

# CHAPTER 22

## THE STRAIGHT LINES

### 22.1 DEFINITION OF A STRAIGHT LINE

A straight line is a curve such that every point on the line segment joining any two points on it lies on it.

**THEOREM** Every first degree equation in  $x, y$  represents a straight line.

[NCERT EXEMPLAR]

**PROOF** Let  $ax + by + c = 0$  be a first degree equation in  $x, y$  where  $a, b, c$  are constants. Let  $P(x_1, y_1)$  and  $Q(x_2, y_2)$  be any two points on the curve represented by  $ax + by + c = 0$ . Then,

$$ax_1 + by_1 + c = 0 \text{ and } ax_2 + by_2 + c = 0 \quad \dots(i)$$

Let  $R$  be any point on the line segment joining  $P$  and  $Q$ . Suppose  $R$  divides  $PQ$  in the ratio  $\lambda : 1$ .

Then, the coordinates of  $R$  are  $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$ . In order to prove that the curve represented by  $ax + by + c = 0$  is a straight line, it is sufficient to show that  $R$  lies on it for all values of  $\lambda$ .

Now,

$$\begin{aligned} a\left(\frac{\lambda x_2 + x_1}{\lambda + 1}\right) + b\left(\frac{\lambda y_2 + y_1}{\lambda + 1}\right) + c &= \frac{\lambda(ax_2 + by_2 + c) + (ax_1 + by_1 + c)}{\lambda + 1} \\ &= \lambda \cdot 0 + 0 = 0 \end{aligned} \quad \text{[Using (i)]}$$

$\therefore R\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$  lies on the curve represented by  $ax + by + c = 0$ .

Thus, every point on the line segment joining  $P$  and  $Q$  lies on  $ax + by + c = 0$ .

Hence,  $ax + by + c = 0$  represents a straight line.

**Q.E.D.**

**NOTE** When we say that a first degree equation in  $x, y$  i.e.,  $ax + by + c = 0$  represents a line, it means that all points  $(x, y)$  satisfying  $ax + by + c = 0$  lie along a line. Thus, a line is also defined as the locus of a point satisfying the condition  $ax + by + c = 0$  where  $a, b, c$  are constants.

It follows from the above discussion that  $ax + by + c = 0$  is the general equation of a line.

It should be noted that there are only two unknowns in the equation of a straight line because equation of every straight line can be put in the form  $ax + by + 1 = 0$  where  $a, b$  are two unknowns. Note that  $x, y$  are not unknowns. In fact these are the coordinates of any point on the line and are known as the current coordinates. Thus, to determine a line we will need two conditions to determine the two unknowns. In the further discussion on straight line you will find that whenever it will be asked to find a straight line there will always be two conditions connecting the two unknowns.

### 22.2 SLOPE (GRADIENT) OF A LINE

The trigonometrical tangent of the angle that a line makes with the positive direction of the  $x$ -axis in anticlockwise sense is called the slope or gradient of the line.

The slope of a line is generally denoted by  $m$ . Thus,  $m = \tan \theta$ .

Since a line parallel to  $x$ -axis makes an angle of  $0^\circ$  with  $x$ -axis, therefore its slope is  $\tan 0^\circ = 0$ . A line parallel to  $y$ -axis i.e., perpendicular to  $x$ -axis makes an angle of  $90^\circ$  with  $x$ -axis, so its slope is



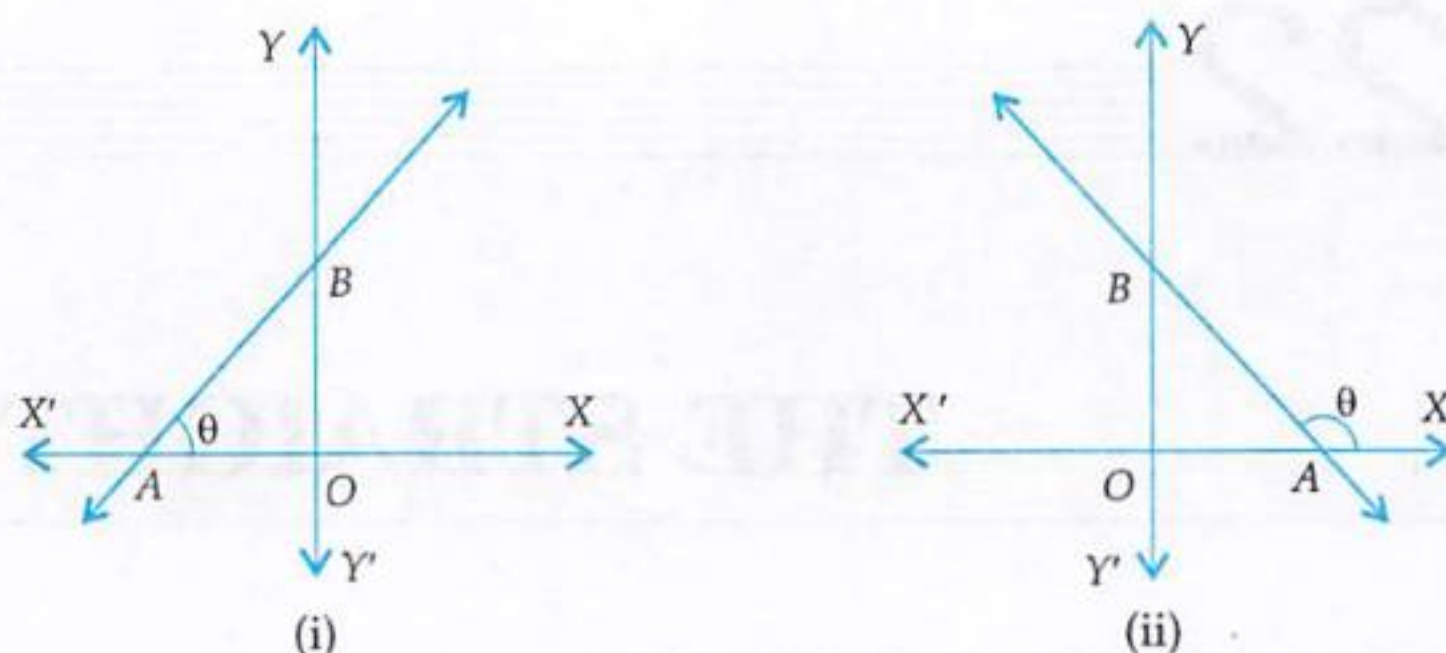


Fig. 22.1

$\tan \pi/2 = \infty$ . Also, the slope of a line equally inclined with axes is 1 or  $-1$  as it makes  $45^\circ$  or  $135^\circ$  angle with  $x$ -axis.

**REMARK** The angle of inclination of a line with the positive direction of  $x$ -axis in anticlockwise sense always lies between  $0^\circ$  and  $180^\circ$ .

**ILLUSTRATION 1** Find the slope of a line whose inclination to the positive direction of  $x$ -axis in anticlockwise sense is (i)  $60^\circ$  (ii)  $0^\circ$  (iii)  $150^\circ$  (iv)  $120^\circ$ . [NCERT]

**SOLUTION** (i) Slope  $= \tan 60^\circ = \sqrt{3}$ . (ii) Slope  $= \tan 0^\circ = 0$ .

(iii) Slope  $= \tan 150^\circ = -\cot 60^\circ = -\frac{1}{\sqrt{3}}$  (iv) Slope  $= \tan 120^\circ = -\cot 30^\circ = -\sqrt{3}$ .

**ILLUSTRATION 2** What can be said regarding a line if its slope is (i) positive (ii) zero (iii) negative?

**SOLUTION** Let  $\theta$  be the angle of inclination of the given line with the positive direction of  $x$ -axis in anticlockwise sense. Then, its slope is given by  $m = \tan \theta$ .

(i) If the slope of the line is positive, then

$$m = \tan \theta > 0 \Rightarrow \theta \text{ lies between } 0^\circ \text{ and } 90^\circ \Rightarrow \theta \text{ is an acute angle.}$$

Thus, a line of positive slope makes an acute angle with the positive direction of  $x$ -axis.

(ii) If the slope of the line is zero, then

$$m = \tan \theta = 0 \Rightarrow \theta = 0^\circ \Rightarrow \text{either the line is } x\text{-axis or it is parallel to } x\text{-axis.}$$

Thus, a line of zero slope is parallel or coincident to  $x$ -axis.

(iii) If the slope of the line is negative, then

$$m = \tan \theta < 0 \Rightarrow \theta \text{ lies between } 90^\circ \text{ and } 180^\circ \Rightarrow \theta \text{ is an obtuse angle.}$$

Thus, a line of negative slope makes an obtuse angle with the positive direction of  $x$ -axis in anticlockwise direction.

### 22.2.1 SLOPE OF A LINE IN TERMS OF COORDINATES OF ANY TWO POINTS ON IT

Let  $P(x_1, y_1)$  and  $Q(x_2, y_2)$  be two points on a line making an angle  $\theta$  with the positive direction of  $x$ -axis. Draw  $PL$ ,  $QM$  perpendiculars on  $x$ -axis and  $PN \perp$  on  $QM$ . Then,

$$PN = LM = OM - OL = x_2 - x_1$$

$$\text{and, } QN = QM - NM = QM - PL = y_2 - y_1$$

In  $\Delta PQN$ , we obtain

$$\tan \theta = \frac{QN}{PN} = \frac{y_2 - y_1}{x_2 - x_1}$$

Thus, if  $(x_1, y_1)$  and  $(x_2, y_2)$  are coordinates of any two points on a line, then its slope is

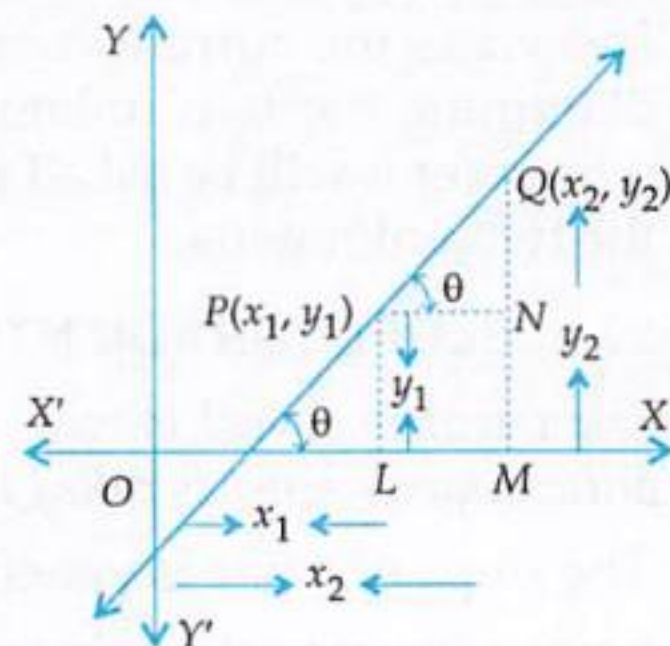


Fig. 22.2



$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Difference of ordinates}}{\text{Difference of abscissae}}$$

**ILLUSTRATION 3** Find the slope of a line which passes through points (3, 2) and (-1, 5).

[NCERT]

**SOLUTION** We know that the slope of a line passing through two points  $(x_1, y_1)$  and  $(x_2, y_2)$  is given by  $m = \frac{y_2 - y_1}{x_2 - x_1}$ . Here, the line passes through (3, 2) and (-1, 5).

$$\text{So, its slope is given by } m = \frac{5 - 2}{-1 - 3} = -\frac{3}{4}.$$

## 22.3 ANGLE BETWEEN TWO LINES

**THEOREM** The angle  $\theta$  between the lines having slopes  $m_1$  and  $m_2$  is given by  $\tan \theta = \pm \frac{m_2 - m_1}{1 + m_1 m_2}$ .

**PROOF** Let  $m_1$  and  $m_2$  be the slopes of two given lines  $AB$  and  $CD$  which intersect at a point  $P$  and make angles  $\theta_1$  and  $\theta_2$  respectively with the positive direction of  $x$ -axis. Then,  $m_1 = \tan \theta_1$  and  $m_2 = \tan \theta_2$ .

Let  $\angle APC = \theta$  be the angle between the given lines. Then,

$$\begin{aligned} \theta_2 &= \theta + \theta_1 && [\text{See Fig. 22.3}] \\ \Rightarrow \theta &= \theta_2 - \theta_1 \\ \Rightarrow \tan \theta &= \tan (\theta_2 - \theta_1) \\ \Rightarrow \tan \theta &= \frac{\tan \theta_2 - \tan \theta_1}{1 + \tan \theta_2 \tan \theta_1} && \left[ \text{Using : } \tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \right] \\ \Rightarrow \tan \theta &= \frac{m_2 - m_1}{1 + m_1 m_2} && \dots(i) \end{aligned}$$

Since  $\angle APD = \pi - \theta$  is also the angle between  $AB$  and  $CD$ . Therefore,

$$\tan \angle APD = \tan (\pi - \theta) = -\tan \theta = -\frac{m_2 - m_1}{1 + m_1 m_2} \quad [\text{Using (i)}] \dots(ii)$$

From (i) and (ii), we find that the angles between two lines of slopes  $m_1$  and  $m_2$  are given by

$$\tan \theta = \pm \frac{m_2 - m_1}{1 + m_1 m_2} \Rightarrow \theta = \tan^{-1} \left( \pm \frac{m_2 - m_1}{1 + m_1 m_2} \right)$$

The acute angle between the lines is given by  $\tan \theta = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right|$ .

Q.E.D.

**ILLUSTRATION** If  $A(-2, 1)$ ,  $B(2, 3)$  and  $C(-2, -4)$  are three points, find the angle between  $BA$  and  $BC$ .

**SOLUTION** Let  $m_1$  and  $m_2$  be the slopes of  $BA$  and  $BC$  respectively. Then,

$$m_1 = \frac{3 - 1}{2 - (-2)} = \frac{2}{4} = \frac{1}{2}, \quad \text{and} \quad m_2 = \frac{-4 - 3}{-2 - 2} = \frac{7}{4}$$

Let  $\theta$  be the acute angle between  $BA$  and  $BC$ . Then,

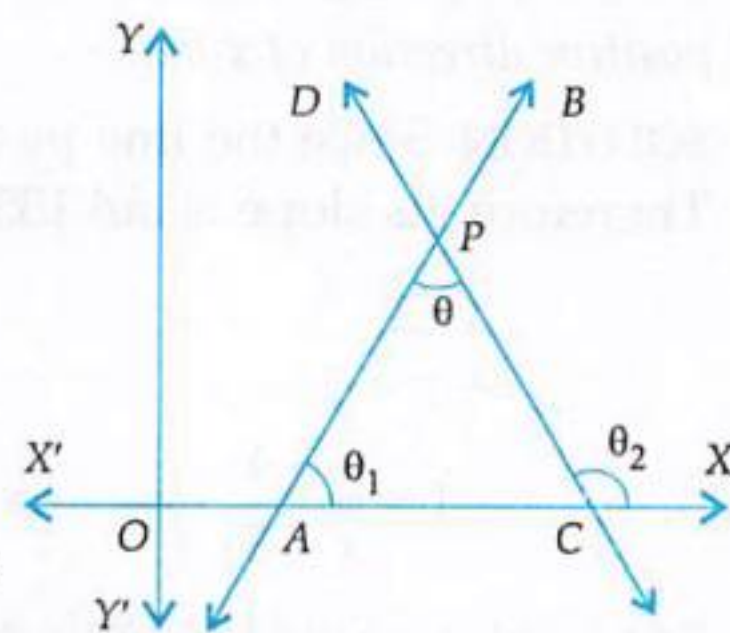


Fig. 22.3



$$\tan \theta = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right| = \left| \frac{\frac{7}{4} - \frac{1}{2}}{1 + \frac{7}{4} \times \frac{1}{2}} \right| = \left| \frac{\frac{10}{8}}{\frac{15}{8}} \right| = \frac{2}{3} \Rightarrow \theta = \tan^{-1} \left( \frac{2}{3} \right).$$

**CONDITION OF PARALLELISM OF LINES** If two lines of slopes  $m_1$  and  $m_2$  are parallel, then the angle  $\theta$  between them is of  $0^\circ$ .

$$\therefore \tan \theta = \tan 0^\circ = 0$$

$$\Rightarrow \frac{m_2 - m_1}{1 + m_1 m_2} = 0 \Rightarrow m_2 = m_1 \quad \left[ \text{Using : } \tan \theta = \pm \frac{m_2 - m_1}{1 + m_1 m_2} \right]$$

Thus, when two lines are parallel, their slopes are equal.

**CONDITION OF PERPENDICULARITY OF TWO LINES** If two lines of slopes  $m_1$  and  $m_2$  are perpendicular, then the angle  $\theta$  between them is of  $90^\circ$ .

From Fig. 22.3, we have

$$\theta_2 = \theta + \theta_1$$

$$\Rightarrow \theta_2 = 90^\circ + \theta_1 \quad [\because \theta = 90^\circ]$$

$$\Rightarrow \tan \theta_2 = \tan (90^\circ + \theta_1) \Rightarrow \tan \theta_2 = -\cot \theta_1 \Rightarrow \tan \theta_1 \tan \theta_2 = -1 \Rightarrow m_1 m_2 = -1$$

Thus, when two lines are perpendicular, the product of their slopes is  $-1$ . If  $m$  is the slope of a line, then the slope of a line perpendicular to it is  $-(1/m)$ .

### ILLUSTRATIVE EXAMPLES

#### BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** Determine  $x$  so that the line passing through  $(3, 4)$  and  $(x, 5)$  makes  $135^\circ$  angle with the positive direction of  $x$ -axis.

**SOLUTION** Since the line passing through  $(3, 4)$  and  $(x, 5)$  makes an angle of  $135^\circ$  with  $x$ -axis. Therefore, its slope is  $\tan 135^\circ = -1$ . But, the slope of the line is also equal to

$$\frac{5-4}{x-3} \quad \left[ \text{Using : } m = \frac{y_2 - y_1}{x_2 - x_1} \right]$$

$$\therefore -1 = \frac{5-4}{x-3} \Rightarrow -x+3=1 \Rightarrow x=2.$$

**EXAMPLE 2** Find the angle between the lines joining the points  $(0, 0)$ ,  $(2, 3)$  and the points  $(2, -2)$ ,  $(3, 5)$ .

**SOLUTION** Let  $\theta$  be the angle between the given lines. We find that

$$m_1 = \text{Slope of the line joining } (0, 0) \text{ and } (2, 3) = \frac{3-0}{2-0} = \frac{3}{2}$$

$$m_2 = \text{Slope of the line joining } (2, -2) \text{ and } (3, 5) = \frac{5+2}{3-2} = 7$$

$$\therefore \tan \theta = \pm \frac{m_2 - m_1}{1 + m_1 m_2} = \pm \frac{7 - 3/2}{1 + 7(3/2)} = \pm \frac{11/2}{23/2} = \pm \frac{11}{23} \Rightarrow \theta = \tan^{-1} \left( \pm \frac{11}{23} \right).$$

**EXAMPLE 3** Let  $A(6, 4)$  and  $B(2, 12)$  be two given points. Find the slope of a line perpendicular to  $AB$ .

**SOLUTION** Let  $m$  be the slope of  $AB$ . Then,

$$m = \frac{12-4}{2-6} = \frac{8}{-4} = -2. \text{ So, the slope of a line perpendicular to } AB \text{ is } -\frac{1}{m} = \frac{1}{2}.$$



**EXAMPLE 4** Determine  $x$  so that 2 is the slope of the line through  $(2, 5)$  and  $(x, 3)$ .

**SOLUTION** The slope of the line through  $(2, 5)$  and  $(x, 3)$  is  $\frac{3-5}{x-2}$ . But, the slope of the line is given as 2.

$$\therefore \frac{3-5}{x-2} = 2 \Rightarrow 2x - 4 = -2 \Rightarrow x = 1$$

**EXAMPLE 5** What is the value of  $y$  so that the line through  $(3, y)$  and  $(2, 7)$  is parallel to the line through  $(-1, 4)$  and  $(0, 6)$ ?

**SOLUTION** Let  $A(3, y)$ ,  $B(2, 7)$ ,  $C(-1, 4)$  and  $D(0, 6)$  be the given points. Then,

$$m_1 = \text{Slope of the line } AB = \frac{7-y}{2-3} = y-7 \text{ and, } m_2 = \text{Slope of the line } CD = \frac{6-4}{0-(-1)} = 2$$

It is given that  $AB$  and  $CD$  are parallel.

$$\therefore m_1 = m_2 \Rightarrow y-7 = 2 \Rightarrow y = 9.$$

**EXAMPLE 6** Without using Pythagoras theorem, show that  $A(4, 4)$ ,  $B(3, 5)$  and  $C(-1, -1)$  are the vertices of a right-angled triangle. [NCERT]

**SOLUTION** In  $\triangle ABC$ , we find that

$$m_1 = \text{Slope of } AB = \frac{4-5}{4-3} = -1 \text{ and, } m_2 = \text{Slope of } AC = \frac{4-(-1)}{4-(-1)} = 1$$

Clearly,  $m_1 m_2 = -1$ . This shows that  $AB$  is perpendicular to  $AC$  i.e.  $\angle CAB = \pi/2$ .

Hence, the given points are the vertices of a right-angled triangle.

**EXAMPLE 7** A quadrilateral has the vertices at the points  $(-4, 2)$ ,  $(2, 6)$ ,  $(8, 5)$  and  $(9, -7)$ . Show that the mid-points of the sides of this quadrilateral are the vertices of a parallelogram.

**SOLUTION** Let  $A(-4, 2)$ ,  $B(2, 6)$ ,  $C(8, 5)$  and  $D(9, -7)$  be the vertices of the given quadrilateral. Let  $P$ ,  $Q$ ,  $R$  and  $S$  be the mid-points of  $AB$ ,  $BC$ ,  $CD$  and  $DA$  respectively. Then, the coordinates of  $P$ ,  $Q$ ,  $R$  and  $S$  are  $P(-1, 4)$ ,  $Q(5, 11/2)$ ,  $R(17/2, -1)$  and  $S(5/2, -5/2)$  respectively. In order to prove that  $PQRS$  is a parallelogram, it is sufficient to show that  $PQ$  is parallel to  $RS$  and  $PQ = RS$ . Let  $m_1$  and  $m_2$  be the slope of  $PQ$  and  $RS$  respectively. Then,

$$m_1 = \frac{11/2 - 4}{5 - (-1)} = \frac{1}{4} \text{ and, } m_2 = \frac{-5/2 + 1}{5/2 - 17/2} = \frac{1}{4}$$

Clearly,  $m_1 = m_2$ . Therefore,  $PQ$  is parallel to  $RS$ .

Now,

$$PQ = \sqrt{(5+1)^2 + \left(\frac{11}{2} - 4\right)^2} = \frac{\sqrt{153}}{2} \text{ and, } RS = \sqrt{\left(\frac{5}{2} - \frac{17}{2}\right)^2 + \left(-\frac{5}{2} + 1\right)^2} = \frac{\sqrt{153}}{2}$$

$$\therefore PQ = RS$$

Thus,  $PQ \parallel RS$  and  $PQ = RS$ . Hence,  $PQRS$  is a parallelogram.

