in the third quadrant, then the point representing its conjugate lies in the $\dots\dots\dots$.
24. The multiplication of a non-zero complex number by i rotates it through $\dots\dots\dots$ in the anti-clockwise direction.
25. The complex number $\cos \theta + i \sin \theta \dots\dots\dots$ be zero for any θ .
26. The argument of the complex number $(-1 + i\sqrt{3})(1+i)(\cos \theta + i \sin \theta)$ is $\dots\dots\dots$.
27. If a complex number coincides with its conjugate, then it lies on $\dots\dots\dots$.
28. The points representing the complex number z for which $|z + 1| < |z - 1|$ lie on the left side of $\dots\dots\dots$.
29. If three complex numbers z_1, z_2 and z_3 are in A.P., then points representing them lie on $\dots\dots\dots$.
30. The principal argument of i^{-1097} is $\dots\dots\dots$.
31. The value of $\frac{i^{4n+1} - i^{4n-1}}{2}$ is $\dots\dots\dots$.
32. If $z_1 = \sqrt{3} + i\sqrt{3}$ and $z_2 = \sqrt{3} + i$, then the point representing $\frac{z_1}{z_2}$ lies in $\dots\dots\dots$.
33. If $0 < \arg(z) < \pi$, then $\arg(z) - \arg(-z) = \dots\dots\dots$.
34. For any two complex numbers z_1, z_2 and any real numbers a, b , $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = \dots\dots\dots$.
35. Let z_1 and z_2 be two complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1) - \arg(z_2) = \dots\dots\dots$.

36. Let z_1 and z_2 be two complex numbers such that $|z_1 + z_2| = |z_1 - z_2|$, then $\arg(z_1) - \arg(z_2) = \dots\dots\dots$.
37. If $|z_1| = |z_2|$ and $\arg\left(\frac{z_1}{z_2}\right) = \pi$, then $z_1 + z_2 = \dots\dots\dots$.

ANSWERS

- | | | | |
|--|--|---|-------------------------------|
| 1. $-\frac{\pi}{4}$ | 2. $\cos \frac{\pi}{2} - i \sin \frac{\pi}{2}$ | 3. -15 | 4. $2(\cos \pi + i \sin \pi)$ |
| 5. 0 | 6. $\frac{1}{2} - \frac{i}{2}$ | 7. $-2\sqrt{3} + 2i$ | 8. $\bar{z}_1$ |
| 9. $2n\pi, n \in \mathbb{Z}$ | 10. 6 and 0 | 11. $2 \sin \frac{\pi}{10}, \frac{\pi}{10}$ | 12. circle |
| 13. perpendicular bisector of the segment joining (0, -2) and (0, 2) | 14. perpendicular bisector of the segment joining (-2, 0) and (2, 0) | 15. $\frac{2\pi}{3}$ | |
| 16. π | 17. -2 | 18. $\sqrt{2}(1+i)$ | 19. -i |
| 20. $y = \sqrt{3}x$ in the first quadrant except the origin | 21. i | 22. $x^2 + y^2$ | 23. Second quadrant |
| 24. a right angle | 25. cannot be | 26. $\frac{11\pi}{12} + \theta$ | 27. x-axis |
| 28. y-axis | 29. a straight line | 30. $-\frac{\pi}{2}$ | 31. i |
| 32. first quadrant | 33. π | 34. $(a^2 + b^2)(z_1 ^2 + z_2 ^2)$ | 35. 0 |
| 36. $\frac{\pi}{2}$ | 37. 0 | | |

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

- Write the values of the square root of i .
- Write the values of the square root of $-i$.
- If $x + iy = \sqrt{\frac{a+ib}{c+id}}$, then write the value of $(x^2 + y^2)^2$.
- If $\pi < \theta < 2\pi$ and $z = 1 + \cos \theta + i \sin \theta$, then write the value of $|z|$.
- If n is any positive integer, write the value of $\frac{i^{4n+1} - i^{4n-1}}{2}$.
- Write the value of $\frac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}}$.
- Write $1 - i$ in polar form.
- Write $-1 + i\sqrt{3}$ in polar form.
- Write the argument of $-i$.
- Write the least positive integral value of n for which $\left(\frac{1+i}{1-i}\right)^n$ is real.