**EXAMPLE 8** Prove that  $A(4, 3)$ ,  $B(6, 4)$ ,  $C(5, 6)$  and  $D(3, 5)$  are the angular points of a square.

**SOLUTION** Clearly,

$$AB = \sqrt{(6-4)^2 + (4-3)^2} = \sqrt{5}, \quad BC = \sqrt{(6-4)^2 + (5-4)^2} = \sqrt{5},$$

$$CD = \sqrt{(5-6)^2 + (3-5)^2} = \sqrt{5} \text{ and, } DA = \sqrt{(5-3)^2 + (3-4)^2} = \sqrt{5}$$

$$\therefore AB = BC = CD = DA.$$

$$\text{Now, } m_1 = \text{Slope of } AB = \frac{4-3}{6-4} = \frac{1}{2}, \quad m_2 = \text{Slope of } BC = \frac{6-4}{5-6} = -2$$



$$\text{and, } m_3 = \text{Slope of } CD = \frac{5-6}{3-5} = \frac{1}{2}$$

Clearly,  $m_1 m_2 = (1/2)(-2) = -1$  and  $m_1 = m_3$ . Therefore,  $AB$  is perpendicular to  $BC$  and it is parallel to  $CD$ . Thus,  $AB = BC = CA = AD$ ,  $AB \perp BC$  and  $AB$  is parallel to  $CD$ . Hence,  $ABCD$  is a square.

**EXAMPLE 9** If the angle between two lines is  $\frac{\pi}{4}$  and slope of one of the line  $\frac{1}{2}$ , find the slope of the other line. [NCERT]

**SOLUTION** We know that the acute angle  $\theta$  between two lines with slopes  $m_1$  and  $m_2$  is given by

$$\tan \theta = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right| \quad \dots(i)$$

Let  $m_1 = \frac{1}{2}$  and  $m_2 = m = \text{slope of the other line}$ . It is given that  $\theta = \frac{\pi}{4}$ . Substituting  $m_1 = \frac{1}{2}$ ,  $m_2 = m$  and  $\theta = \frac{\pi}{4}$  in (i), we obtain

$$\therefore \tan \frac{\pi}{4} = \left| \frac{m - \frac{1}{2}}{1 + m \times \frac{1}{2}} \right| \Rightarrow 1 = \left| \frac{2m - 1}{2 + m} \right|$$

$$\Rightarrow \frac{2m - 1}{m + 2} = \pm 1 \Rightarrow 2m - 1 = m + 2 \text{ or, } 2m - 1 = -(m + 2) \Rightarrow m = 3 \text{ or, } m = -\frac{1}{3}$$

Hence, the slope of the other line is 3 or,  $-\frac{1}{3}$ .

#### BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 10** If the points  $P(h, k)$ ,  $Q(x_1, y_1)$  and  $R(x_2, y_2)$  lie on a line. Show that:  
 $(h - x_1)(y_2 - y_1) = (k - y_1)(x_2 - x_1)$  [NCERT]

**SOLUTION** It is given that the points  $P(h, k)$ ,  $Q(x_1, y_1)$  and  $R(x_2, y_2)$  are collinear.

$\therefore$  Slope of  $PQ$  = Slope of  $QR$

$$\Rightarrow \frac{k - y_1}{h - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \Rightarrow (k - y_1)(x_2 - x_1) = (h - x_1)(y_2 - y_1)$$

**EXAMPLE 11** In Fig. 22.4, time and distance graph of a linear motion is given. Two positions of time and distance recorded as, when  $T = 0$ ,  $D = 2$  and when  $T = 3$ ,  $D = 8$ . Using the concept of slope, find law of motion i.e. how distance depends upon time. [NCERT]

**SOLUTION** Let  $P(T, D)$  be any point on the line, where  $D$  denotes the distance at any time  $T$ .

Clearly, points  $A(0, 2)$ ,  $B(3, 8)$  and  $P(T, D)$  are collinear.

$\therefore$  Slope of  $AB$  = Slope of  $BP$

$$\Rightarrow \frac{8 - 2}{3 - 0} = \frac{D - 8}{T - 3}$$

$$\Rightarrow 2 = \frac{D - 8}{T - 3} \Rightarrow D = 2(T + 1), \text{ which is the required relation.}$$

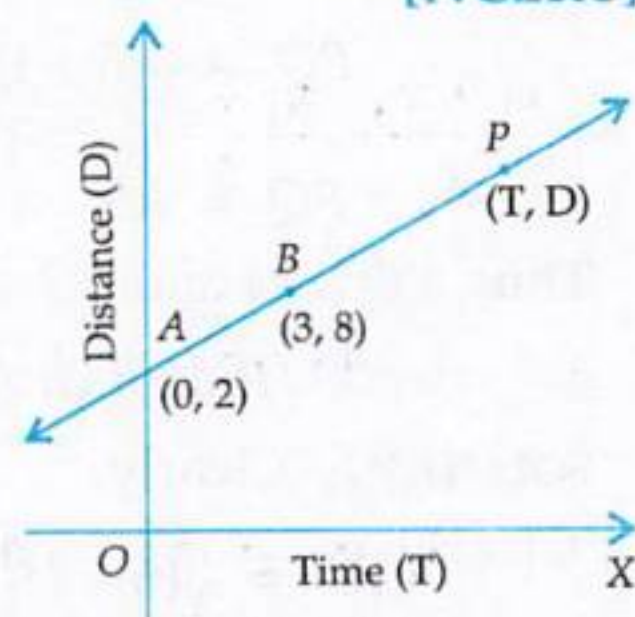


Fig. 22.4

**EXAMPLE 12** If points  $(a, 0)$ ,  $(0, b)$  and  $(x, y)$  are collinear, using the concept of slope, prove that  $\frac{x}{a} + \frac{y}{b} = 1$ .

**SOLUTION** Let  $A(a, 0)$ ,  $B(0, b)$  and  $P(x, y)$  be the given collinear points. Then,



$$\Rightarrow \text{Slope of } AB = \text{Slope of } BP$$

$$\Rightarrow \frac{b-0}{0-a} = \frac{y-b}{x-0} \Rightarrow \frac{-b}{a} = \frac{y-b}{x}$$

$$\Rightarrow -bx = ay - ab \Rightarrow bx + ay = ab \Rightarrow \frac{x}{a} + \frac{y}{b} = 1 \quad [\text{On dividing both sides by } ab]$$

**EXAMPLE 13** A ray of light passing through the point  $(1, 2)$  reflects on the  $x$ -axis at point  $A$  and the reflected ray passes through the point  $(5, 3)$ . Find the co-ordinates of  $A$ .

[NCERT EXEMPLAR]

**SOLUTION** Let the coordinates of  $A$  be  $(h, 0)$ . Let  $AN$  be the normal at  $A$ . Then,

$$\angle PAN = \angle QAN = \theta \text{ (say)}$$

Clearly,  $AQ$  makes angle  $90^\circ - \theta$  with  $OX$ . Therefore, slope of  $AQ$  is  $\tan(90^\circ - \theta) = \cot \theta$ . Also, the coordinates of  $A$  and  $Q$  are  $(h, 0)$  and  $(5, 3)$  respectively.

$$\therefore \text{Slope of } AQ = \frac{3-0}{5-h}$$

$$\text{Thus, } \cot \theta = \frac{3-0}{5-h} \Rightarrow \cot \theta = \frac{3}{5-h} \quad \dots(i)$$

$AP$  makes angle  $90^\circ + \theta$  with  $OX$ . Therefore, slope of  $AP$  is  $\tan(90^\circ + \theta) = -\cot \theta$ . Also,  $AP$  passes through  $A(h, 0)$  and  $P(1, 2)$ .

$$\therefore \text{Slope of } AP = \frac{2-0}{1-h} \Rightarrow -\cot \theta = \frac{2}{1-h} \Rightarrow \cot \theta = \frac{2}{h-1} \quad \dots(ii)$$

From (i) and (ii), we obtain

$$\frac{3}{5-h} = \frac{2}{h-1} \Rightarrow 3h-3 = 10-2h \Rightarrow 5h = 13 \Rightarrow h = \frac{13}{5}$$

Hence, the coordinates of  $A$  are  $(13/5, 0)$ .

**EXAMPLE 14** Prove that the line joining the mid-points of the two sides of a triangle is parallel to the third side.

**SOLUTION** Let  $A(x_1, y_1)$ ,  $B(x_2, y_2)$  and  $C(x_3, y_3)$  be the vertices of a  $\triangle ABC$  and  $D$  and  $E$  be the mid-points of sides  $AB$  and  $AC$  respectively. Then, the

coordinates of  $D$  and  $E$  are  $\left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$  and  $\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)$  respectively.

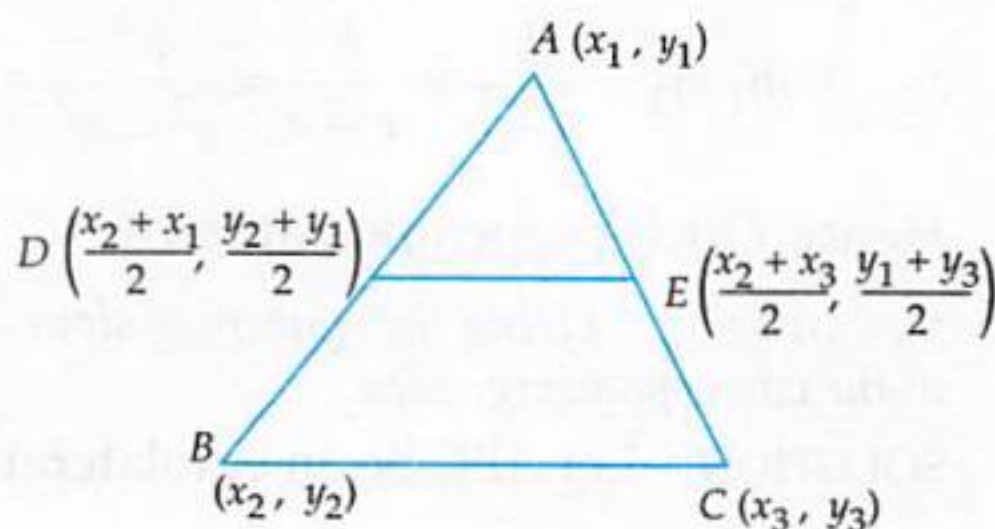


Fig. 22.6

$$\therefore m_1 = \text{Slope of } DE = \frac{\frac{y_1+y_3}{2} - \frac{y_2+y_1}{2}}{\frac{x_1+x_3}{2} - \frac{x_2+x_1}{2}} = \frac{y_3-y_2}{x_3-x_2} \text{ and, } m_2 = \text{Slope of } BC = \frac{y_3-y_2}{x_3-x_2}$$

Clearly,  $m_1 = m_2$ . Hence,  $DE$  is parallel to  $BC$ .

**EXAMPLE 15** If  $A(2, 0)$ ,  $B(0, 2)$  and  $C(0, 7)$  are three vertices, taken in order, of an isosceles trapezium  $ABCD$  in which  $AB \parallel DC$ . Find the coordinates of  $D$ .

**SOLUTION** Let the coordinates of  $D$  be  $(x, y)$ . It is given that  $AB \parallel DC$ .

$$\therefore \text{Slope of } AB = \text{Slope of } DC$$

$$\Rightarrow \frac{2-0}{0-2} = \frac{7-y}{0-x} \Rightarrow -1 = \frac{y-7}{x} \Rightarrow x+y-7=0 \quad \dots(i)$$



Since  $ABCD$  is an isosceles trapezium. Therefore,

$$AD = BC$$

$$\Rightarrow AD^2 = BC^2$$

$$\Rightarrow (x-2)^2 + (y-0)^2 = (0-0)^2 + (2-7)^2$$

$$\Rightarrow (x-2)^2 + y^2 = 25$$

$$\Rightarrow (x-2)^2 + (7-x)^2 = 25$$

[From (i),  $y = 7 - x$ ]

$$\Rightarrow 2x^2 - 18x + 28 = 0 \Rightarrow x^2 - 9x + 14 = 0 \Rightarrow (x-2)(x-7) = 0 \Rightarrow x = 2, 7$$

From (i),  $x = 2 \Rightarrow y = 5$  and  $x = 7 \Rightarrow y = 0$ . Hence, the coordinates of  $D$  are  $(2, 5)$  or  $(7, 0)$ .

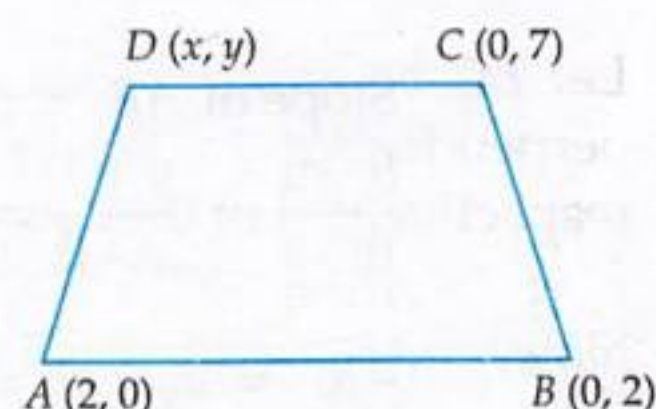


Fig. 22.7

**EXAMPLE 16** By using the concept of slope, prove that the diagonals of a rhombus are at right angles.

**SOLUTION** Let  $OABC$  be a rhombus whose each side is of length  $a$  such that  $O$  is the origin and  $OA$  is along  $x$ -axis. Let  $b$  be the height of the rhombus. Let  $BL$  and  $CM$  be perpendiculars drawn from  $B$  and  $C$  respectively on  $x$ -axis.

Further, let  $OM = c$ . Clearly,  $\triangle OMC \cong \triangle ALB$ . Therefore,  $OM = AL \Rightarrow AL = c$ . Thus, we have

$$OM = c, CM = b, OA = a, OL = a + c \text{ and } LB = b$$

So, the coordinates of the vertices of the rhombus are  $O(0, 0)$ ,  $A(a, 0)$ ,  $B(a + c, b)$  and  $C(c, b)$ . In right triangle  $OMC$ , we obtain

$$OC^2 = OM^2 + MC^2 \Rightarrow a^2 = c^2 + b^2 \quad \dots (i)$$

Now,

$$m_1 = \text{Slope of diagonal } OB = \frac{b-0}{a+c-0} = \frac{b}{a+c}$$

$$m_2 = \text{Slope of diagonal } AC = \frac{b-0}{c-a} = \frac{b}{c-a}$$

$$\therefore m_1 m_2 = \frac{b}{a+c} \times \frac{b}{c-a} = \frac{b^2}{c^2 - a^2} = \frac{b^2}{-b^2} = -1 \quad [\text{Using (i)}]$$

Hence,  $OB$  is perpendicular to  $AC$ .

**EXAMPLE 17** Using the concept of slope, prove that medians of an equilateral triangle are perpendicular to the corresponding sides.

**SOLUTION** Let  $ABC$  be an equilateral triangle such that  $AB = AC = BC = 2a$ .

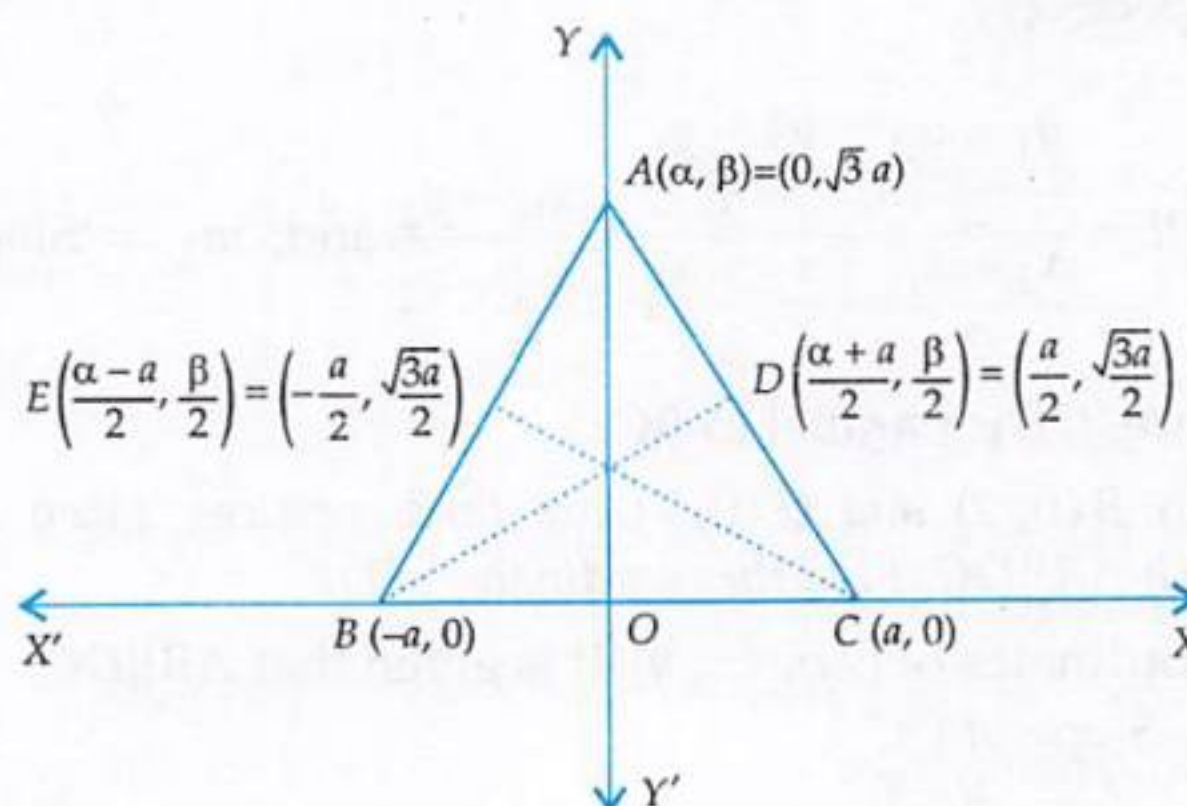


Fig. 22.9



Let  $BC$  be along  $X$ -axis, mid-point of  $BC$  be the origin and a line passing through  $O$  and perpendicular to  $BC$  be  $Y$ -axis. Then, the coordinates of  $B$  and  $C$  are  $(-a, 0)$  and  $(a, 0)$  respectively. Let the coordinates of  $A$  be  $(\alpha, \beta)$ . Since  $ABC$  is an equilateral triangle.

$$\therefore AB = AC = BC$$

$$\text{Now, } AB = AC$$

$$\Rightarrow AB^2 = AC^2 \Rightarrow (\alpha + a)^2 + (\beta - 0)^2 = (\alpha - a)^2 + (\beta - 0)^2 \Rightarrow 4a\alpha = 0 \Rightarrow \alpha = 0.$$

In right triangle  $COA$ , we obtain

$$AC^2 = OA^2 + OC^2 \Rightarrow (2a)^2 = \beta^2 + a^2 \Rightarrow \beta = \sqrt{3}a$$

Thus, the coordinates of  $A$  are  $(0, \sqrt{3}a)$ . Consequently,  $A$  lies on  $y$ -axis.

$$\therefore OA \perp BC$$

[ $\because$   $OA$  is along  $y$ -axis and  $BC$  is along  $x$ -axis]

Let  $D$  and  $E$  be the mid-points of  $AC$  and  $AB$  respectively. Then, the coordinates of  $D$  and  $E$  are

$$\left(\frac{\alpha + a}{2}, \frac{\beta}{2}\right) = \left(\frac{a}{2}, \frac{\sqrt{3}a}{2}\right) \text{ and } \left(\frac{\alpha - a}{2}, \frac{\beta}{2}\right) = \left(-\frac{a}{2}, \frac{\sqrt{3}a}{2}\right) \text{ respectively.}$$

$$\text{Now, } m_1 = \text{Slope of } AC = \frac{\sqrt{3}a - 0}{0 - a} = -\sqrt{3}, m_2 = \text{Slope of } BD = \frac{\frac{\sqrt{3}a}{2} - 0}{\frac{a}{2} + a} = \frac{1}{\sqrt{3}}$$

Clearly,  $m_1 m_2 = -1$ . Therefore,  $BD \perp AC$ .

$$\text{Also Slope of } AB \times \text{Slope of } CE = \frac{\sqrt{3}a - 0}{0 + a} \times \frac{\frac{\sqrt{3}a}{2} - 0}{-\frac{a}{2} - a} = \sqrt{3} \times -\frac{\sqrt{3}}{3} = -1 \Rightarrow AB \perp CE$$

Thus,  $AO$ ,  $BD$  and  $CE$  are medians of an equilateral triangle  $ABC$  such that  $AO \perp BC$ ,  $BD \perp CA$  and  $CE \perp AB$ . Hence, medians of an equilateral triangle are perpendicular to the corresponding sides.

**EXAMPLE 18** Prove that a triangle which has one of the angle as  $30^\circ$ , cannot have all vertices with integral coordinates.

**SOLUTION** Let  $ABC$  be a triangle the coordinates of whose vertices are  $A(x_1, y_1)$ ,  $B(x_2, y_2)$  and  $C(x_3, y_3)$ , where  $x_1, x_2, x_3, y_1, y_2, y_3$  are integers. Let  $\angle BAC = 30^\circ$ . We find that

$$m_1 = \text{Slope of } AB = \frac{y_1 - y_2}{x_1 - x_2}, \text{ and } m_2 = \text{Slope of } AC = \frac{y_1 - y_3}{x_1 - x_3}$$

$$\text{Now, } \angle BAC = 30^\circ$$

$$\Rightarrow \tan 30^\circ = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \left| \frac{\frac{y_1 - y_2}{x_1 - x_2} - \frac{y_1 - y_3}{x_1 - x_3}}{1 + \frac{y_1 - y_2}{x_1 - x_2} \times \frac{y_1 - y_3}{x_1 - x_3}} \right| \Rightarrow \frac{1}{\sqrt{3}} = \left| \frac{(y_1 - y_2)(x_1 - x_3) - (y_1 - y_3)(x_1 - x_2)}{(x_1 - x_2)(x_1 - x_3) + (y_1 - y_2)(y_1 - y_3)} \right|$$

This is not possible as LHS is an irrational number and RHS is a rational number. Hence,  $x_1, x_2, x_3, y_1, y_2, y_3$  cannot be all integers.

**EXAMPLE 19** The vertices of a triangle are  $A(x_1, x_1 \tan \theta_1)$ ,  $B(x_2, x_2 \tan \theta_2)$  and  $C(x_3, x_3 \tan \theta_3)$ . If the circumcentre of  $\triangle ABC$  coincides with the origin and  $H(\bar{x}, \bar{y})$  is the orthocentre, show that



$$\frac{\bar{y}}{\bar{x}} = \frac{\sin \theta_1 + \sin \theta_2 + \sin \theta_3}{\cos \theta_1 + \cos \theta_2 + \cos \theta_3}$$

**SOLUTION** The circumcentre of  $\Delta ABC$  is at the origin  $O$ . Let the circum-radius be  $r$ . Then,

$$OA = OB = OC = r$$

$$\Rightarrow OA^2 = OB^2 = OC^2 = r^2$$

$$\Rightarrow x_1^2 + x_1^2 \tan^2 \theta_1 = x_2^2 + x_2^2 \tan^2 \theta_2 = x_3^2 + x_3^2 \tan^2 \theta_3 = r^2$$

$$\Rightarrow x_1^2 \sec^2 \theta_1 = x_2^2 \sec^2 \theta_2 = x_3^2 \sec^2 \theta_3 = r^2 \Rightarrow x_1 = r \cos \theta_1, x_2 = r \cos \theta_2, x_3 = r \cos \theta_3$$

Thus, the coordinates of the vertices of  $\Delta ABC$  are

$$A \equiv (x_1, x_1 \tan \theta_1) = (r \cos \theta_1, r \sin \theta_1), B \equiv (x_2, x_2 \tan \theta_2) = (r \cos \theta_2, r \sin \theta_2)$$

$$\text{and, } C \equiv (x_3, x_3 \tan \theta_3) = (r \cos \theta_3, r \sin \theta_3).$$

So, the coordinates of the centroid  $G$  are

$$\left( \frac{r \cos \theta_1 + r \cos \theta_2 + r \cos \theta_3}{3}, \frac{r \sin \theta_1 + r \sin \theta_2 + r \sin \theta_3}{3} \right)$$

We know that the circumcentre ( $O$ ), Centroid ( $G$ ) and orthocentric ( $H$ ) of a triangle are collinear.

$\therefore$  Slope of  $OH$  = Slope of  $OG$

$$\Rightarrow \frac{\bar{y} - 0}{\bar{x} - 0} = \frac{\frac{r \sin \theta_1 + r \sin \theta_2 + r \sin \theta_3}{3} - 0}{\frac{r \cos \theta_1 + r \cos \theta_2 + r \cos \theta_3}{3} - 0} \Rightarrow \frac{\bar{y}}{\bar{x}} = \frac{\sin \theta_1 + \sin \theta_2 + \sin \theta_3}{\cos \theta_1 + \cos \theta_2 + \cos \theta_3}$$

### EXERCISE 22.1

#### BASIC

- Find the slopes of the lines which make the following angles with the positive direction of  $x$ -axis:
  - $-\frac{\pi}{4}$
  - $\frac{2\pi}{3}$
  - $\frac{3\pi}{4}$
  - $\frac{\pi}{3}$
- Find the slope of a line passing through the following points:
  - $(-3, 2)$  and  $(1, 4)$
  - $(at_1^2, 2at_1)$  and  $(at_2^2, 2at_2)$
  - $(3, -5)$ , and  $(1, 2)$
- State whether the two lines in each of the following are parallel, perpendicular or neither:
  - Through  $(5, 6)$  and  $(2, 3)$ ; through  $(9, -2)$  and  $(6, -5)$
  - Through  $(9, 5)$  and  $(-1, 1)$ ; through  $(3, -5)$  and  $(8, -3)$
  - Through  $(6, 3)$  and  $(1, 1)$ ; through  $(-2, 5)$  and  $(2, -5)$
  - Through  $(3, 15)$  and  $(16, 6)$ ; through  $(-5, 3)$  and  $(8, 2)$ .
- Find the slope of a line (i) which bisects the first quadrant angle (ii) which makes an angle of  $30^\circ$  with the positive direction of  $y$ -axis measured anticlockwise. [NCERT]
- Using the method of slope, show that the following points are collinear:
  - $A(4, 8), B(5, 12), C(9, 28)$
  - $A(16, -18), B(3, -6), C(-10, 6)$
- What is the value of  $y$  so that the line through  $(3, y)$  and  $(2, 7)$  is parallel to the line through  $(-1, 4)$  and  $(0, 6)$ ?



7. What can be said regarding a line if its slope is  
(i) zero                      (ii) positive                      (iii) negative?
8. Show that the line joining  $(2, -3)$  and  $(-5, 1)$  is parallel to the line joining  $(7, -1)$  and  $(0, 3)$ .
9. Show that the line joining  $(2, -5)$  and  $(-2, 5)$  is perpendicular to the line joining  $(6, 3)$  and  $(1, 1)$ .

**BASED ON LOTS**

10. Without using Pythagoras theorem, show that the points  $A (0, 4)$ ,  $B (1, 2)$  and  $C (3, 3)$  are the vertices of a right angled triangle.
11. Prove that the points  $(-4, -1)$ ,  $(-2, -4)$ ,  $(4, 0)$  and  $(2, 3)$  are the vertices of a rectangle.
12. If three points  $A (h, 0)$ ,  $P (a, b)$  and  $B (0, k)$  lie on a line, show that:  $\frac{a}{h} + \frac{b}{k} = 1$ . [NCERT]
13. The slope of a line is double of the slope of another line. If tangents of the angle between them is  $\frac{1}{3}$ , find the slopes of the other line. [NCERT]
14. Consider the following population and year graph:  
Find the slope of the line  $AB$  and using it, find what will be the population in the year 2020. [NCERT]



Fig. 22.10

15. Without using the distance formula, show that points  $(-2, -1)$ ,  $(4, 0)$ ,  $(3, 3)$  and  $(-3, 2)$  are the vertices of a parallelogram.
16. Find the angle between the X-axis and the line joining the points  $(3, -1)$  and  $(4, -2)$ .  
[NCERT]
17. Line through the points  $(-2, 6)$  and  $(4, 8)$  is perpendicular to the line through the points  $(8, 12)$  and  $(x, 24)$ . Find the value of  $x$ .  
[NCERT]
18. Find the value of  $x$  for which the points  $(x, -1)$ ,  $(2, 1)$  and  $(4, 5)$  are collinear.  
[NCERT]
19. By using the concept of slope, show that the points  $(-2, -1)$ ,  $(4, 0)$ ,  $(3, 3)$  and  $(-3, 2)$  are the vertices of a parallelogram.  
[NCERT]
20. A quadrilateral has vertices  $(4, 1)$ ,  $(1, 7)$ ,  $(-6, 0)$  and  $(-1, -9)$ . Show that the mid-points of the sides of this quadrilateral form a parallelogram.

## ANSWERS

1. (i)  $-1$  (ii)  $-\sqrt{3}$  (iii)  $-1$  (iv)  $\sqrt{3}$   
2. (i)  $\frac{1}{2}$  (ii)  $\frac{2}{t_2 + t_1}$  (iii)  $-\frac{7}{2}$



3. (i) parallel (ii) parallel (iii) perpendicular (iv) neither.  
 4. (i) 1 (ii)  $-\sqrt{3}$  6. 9  
 7. (i) The line is either  $x$ -axis or it is parallel to  $x$ -axis.  
 (ii) The line makes an acute angle with positive direction of  $x$ -axis.  
 (iii) The line makes an obtuse angle with the positive direction of  $x$ -axis.  
 13.  $1, \frac{1}{2}$  14. 104.50 crores 16.  $135^\circ$  17. 4 18. 1

### HINTS TO SELECTED PROBLEMS

12. It is given that points  $A(h, 0)$ ,  $B(0, k)$  and  $P(a, b)$  are collinear. Therefore,

Slope of  $PA$  = Slope of  $PB$

$$\Rightarrow \frac{b-0}{a-h} = \frac{b-k}{a-0}$$

$$\Rightarrow ab = (a-h)(b-k) \Rightarrow ab = ab - ak - bh + hk \Rightarrow hk = ak + bh \Rightarrow \frac{a}{h} + \frac{b}{k} = 1$$

13. Let  $m$  be the slope of first line. Then the slope of the second line is  $2m$ . Let  $\theta$  be the angle between the lines. Then,

$$\tan \theta = \left| \frac{2m-m}{1+2m^2} \right|$$

$$\Rightarrow \frac{1}{3} = \pm \frac{m}{1+2m^2} \Rightarrow 2m^2 \pm 3m + 1 = 0 \Rightarrow 2m^2 + 3m + 1 = 0 \text{ or } 2m^2 - 3m + 1 = 0$$

$$\Rightarrow (2m+1)(m+1) = 0 \text{ or } (2m-1)(m-1) = 0 \Rightarrow m = \pm \frac{1}{2}, \pm 1.$$

14. Let  $P$  be the population in the year 2020. Then,  $C(2020, P)$  lies on the line.

Since  $A, B, C$  are collinear points. Therefore,

Slope of  $AB$  = Slope of  $BC$

$$\Rightarrow \frac{97-92}{2005-1995} = \frac{P-97}{2020-2005} \Rightarrow \frac{5}{10} = \frac{P-97}{15} \Rightarrow 7.5 = P-97 \Rightarrow P = 104.5$$

16. Let  $\theta$  be the angle between the line joining the points  $(3, -1)$  and  $(4, -2)$  and  $x$ -axis. Then,

$$\tan \theta = \text{Slope of the line} \Rightarrow \tan \theta = \frac{-2+1}{4-3} \Rightarrow \tan \theta = -1 \Rightarrow \theta = 135^\circ.$$

17. It is given that the line through the points  $A(-2, 6)$  and  $B(4, 8)$  is perpendicular to the line through the points  $C(8, 12)$  and  $D(x, 24)$ . Therefore,

Slope of  $AB \times$  Slope of  $CD = -1$

$$\Rightarrow \frac{8-6}{4+2} \times \frac{24-12}{x-8} = -1 \Rightarrow \frac{1}{3} \times \frac{12}{x-8} = -1 \Rightarrow -4 = x-8 \Rightarrow x = 4$$

18. It is given that points  $A(x, -1)$ ,  $B(2, 1)$  and  $C(4, 5)$  are collinear.

$$\therefore \text{Slope of } AB = \text{Slope of } BC \Rightarrow \frac{1+1}{2-x} = \frac{5-1}{4-2} \Rightarrow \frac{2}{2-x} = 2 \Rightarrow 2-x=1 \Rightarrow x=1.$$



20. Given points are  $A(-2, -1)$ ,  $B(4, 0)$ ,  $C(3, 3)$  and  $D(-3, 2)$ . Therefore,

$$m_1 = \text{Slope of } AB = \frac{0 + 1}{4 + 2} = \frac{1}{6}, \quad m_2 = \text{Slope of } CD = \frac{2 - 3}{-3 - 2} = \frac{1}{6}$$

$$m_3 = \text{Slope of } BC = \frac{3 - 0}{3 - 4} = -3 \text{ and, } m_4 = \text{Slope of } AD = \frac{2 + 1}{-3 + 2} = -3$$

Clearly,  $m_1 = m_2$  and  $m_3 = m_4$ . Therefore,  $AB \parallel CD$  and  $BC \parallel AD$ .

Hence,  $ABCD$  is a parallelogram.

## 22.4 INTERCEPTS OF A LINE ON THE AXES

If a straight line cuts  $x$ -axis at  $A$  and the  $y$ -axis at  $B$ , then  $OA$  and  $OB$  are known as the intercepts of the line on  $x$ -axis and  $y$ -axis respectively.

The intercepts are positive or negative according as the line meets with positive or negative directions of the coordinate axes.

In Fig. 22.11,  $OA = x$ -intercept,  $OB = y$ -intercept.

$OA$  is positive or negative according as  $A$  lies on  $OX$  or  $OX'$  respectively. Similarly,  $OB$  is positive or negative according as  $B$  lies on  $OY$  or  $OY'$  respectively.

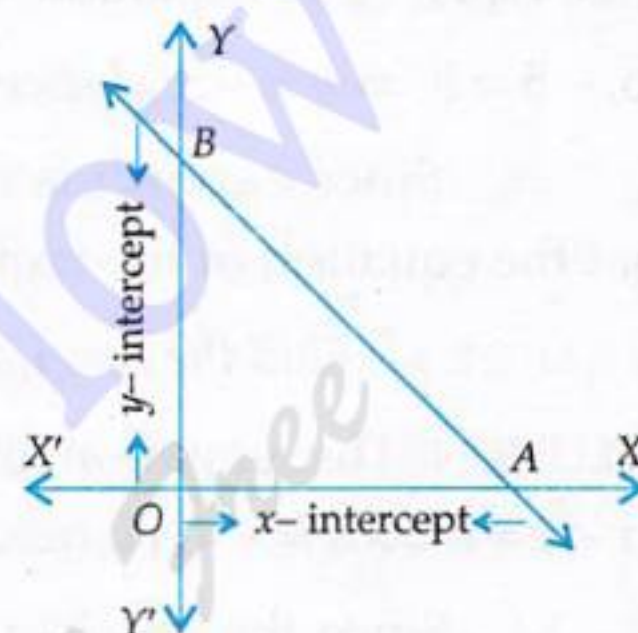


Fig. 22.11

## 22.5 EQUATIONS OF LINES PARALLEL TO THE COORDINATE AXES

### 22.5.1 EQUATION OF A LINE PARALLEL TO $x$ -AXIS

Let  $AB$  be a straight line parallel to  $x$ -axis at a distance  $b$  from it. Then, clearly the ordinate of each point on  $AB$  is  $b$ . Thus,  $AB$  can be considered as the locus of a point at a distance  $b$  from  $x$ -axis. Thus, if  $P(x, y)$  is any point on  $AB$ , then  $y = b$ . (See Fig. 22.12).

Hence, the equation of a line parallel to  $x$ -axis at a distance  $b$  from it, is  $y = b$ .

Since  $x$ -axis is a parallel to itself at a distance 0 from it, therefore the equation of  $x$ -axis is  $y = 0$ .

If a line is parallel to  $x$ -axis at a distance  $b$  and below  $x$ -axis, then its equation is  $y = -b$ .

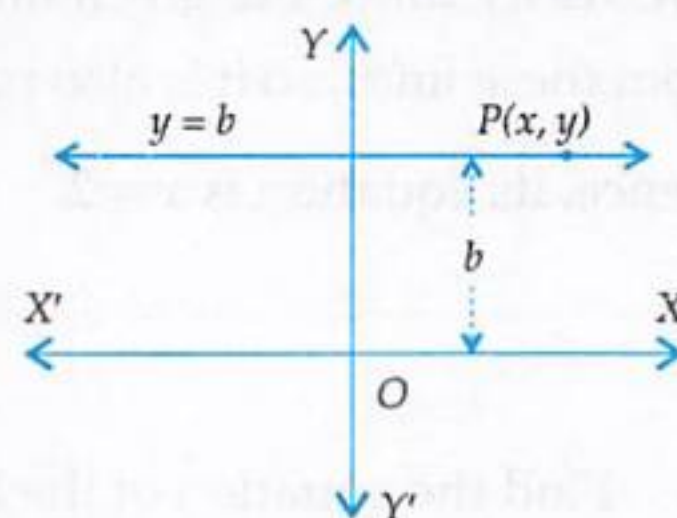


Fig. 22.12

### 22.5.2 EQUATION OF A LINE PARALLEL TO $y$ -AXIS

Let  $AB$  be a line parallel to  $y$ -axis and at a distance  $a$  from it. Then, the abscissa of every point on  $AB$  is  $a$ . So it can be treated as the locus of a point a distance  $a$  from  $y$ -axis. Thus, if  $P(x, y)$  is any point on  $AB$ , then  $x = a$ . (See Fig. 22.13)

Hence, the equation of a line parallel to  $y$ -axis at a distance  $a$  from it, is  $x = a$ .

Since  $y$ -axis is parallel to itself at a distance 0 from it, therefore the equation of  $y$ -axis is  $x = 0$ .

If a line is parallel to  $y$ -axis at a distance  $a$  and to the left of  $y$ -axis, then its equation is  $x = -a$ .

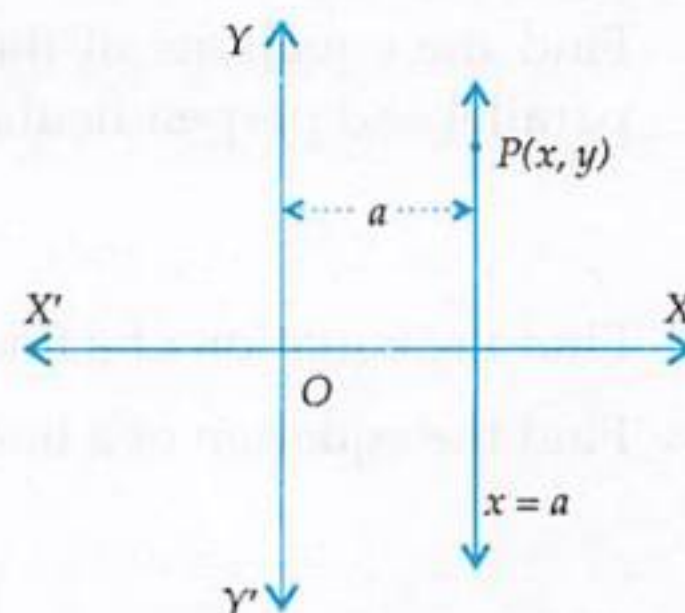


Fig. 22.13



## ILLUSTRATIVE EXAMPLES

## BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** Write down the equations of the following lines:

- (i)  $x$ -axis (ii)  $y$ -axis [NCERT]  
 (iii) A line parallel to  $x$ -axis at a distance of 3 units below  $x$ -axis.  
 (iv) A line parallel to  $y$ -axis at a distance of 5 units on the left hand side of it.

**SOLUTION** (i)  $y = 0$  (ii)  $x = 0$  (iii)  $y = -3$  (iv)  $x = -5$ .

**EXAMPLE 2** Find the equation of a line which is parallel to  $x$ -axis and passes through  $(3, -5)$ .

**SOLUTION** The equation of a line parallel to  $x$ -axis is  $y = b$ . Since it passes through  $(3, -5)$ .  
 So,  $-5 = b \Rightarrow b = -5$ . Hence, the equation of the required line is  $y = -5$ .

**ALITER** Since  $y$ -coordinate of every point on a line parallel to  $x$ -axis is always same, it follows that the equation of the required line is  $y = -5$ .

**EXAMPLE 3** Find the equation of a line which is parallel to  $y$ -axis and passes through  $(-4, 3)$ .

**SOLUTION** The equation of a line parallel to  $y$ -axis is  $x = a$ . Since, it passes through  $(-4, 3)$ .  
 So  $-4 = a \Rightarrow a = -4$ . Hence, the equation of the required line is  $x = -4$ .

**ALITER** Since the abscissa of every point on a line parallel to  $y$ -axis is always same. So, the equation of the required line is  $x = -4$ .

## BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 4** Find the equation of a line which is equidistant from the lines  $x = -4$  and  $x = 8$ .

**SOLUTION** Since the given lines are both parallel to  $y$ -axis and the required line is equidistant from these lines, so it is also parallel to  $y$ -axis and its distance from  $y$ -axis is  $\frac{1}{2}(-4 + 8) = 2$  units.

Hence, its equation is  $x = 2$ .

## EXERCISE 22.2

## BASIC

- Find the equation of the line parallel to  $x$ -axis and passing through  $(3, -5)$ .
- Find the equation of the line perpendicular to  $x$ -axis and having intercept  $-2$  on  $x$ -axis.
- Find the equation of the line parallel to  $x$ -axis and having intercept  $-2$  on  $y$ -axis.
- Draw the lines  $x = -3$ ,  $x = 2$ ,  $y = -2$ ,  $y = 3$  and write the coordinates of the vertices of the square so formed.
- Find the equations of the straight lines which pass through  $(4, 3)$  and are respectively parallel and perpendicular to the  $x$ -axis.

## BASED ON LOTS

- Find the equation of a line which is equidistant from the lines  $x = -2$  and  $x = 6$ .
- Find the equation of a line equidistant from the lines  $y = 10$  and  $y = -2$ .

## ANSWERS

- |                   |             |             |                                         |
|-------------------|-------------|-------------|-----------------------------------------|
| 1. $y = -5$       | 2. $x = -2$ | 3. $y = -2$ | 4. $(2, 3), (-3, 3), (-3, -2), (2, -2)$ |
| 5. $y = 3, x = 4$ | 6. $x = 2$  | 7. $y = 4$  |                                         |



## HINTS TO SELECTED PROBLEMS

6. Since the given lines are parallel to  $y$ -axis and the required line is equidistant from the given lines. Therefore, it is parallel to  $y$ -axis at a distance  $\frac{1}{2}(-2 + 6) = 2$  units from it. So its equation is  $x = 2$ .

## 22.6 DIFFERENT FORMS OF THE EQUATION OF A STRAIGHT LINE

In section 22.1, we have seen that a first degree equation in  $x, y$  represents a straight line. The equation of a straight line can be written in different forms depending on the data given. In this section, we shall learn about these forms.

## 22.6.1 SLOPE INTERCEPT FORM OF A LINE

**THEOREM** The equation of a line with slope  $m$  and making an intercept  $c$  on  $y$ -axis is  $y = mx + c$ .

**PROOF** Let the given line intersects  $y$ -axis at  $Q$  and makes an angle  $\theta$  with  $x$ -axis. Then,  $m = \tan \theta$ . Let  $P(x, y)$  be any point on the line. Draw  $PL$  perpendicular to  $x$ -axis and  $QM \perp PL$ .

Clearly,  $\angle MQP = \theta$ ,  $QM = OL = x$

and,  $PM = PL - ML = PL - OQ = y - c$ .

In triangle  $PMQ$ , we obtain

$$\tan \theta = \frac{PM}{QM} = \frac{y - c}{x}$$

$$\Rightarrow m = \frac{y - c}{x}$$

$$\Rightarrow y = mx + c, \text{ which is the required equation of the line.}$$

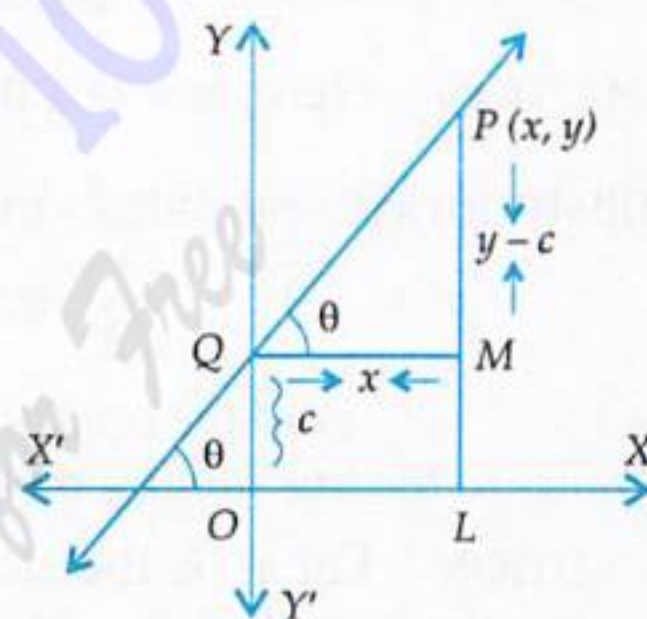


Fig. 22.14

Q.E.D.

**REMARK 1** If the line passes through the origin, then  $0 = m \cdot 0 + c \Rightarrow c = 0$ . Therefore, the equation of a line passing through the origin is  $y = mx$ , where  $m$  is the slope of the line.

**REMARK 2** If the line is parallel to  $x$ -axis, then  $m = 0$ , therefore the equation of a line parallel to  $x$ -axis is  $y = c$ .

## ILLUSTRATIVE EXAMPLES

## BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** Find the equation of a line with slope  $-1$  and cutting off an intercept of 4 units on negative direction of  $y$ -axis.

**SOLUTION** Here,  $m = -1$  and  $c = -4$

Substituting these values in  $y = mx + c$ , we obtain that the equation of the line is

$$y = -x - 4 \text{ or, } x + y + 4 = 0$$

**EXAMPLE 2** Find the equation of a straight line which cuts off an intercept of 5 units on negative direction of  $y$ -axis and makes an angle of  $120^\circ$  with the positive direction of  $x$ -axis.

**SOLUTION** Here,  $m = \tan 120^\circ = \tan (90 + 30^\circ) = -\cot 30^\circ = -\sqrt{3}$  and  $c = -5$ .

Substituting these values in  $y = mx + c$ , we obtain that the equation of the line is

$$y = -\sqrt{3}x - 5 \text{ or, } \sqrt{3}x + y + 5 = 0$$



**EXAMPLE 3** Find the equation of a straight line cutting off an intercept  $-1$  from  $y$ -axis and being equally inclined to the axes.

**SOLUTION** Since the required line is equally inclined with the coordinate axes, therefore it makes either an angle of  $45^\circ$  or  $135^\circ$  with the  $x$ -axis.