[NCERT]

11. Find the principal argument of $(1 + i\sqrt{3})^2$.
12. Find z , if $|z| = 4$ and $\arg(z) = \frac{5\pi}{6}$.
13. If $|z - 5i| = |z + 5i|$, then find the locus of z .
14. If $\frac{(a^2 + 1)^2}{2a - i} = x + iy$, find the value of $x^2 + y^2$.
15. Write the value of $\sqrt{-25} \times \sqrt{-9}$.
16. Write the sum of the series $i + i^2 + i^3 + \dots$ upto 1000 terms.
17. Write the value of $\arg(z) + \arg(\bar{z})$.
18. If $|z + 4| \leq 3$, then find the greatest and least values of $|z + 1|$.
19. For any two complex numbers z_1 and z_2 and any two real numbers a, b , find the value of $|az_1 - bz_2|^2 + |az_2 + bz_1|^2$.
20. Write the conjugate of $\frac{2-i}{(1-2i)^2}$.
21. If $n \in N$, then find the value of $i^n + i^{n+1} + i^{n+2} + i^{n+3}$.
22. Find the real value of a for which $3i^3 - 2ai^2 + (1-a)i + 5$ is real.
23. If $|z| = 2$ and $\arg(z) = \frac{\pi}{4}$, find z .
24. Write the argument of $(1 + \sqrt{3})(1 + i)(\cos \theta + i \sin \theta)$.

ANSWERS

- | | | | |
|--|---|--|-------------------------------|
| 1. $\pm \frac{1}{\sqrt{2}}(1 + i)$ | 2. $\pm \frac{1}{\sqrt{2}}(1 - i)$ | 3. $\frac{a^2 + b^2}{c^2 + d^2}$ | 4. $-2 \cos \frac{\theta}{2}$ |
| 5. i | 6. -2 | 7. $\sqrt{2} \left(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right)$ | |
| 8. $\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$ | 9. $\frac{3\pi}{2}$ or $-\frac{\pi}{2}$ | 10. 2 | 11. $\frac{2\pi}{3}$ |
| 12. $-2\sqrt{3} + 2i$ | 13. Real axis | 14. $\frac{(a^2 + 1)^4}{4a^2 + 1}$ | 15. -15 |
| 16. 0 | 17. 0 | 18. 6 and 0 | |
| 19. $(a^2 + b^2)(z_1 ^2 + z_2 ^2)$ | | 20. $-\frac{2}{25} - \frac{11}{25}i$ | 21. 0 |
| 22. $a = 2$ | 23. $\sqrt{2}(1 + i)$ | 24. $\frac{7\pi}{12} + \theta$ | |

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The value of $(1 + i)(1 + i^2)(1 + i^3)(1 + i^4)$ is
 (a) 2 (b) 0 (c) 1 (d) i
2. If $\frac{3 + 2i \sin \theta}{1 - 2i \sin \theta}$ is a real number and $0 < \theta < 2\pi$, then $\theta =$
 (a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$