So, its slope is either  $m = \tan 45^\circ$  or,  $m = \tan 135^\circ$   
i. e.  $m = 1$  or,  $-1$ . It is given that  $c = -1$ .

Substituting these values in  $y = mx + c$ , we obtain that the equations of the lines are  $y = x - 1$  and  $y = -x - 1$ .

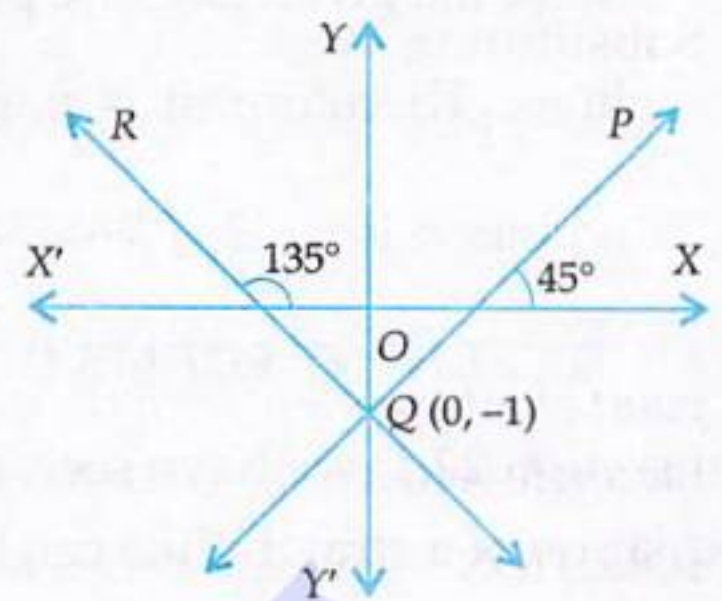


Fig. 22.15

**EXAMPLE 4** Find the equation of a straight line which makes an angle of  $\tan^{-1} \sqrt{2}$  with the  $x$ -axis and cuts off an intercept of  $-\frac{3}{\sqrt{2}}$  with the  $y$ -axis.

**SOLUTION** Here,  $m = \tan \theta = \tan (\tan^{-1} \sqrt{2}) = \sqrt{2}$  and  $c = -\frac{3}{\sqrt{2}}$ .

Substituting these values in  $y = mx + c$ , we obtain that the equation of the required line is

$$y = \sqrt{2}x - \frac{3}{\sqrt{2}} \text{ or, } \sqrt{2}y = 2x - 3$$

**EXAMPLE 5** Find the equation of a straight line which cuts off an intercept of length 3 on  $y$ -axis and is parallel to the line joining the points  $(3, -2)$  and  $(1, 4)$ .

**SOLUTION** Let  $m$  be the slope of the required line. Since the required line is parallel to the line joining the points  $A(3, -2)$  and  $B(1, 4)$ .

$$\therefore m = \text{Slope of the line } AB = \frac{4 - (-2)}{1 - 3} = -3$$

It is given that  $c = 3$ . Substituting these values in  $y = mx + c$ , we obtain that the equation of the required line is  $y = -3x + 3$  or,  $3x + y - 3 = 0$ .

**EXAMPLE 6** Find the equation of a line that has  $y$ -intercept 4 and is perpendicular to the line joining  $(2, -3)$  and  $(4, 2)$ .

**SOLUTION** Let  $m$  be the slope of the required line. Since the required line is perpendicular to the line joining  $A(2, -3)$  and  $B(4, 2)$ .

$$\therefore m \times \text{Slope of } AB = -1 \Rightarrow m \times \frac{2 + 3}{4 - 2} = -1 \Rightarrow m = -\frac{2}{5}$$

The required line cuts off an intercept of length 4 on  $y$ -axis. So,  $c = 4$

Substituting these values in  $y = mx + c$ , we obtain that the equation of the required line is

$$y = -\frac{2}{5}x + 4 \text{ or, } 2x + 5y - 20 = 0.$$

**EXAMPLE 7** Find the equation of the straight line which makes angle of  $15^\circ$  with the positive direction of  $x$ -axis and which cuts an intercept of length 4 on the negative direction of  $Y$ -axis.

**SOLUTION** Let  $m$  be the slope of the line. Then,

$$m = \tan 15^\circ = \tan (45^\circ - 30^\circ) = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

$$\Rightarrow m = \frac{(\sqrt{3} - 1)^2}{(\sqrt{3} + 1)(\sqrt{3} - 1)} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$



It is given that the line cuts an intercept of length 4 on the negative direction of  $y$ -axis.

$$\therefore c = -4$$

Substituting these values in  $y = mx + c$ , we get

$$y = (2 - \sqrt{3})x - 4 \text{ as the equation of the required line.}$$

### BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

**EXAMPLE 8**  $P_1, P_2$  are points on either of the two lines  $y - \sqrt{3}|x| = 2$  at a distance of 5 units from their point of intersection. Find the coordinates of the foot of perpendiculars drawn from  $P_1, P_2$  on the bisector of the angle between the given lines. [NCERT EXEMPLAR]

**SOLUTION** The equations of the given lines are

$$\therefore y - \sqrt{3}x = 2 \text{ for } x \geq 0 \quad \dots (i) \quad \text{and,} \quad y + \sqrt{3}x = 2 \text{ for } x < 0 \quad \dots (ii)$$

The slopes of these two lines are  $\sqrt{3}$  and  $-\sqrt{3}$  respectively. So, they make angles of  $60^\circ$  and  $120^\circ$  respectively with  $x$ -axis. Consequently, each makes  $30^\circ$  angles with the positive direction of  $y$ -axis as shown in Fig. 22.16.

Clearly, lines (i) and (ii) intersect at  $P(0, 2)$  and  $y$ -axis is the bisector of the acute angle between them. It is given that  $PP_1 = 5 = PP_2$ .

Let  $M$  be the foot of the perpendiculars drawn from  $P_1$  and  $P_2$  on  $y$ -axis.

In right triangle  $PMP_1$ , we have

$$\cos 30^\circ = \frac{PM}{PP_1} \Rightarrow \frac{\sqrt{3}}{2} = \frac{PM}{5} \Rightarrow PM = \frac{5\sqrt{3}}{2}$$

$$\therefore OM = OP + PM = 2 + \frac{5\sqrt{3}}{2} = \frac{4 + 5\sqrt{3}}{2}$$

Hence, the coordinates of  $M$  are  $\left(0, \frac{4 + 5\sqrt{3}}{2}\right)$ .

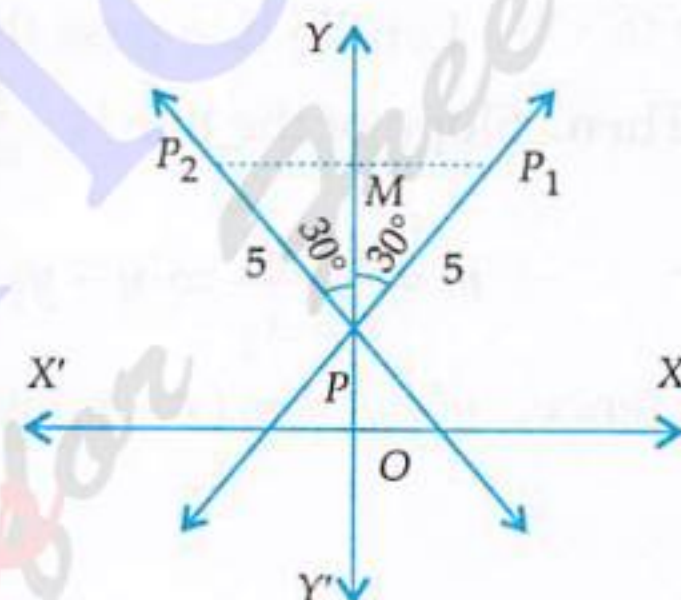


Fig. 22.16

### EXERCISE 22.3

#### BASIC

- Find the equation of a line making an angle of  $150^\circ$  with the  $x$ -axis and cutting off an intercept 2 from  $y$ -axis.
- Find the equation of a straight line:
  - with slope 2 and  $y$ -intercept 3;
  - with slope  $-1/3$  and  $y$ -intercept  $-4$ .
  - with slope  $-2$  and intersecting the  $x$ -axis at a distance of 3 units to the left of origin.
- Find the equations of the bisectors of the angles between the coordinate axes.
- Find the equation of a line which makes an angle of  $\tan^{-1}(3)$  with the  $x$ -axis and cuts off an intercept of 4 units on negative direction of  $y$ -axis.
- Find the equation of a line that has  $y$ -intercept  $-4$  and is parallel to the line joining  $(2, -5)$  and  $(1, 2)$ .
- Find the equation of a line which is perpendicular to the line joining  $(4, 2)$  and  $(3, 5)$  and cuts off an intercept of length 3 on  $y$ -axis.
- Find the equation of the perpendicular to the line segment joining  $(4, 3)$  and  $(-1, 1)$  if it cuts off an intercept  $-3$  from  $y$ -axis.
- Find the equation of the straight line intersecting  $y$ -axis at a distance of 2 units above the origin and making an angle of  $30^\circ$  with the positive direction of the  $x$ -axis. [NCERT]



## ANSWERS

1.  $x + \sqrt{3}y = 2\sqrt{3}$     2. (i)  $y = 2x + 3$     (ii)  $x + 3y + 12 = 0$     (iii)  $2x + y + 6 = 0$   
 3.  $x \pm y = 0$     4.  $y = 3x - 4$     5.  $7x + y + 4 = 0$     6.  $x - 3y + 9 = 0$   
 7.  $5x + 2y + 6 = 0$     8.  $x - \sqrt{3}y + 2\sqrt{3} = 0$

## HINTS TO SELECTED PROBLEM

8. Clearly,  $c = y\text{-intercept} = 2$  and,  $m = \tan 30^\circ = \frac{1}{\sqrt{3}}$ .

So, the equation of the line is  $y = \frac{1}{\sqrt{3}}x + 2$  or,  $x - \sqrt{3}y + 2\sqrt{3} = 0$ .

## 22.6.2 POINT-SLOPE FORM OF A LINE

**THEOREM** The equation of a line which passes through the point  $(x_1, y_1)$  and has the slope ' $m$ ' is  $y - y_1 = m(x - x_1)$ .

**PROOF** Let the line pass through the point  $Q(x_1, y_1)$  and let  $P(x, y)$  be any point on the line. Then, Slope of the line is  $= \frac{y - y_1}{x - x_1}$ . But, the slope of the line is  $m$ .

$$\therefore m = \frac{y - y_1}{x - x_1} \Rightarrow y - y_1 = m(x - x_1)$$

Hence,  $y - y_1 = m(x - x_1)$  is the required equation of the line.

Q.E.D.

## ILLUSTRATIVE EXAMPLES

## BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** Find the equation of a line passing through  $(2, -3)$  and inclined at an angle of  $135^\circ$  with the positive direction of  $x$ -axis.

**SOLUTION** Here,  $m = \text{Slope of the line} = \tan 135^\circ = \tan (90^\circ + 45^\circ) = -\cot 45^\circ = -1$ .

$$x_1 = 2, y_1 = -3$$

So, the equation of the line is  $y - y_1 = m(x - x_1)$

$$\text{i.e. } y - (-3) = -1(x - 2) \Rightarrow y + 3 = -x + 2 \Rightarrow x + y + 1 = 0.$$

**EXAMPLE 2** Determine the equation of line through the point  $(-4, -3)$  and parallel to  $x$ -axis.

**SOLUTION** Here,  $m = \text{Slope} = 0$ ,  $x_1 = -4$ ,  $y_1 = -3$ .

So, the equation of the line is  $y - y_1 = m(x - x_1)$  or,  $y + 3 = 0(x + 4) \Rightarrow y + 3 = 0$ .

**EXAMPLE 3** Find the equation of the line passing through  $(1, 2)$  and making angle of  $30^\circ$  with  $y$ -axis.

[NCERT EXEMPLAR]

**SOLUTION** The required line makes  $30^\circ$  with the positive direction of  $y$ -axis as shown in Fig. 22.17. So, it makes  $60^\circ$  with the positive direction of  $x$ -axis. Therefore, its slope  $m$  is given by  $m = \tan 60^\circ = \sqrt{3}$ . Thus, the required line passes through  $(1, 2)$  and has  $\sqrt{3}$  as its slope. Hence, its equation is  $y - 2 = \sqrt{3}(x - 1)$  or  $\sqrt{3}x - y + 2 - \sqrt{3} = 0$

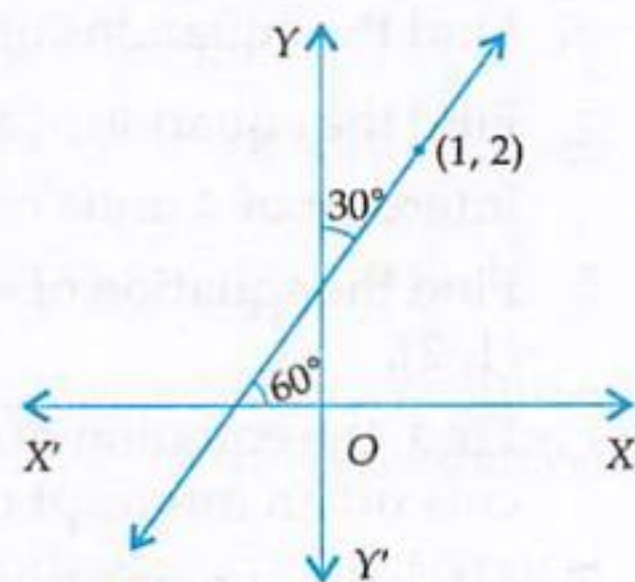


Fig. 22.17

**EXAMPLE 4** Find the equation of the perpendicular bisector of the line segment joining the points  $A(2, 3)$  and  $B(6, -5)$ .



**SOLUTION** The slope of  $AB$  is given by  $m = \frac{-5-3}{6-2} = -2$  [Using:  $m = \frac{y_2 - y_1}{x_2 - x_1}$ ]

$\therefore$  Slope of a line perpendicular to  $AB = -\frac{1}{m} = \frac{1}{2}$

Let  $P$  be the mid-point of  $AB$ . Then, the coordinates of  $P$  are  $\left(\frac{2+6}{2}, \frac{3-5}{2}\right)$  i.e.  $(4, -1)$ .

Thus, the required line passes through  $P(4, -1)$  and has slope  $\frac{1}{2}$ . So its equation is

$$y + 1 = \frac{1}{2}(x - 4) \Rightarrow x - 2y - 6 = 0. \quad [\text{Using: } y - y_1 = m(x - x_1)]$$

**EXAMPLE 5** Find the equation of the line for which  $\tan \theta = \frac{1}{2}$ , where  $\theta$  is the inclination of the line and

(i)  $x$ -intercept equal to 4. (ii)  $y$ -intercept is  $-\frac{3}{2}$ .

[NCERT]

**SOLUTION** (i) Clearly, the line passes through  $(4, 0)$  and has slope  $= \frac{1}{2}$ . So, its equation is

$$y - 0 = \frac{1}{2}(x - 4) \quad \left[ \text{Putting } x_1 = 4, y_1 = 0 \text{ and } m = \frac{1}{2} \text{ in } y - y_1 = m(x - x_1) \right]$$

$$\Rightarrow x - 2y - 4 = 0$$

(ii) The line passes through  $\left(0, -\frac{3}{2}\right)$  and has slope  $= \frac{1}{2}$ . So, its equation is

$$y - \left(-\frac{3}{2}\right) = \frac{1}{2}(x - 0) \quad \left[ \text{Putting } x_1 = 0, y_1 = -\frac{3}{2} \text{ and } m = \frac{1}{2} \text{ in } y - y_1 = m(x - x_1) \right]$$

$$\Rightarrow 2y + 3 = x \text{ or, } x - 2y - 3 = 0$$

**EXAMPLE 6** The perpendicular from the origin to a line meets it at the point  $(-2, 9)$ , find the equation of the line. [NCERT]

**SOLUTION** We find that:  $m_1 = \text{Slope of } OP = \frac{9-0}{-2-0} = -\frac{9}{2}$ .

Let  $m$  be the slope of the line  $AB$ . Since  $AB \perp OP$ .

$\therefore$  Slope of  $AB \times \text{Slope of } OP = -1$

$$\Rightarrow m \times -\frac{9}{2} = -1 \Rightarrow m = \frac{2}{9}$$

The equation of  $AB$  is

$$y - 9 = \frac{2}{9}\{x - (-2)\}$$

$$\Rightarrow 9y - 81 = 2x + 4 \Rightarrow 2x - 9y + 85 = 0$$

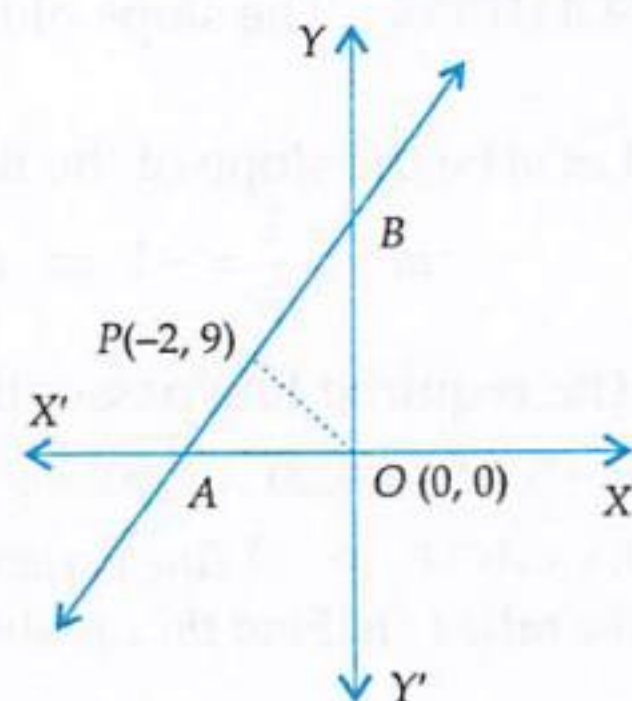


Fig. 22.18

**EXAMPLE 7** Find the equation of the line passing through the point  $(0, 2)$  making an angle  $\frac{2\pi}{3}$  with the positive  $x$ -axis. Also, find the equation of line parallel to it and crossing the  $y$ -axis at a distance of 2 units below the origin. [NCERT]

**SOLUTION** The equation of the line passing through  $(0, 2)$  and making an angle  $\frac{2\pi}{3}$  with the positive  $x$ -axis is

$$y - 2 = \tan \frac{2\pi}{3}(x - 0) \quad [\text{Using: } y - y_1 = m(x - x_1)]$$



$$\Rightarrow y - 2 = -\sqrt{3}x \Rightarrow \sqrt{3}x + y - 2 = 0$$

A line parallel to this line crosses  $y$ -axis at a distance of 2 units below the origin. So, it passes through  $(0, -2)$  and makes an angle  $\frac{2\pi}{3}$  with the  $x$ -axis. Hence, its equation is

$$y + 2 = \tan \frac{2\pi}{3} (x - 0) \Rightarrow y + 2 = -\sqrt{3}x \Rightarrow \sqrt{3}x + y + 2 = 0$$

#### BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 8** Two lines passing through the point  $(2, 3)$  intersect each other at an angle of  $60^\circ$ . If slope of one line is 2, find the equation of the other line. [NCERT]

**SOLUTION** Let the slope of the other line be  $m$ . It is given that the angle between the two lines is  $60^\circ$ .

$$\therefore \tan 60^\circ = \left| \frac{m - 2}{1 + 2m} \right|$$

$$\Rightarrow \sqrt{3} = \left| \frac{m - 2}{1 + 2m} \right| \Rightarrow \frac{m - 2}{1 + 2m} = \pm \sqrt{3} \Rightarrow m - 2 = \pm \sqrt{3} + 2\sqrt{3}m \Rightarrow m(1 - 2\sqrt{3}) = 2 \pm \sqrt{3}$$

$$\Rightarrow m = \frac{2 \pm \sqrt{3}}{1 - 2\sqrt{3}} \Rightarrow m = \frac{2 + \sqrt{3}}{1 - 2\sqrt{3}}, \frac{2 - \sqrt{3}}{1 + 2\sqrt{3}} \Rightarrow m = -\frac{2 + \sqrt{3}}{2\sqrt{3} - 1}, \frac{2 - \sqrt{3}}{2\sqrt{3} + 1}$$

Substituting  $x_1 = 2, y_1 = 3$  and the values of  $m$  in  $y - y_1 = m(x - x_1)$ , we obtain that the equations of the required lines are

$$y - 3 = -\frac{2 + \sqrt{3}}{2\sqrt{3} - 1}(x - 2) \text{ and } y - 3 = \frac{2 - \sqrt{3}}{2\sqrt{3} + 1}(x - 2)$$

**EXAMPLE 9** Find the equation of the line passing through  $(-3, 5)$  and perpendicular to the line through the points  $(2, 5)$  and  $(-3, 6)$ . [NCERT]

**SOLUTION** The slope of the line passing through  $(2, 5)$  and  $(-3, 6)$  is  $\frac{6 - 5}{-3 - 2} = \frac{-1}{5}$

Let  $m$  be the slope of the line perpendicular to the line passing through  $(2, 5)$  and  $(-3, 6)$ . Then,

$$m \times -\frac{1}{5} = -1 \Rightarrow m = 5$$

The required line passes through  $(-3, 5)$  and has slope  $m = 5$ . So, its equation is

$$y - 5 = 5(x + 3) \text{ or, } 5x - y + 20 = 0$$

**EXAMPLE 10** A line perpendicular to the line segment joining the points  $(1, 0)$  and  $(2, 3)$  divides it in the ratio  $1 : n$ . Find the equation of the line. [NCERT]

**SOLUTION** The slope of the line joining  $A(1, 0)$  and  $B(2, 3)$  is  $\frac{3 - 0}{2 - 1} = 3$  and the coordinates of

the point dividing it in the ratio  $1 : n$  are  $\left( \frac{n + 2}{n + 1}, \frac{3}{n + 1} \right)$ . The slope of the line perpendicular to the

line segment  $AB$  is  $-\frac{1}{3}$ . Hence, the equation of the required line is

$$y - \frac{3}{n + 1} = -\frac{1}{3} \left( x - \frac{n + 2}{n + 1} \right) \text{ or, } (n + 1)x + 3(n + 1)y = n + 11$$

**EXAMPLE 11** Find the equation of a line which divides the join of  $(1, 0)$  and  $(3, 0)$  in the ratio  $2 : 1$  and perpendicular to it.



**SOLUTION** Let  $C$  be the point which divides the join of  $A(1, 0)$  and  $B(3, 0)$  in the ratio  $2:1$ . Then, the coordinates of  $C$  are

$$\left( \frac{2 \times 3 + 1 \times 1}{2 + 1}, \frac{2 \times 0 + 1 \times 0}{2 + 1} \right) = \left( \frac{7}{3}, 0 \right)$$

Since  $AB$  is along  $x$ -axis, therefore a line perpendicular to  $AB$  is parallel to  $y$ -axis. As it passes through  $C(7/3, 0)$ , therefore its equation is  $x = \frac{7}{3}$  or,  $3x = 7$

Hence, the equation of the required line is  $3x = 7$ .

**EXAMPLE 12** The vertices of a triangle are  $A(10, 4)$ ,  $B(-4, 9)$  and  $C(-2, -1)$ . Find the equation of its altitudes. Also, find its orthocentre.