3. If $(1+i)(1+2i)(1+3i)\dots(1+ni) = a+ib$, then $2 \times 5 \times 10 \times \dots \times (1+n^2)$ is equal to
 (a) $\sqrt{a^2+b^2}$ (b) $\sqrt{a^2-b^2}$ (c) a^2+b^2 (d) a^2-b^2 (e) $a+b$
4. If $\sqrt{a+ib} = x+iy$, then possible value of $\sqrt{a-ib}$ is
 (a) x^2+y^2 (b) $\sqrt{x^2+y^2}$ (c) $x+iy$ (d) $x-iy$ (e) $\sqrt{x^2-y^2}$
5. If $z = \cos \frac{\pi}{4} + i \sin \frac{\pi}{6}$, then
 (a) $|z| = 1, \arg(z) = \frac{\pi}{4}$ (b) $|z| = 1, \arg(z) = \frac{\pi}{6}$
 (c) $|z| = \frac{\sqrt{3}}{2}, \arg(z) = \frac{5\pi}{24}$ (d) $|z| = \frac{\sqrt{3}}{2}, \arg(z) = \tan^{-1} \frac{1}{\sqrt{2}}$
6. The polar form of $(i^{25})^3$ is
 (a) $\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$ (b) $\cos \pi + i \sin \pi$ (c) $\cos \pi - i \sin \pi$ (d) $\cos \frac{\pi}{2} - i \sin \frac{\pi}{2}$
7. If $i^2 = -1$, then the sum $i + i^2 + i^3 + \dots$ upto 1000 terms is equal to
 (a) 1 (b) -1 (c) i (d) 0
8. If $z = \frac{-2}{1+i\sqrt{3}}$, then the value of $\arg(z)$ is
 (a) π (b) $\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{4}$
9. If $a = \cos \theta + i \sin \theta$, then $\frac{1+a}{1-a} =$
 (a) $\cot \frac{\theta}{2}$ (b) $\cot \theta$ (c) $i \cot \frac{\theta}{2}$ (d) $i \tan \frac{\theta}{2}$
10. If $(1+i)(1+2i)(1+3i)\dots(1+ni) = a+ib$, then $2 \cdot 5 \cdot 10 \cdot 17 \dots (1+n^2) =$
 (a) $a-ib$ (b) a^2-b^2 (c) a^2+b^2 (d) none of these
11. If $\frac{(a^2+1)^2}{2a-i} = x+iy$, then x^2+y^2 is equal to
 (a) $\frac{(a^2+1)^4}{4a^2+1}$ (b) $\frac{(a+1)^2}{4a^2+1}$ (c) $\frac{(a^2-1)^2}{(4a^2-1)^2}$ (d) none of these
12. The principal value of the amplitude of $(1+i)$ is
 (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{12}$ (c) $\frac{3\pi}{4}$ (d) π
13. The least positive integer n such that $\left(\frac{2i}{1+i}\right)^n$ is a positive integer, is
 (a) 16 (b) 8 (c) 4 (d) 2
14. If z is a non-zero complex number, then $\left|\frac{\bar{z}}{z}\right|^2$ is equal to
 (a) $\left|\frac{\bar{z}}{z}\right|$ (b) $|z|$ (c) $|\bar{z}|$ (d) none of these
15. If $a = 1+i$, then a^2 equals
 (a) $1-i$ (b) $2i$ (c) $(1+i)(1-i)$ (d) $i-1$

16. If $(x + iy)^{1/3} = a + ib$, then $\frac{x}{a} + \frac{y}{b} =$
 (a) 0 (b) 1 (c) -1 (d) none of these
17. $(\sqrt{-2})(\sqrt{-3})$ is equal to
 (a) $\sqrt{6}$ (b) $-\sqrt{6}$ (c) $i\sqrt{6}$ (d) none of these
18. The argument of $\frac{1-i\sqrt{3}}{1+i\sqrt{3}}$ is
 (a) 60° (b) 120° (c) 210° (d) 240°
19. If $z = \left(\frac{1+i}{1-i}\right)$, then z^4 equals
 (a) 1 (b) -1 (c) 0 (d) none of these
20. If $z = \frac{1+2i}{1-(1-i)^2}$, then $\arg(z)$ equals
 (a) 0 (b) $\frac{\pi}{2}$ (c) π (d) none of these
21. If $z = \frac{1}{(2+3i)^2}$, then $|z| =$
 (a) $\frac{1}{13}$ (b) $\frac{1}{5}$ (c) $\frac{1}{12}$ (d) none of these
22. If $z = \frac{1}{(1-i)(2+3i)}$, then $|z| =$
 (a) 1 (b) $1/\sqrt{26}$ (c) $5/\sqrt{26}$ (d) none of these
23. If $z = 1 - \cos \theta + i \sin \theta$, then $|z| =$
 (a) $2 \sin \frac{\theta}{2}$ (b) $2 \cos \frac{\theta}{2}$ (c) $2 \left| \sin \frac{\theta}{2} \right|$ (d) $2 \left| \cos \frac{\theta}{2} \right|$
24. If $x + iy = (1+i)(1+2i)(1+3i)$, then $x^2 + y^2 =$
 (a) 0 (b) 1 (c) 100 (d) none of these
25. If $z = \frac{1}{1 - \cos \theta - i \sin \theta}$, then $\operatorname{Re}(z) =$
 (a) 0 (b) $\frac{1}{2}$ (c) $\cot \frac{\theta}{2}$ (d) $\frac{1}{2} \cot \frac{\theta}{2}$
26. If $x + iy = \frac{3+5i}{7-6i}$, then $y =$
 (a) $9/85$ (b) $-9/85$ (c) $53/85$ (d) none of these
27. If $\frac{1-ix}{1+ix} = a + ib$, then $a^2 + b^2 =$
 (a) 1 (b) -1 (c) 0 (d) none of these
28. If θ is the amplitude of $\frac{a+ib}{a-ib}$, then $\tan \theta =$
 (a) $\frac{2a}{a^2+b^2}$ (b) $\frac{2ab}{a^2-b^2}$ (c) $\frac{a^2-b^2}{a^2+b^2}$ (d) none of these
29. If $z = \frac{1+7i}{(2-i)^2}$, then