**SOLUTION** Let  $AD$ ,  $BE$  and  $CF$  be three altitudes of  $\triangle ABC$ . Clearly,  $AD \perp BC$ ,  $BE \perp CA$  and  $CF \perp AB$ . We find that

$$\text{Slope of } BC = \frac{-1 - 9}{-2 + 4} = -5 \Rightarrow \text{Slope of } AD = \frac{1}{5} [\because AD \perp BC]$$

Since  $AD$  passes through  $A(10, 4)$ . Therefore, equation of  $AD$  is

$$y - 4 = \frac{1}{5}(x - 10) \Rightarrow x - 5y + 10 = 0 \quad \dots(i)$$

$$\text{Slope of } AC = \frac{4 + 1}{10 + 2} = \frac{5}{12} \Rightarrow \text{Slope of } BE = -\frac{12}{5} [\because BE \perp AC]$$

Clearly,  $BE$  passes through  $B(-4, 9)$  and has slope  $-12/5$ . So, the equation of  $BE$  is

$$y - 9 = -\frac{12}{5}(x + 4) \Rightarrow 12x + 5y + 3 = 0 \quad \dots(ii)$$

$$\text{Slope of } AB = \frac{4 - 9}{10 + 4} = -\frac{5}{14} \Rightarrow \text{Slope of } CF = \frac{14}{5} \quad [\because CF \perp AB]$$

Clearly,  $CF$  passes through  $C(-2, -1)$  and has slope  $\frac{14}{5}$ . So, the equation of  $CF$  is

$$y + 1 = \frac{14}{5}(x + 2) \Rightarrow 14x - 5y + 23 = 0 \quad \dots(iii)$$

Thus, the altitudes of  $\triangle ABC$  are

$$x - 5y + 10 = 0, 12x + 5y + 3 = 0 \text{ and } 14x - 5y + 23 = 0.$$

The orthocentre of  $\triangle ABC$  is the point of intersection of its altitudes. Solving (i) and (ii) by cross-multiplication, we get

$$\frac{x}{-65} = \frac{y}{117} = \frac{1}{65} \Rightarrow x = -1, y = \frac{9}{5}$$

Hence, the coordinates of the orthocentre are  $(-1, 9/5)$ .

**EXAMPLE 13** Find the equations of the altitudes of the triangle whose vertices are  $A(7, -1)$ ,  $B(-2, 8)$  and  $C(1, 2)$ .

**SOLUTION** Let  $AD$ ,  $BE$  and  $CF$  be three altitudes of triangle  $ABC$ . Let  $m_1, m_2$  and  $m_3$  be the slopes of  $AD, BE$  and  $CF$  respectively. Then,

$$AD \perp BC \Rightarrow \text{Slope of } AD \times \text{Slope of } BC = -1$$

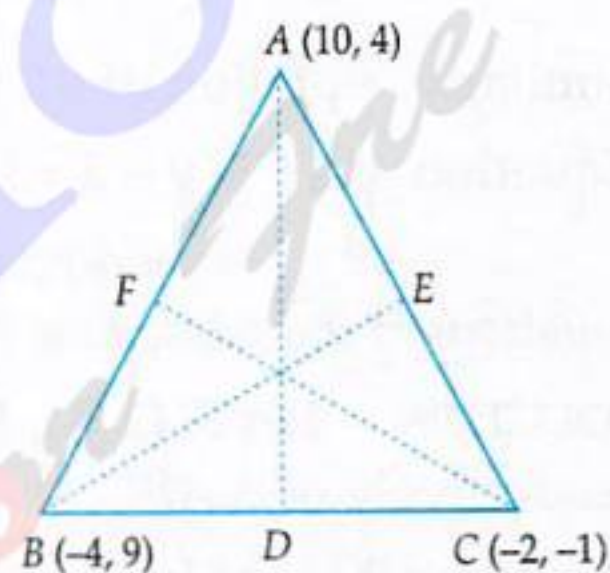


Fig. 22.19



$$\Rightarrow m_1 \times \left( \frac{2-8}{1+2} \right) = -1 \Rightarrow m_1 = \frac{1}{2}$$

$$BE \perp AC \Rightarrow \text{Slope of } BE \times \text{Slope of } AC = -1$$

$$\Rightarrow m_2 \times \left( \frac{-1-2}{7-1} \right) = -1 \Rightarrow m_2 = 2$$

$$CF \perp AB \Rightarrow \text{Slope of } CF \times \text{Slope of } AB = -1$$

$$\Rightarrow m_3 \times \frac{-1-8}{7+2} = -1 \Rightarrow m_3 = 1.$$

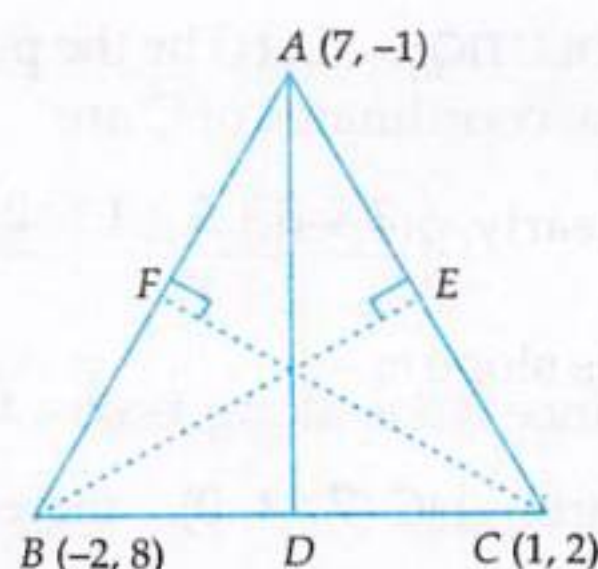


Fig. 22.20

Since  $AD$  passes through  $A(7, -1)$  and has slope  $m_1 = \frac{1}{2}$ . So, its equation is

$$y + 1 = \frac{1}{2}(x - 7) \Rightarrow x - 2y - 9 = 0$$

Similarly, equation of  $BE$  is  $y - 8 = 2(x + 2)$  or,  $2x - y + 12 = 0$ .

Equation of  $CF$  is  $y - 2 = 1(x - 1)$  or,  $x - y + 1 = 0$ .

**EXAMPLE 14** The mid-points of the sides of a triangle are  $(2, 1)$ ,  $(-5, 7)$  and  $(-5, -5)$ . Find the equations of the sides of the triangle.

**SOLUTION** Let  $D(2, 1)$ ,  $E(-5, 7)$  and  $F(-5, -5)$  be the mid-points of sides  $BC$ ,  $CA$  and  $AB$  respectively of  $\triangle ABC$ .

We know that the line joining the mid-points of two sides of a triangle is parallel to the third two side. Therefore,  $DE \parallel AB$ ,  $EF \parallel BC$  and  $DF \parallel AC$ .

$\therefore$  Slope of  $AB$  = Slope of  $DE$

Slope of  $BC$  = Slope of  $EF$  and, Slope of  $AC$  = Slope of  $DF$

Let  $m_1$ ,  $m_2$  and  $m_3$  be the slopes of  $AB$ ,  $BC$  and  $CA$  respectively. Then,

$$m_1 = \text{Slope of } AB = \text{Slope of } DE = \frac{7-1}{-5-2} = \frac{-6}{7}$$

$$m_2 = \text{Slope of } BC = \text{Slope of } EF = \frac{7+5}{-5+5} \quad (\text{Undefined})$$

$$m_3 = \text{Slope of } CA = \text{Slope of } DF = \frac{1+5}{2+5} = \frac{6}{7}$$

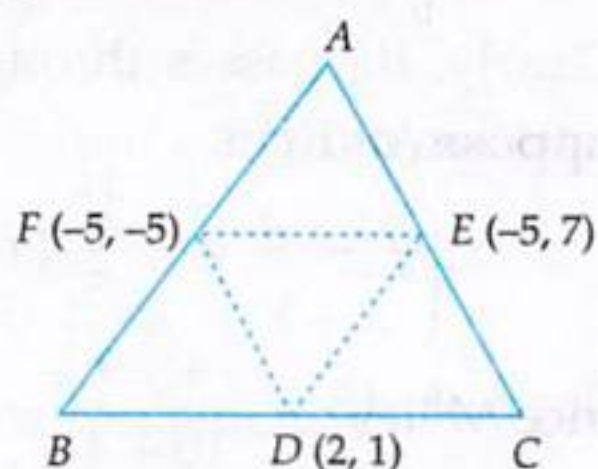


Fig. 22.21

Side  $AB$  passes through  $F(-5, -5)$  and has slope  $m_1 = \frac{-6}{7}$ . So, its equation is

$$y + 5 = \frac{-6}{7}(x + 5) \text{ or, } 6x + 7y + 65 = 0$$

Side  $BC$  is parallel to  $Y$ -axis and passes through  $D(2, 1)$ . So, its equation is  $x = k$ . As it passes through  $(2, 1)$ . Therefore,  $2 = k$ . Hence, equation of  $BC$  is  $x = 2$ .

Side  $CA$  passes through  $E(-5, 7)$  and has slope  $m_3 = \frac{6}{7}$ . So, its equation is

$$y - 7 = \frac{6}{7}(x + 5) \text{ or, } 6x - 7y + 79 = 0$$

**EXAMPLE 15** Find the equation of the perpendicular bisector of the line segment joining the points  $(1, 1)$  and  $(2, 3)$ . [NCERT]

**SOLUTION** Let  $P$  be the mid-point of the line segment joining points  $A(1, 1)$  and  $B(2, 3)$ . Then, the coordinates of  $P$  are  $\left(\frac{3}{2}, 2\right)$ . Let  $m$  be the slope of perpendicular bisector of  $AB$ . Then,



$$m \times \text{Slope of } AB = -1 \Rightarrow m \times \frac{3-1}{2-1} = -1 \Rightarrow m = -\frac{1}{2}$$

Clearly, perpendicular bisector of  $AB$  passes through  $P\left(\frac{3}{2}, 2\right)$  and

has slope  $m = -\frac{1}{2}$ . So, its equation is

$$y - 2 = -\frac{1}{2}\left(x - \frac{3}{2}\right) \text{ or, } 2x + 4y - 11 = 0.$$

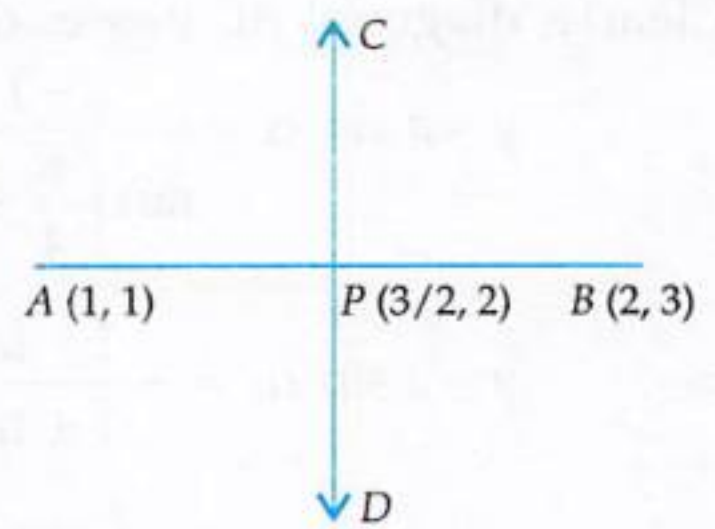


Fig. 22.22

**EXAMPLE 16** Show that the perpendicular drawn from the point  $(4, 1)$  on the line segment joining  $(6, 5)$  and  $(2, -1)$  divides it internally in the ratio  $8:5$ .

**SOLUTION** Suppose perpendicular drawn from  $P(4, 1)$  on the line joining  $A(6, 5)$  and  $B(2, -1)$  meets  $AB$  at  $M$ . Let  $m$  be the slope of  $PM$ . Then,

$$PM \perp AB$$

$$\Rightarrow m \times \text{Slope of } AB = -1$$

$$\Rightarrow m \times \frac{-1-5}{2-6} = -1 \Rightarrow m \times \frac{3}{2} = -1 \Rightarrow m = -\frac{2}{3}$$

Clearly,  $PM$  passes through  $P(4, 1)$  and has slope  $m = -\frac{2}{3}$ . So, its

equation is

$$y - 1 = -\frac{2}{3}(x - 4) \text{ or, } 2x + 3y - 11 = 0 \quad \dots(i)$$

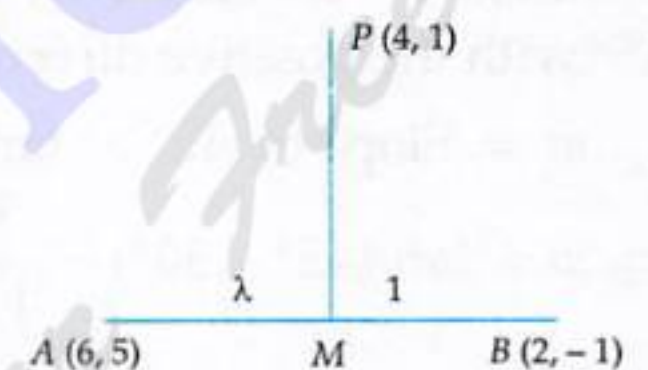


Fig. 22.23

Suppose  $M$  divides line segment  $AB$  in the ratio  $\lambda:1$ . Then, coordinates of  $M$  are

$$\left(\frac{2\lambda + 6}{\lambda + 1}, \frac{-\lambda + 5}{\lambda + 1}\right)$$

Since  $M$  lies on line  $PM$  whose equation is  $2x + 3y - 11 = 0$ .

$$\therefore 2\left(\frac{2\lambda + 6}{\lambda + 1}\right) + 3\left(\frac{-\lambda + 5}{\lambda + 1}\right) - 11 = 0$$

$$\Rightarrow 4\lambda + 12 - 3\lambda + 15 - 11\lambda - 11 = 0 \Rightarrow -10\lambda + 16 = 0 \Rightarrow \lambda = \frac{8}{5}$$

Hence,  $M$  divides  $AB$  internally in the ratio  $8:5$ .

#### BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

**EXAMPLE 17** One side of a square makes an angle  $\alpha$  with  $x$ -axis and one vertex of the square is at the origin. Prove that the equations of its diagonals are  $x(\sin \alpha + \cos \alpha) = y(\cos \alpha - \sin \alpha)$  and  $x(\cos \alpha - \sin \alpha) + y(\sin \alpha + \cos \alpha) = a$ , where  $a$  is the length of the side of the square.

**SOLUTION** Let  $OABC$  be the square such that its side  $OA$  makes an angle  $\alpha$  with  $x$ -axis. Since  $OA = a$ , therefore coordinates of  $A$  are  $(a \cos \alpha, a \sin \alpha)$ . Clearly, the diagonal  $OB$  makes an angle  $(\pi/4 + \alpha)$  with  $x$ -axis and passes through  $(0, 0)$ . So, equation of  $OB$  is

$$y - 0 = \tan\left(\frac{\pi}{4} + \alpha\right)(x - 0) \text{ or, } y = \frac{1 + \tan \alpha}{1 - \tan \alpha} x$$

$$\Rightarrow y(\cos \alpha - \sin \alpha) = x(\cos \alpha + \sin \alpha) \quad \dots(i)$$

Since the diagonal  $AC$  is perpendicular to  $OB$ . Therefore, Slope of  $AC = \frac{-1}{\tan\left(\frac{\pi}{4} + \alpha\right)}.$



Clearly, diagonal AC passes through  $(a \cos \alpha, a \sin \alpha)$ . So, equation of AC is

$$y - a \sin \alpha = \frac{-1}{\tan \left( \frac{\pi}{4} + \alpha \right)} (x - a \cos \alpha)$$

$$\Rightarrow y - a \sin \alpha = -\frac{1 - \tan \alpha}{1 + \tan \alpha} (x - a \cos \alpha)$$

$$\Rightarrow y - a \sin \alpha = -\left( \frac{\cos \alpha - \sin \alpha}{\cos \alpha + \sin \alpha} \right) (x - a \cos \alpha)$$

$$\Rightarrow x(\cos \alpha - \sin \alpha) + y(\cos \alpha + \sin \alpha) = a.$$

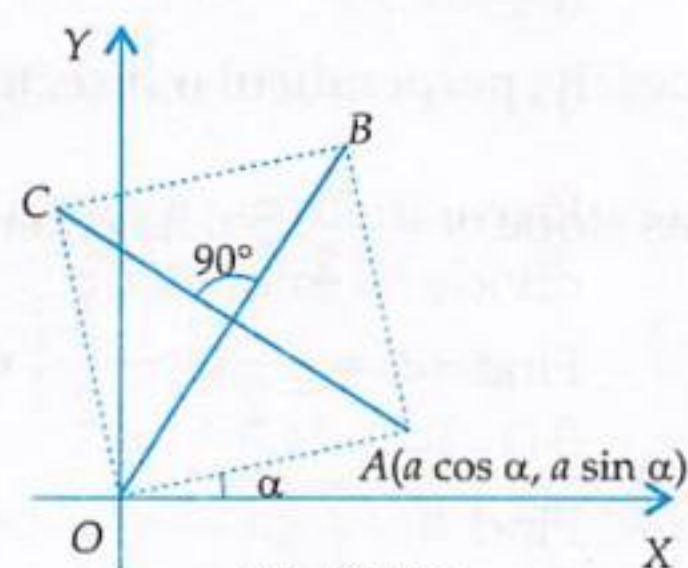


Fig. 22.24

**EXAMPLE 18** A line passing through the point  $A(3, 0)$  makes  $30^\circ$  angle with the positive direction of  $x$ -axis. If this line is rotated through an angle of  $15^\circ$  in clockwise direction, find its equation in new position. [NCERT EXEMPLAR]

**SOLUTION** Let AB be the given line and AC be its new position. Clearly, AC makes an angle of  $15^\circ$  with the positive direction of  $X$ -axis.

$$\therefore m = \text{Slope of AC} = \tan 15^\circ$$

$$\Rightarrow m = \tan(45^\circ - 30^\circ) = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}$$

$$\Rightarrow m = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{(\sqrt{3} - 1)^2}{(\sqrt{3} + 1)(\sqrt{3} - 1)} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$

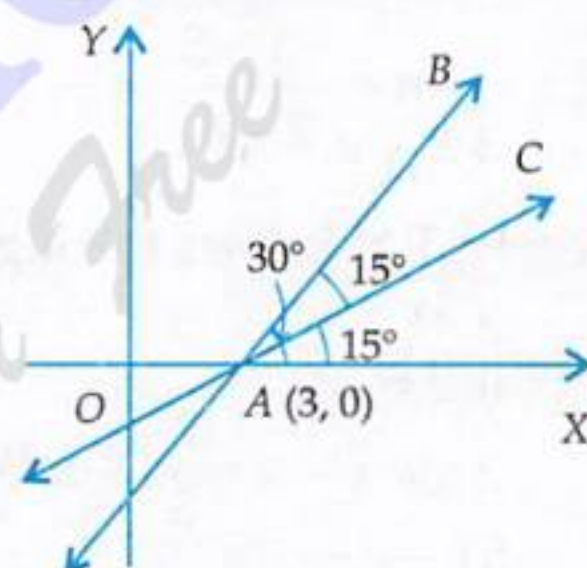


Fig. 22.25

Clearly, AC passes through  $A(3, 0)$  and has slope  $m = 2 - \sqrt{3}$ . So, its equation is

$$y - 0 = (2 - \sqrt{3})(x - 3) \text{ or, } (2 - \sqrt{3})x - y - 3(2 - \sqrt{3}) = 0$$

### EXERCISE 22.4

#### BASIC

- Find the equation of the straight line passing through the point  $(6, 2)$  and having slope  $-3$ .
- Find the equation of the straight line passing through  $(-2, 3)$  and inclined at an angle of  $45^\circ$  with the  $x$ -axis.
- Find the equation of the line passing through  $(0, 0)$  with slope  $m$ . [NCERT]
- Find the equation of the line passing through  $(2, 2\sqrt{3})$  and inclined with  $x$ -axis at an angle of  $75^\circ$ . [NCERT]
- Find the equation of the straight line which passes through the point  $(1, 2)$  and makes such an angle with the positive direction of  $x$ -axis whose sine is  $\frac{3}{5}$ .
- Find the equation of the straight line passing through  $(3, -2)$  and making an angle of  $60^\circ$  with the positive direction of  $y$ -axis.
- Find the lines through the point  $(0, 2)$  making angles  $\frac{\pi}{3}$  and  $\frac{2\pi}{3}$  with the  $x$ -axis. Also, find the lines parallel to them cutting the  $y$ -axis at a distance of 2 units below the origin.
- Find the equations of the straight lines which cut off an intercept 5 from the  $y$ -axis and are equally inclined to the axes.
- Find the equation of the line which intercepts a length 2 on the positive direction of the  $x$ -axis and is inclined at an angle of  $135^\circ$  with the positive direction of  $y$ -axis.



10. Find the equation of the straight line which divides the join of the points (2, 3) and (−5, 8) in the ratio 3 : 4 and is also perpendicular to it.

### BASED ON LOTS

11. Prove that the perpendicular drawn from the point (4, 1) on the join of (2, −1) and (6, 5) divides it in the ratio 5 : 8.
12. Find the equations to the altitudes of the triangle whose angular points are A (2, −2), B (1, 1) and C (−1, 0).
13. Find the equation of the right bisector of the line segment joining the points (3, 4) and (−1, 2). [NCERT]
14. Find the equation of the line passing through the point (−3, 5) and perpendicular to the line joining (2, 5) and (−3, 6).
15. Find the equation of the right bisector of the line segment joining the points A (1, 0) and B (2, 3).

### ANSWERS

- |                                                                                                 |                      |                                        |
|-------------------------------------------------------------------------------------------------|----------------------|----------------------------------------|
| 1. $3x + y - 20 = 0$                                                                            | 2. $x - y + 5 = 0$   | 3. $y = mx$                            |
| 4. $(2 + \sqrt{3})x - y - 4 = 0$                                                                | 5. $3x - 4y + 5 = 0$ | 6. $x - \sqrt{3}y - 3 - 2\sqrt{3} = 0$ |
| 7. $\sqrt{3}x - y + 2 = 0, \sqrt{3}x + y - 2 = 0, \sqrt{3}x + y + 2 = 0, \sqrt{3}x - y - 2 = 0$ |                      |                                        |
| 8. $y = x + 5$ or $x + y = 5$                                                                   | 9. $x - y - 2 = 0$   | 10. $49x - 35y + 229 = 0$              |
| 12. $2x + y - 2 = 0, 3x - 2y - 1 = 0, x - 3y + 1 = 0$                                           |                      | 13. $2x + y = 5$                       |
| 14. $5x - y + 20 = 0$                                                                           | 15. $x + 3y - 6 = 0$ |                                        |

### HINTS TO SELECTED PROBLEMS

3. The equation of the line passing through (0, 0) and slope  $m$  is  $y - 0 = m(x - 0)$  or,  $y = mx$ .
4. The equation of the required line is

$$y - 2\sqrt{3} = \tan 75^\circ(x - 2)$$

$$\text{or, } y - 2\sqrt{3} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}(x - 2) \quad \left[ \because \tan 75^\circ = \tan (45^\circ + 30^\circ) = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} \right]$$

$$\text{or, } y - 2\sqrt{3} = (2 + \sqrt{3})(x - 2)$$

13. The right bisector of the segment joining A (3, 4) and B (−1, 2) passes through the mid-point C (1, 3) of AB and is perpendicular to AB. Let  $m$  be the slope of AB. Then,

$$m = \frac{2 - 4}{-1 - 3} = \frac{1}{2}$$

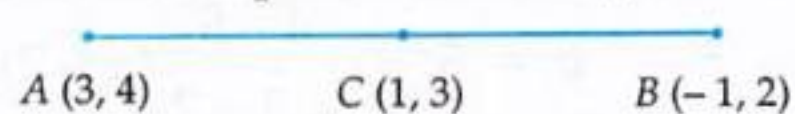


Fig. 22.26

So,  $m_1 = \text{Slope of a line perpendicular to } AB = -2$ .