$$(a) |z| = 2 \quad (b) |z| = \frac{1}{2} \quad (c) \operatorname{amp}(z) = \frac{\pi}{4} \quad (d) \operatorname{amp}(z) = \frac{3\pi}{4}$$

30. The amplitude of $\frac{1}{i}$ is equal to

$$(a) 0 \quad (b) \frac{\pi}{2} \quad (c) -\frac{\pi}{2} \quad (d) \pi$$

31. The argument of $\frac{1-i}{1+i}$ is

$$(a) -\frac{\pi}{2} \quad (b) \frac{\pi}{2} \quad (c) \frac{3\pi}{2} \quad (d) \frac{5\pi}{2}$$

32. The amplitude of $\frac{1+i\sqrt{3}}{\sqrt{3}+i}$ is

$$(a) \frac{\pi}{3} \quad (b) -\frac{\pi}{3} \quad (c) \frac{\pi}{6} \quad (d) -\frac{\pi}{6}$$

33. The value of $(i^5 + i^6 + i^7 + i^8 + i^9)/(1+i)$ is

$$(a) \frac{1}{2}(1+i) \quad (b) \frac{1}{2}(1-i) \quad (c) 1 \quad (d) \frac{1}{2}$$

34. $\frac{1+2i+3i^2}{1-2i+3i^2}$ equals

$$(a) i \quad (b) -1 \quad (c) -i \quad (d) 4$$

35. The value of $\frac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}} - 1$ is

$$(a) -1 \quad (b) -2 \quad (c) -3 \quad (d) -4$$

36. The value of $(1+i)^4 + (1-i)^4$ is

$$(a) 8 \quad (b) 4 \quad (c) -8 \quad (d) -4$$

37. If $z = a+ib$ lies in third quadrant, then $\frac{\bar{z}}{z}$ also lies in the third quadrant if

$$(a) a > b > 0 \quad (b) a < b < 0 \quad (c) b < a < 0 \quad (d) b > a > 0$$

[NCERT EXEMPLAR]

38. If $f(z) = \frac{7-z}{1-z^2}$, where $z = 1+2i$, then $|f(z)|$ is

$$(a) \frac{|z|}{2} \quad (b) |z| \quad (c) 2|z| \quad (d) \text{none of these}$$

[NCERT EXEMPLAR]

39. A real value of x satisfies the equation $\frac{3-4ix}{3+4ix} = a-ib$ ($a, b \in \mathbb{R}$), if $a^2 + b^2 =$

$$(a) 1 \quad (b) -1 \quad (c) 2 \quad (d) -2$$

[NCERT EXEMPLAR]

40. The complex number z which satisfies the condition $\left| \frac{i+z}{i-z} \right| = 1$ lies on

$$(a) \text{circle } x^2 + y^2 = 1 \quad (b) \text{the } x\text{-axis} \quad (c) \text{the } y\text{-axis} \quad (d) \text{the line } x+y=1$$

[NCERT EXEMPLAR]

41. If z is a complex number, then

- (a) $|z|^2 > |z|^2$ (b) $|z|^2 = |z|^2$ (c) $|z|^2 < |z|^2$ (d) $|z|^2 \geq |z|^2$

[NCERT EXEMPLAR]

42. Which of the following is correct for any two complex numbers z_1 and z_2 ?

- (a) $|z_1 z_2| = |z_1| |z_2|$ (b) $\arg(z_1 z_2) = \arg(z_1) \arg(z_2)$
(c) $|z_1 + z_2| = |z_1| + |z_2|$ (d) $|z_1 + z_2| \geq |z_1| + |z_2|$

[NCERT EXEMPLAR]

43. If the complex number $z = x + iy$ satisfies the condition $|z + 1| = 1$, then z lies on

- (a) x -axis (b) circle with centre $(-1, 0)$ and radius 1
(c) y -axis (d) none of these

[NCERT EXEMPLAR]

44. $\sin x + i \cos 2x$ and $\cos x - i \sin 2x$ are conjugate to each other for

- (a) $x = n\pi$ (b) $x = \left(n + \frac{1}{2}\right) \frac{\pi}{2}$ (c) $x = 0$ (d) No value of x

[NCERT EXEMPLAR]

45. The real value of α for which the expression $\frac{1 - i \sin \alpha}{1 + 2i \sin \alpha}$ is purely real, is

- (a) $(n+1) \frac{\pi}{2}$ (b) $(2n+1) \frac{\pi}{2}$ (c) $n\pi$ (d) none of these

where $n \in \mathbb{N}$.