Hence, the equation of the right bisector of AB is  $y - 3 = -2(x - 1)$  or,  $2x + y - 5 = 0$ .

### 22.6.3 TWO-POINT FORM OF A LINE

**THEOREM** The equation of a line passing through two points  $(x_1, y_1)$  and  $(x_2, y_2)$  is

$$y - y_1 = \left( \frac{y_2 - y_1}{x_2 - x_1} \right) (x - x_1).$$

**PROOF** Let  $m$  be the slope of the line passing through  $(x_1, y_1)$  and  $(x_2, y_2)$ . Then,



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

So, the equation of the line is

$$y - y_1 = m(x - x_1)$$

[Using point-slope form]

Substituting the value of  $m$ , we obtain

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

This is the required equation of the line in two point form.

**Q.E.D.**

### ILLUSTRATIVE EXAMPLES

#### BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** Find the equation of the line joining the points  $(-1, 3)$  and  $(4, -2)$ .

**SOLUTION** Here, the two points are  $(x_1, y_1) \equiv (-1, 3)$  and  $(x_2, y_2) \equiv (4, -2)$ .

So, the equation of the line in two-point form is

$$y - 3 = \frac{3 - (-2)}{-1 - 4} (x + 1) \Rightarrow y - 3 = -x - 1 \Rightarrow x + y - 2 = 0.$$

**EXAMPLE 2** Find the equation of the line joining the points  $(at_1^2, 2at_1)$  and  $(at_2^2, 2at_2)$ .

**SOLUTION** Here,  $x_1 = at_1^2$ ,  $y_1 = 2at_1$ ,  $x_2 = at_2^2$ ,  $y_2 = 2at_2$ . So, the equation of the required line is

$$y - 2at_1 = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2} (x - at_1^2)$$

$$\Rightarrow y - 2at_1 = \frac{2}{t_1 + t_2} (x - at_1^2)$$

$$\Rightarrow y(t_1 + t_2) - 2at_1^2 - 2at_1 t_2 = 2x - 2at_1^2 \Rightarrow y(t_1 + t_2) = 2x + 2at_1 t_2.$$

**EXAMPLE 3** Find the equations of the medians of the triangle  $ABC$  whose vertices are  $A(2, 5)$ ,  $B(-4, 9)$  and  $C(-2, -1)$ .

**SOLUTION** Let  $D, E, F$  be the mid-points of  $BC, CA$  and  $AB$  respectively. Then, the coordinates of these points are  $D(-3, 4)$ ,  $E(0, 2)$  and  $F(-1, 7)$  respectively. The median  $AD$  passes through points  $A(2, 5)$  and  $D(-3, 4)$ .

So, equation of  $AD$  is

$$y - 5 = \frac{4 - 5}{-3 - 2} (x - 2)$$

$$\Rightarrow y - 5 = \frac{1}{5} (x - 2) \Rightarrow x - 5y + 23 = 0$$

The median  $BE$  passes through points  $B(-4, 9)$  and  $E(0, 2)$ .

So, equation of median  $BE$  is

$$(y - 9) = \left( \frac{2 - 9}{0 + 4} \right) (x + 4) \Rightarrow 7x + 4y - 8 = 0.$$

Similarly, the equation of the median  $CF$  is

$$(y + 1) = \frac{7 + 1}{-1 + 2} (x + 2) \Rightarrow 8x - y + 15 = 0.$$

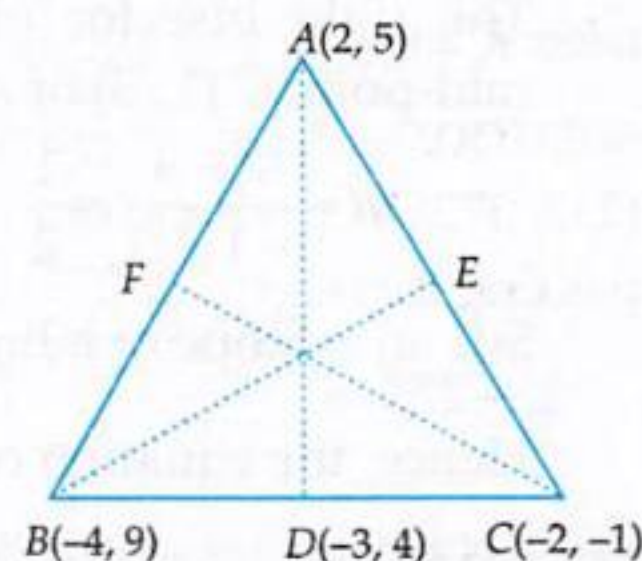


Fig. 22.27



### BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 4** In what ratio is the line joining the points (2, 3) and (4, 1) divides the segment joining the points (1, 2) and (4, 3)?

**SOLUTION** The equation of the line joining the points (2, 3) and (4, 1) is

$$y - 3 = \frac{1-3}{4-2} (x - 2) \Rightarrow y - 3 = -x + 2 \Rightarrow x + y - 5 = 0 \quad \dots(i)$$

Suppose the line joining (2, 3) and (4, 1) divides the segment joining (1, 2) and (4, 3) at point P in the ratio  $\lambda : 1$ . Then, the coordinates of P are  $\left( \frac{4\lambda + 1}{\lambda + 1}, \frac{3\lambda + 2}{\lambda + 1} \right)$ . Clearly P lies on line (i).

$$\therefore \frac{4\lambda + 1}{\lambda + 1} + \frac{3\lambda + 2}{\lambda + 1} - 5 = 0 \Rightarrow \lambda = 1. \text{ Hence, the required ratio is } \lambda : 1 \text{ i.e., } 1 : 1.$$

**EXAMPLE 5** In what ratio, the line joining (-1, 1) and (5, 7) is divided by the line  $x + y = 4$ ?

**SOLUTION** Suppose the line  $x + y = 4$  divides the join of A (-1, 1) and B (5, 7) in the ratio  $\lambda : 1$ .

The coordinates of the point of division are  $\left( \frac{5\lambda + 1}{\lambda + 1}, \frac{7\lambda + 1}{\lambda + 1} \right)$ . It lies on  $x + y = 4$ .

$$\therefore \frac{5\lambda + 1}{\lambda + 1} + \frac{7\lambda + 1}{\lambda + 1} = 4 \Rightarrow 5\lambda + 1 + 7\lambda + 1 = 4(\lambda + 1) \Rightarrow 12\lambda = 4\lambda + 4 \Rightarrow 8\lambda = 4 \Rightarrow \lambda = \frac{1}{2}$$

Hence, the required ratio are 1 : 2.

**EXAMPLE 6** Prove that the points (5, 1), (1, -1) and (11, 4) are collinear. Also find the equation of the straight line on which these points lie.

**SOLUTION** Let the given points be A (5, 1), B (1, -1) and C (11, 4). Then, the equation of the line passing through A and B is

$$y - 1 = \frac{-1 - 1}{1 - 5} (x - 5) \Rightarrow x - 2y - 3 = 0$$

Clearly, point C (11, 4) satisfies the equation  $x - 2y - 3 = 0$ . Hence, the given points lie on the same straight line, whose equation is  $x - 2y - 3 = 0$ .

**EXAMPLE 7** The Fahrenheit temperature  $F$  and absolute temperature  $K$  satisfy a linear equation. Given that  $K = 273$  when  $F = 32$  and that  $K = 373$  when  $F = 212$ . Express  $K$  in terms of  $F$  and find the value of  $F$ , when  $K = 0$ . [NCERT]

**SOLUTION** Assuming  $F$  along X-axis and  $K$  along Y-axis, we have two points (32, 273) and (212, 373) in  $xy$ -plane or  $FK$ -plane. As  $F$  and  $K$  satisfy a linear equation. The equation of the line passing through (32, 273) and (212, 373) is

$$K - 273 = \frac{373 - 273}{212 - 32} (F - 32) \Rightarrow K - 273 = \frac{100}{180} (F - 32) \Rightarrow K = \frac{5}{9} (F - 32) + 273 \quad \dots(i)$$

Putting  $K = 0$  in (i), we obtain

$$0 = \frac{5}{9} (F - 32) + 273 \Rightarrow F - 32 = -\frac{273 \times 9}{5} \Rightarrow F = 32 - 491.4 \Rightarrow F = -459.4$$

### BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

**EXAMPLE 8** Find the equation of the internal bisector of angle BAC of the triangle ABC whose vertices A, B, C are (5, 2), (2, 3) and (6, 5) respectively.



**SOLUTION** We have,  $AB = \sqrt{(5-2)^2 + (2-3)^2} = \sqrt{10}$

and,  $AC = \sqrt{(5-6)^2 + (2-5)^2} = \sqrt{10}$

$\therefore AB:AC = \sqrt{10}:\sqrt{10} = 1:1$

The internal bisector  $AD$  of  $\angle BAC$  divides  $BC$  in the ratio  $AB:AC$

i.e.  $1:1$ . So, coordinates of  $D$  are

$$\left( \frac{2+6}{2}, \frac{3+5}{2} \right) = (4, 4).$$

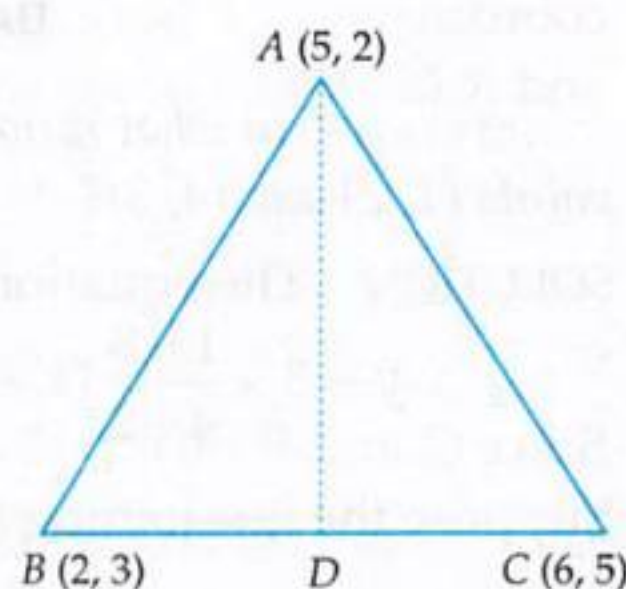


Fig. 22.28

Equation of  $AD$  is  $y - 2 = \frac{4-2}{4-5}(x-5)$  or,  $2x + y - 12 = 0$

**EXAMPLE 9** A rectangle has two opposite vertices at the points  $(1, 2)$  and  $(5, 5)$ . If the other vertices lie on the line  $x = 3$ , find the equations of the sides of the rectangle.

**SOLUTION** Let  $ABCD$  be a rectangle whose two opposite vertices are  $A(1, 2)$  and  $C(5, 5)$ .

Let the coordinates of other two vertices  $B$  and  $D$  of rectangle  $ABCD$  be  $B(3, y_1)$  and  $D(3, y_2)$ .

Since diagonals  $AC$  and  $BD$  bisect each other. Therefore, mid-points of  $AC$  and  $BD$  are same.

$$\therefore \frac{y_1 + y_2}{2} = \frac{2+5}{2} \Rightarrow y_1 + y_2 = 7 \quad \dots(i)$$

Since  $ABCD$  is a rectangle.

$$\therefore AC = BD$$

$$\Rightarrow AC^2 = BD^2$$

$$\Rightarrow (1-5)^2 + (2-5)^2 = (3-3)^2 + (y_1 - y_2)^2$$

$$\Rightarrow 16 + 9 = (y_1 - y_2)^2$$

$$\Rightarrow y_1 - y_2 = \pm 5 \quad \dots(ii)$$

Solving (i) and (ii), we get

$$y_1 = 6 \text{ and } y_2 = 1 \text{ or, } y_1 = 1 \text{ and } y_2 = 6$$

Thus, the coordinates of  $B$  and  $D$  are  $B(3, 1)$  and  $D(3, 6)$ .

The equation of side  $AB$  is:  $y - 2 = \frac{1-2}{3-1}(x-1)$  or,  $y - 2 = -\frac{1}{2}(x-1)$  or,  $x + 2y - 5 = 0$

The equation of side  $BC$  is:  $y - 1 = \frac{5-1}{5-3}(x-3)$  or,  $y - 1 = 2(x-3)$  or,  $2x - y - 5 = 0$

The equation of side  $CD$  is:  $y - 5 = \frac{6-5}{3-5}(x-5)$  or,  $y - 5 = -\frac{1}{2}(x-5)$  or,  $x + 2y - 15 = 0$

The equation of side  $AD$  is:  $y - 2 = \frac{6-2}{3-1}(x-1)$  or,  $y - 2 = 2(x-1)$  or,  $2x - y = 0$

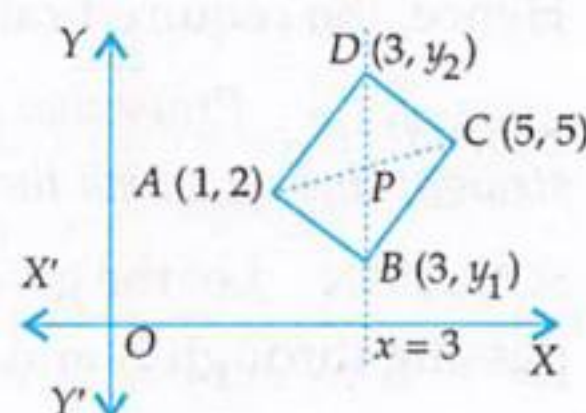


Fig. 22.29

**EXAMPLE 10** Find the coordinates of the vertices of a square inscribed in the triangle with vertices  $A(0, 0)$ ,  $B(2, 1)$  and  $C(3, 0)$ ; given that two of its vertices are on the side  $AC$ .

**SOLUTION** Let  $PQRS$  be the square inscribed in the triangle  $ABC$  such that its vertices  $P$  and  $S$  lie on side  $AC$  which is along  $X$ -axis. Let the length of each side of the square be  $l$  and the



coordinates of  $P$  be  $(a, 0)$ . Then, the coordinates of other vertices are  $P(a, 0)$ ,  $S(a + l, 0)$ ,  $Q(a, l)$  and  $R(a + l, l)$ . The equations of sides  $AB$  and  $BC$  are

$$y - 0 = \frac{1 - 0}{2 - 0}(x - 0) \text{ and } y - 0 = \frac{1 - 0}{2 - 3}(x - 3) \text{ respectively}$$

or,  $x - 2y = 0$  and  $x + y - 3 = 0$  respectively.

Since  $Q$  and  $R$  lie on  $AB$  and  $BC$  respectively.

$$\therefore a - 2l = 0 \text{ and } a + l + l - 3 = 0$$

$$\Rightarrow a = 2l \text{ and } a + 2l - 3 = 0$$

$$\Rightarrow a = \frac{3}{2} \text{ and } l = \frac{3}{4}$$

Hence, the coordinates of the vertices of the square are

$$P(3/2, 0), Q(3/2, 3/4), R(9/4, 3/4) \text{ and } S(9/4, 0)$$

**EXAMPLE 11** A line is such that its segment between the lines  $5x - y + 4 = 0$  and  $3x + 4y - 4 = 0$  is bisected at the point  $(1, 5)$ . Obtain its equation. [NCERT]

**SOLUTION** Suppose the required line intersects the lines  $5x - y + 4 = 0$  and  $3x + 4y - 4 = 0$  at  $P(x_1, y_1)$  and  $Q(x_2, y_2)$  respectively. Clearly,  $P(x_1, y_1)$  lies on  $5x - y + 4 = 0$  and  $Q(x_2, y_2)$  lies on  $3x + 4y - 4 = 0$ .

$$\therefore 5x_1 - y_1 + 4 = 0 \text{ and } 3x_2 + 4y_2 - 4 = 0 \Rightarrow y_1 = 5x_1 + 4 \text{ and } y_2 = \frac{4 - 3x_2}{4} \dots(i)$$

Since  $R$  is the mid-point of  $PQ$ . Therefore,

$$\frac{x_1 + x_2}{2} = 1 \text{ and } \frac{y_1 + y_2}{2} = 5$$

$$\Rightarrow x_1 + x_2 = 2 \text{ and } y_1 + y_2 = 10$$

$$\Rightarrow x_1 + x_2 = 2 \text{ and } 5x_1 + 4 + \frac{4 - 3x_2}{4} = 10 \quad [\text{Using (i)}]$$

$$\Rightarrow x_1 + x_2 = 2 \text{ and } 20x_1 - 3x_2 = 20$$

$$\text{Solving these two equations, we obtain: } x_1 = \frac{26}{23} \text{ and } x_2 = \frac{20}{23}.$$

$$\text{Substituting these values in (i), we get: } y_1 = \frac{222}{23} \text{ and } y_2 = \frac{8}{23}.$$

Thus, the coordinates of  $P$  and  $Q$  are  $\left(\frac{26}{23}, \frac{222}{23}\right)$  and  $\left(\frac{20}{23}, \frac{8}{23}\right)$  respectively. Hence, the equation of  $PQ$  is

$$y - \frac{222}{23} = \frac{\frac{8}{23} - \frac{222}{23}}{\frac{20}{23} - \frac{26}{23}} \left(x - \frac{26}{23}\right)$$

$$\Rightarrow 23y - 222 = \frac{-214}{-6} (23x - 26) \Rightarrow 23y - 222 = \frac{107}{3} (23x - 26) \Rightarrow 107x - 3y - 92 = 0.$$

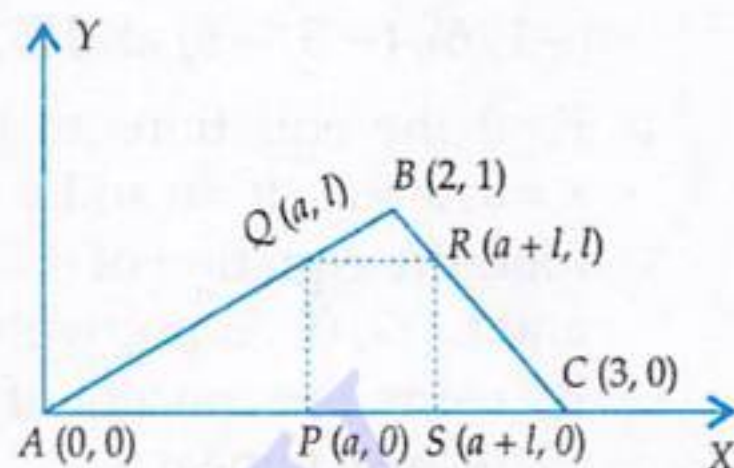


Fig. 22.30

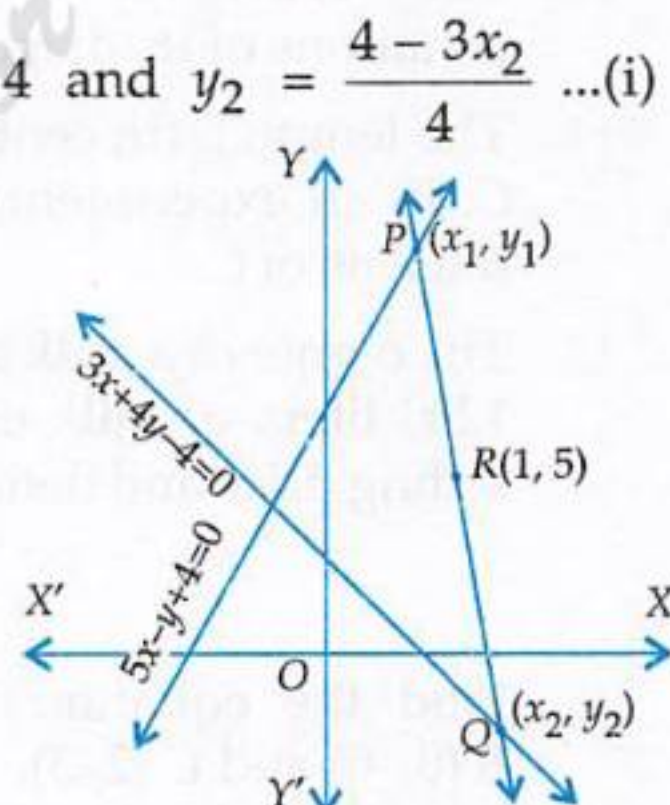


Fig. 22.31

## EXERCISE 22.5

### BASIC

- Find the equation of the straight lines passing through the following pair of points:
  - $(0, 0)$  and  $(2, -2)$
  - $(a, b)$  and  $(a + c \sin \alpha, b + c \cos \alpha)$



- (iii)  $(0, -a)$  and  $(b, 0)$  (iv)  $(a, b)$  and  $(a+b, a-b)$   
 (v)  $(at_1, a/t_1)$  and  $(at_2, a/t_2)$  (vi)  $(a \cos \alpha, a \sin \alpha)$  and  $(a \cos \beta, a \sin \beta)$

- Find the equations to the sides of the triangles the coordinates of whose angular points are respectively: (i)  $(1, 4)$ ,  $(2, -3)$  and  $(-1, -2)$  (ii)  $(0, 1)$ ,  $(2, 0)$  and  $(-1, -2)$ .
- Find the equations of the medians of a triangle, the coordinates of whose vertices are  $(-1, 6)$ ,  $(-3, -9)$  and  $(5, -8)$ .
- Find the equations to the diagonals of the rectangle the equations of whose sides are  $x = a$ ,  $x = a'$ ,  $y = b$  and  $y = b'$ .
- Find the equation of the side  $BC$  of the triangle  $ABC$  whose vertices are  $A(-1, -2)$ ,  $B(0, 1)$  and  $C(2, 0)$  respectively. Also, find the equation of the median through  $A(-1, -2)$ .
- By using the concept of equation of a line, prove that the three points  $(-2, -2)$ ,  $(8, 2)$  and  $(3, 0)$  are collinear. [NCERT]

#### BASED ON LOTS

- Prove that the line  $y - x + 2 = 0$  divides the join of points  $(3, -1)$  and  $(8, 9)$  in the ratio  $2:3$ .
- Find the equation to the straight line which bisects the distance between the points  $(a, b)$ ,  $(a', b')$  and also bisects the distance between the points  $(-a, b)$  and  $(a', -b')$ .
- In what ratio is the line joining the points  $(2, 3)$  and  $(4, -5)$  divided by the line passing through the points  $(6, 8)$  and  $(-3, -2)$ . [NCERT]
- The vertices of a quadrilateral are  $A(-2, 6)$ ,  $B(1, 2)$ ,  $C(10, 4)$  and  $D(7, 8)$ . Find the equations of its diagonals.
- The length  $L$  (in centimeters) of a copper rod is a linear function of its Celsius temperature  $C$ . In an experiment, if  $L = 124.942$  when  $C = 20$  and  $L = 125.134$  when  $C = 110$ , express  $L$  in terms of  $C$ . [NCERT]
- The owner of a milk store finds that he can sell 980 liters milk each week at ₹ 14 per liter and 1220 liters of milk each week at ₹ 16 per liter. Assuming a linear relationship between selling price and demand, how many liters could he sell weekly at ₹ 17 per liter. [NCERT]

#### BASED ON HOTS

- Find the equation of the bisector of angle  $A$  of the triangle whose vertices are  $A(4, 3)$ ,  $B(0, 0)$  and  $C(2, 3)$ .
- Find the equations to the straight lines which go through the origin and trisect the portion of the straight line  $3x + y = 12$  which is intercepted between the axes of coordinates.
- Find the equations of the diagonals of the square formed by the lines  $x = 0$ ,  $y = 0$ ,  $x = 1$  and  $y = 1$ . [NCERT EXEMPLAR]

#### ANSWERS

- (i)  $y = -x$  (ii)  $y - b = \cot \alpha (x - a)$  (iii)  $ax - by = ab$   
 (iv)  $(a - 2b)x - by + b^2 + 2ab - a^2 = 0$  (v)  $t_1 t_2 y + x = a(t_1 + t_2)$   
 (vi)  $x \cos \left( \frac{\alpha + \beta}{2} \right) + y \sin \left( \frac{\alpha + \beta}{2} \right) = a \cos \left( \frac{\alpha - \beta}{2} \right)$
- (i)  $x + 3y + 7 = 0$ ,  $y - 3x = 1$ ,  $y + 7x = 11$  (ii)  $2x - 3y = 4$ ,  $y - 3x = 1$ ,  $x + 2y = 2$ .
- $29x + 4y + 5 = 0$ ,  $8x - 5y - 21 = 0$ ,  $13x + 14y + 47 = 0$
- $y(a' - a) - x(b' - b) = a'b - ab'$ ,  $y(a' - a) + x(b' - b) = a'b' - ab$
- $x + 2y - 2 = 0$ , Median:  $5x - 4y - 3 = 0$
- $2ay - 2b'x = ab - a'b'$  9.  $5:97$  10.  $x + 6y - 34 = 0$ ,  $x - y + 1 = 0$
- $L = \frac{4}{1875}C + 124.899$  12. 1340 liters 13.  $x - 3y + 5 = 0$



14.  $y = 6x, 2y = 3x$

15.  $y = x, x + y = 1$

## HINTS TO SELECTED PROBLEMS

6. The equation of the line passing through points
- $(-2, -2)$
- and
- $(8, 2)$
- is

$$y + 2 = \frac{2 + 2}{8 + 2}(x + 2) \text{ or, } 2x - 5y - 6 = 0$$

Clearly,  $(3, 0)$  satisfies this equation which means that the line passing through  $(-2, -2)$  and  $(8, 2)$  also passes through  $(3, 0)$ . Hence, these points are collinear.

9. The equation of the line passing through
- $(6, 8)$
- and
- $(-3, -2)$
- is

$$y + 2 = \frac{8 + 2}{6 + 3}(x + 3) \text{ or, } 10x - 9y + 12 = 0 \quad \dots(i)$$

Suppose this line divides the line segment joining  $(2, 3)$  and  $(4, -5)$  in the ratio  $\lambda : 1$ , then the point of division  $\left(\frac{4\lambda + 2}{\lambda + 1}, \frac{-5\lambda + 3}{\lambda + 1}\right)$  lies on (i).

$$\therefore 10\left(\frac{4\lambda + 2}{\lambda + 1}\right) - 9\left(\frac{-5\lambda + 3}{\lambda + 1}\right) + 12 = 0$$

$$\Rightarrow 40\lambda + 20 + 45\lambda - 27 + 12\lambda + 12 = 0 \Rightarrow 97\lambda + 5 = 0 \Rightarrow \lambda = \frac{-5}{97}$$

Hence, the required ratio is  $5 : 97$  externally.

11. The equation of the line passing through
- $(C_1 = 20, L_1 = 124.942)$
- and
- $(C_2 = 110, L_2 = 125.134)$
- is

$$L - 124.942 = \frac{125.134 - 124.942}{110 - 20}(C - 20) \Rightarrow L = \frac{4}{1875}C + 124.899$$

12. Let
- $x$
- denote the price per liter and
- $y$
- denote the quantity of the milk sold at this price. Since there is linear relationship between the price per liter and quantity sold. So, the line representing the relationship passes through
- $(14, 980)$
- and
- $(16, 1220)$
- . So, its equation is

$$y - 980 = \frac{1220 - 980}{16 - 14}(x - 14) \Rightarrow y - 980 = 120(x - 14) \Rightarrow 120x - y - 700 = 0$$

When  $x = 17$ , we obtain:  $120 \times 17 - y - 700 \Rightarrow y = 1340$

## 22.6.4 THE INTERCEPT FORM OF A LINE

**THEOREM** The equation of a line which cuts off intercepts  $a$  and  $b$  respectively from the  $x$  and  $y$ -axes is  $\frac{x}{a} + \frac{y}{b} = 1$ .

**PROOF** Let  $AB$  be the line which cuts off intercepts  $OA = a$  and  $OB = b$  on the  $x$  and  $y$  axes respectively. Let  $P(x, y)$  be any point on the line. Draw  $PL \perp OX$ . Then,  $OL = x$  and  $PL = y$ .

Clearly,

$$\text{Area of } \triangle OAB = \text{Area of } \triangle OPA + \text{Area of } \triangle OPB$$

$$\Rightarrow \frac{1}{2} OA \cdot OB = \frac{1}{2} OA \cdot PL + \frac{1}{2} OB \cdot PM$$

$$\Rightarrow \frac{1}{2} ab = \frac{1}{2} ay + \frac{1}{2} bx$$

$$\Rightarrow ab = ay + bx$$

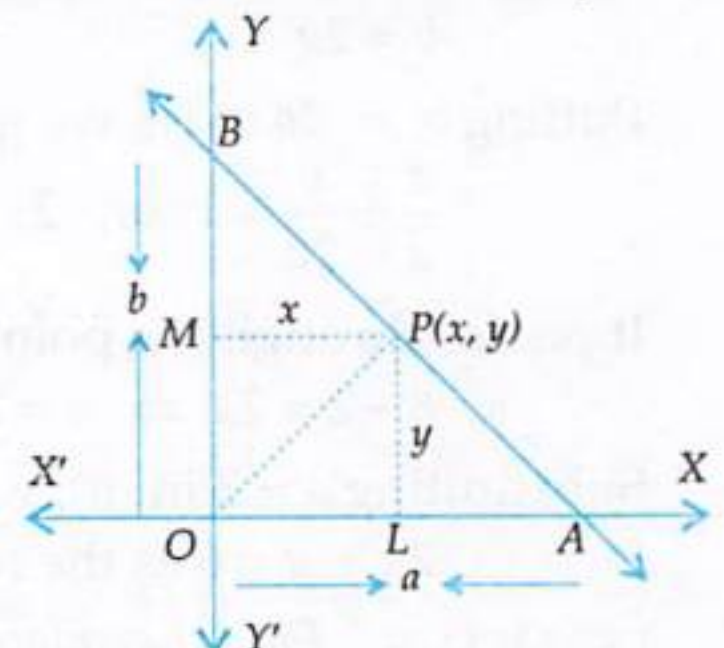


Fig. 22.32



$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1$$

This is the equation of the line in the intercept form.

Q.E.D.

### ILLUSTRATIVE EXAMPLES

#### BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** Find the equation of the line which cuts off an intercept 4 on the positive direction of x-axis and an intercept 3 on the negative direction of y-axis.

**SOLUTION** Here  $a = 4$ ,  $b = -3$ . So, the equation of the line is

$$\frac{x}{a} + \frac{y}{b} = 1 \text{ or, } \frac{x}{4} + \frac{y}{-3} = 1 \text{ or, } 3x - 4y = 12.$$

**EXAMPLE 2** Find the equation of the straight line which makes equal intercepts on the axes and passes through the point (2, 3). [NCERT]

**SOLUTION** Let the equation of the line be  $\frac{x}{a} + \frac{y}{b} = 1$ . Since it makes equal intercepts on the coordinate axes, therefore  $a = b$ . So, the equation of the line is

$$\frac{x}{a} + \frac{y}{a} = 1 \text{ or, } x + y = a \quad \dots (i)$$

This passes through the point (2, 3).

$$\therefore 2 + 3 = a \Rightarrow a = 5.$$

Thus, the equation of the required line is  $x + y = 5$ .

[Putting  $a = 5$  in (i)]

**EXAMPLE 3** Find the equation of the line which cuts off equal and positive intercepts from the axes and passes through the point  $(\alpha, \beta)$ .

**SOLUTION** Let the equation of the line be  $\frac{x}{a} + \frac{y}{b} = 1$  which cuts off intercepts  $a$  and  $b$  with the coordinate axes. It is given that  $a = b$ . Therefore, the equation of the line is

$$\frac{x}{a} + \frac{y}{a} = 1 \Rightarrow x + y = a \quad \dots (i)$$

It is given that the line (i) passes through  $(\alpha, \beta)$ .

$$\therefore \alpha + \beta = a.$$

Putting the value of  $a$  in (i), we obtain the equation of the line as  $x + y = \alpha + \beta$ .

**EXAMPLE 4** Find the equation of a straight line which passes through the point (4, -2) and whose intercept on y-axis is twice that on x-axis.

**SOLUTION** Let the equation of the line be

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \dots (i)$$

It is given that its y-intercept is twice the x-intercept.

$$\therefore b = 2a$$

Putting  $b = 2a$  in (i), we get

$$\frac{x}{a} + \frac{y}{2a} = 1 \text{ or, } 2x + y = 2a \quad \dots (ii)$$

It passes through the point (4, -2). Therefore, putting  $x = 4$ ,  $y = -2$  in (ii), we get

$$8 - 2 = 2a \Rightarrow a = 3.$$

Substituting  $a = 3$  in (ii), we get

$$2x + y = 6 \text{ as the required equation of the line.}$$

**EXAMPLE 5** Find the equation of the straight line whose intercepts on X-axis and Y-axis are respectively twice and thrice of those by the line  $3x + 4y = 12$ .



**SOLUTION** The equation of the given line is  $3x + 4y = 12$ . This can be written as  $\frac{x}{4} + \frac{y}{3} = 1$ .

Clearly, its intercepts on  $X$  and  $Y$ -axes are 4 and 3 respectively.

$\therefore$   $x$ -intercept of the required line  $= 2 \times 4 = 8$  and,  $y$ -intercept of the required line  $= 3 \times 3 = 9$

Hence, the equation of the required line is  $\frac{x}{8} + \frac{y}{9} = 1$  or,  $9x + 8y = 72$ .

**EXAMPLE 6** Find the equation of the line through  $(2, 3)$  so that the segment of the line intercepted between the axes is bisected at this point.

**SOLUTION** Let the equation of the line be  $\frac{x}{a} + \frac{y}{b} = 1$  which meets the  $x$  and  $y$  axes at  $A(a, 0)$  and  $B(0, b)$  respectively. The coordinates of the mid-point of  $AB$  are  $(a/2, b/2)$ . It is given that the point  $(2, 3)$  bisects  $AB$ .

$$\therefore \frac{a}{2} = 2 \text{ and } \frac{b}{2} = 3 \Rightarrow a = 4 \text{ and } b = 6.$$

Putting  $a = 4$  and  $b = 6$  in  $\frac{x}{a} + \frac{y}{b} = 1$ , we obtain

$$\frac{x}{4} + \frac{y}{6} = 1 \text{ or, } 3x + 2y = 12$$

Hence, the equation of the required line is  $3x + 2y = 12$ .

#### BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 7** If the intercept of a line between the coordinate axes is divided by the point  $(-5, 4)$  in the ratio  $1 : 2$ , then find the equation of the line.

**SOLUTION** Let the equation of the line be  $\frac{x}{a} + \frac{y}{b} = 1$ . It meets the coordinate axes at  $A(a, 0)$  and  $B(0, b)$ . It is given that  $P(-5, 4)$  divides  $AB$  in the ratio  $1 : 2$ .

Using section formula the coordinates of  $P$  are

$$\left( \frac{1 \times 0 + 2 \times a}{1 + 2}, \frac{1 \times b + 2 \times 0}{1 + 2} \right) = \left( \frac{2a}{3}, \frac{b}{3} \right).$$

$$\therefore -5 = \frac{2a}{3}, 4 = \frac{b}{3} \Rightarrow a = -\frac{15}{2}, b = 12.$$

Substituting the values of  $a$  and  $b$  in  $\frac{x}{a} + \frac{y}{b} = 1$ , we obtain

$$-\frac{2x}{15} + \frac{y}{12} = 1 \text{ or, } -8x + 5y = 60 \text{ or, } 8x - 5y + 60 = 0 \text{ as the equation of the line.}$$

**EXAMPLE 8** A straight line cuts intercepts from the axes of coordinates the sum of whose reciprocals is a constant. Show that it always passes through a fixed point. **[NCERT EXEMPLAR]**

**SOLUTION** Let the equation of the line be  $\frac{x}{a} + \frac{y}{b} = 1$  ... (i)

Its intercepts on  $x$  and  $y$  axes are  $a$  and  $b$  respectively. It is given that

$$\frac{1}{a} + \frac{1}{b} = \text{Constant} = k \text{ (say)}$$

$$\therefore \frac{1}{ka} + \frac{1}{kb} = 1 \Rightarrow \frac{1}{k} + \frac{1}{k} = 1 \Rightarrow \left( \frac{1}{k}, \frac{1}{k} \right) \text{ satisfies the equation } \frac{x}{a} + \frac{y}{b} = 1$$

Hence, line (i) passes through the fixed point  $\left( \frac{1}{k}, \frac{1}{k} \right)$ .

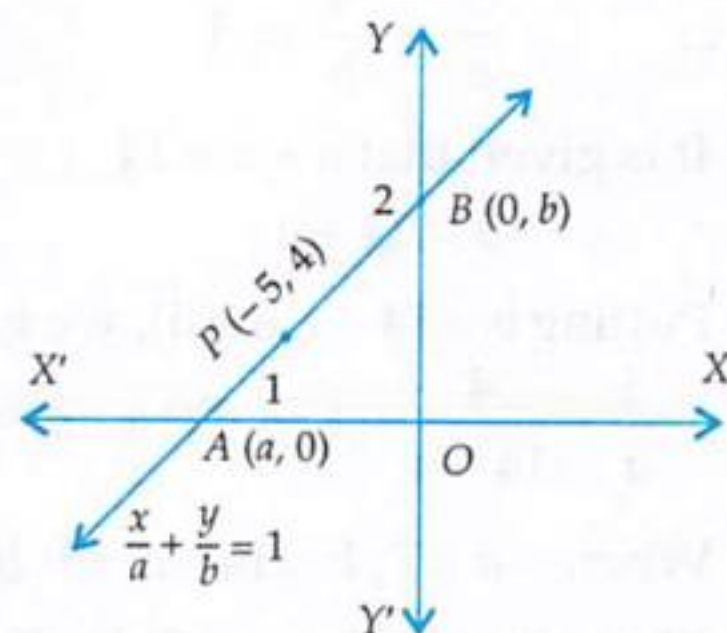


Fig. 22.33



**EXAMPLE 9** A line passes through the point  $(3, -2)$ . Find the locus of the middle point of the portion of the line intercepted between the axes.

**SOLUTION** Let the equation of the line be  $\frac{x}{a} + \frac{y}{b} = 1$  ... (i)

It passes through  $(3, -2)$ .

$$\therefore \frac{3}{a} - \frac{2}{b} = 1 \quad \dots (ii)$$

The line (i) cuts the coordinate axes at  $A(a, 0)$  and  $B(0, b)$ . Let  $P(h, k)$  be the mid-point of the portion  $AB$ . Then,

$$h = \frac{a+0}{2}, k = \frac{0+b}{2} \Rightarrow a = 2h \text{ and } b = 2k$$

Substituting the values of  $a$  and  $b$  in (ii), we get

$$\frac{3}{2h} - \frac{2}{2k} = 1$$

Hence, locus of  $P(h, k)$  is  $\frac{3}{2x} - \frac{1}{y} = 1$  or,  $3y - 2x = 2xy$ .

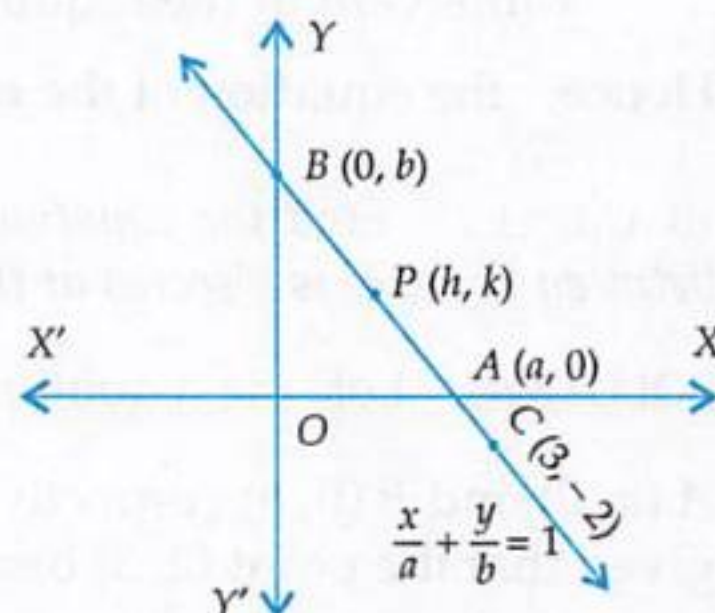


Fig. 22.34

### BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

**EXAMPLE 10** Find the equation of the line which passes through the point  $(3, 4)$  and the sum of its intercepts on the axes is 14.

**SOLUTION** Let the equation of the line be  $\frac{x}{a} + \frac{y}{b} = 1$  ... (i)

This passes through  $(3, 4)$ .

$$\therefore \frac{3}{a} + \frac{4}{b} = 1 \quad \dots (ii)$$

It is given that  $a + b = 14$

$$\therefore b = 14 - a$$

Putting  $b = 14 - a$  in (ii), we get

$$\frac{3}{a} + \frac{4}{14-a} = 1 \Rightarrow 3(14-a) + 4a = a(14-a) \Rightarrow a^2 - 13a + 42 = 0 \Rightarrow (a-7)(a-6) = 0 \Rightarrow a = 7, 6$$

When  $a = 7, b = 14 - a \Rightarrow b = 14 - 7 = 7$  and for  $a = 6, b = 14 - a \Rightarrow b = 14 - 6 = 8$ .

Thus, we obtain:  $a = 7, b = 7$  or,  $a = 6, b = 8$ .

Putting the values of  $a$  and  $b$  in (i), we obtain that the equations of the lines are

$$\frac{x}{7} + \frac{y}{7} = 1 \text{ and } \frac{x}{6} + \frac{y}{8} = 1 \text{ or, } x + y = 7 \text{ and } 4x + 3y = 24.$$

**EXAMPLE 11** Find the equations of the lines which cut-off intercepts on the axes whose sum and product are 1 and  $-6$  respectively. [NCERT]

**SOLUTION** Let the equation of the line be

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \dots (i)$$

Clearly, it cuts off intercepts  $a$  and  $b$  on  $x$  and  $y$ -axes respectively. It is given that

$$a + b = 1 \text{ and } ab = -6$$

$$\therefore (a-b)^2 = (a+b)^2 - 4ab \Rightarrow (a-b)^2 = 1 - 4 \times -6 = 25 \Rightarrow a-b = \pm 5$$

Solving  $a + b = 1$  and  $a - b = 5$ , we get:  $a = 3$  and  $b = -2$

Solving  $a + b = 1$  and  $a - b = -5$ , we get:  $a = -2$  and  $b = 3$ .

Substituting these values in (i), we obtain the equations of the required line as



$$\frac{x}{3} - \frac{y}{2} = 1 \text{ and } -\frac{x}{2} + \frac{y}{3} = 1 \text{ or, } 2x - 3y - 6 = 0 \text{ and } -3x + 2y - 6 = 0$$

**EXAMPLE 12** Find the equations of the straight lines which pass through the origin and trisect the intercept of the line  $3x + 4y = 12$  between the axes.

**SOLUTION** The equation of the given line is  $3x + 4y = 12$  or,  $\frac{x}{4} + \frac{y}{3} = 1$ . It cuts the coordinate axes at  $A(4, 0)$  and  $B(0, 3)$ . The portion  $AB$  of the given line intercepted between the axes is trisected by points  $P$  and  $Q$ .

$$\therefore \frac{AP}{PB} = \frac{1}{2} \text{ and } \frac{AQ}{QB} = \frac{2}{1}$$

$\Rightarrow$   $P$  and  $Q$  divide  $AB$  internally in the ratio  $1:2$  and  $2:1$  respectively.

So, coordinates  $P$  and  $Q$  are

$$P\left(\frac{1 \times 0 + 2 \times 4}{1 + 2}, \frac{1 \times 3 + 2 \times 0}{1 + 2}\right) \equiv P\left(\frac{8}{3}, 1\right),$$

$$Q\left(\frac{2 \times 0 + 1 \times 4}{2 + 1}, \frac{2 \times 3 + 1 \times 0}{2 + 1}\right) \equiv Q\left(\frac{4}{3}, 2\right)$$

Hence, the equation of  $OQ$  is

$$y - 0 = \frac{2 - 0}{\frac{4}{3} - 0}(x - 0) \text{ or, } 3x - 2y = 0.$$

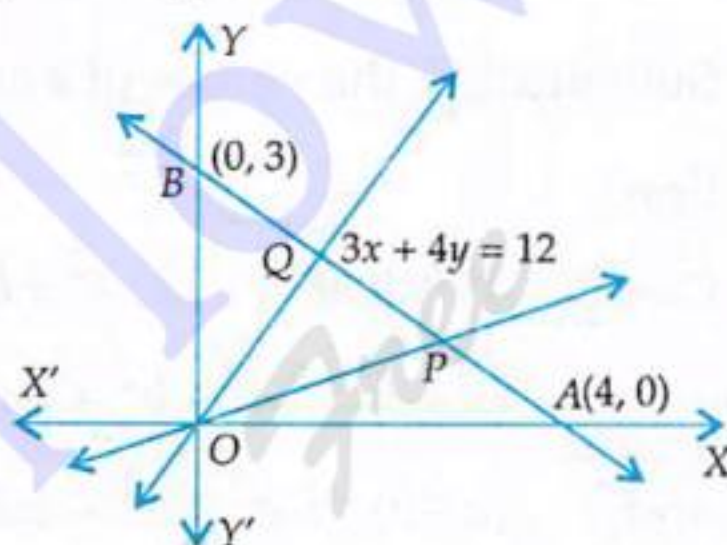


Fig. 22.35

**EXAMPLE 13** The area of the triangle formed by the coordinates axes and a line is 6 square units and the length of the hypotenuse is 5 units. Find the equation of the line.

**SOLUTION** Let the equation of the line be  $\frac{x}{a} + \frac{y}{b} = 1$  ... (i)

It cuts the coordinates axes at  $A(a, 0)$  and  $B(0, b)$  such that area of  $\triangle OAB$  is 6 square units and  $AB = 5$  units. It is given that

Area of  $\triangle OAB = 6$  sq. units

$$\Rightarrow \frac{1}{2}(OA \times OB) = 6 \Rightarrow |a||b| = 12 \Rightarrow |ab| = 12 \Rightarrow ab = \pm 12 \quad \dots (ii)$$

It is also given that

$$AB = 5 \Rightarrow AB^2 = 25 \Rightarrow OA^2 + OB^2 = 25 \Rightarrow a^2 + b^2 = 25 \quad \dots (iii)$$

Following cases arise:

Case I When  $ab = 12$ ,  $a^2 + b^2 = 25$  and  $a > 0$ ,  $b > 0$ : In this case,

$$(a + b)^2 = a^2 + b^2 + 2ab \Rightarrow (a + b)^2 = 25 + 2 \times 12 = 49 \Rightarrow a + b = 7$$

$$\text{and, } (a - b)^2 = a^2 + b^2 - 2ab \Rightarrow (a - b)^2 = 25 - 24 = 1 \Rightarrow a - b = \pm 1$$

Thus, we obtain

$$(a + b = 7 \text{ and } a - b = 1) \text{ or, } (a + b = 7 \text{ and } a - b = -1) \Rightarrow (a = 4, b = 3) \text{ or, } (a = 3, b = 4)$$

Substituting the values of  $a$  and  $b$  in (i), we obtain  $\frac{x}{4} + \frac{y}{3} = 1$  or  $\frac{x}{3} + \frac{y}{4} = 1$  as the equations of the line.