[NCERT EXEMPLAR]

46. The value of $(z + 3)(\bar{z} + 3)$ is equivalent to

- (a) $|z + 3|^2$ (b) $|z - 3|$ (c) $z^2 + 3$ (d) none of these

[NCERT EXEMPLAR]

47. If $\left(\frac{1+i}{1-i}\right)^n = 1$, then $n =$

- (a) $2m+1$ (b) $4m$ (c) $2m$ (d) $4m+1$

where $m \in \mathbb{N}$

[NCERT EXEMPLAR]

48. The vector represented by the complex number $2 - i$ is rotated about the origin through an angle $\frac{\pi}{2}$ in the clockwise direction, the new position of point is

- (a) $1 + 2i$ (b) $-1 - 2i$ (c) $2 + i$ (d) $-1 + 2i$

[NCERT EXEMPLAR]

49. The real value of θ for which the expression $\frac{1 + i \cos \theta}{1 - 2i \cos \theta}$ is a real number, is

- (a) $n\pi + \frac{\pi}{4}$ (b) $n\pi + (-1)^n \frac{\pi}{4}$ (c) $2n\pi \pm \frac{\pi}{2}$ (d) none of these

[NCERT EXEMPLAR]

50. $|z_1 + z_2| = |z_1| + |z_2|$ is possible if

- (a) $z_2 = \bar{z}_1$ (b) $z_2 = \frac{1}{z_1}$
(c) $\arg(z_1) = \arg(z_2)$ (d) $|z_1| = |z_2|$

[NCERT EXEMPLAR]

51. The equation $|z + 1 - i| = |z - 1 + i|$ represents a

- (a) straight line (b) circle (c) parabola (d) hyperbola

[NCERT EXEMPLAR]

52. The area of the triangle on the complex plane formed by the complex numbers z , $-iz$ and $z + iz$ is

- (a) $|z|^2$ (b) $|\bar{z}|^2$ (c) $\frac{1}{2}|z|^2$ (d) none of these

[NCERT EXEMPLAR]

ANSWERS

1. (b) 2. (a) 3. (c) 4. (d) 5. (d) 6. (d) 7. (d) 8. (c)
 9. (c) 10. (c) 11. (a) 12. (a) 13. (b) 14. (a) 15. (b) 16. (d)
 17. (b) 18. (d) 19. (a) 20. (a) 21. (a) 22. (b) 23. (c) 24. (c)
 25. (b) 26. (c) 27. (a) 28. (b) 29. (d) 30. (c) 31. (a) 32. (c)
 33. (a) 34. (c) 35. (b) 36. (c) 37. (c) 38. (a) 39. (a) 40. (b)
 41. (b) 42. (a) 43. (b) 44. (d) 45. (c) 46. (a) 47. (b) 48. (b)
 49. (c) 50. (c) 51. (a) 52. (c)

SUMMARY

1. $\sqrt{-1}$ is an imaginary quantity and is denoted by i which has the following properties:

$$i^2 = -1, i^3 = -i, i^4 = 1 \text{ and } i^{\pm n} = i^{\pm k}, n \in N$$

where k is the remainder when n is denoted by 4.

2. For any positive real number a , $\sqrt{-a} = i\sqrt{a}$.

3. For any two real numbers a and b , we have

$$\sqrt{a}\sqrt{b} = \begin{cases} \sqrt{ab}, & \text{if at least one of } a \text{ and } b \text{ is positive} \\ -\sqrt{ab}, & \text{if } a < 0, b < 0. \end{cases}$$

4. If a, b are real numbers, then a number $z = a + ib$ is called a complex number, real number a is known as the real part of z and b is known as its imaginary part. We write $a = \operatorname{Re}(z)$, $b = \operatorname{Im}(z)$.

A complex number z is purely real iff $\operatorname{Im}(z) = 0$ and z is purely imaginary iff $\operatorname{Re}(z) = 0$

5. For any two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, we define

$$\text{Addition: } z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

$$\text{Subtraction: } z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$$

$$\text{Multiplication: } z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

$$\text{Reciprocal: } \frac{1}{z_1} = \frac{a_1}{a_1^2 + b_1^2} - i \frac{b_1}{a_1^2 + b_1^2}$$

$$\text{Division: } \frac{z_1}{z_2} = z_1 \left(\frac{1}{z_2} \right) = (a_1 + ib_1) \left(\frac{a_2}{a_2^2 + b_2^2} - i \frac{b_2}{a_2^2 + b_2^2} \right) = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}$$

Addition is commutative and associative. Complex number $0 = 0 + i0$ is the identity element for addition and every complex number $z = a + ib$ has its additive inverse $-z = -a - ib$.

Multiplication is also commutative and associative. Complex number $1 = 1 + 0i$ is the identity element for multiplication. Every non-zero complex number $z = a + ib$ has its multiplicative inverse $1/z$ (also known as reciprocal of z) such that $\frac{1}{z} = \frac{a - ib}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$.

6. The conjugate of a complex number $z = a + ib$ is denoted by $\bar{z}$ and is equal to $a - ib$.

For any three complex numbers z, z_1, z_2 , we have

- (i) $(\bar{\bar{z}}) = z$ (ii) $z + \bar{z} = 2 \operatorname{Re}(z)$
 (iii) $z - \bar{z} = 2i \operatorname{Im}(z)$ (iv) $z = \bar{z} \Leftrightarrow z$ is purely real
 (v) $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary (vi) $z\bar{z} = \{\operatorname{Re}(z)\}^2 + \{\operatorname{Im}(z)\}^2 = |z|^2$
 (vii) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$ (viii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
 (ix) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

7. The modulus of a complex number $z = a + ib$ is denoted by $|z|$ and is defined as

$$|z| = \sqrt{a^2 + b^2} = \sqrt{\{\operatorname{Re}(z)\}^2 + \{\operatorname{Im}(z)\}^2}$$

If z, z_1, z_2 are three complex numbers, then

- (i) $|z| = 0 \Leftrightarrow z = 0$ i.e. $\operatorname{Re}(z) = \operatorname{Im}(z) = 0$ (ii) $|z| = |\bar{z}| = |-z|$
 (iii) $-|z| \leq \operatorname{Re}(z) \leq |z|$; $-|z| \leq \operatorname{Im}(z) \leq |z|$ (iv) $z\bar{z} = |z|^2$
 (v) $|\operatorname{Im}(z^n)| \leq n|\operatorname{Im}(z)| |z|^{n-1}, n \in \mathbb{N}$ (vi) $|\operatorname{Re}(z)| + |\operatorname{Im}(z)| \leq \sqrt{2}|z|$

8. A complex number $z = x + iy$ can be represented by a point $P(x, y)$ (see Fig. 13.8) on the plane which is known as the Argand or Gaussian or Complex plane. The length of the line segment OP is called the modulus of z and is denoted by $|z|$.

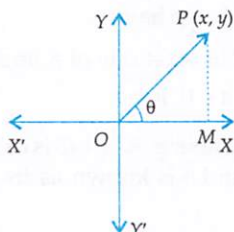


Fig.12.8

$$\text{Clearly, } |z| = \sqrt{x^2 + y^2} = \sqrt{\{\operatorname{Re}(z)\}^2 + \{\operatorname{Im}(z)\}^2}$$

The angle θ which OP makes with the positive direction of x -axis in anti-clockwise sense is called the argument or amplitude of z and is denoted by $\arg(z)$ or $\operatorname{amp}(z)$.

$$\text{Clearly, } \tan \theta = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}.$$

Let $OP = r$ and $\angle XOP = \theta$. Then, $x = r \cos \theta$ and $y = r \sin \theta$

$$\therefore z = x + iy = r(\cos \theta + i \sin \theta)$$

This is known as the polar form of complex number z . The Euler's notations are

$$e^{\pm i\theta} = \cos \theta \pm i \sin \theta$$

$$\therefore z = r(\cos \theta + i \sin \theta)$$

or, $z = re^{i\theta}$, which is known as the Eulerian form of z .