Case II When  $ab = 12$ ,  $a^2 + b^2 = 25$  and  $a < 0$ ,  $b < 0$ : In this case, we have

$$(a + b)^2 = a^2 + b^2 + 2ab = 25 + 24 = 49 \Rightarrow a + b = -7$$

$$[\because a < 0, b < 0]$$

$$\text{and, } (a - b)^2 = a^2 + b^2 - 2ab = 25 - 24 = 1 \Rightarrow a - b = \pm 1$$

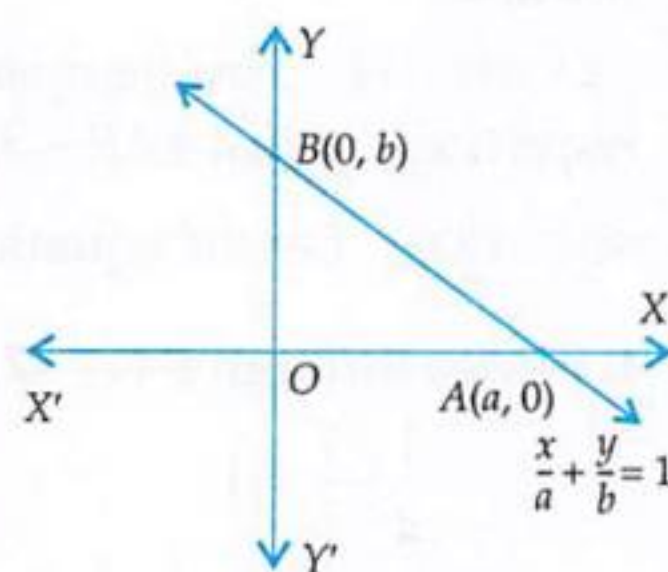


Fig. 22.36



Thus, we have

$$(a+b=-7 \text{ and } a-b=1) \text{ or } (a+b=-7 \text{ and } a-b=-1) \Rightarrow (a=-3, b=-4) \text{ or } (a=-4, b=-3)$$

Substituting the values of  $a$  and  $b$  in (i), we obtain  $\frac{x}{-3} + \frac{y}{-4} = 1$  or  $\frac{x}{-4} + \frac{y}{-3} = 1$  as the equations of the line.

Case III When  $ab = -12$ ,  $a^2 + b^2 = 25$  and  $a > 0$ ,  $b < 0$ : In this case, we have

$$(a+b)^2 = a^2 + b^2 + 2ab = 25 - 24 = 1 \Rightarrow a+b = \pm 1$$

$$\text{and, } (a-b)^2 = a^2 + b^2 - 2ab = 25 + 24 = 49 \Rightarrow a-b = 7 \quad [\because a > 0, b < 0 \therefore a-b > 0]$$

Thus, we have

$$(a+b=1 \text{ and } a-b=7) \text{ or } (a+b=-1 \text{ and } a-b=7) \Rightarrow (a=4, b=-3) \text{ or } (a=3, b=-4)$$

Substituting the values of  $a$  and  $b$  in (i), we obtain  $\frac{x}{4} - \frac{y}{3} = 1$  or  $\frac{x}{3} - \frac{y}{4} = 1$  as the equations of the line.

Case IV When  $ab = -12$ ,  $a^2 + b^2 = 25$  and  $a < 0$ ,  $b > 0$ : In this case, we have

$$(a+b)^2 = a^2 + b^2 + 2ab = 25 - 24 = 1 \Rightarrow a+b = \pm 1$$

$$\text{and, } (a-b)^2 = a^2 + b^2 - 2ab = 25 + 24 = 49 \Rightarrow a-b = -7 \quad [a < 0, b > 0 \therefore a-b < 0]$$

Thus, we have

$$(a+b=1 \text{ and } a-b=-7) \text{ or, } (a+b=-1 \text{ and } a-b=-7) \Rightarrow (a=-3 \text{ and } b=4) \text{ or } (a=-4, b=3)$$

Substituting the values of  $a$  and  $b$  in (i), we obtain  $\frac{x}{-3} + \frac{y}{4} = 1$  or  $\frac{x}{-4} - \frac{y}{3} = 1$  as the equations of the line.

**EXAMPLE 14** Find the equation of the line which passes through  $P(1, -7)$  and meets the axes at  $A$  and  $B$  respectively so that  $4AP - 3BP = 0$ .

**SOLUTION** Let the equation of the required line be  $\frac{x}{a} + \frac{y}{b} = 1$  ... (i)

It passes through  $P(1, -7)$ .

$$\therefore \frac{1}{a} - \frac{7}{b} = 1 \quad \dots (ii)$$

It is given that the point  $P(1, -7)$  divides segment  $AB$  in such a way that

$$4AP - 3BP = 0 \text{ i.e. } \frac{AP}{BP} = \frac{3}{4} \text{ or, } AP : BP = 3 : 4$$

This means that  $P$  divides  $AB$  internally in the ratio  $3 : 4$ . So, the coordinates of  $P$  are

$$\left( \frac{3 \times 0 + 4 \times a}{3 + 4}, \frac{3 \times b + 4 \times 0}{3 + 4} \right) = \left( \frac{4a}{7}, \frac{3b}{7} \right)$$

But, the coordinates of  $P$  are given as  $(1, -7)$ .

$$\therefore \frac{4a}{7} = 1 \text{ and } \frac{3b}{7} = -7 \Rightarrow a = \frac{7}{4} \text{ and } b = -\frac{49}{3}$$

Substituting the values of  $a$  and  $b$  in (i), we obtain

$$\frac{4x}{7} - \frac{3y}{49} = 1 \text{ or } 28x - 3y = 49 \text{ as the required equation.}$$

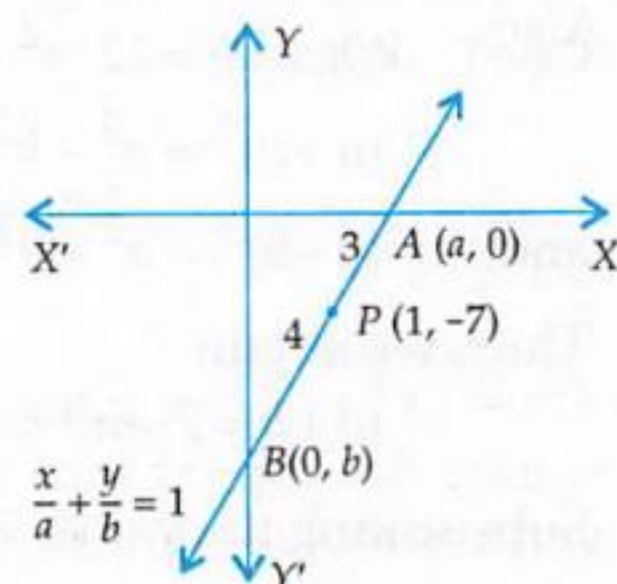


Fig. 22.37



**EXAMPLE 15** Show that the locus of the mid-point of the segment intercepted between the axes of the variable line  $x \cos \alpha + y \sin \alpha = p$  is  $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$ , where  $p$  is a constant. [NCERT EXEMPLAR]

**SOLUTION** The given equation is  $x \cos \alpha + y \sin \alpha = p$  or,  $\frac{x}{p/\cos \alpha} + \frac{y}{p/\sin \alpha} = 1$  ... (i)

This cuts the coordinate axes at  $A(p/\cos \alpha, 0)$  and  $B(0, p/\sin \alpha)$ . Let  $P(h, k)$  be the mid-point of the intercept  $AB$ . Then,

$$h = \frac{p/\cos \alpha + 0}{2}, k = \frac{0 + p/\sin \alpha}{2}$$

$$\Rightarrow h = \frac{p}{2 \cos \alpha}, k = \frac{p}{2 \sin \alpha}$$

$$\Rightarrow \cos \alpha = \frac{p}{2h}, \sin \alpha = \frac{p}{2k} \quad \dots (i)$$

Here,  $\alpha$  is a variable. To find the locus of  $P(h, k)$ , we have to eliminate  $\alpha$ . From (i), we obtain

$$\cos^2 \alpha + \sin^2 \alpha = \frac{p^2}{4h^2} + \frac{p^2}{4k^2} \Rightarrow 1 = \frac{p^2}{4h^2} + \frac{p^2}{4k^2} \Rightarrow \frac{4}{p^2} = \frac{1}{h^2} + \frac{1}{k^2}$$

Hence, the locus of  $(h, k)$  is  $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$ .

**EXAMPLE 16** If the sum of the distances of a moving point in a plane from the axes is 1, then find the locus of the point. [NCERT EXEMPLAR]

**SOLUTION** Let  $P(h, k)$  be a moving point in the  $xy$ -plane. Let  $PL$  and  $PM$  be perpendiculars from  $P$  on  $OX$  and  $OY$  respectively. Then,

$$PL = |k| \text{ and } PM = |h|.$$

It is given that  $P(h, k)$  moves in the  $xy$ -plane such that

$$PL + PM = 1 \Rightarrow |k| + |h| = 1$$

Hence, the locus of  $P(h, k)$  is  $|y| + |x| = 1$  or,  $|x| + |y| = 1$ .

Now,

$$|x| + |y| = 1 \Rightarrow 0 \leq |x| \leq 1, 0 \leq |y| \leq 1$$

Also,

$$|x| + |y| = 1 \Rightarrow \begin{cases} x + y = 1, & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 1 \\ -x + y = 1, & \text{if } -1 \leq x < 0, 0 \leq y \leq 1 \\ -x - y = 1, & \text{if } -1 < x \leq 0, -1 < y \leq 0 \\ x - y = 1, & \text{if } 0 \leq x \leq 1, -1 < y \leq 0 \end{cases}$$

Thus,  $|x| + |y| = 1$  gives four line segments  $AB$ ,  $BC$ ,  $CD$  and  $DA$ . These line segments form a square  $ABCD$  as shown in Fig. 22.38. Thus, the locus of the variable point  $P$  is the square having vertices at  $A(1, 0)$ ,  $B(0, 1)$ ,  $C(-1, 0)$  and  $D(0, -1)$ .

**EXAMPLE 17** The line  $\frac{x}{a} + \frac{y}{b} = 1$  moves in such a way that  $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$ , where  $c$  is a constant. Find the locus of the foot of the perpendicular from the origin on the given line.

**SOLUTION** Let  $P(h, k)$  be the foot of the perpendicular from the origin  $O$  on the line  $\frac{x}{a} + \frac{y}{b} = 1$  which cuts the coordinates axes at  $A(a, 0)$  and  $B(0, b)$ . Then,

$$\text{Slope of } OP \times \text{Slope of } AB = -1$$

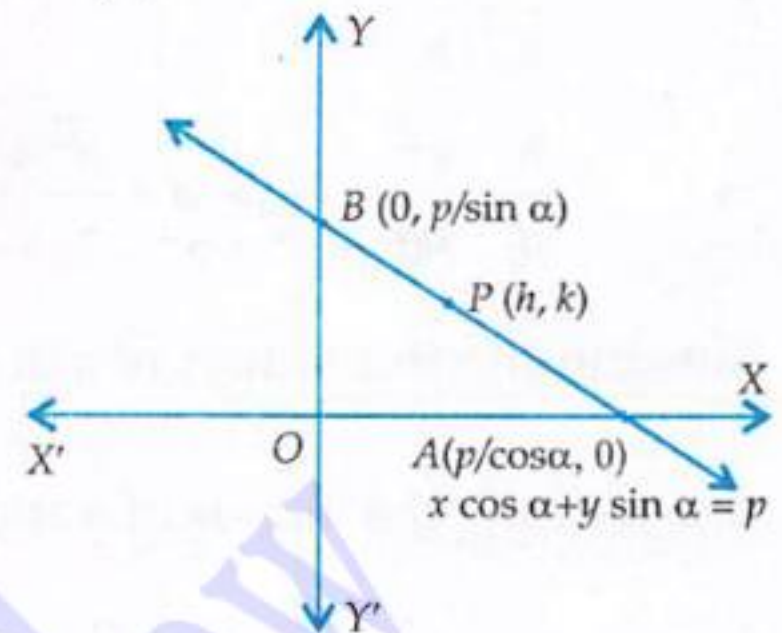


Fig. 22.38

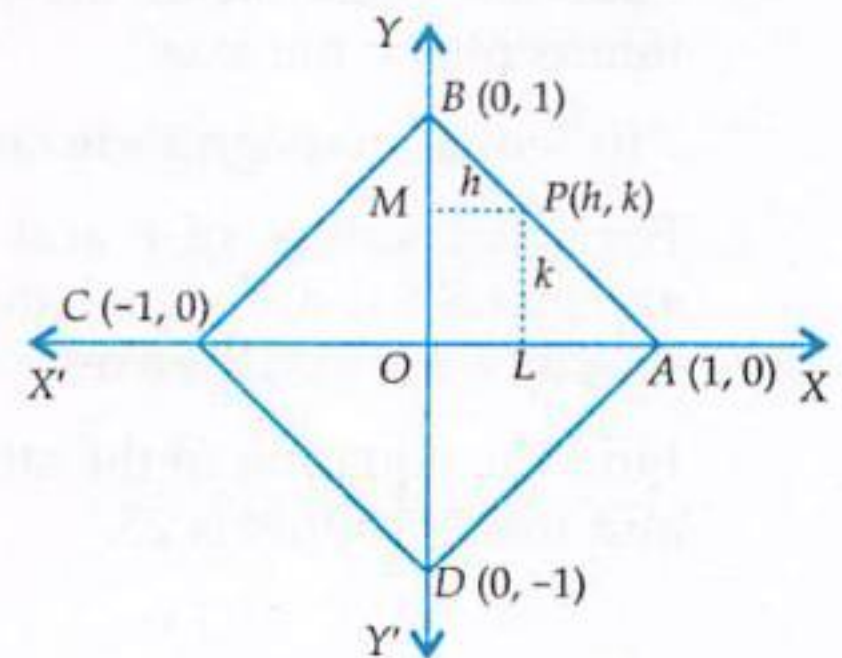


Fig. 22.39



$$\Rightarrow \frac{k-0}{h-0} \times \frac{b-0}{0-a} = -1 \Rightarrow bk = ah \Rightarrow b = \frac{ah}{k} \quad \dots(i)$$

Also,  $P(h, k)$  lies on  $\frac{x}{a} + \frac{y}{b} = 1$ .

$$\therefore \frac{h}{a} + \frac{k}{b} = 1 \quad \dots(ii)$$

$$\Rightarrow \frac{h}{a} + \frac{k^2}{ah} = 1 \Rightarrow a = \frac{h^2 + k^2}{h} \quad [\text{Using (i)}]$$

Substituting this values of  $a$  in (i), we obtain  $b = \frac{h^2 + k^2}{k}$ .

Substituting the values of  $a$  and  $b$  in  $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$ , we obtain

$$\frac{h^2}{(h^2 + k^2)^2} + \frac{k^2}{(h^2 + k^2)^2} = \frac{1}{c^2} \text{ or, } h^2 + k^2 = c^2$$

Hence, the locus of  $(h, k)$  is  $x^2 + y^2 = c^2$ .

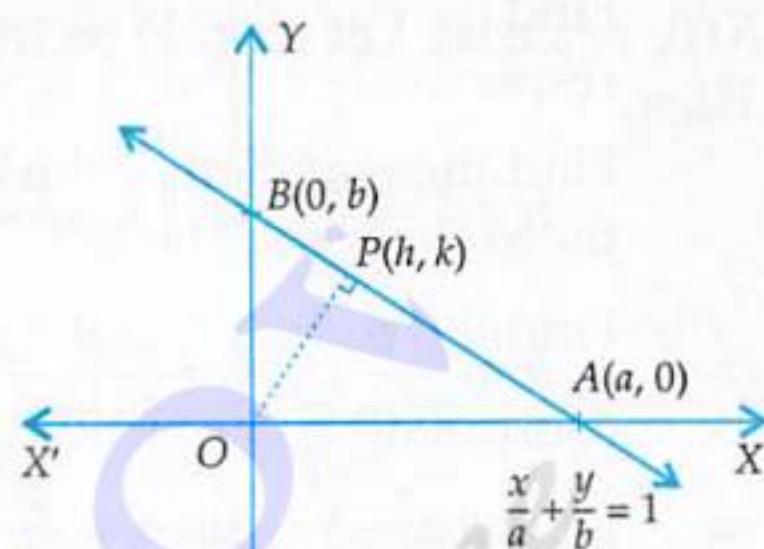


Fig. 22.40

### EXERCISE 22.6

#### BASIC

- Find the equation to the straight line
  - cutting off intercepts 3 and 2 from the axes.
  - cutting off intercepts -5 and 6 from the axes.
- Find the equation of the straight line which passes through  $(1, -2)$  and cuts off equal intercepts on the axes [NCERT EXEMPLAR]
- Find the equation to the straight line which passes through the point  $(5, 6)$  and has intercepts on the axes
  - equal in magnitude and both positive.
  - equal in magnitude but opposite in sign.
- For what values of  $a$  and  $b$  the intercepts cut off on the coordinate axes by the line  $ax + by + 8 = 0$  are equal in length but opposite in signs to those cut off by the line  $2x - 3y + 6 = 0$  on the axes. [NCERT EXEMPLAR]
- Find the equation to the straight line which cuts off equal positive intercepts on the axes and their product is 25.

#### BASED ON LOTS

- Find the equation of the line which passes through the point  $(-4, 3)$  and the portion of the line intercepted between the axes is divided internally in the ratio 5 : 3 by this point. [NCERT EXEMPLAR]
- A straight line passes through the point  $(\alpha, \beta)$  and this point bisects the portion of the line intercepted between the axes. Show that the equation of the straight line is  $\frac{x}{2\alpha} + \frac{y}{2\beta} = 1$ . [NCERT]
- Find the equation of the line which passes through the point  $(3, 4)$  and is such that the portion of it intercepted between the axes is divided by the point in the ratio 2 : 3.
- Point  $R(h, k)$  divides a line segment between the axes in the ratio 1 : 2. Find the equation of the line. [NCERT]



10. Find the equation of the straight line which passes through the point  $(-3, 8)$  and cuts off positive intercepts on the coordinate axes whose sum is 7.
11. Find the equation to the straight line which passes through the point  $(-4, 3)$  and is such that the portion of it between the axes is divided by the point in the ratio 5 : 3.
12. Find the equation of a line which passes through the point  $(22, -6)$  and is such that the intercept on  $x$ -axis exceeds the intercept on  $y$ -axis by 5.
13. Find the equation of the line, which passes through  $P(1, -7)$  and meets the axes at  $A$  and  $B$  respectively so that  $4AP - 3BP = 0$ .
14. Find the equation of the line passing through the point  $(2, 2)$  and cutting off intercepts on the axes whose sum is 9. [NCERT]
15. Find the equation of the straight line which passes through the point  $P(2, 6)$  and cuts the coordinate axes at the point  $A$  and  $B$  respectively so that  $\frac{AP}{BP} = \frac{2}{3}$ .
16. Find the equations of the straight lines each of which passes through the point  $(3, 2)$  and cuts off intercepts  $a$  and  $b$  respectively on  $x$  and  $y$ -axes such that  $a - b = 2$ .
17. Find the equations of the straight lines which pass through the origin and trisect the portion of the straight line  $2x + 3y = 6$  which is intercepted between the axes.
18. Find the equation of the straight line passing through the point  $(2, 1)$  and bisecting the portion of the straight line  $3x - 5y = 15$  lying between the axes.
19. Find the equation of the straight line passing through the origin and bisecting the portion of the line  $ax + by + c = 0$  intercepted between the coordinate axes.

**ANSWERS**

1. (i)  $2x + 3y = 6$       (ii)  $-6x + 5y = 30$       2.  $x + y = -1$
3. (i)  $x + y = 11$       (ii)  $x - y = -1$
4.  $a = -\frac{8}{3}, b = 4$       5.  $x + y = 5$       6.  $9x - 20y + 96 = 0$       8.  $2x + y = 10$
9.  $2kx + hy = 3hk$       10.  $4x + 3y = 12$       11.  $9x - 20y + 96 = 0$
12.  $6x + 11y - 66 = 0$  or  $x + 2y - 10 = 0$       13.  $28x - 3y = 49$
14.  $x + 2y - 6 = 0, 2x + y - 6 = 0$       15.  $9x + 2y = 30$
16.  $2x + 3y = 12, x - y = 1$       17.  $x - 3y = 0, 4x - 3y = 0$
18.  $5x + y = 11$       19.  $ax - by = 0$

**HINTS TO SELECTED PROBLEMS**

2. The equation of a line cutting off equal intercepts ' $a$ ' on the coordinate axes is

$$\frac{x}{a} + \frac{y}{a} = 1 \text{ or, } x + y = a \quad \dots(i)$$

If it passes through  $(1, -2)$ , then  $1 - 2 = a \Rightarrow a = -1$ . Substituting  $a = -1$  in (i), we get  $x + y = -1$  as the equation of the line.

7. Let the equation of the line be  $\frac{x}{a} + \frac{y}{b} = 1$  ...(i)

This line cuts the coordinate axes at  $A(a, 0)$  and  $B(0, b)$ . It is given that  $(\alpha, \beta)$  bisects the segment  $AB$ .

$$\therefore \alpha = \frac{a + 0}{2}, \beta = \frac{0 + b}{2} \Rightarrow a = 2\alpha, b = 2\beta$$

Substituting these values in (i), we get  $\frac{x}{2\alpha} + \frac{y}{2\beta} = 1$ .



9. Let the equation of the line be  $\frac{x}{a} + \frac{y}{b} = 1$  ... (i)

It cuts the axes at  $A(a, 0)$  and  $B(0, b)$ . It is given that the point  $R(h, k)$  divides segment  $AB$  in the ratio  $1 : 2$ .

$$\therefore h = \frac{2a + 0}{3} \text{ and } k = \frac{0 + b}{3} \Rightarrow a = \frac{3h}{2}, b = 3k$$

Substituting these values in (i), we obtain  $\frac{2x}{h} + \frac{y}{k} = 3$  or,  $2kx + hy = 3hk$  as the equation of the line.

14. Let the equation of the line be  $\frac{x}{a} + \frac{y}{b} = 1$  ... (i)

It passes through  $(2, 2)$  and the sum of the intercepts on the axes is 9. Therefore,

$$\frac{2}{a} + \frac{2}{b} = 1 \text{ and } a + b = 9$$

$$\Rightarrow 2b + 2a = ab \text{ and } a + b = 9$$

$$\Rightarrow 2(9 - a) + 2a = a(9 - a)$$

[On eliminating  $b$ ]

$$\Rightarrow a^2 - 9a + 18 = 0 \Rightarrow (a - 6)(a - 3) = 0 \Rightarrow a = 3, 6.$$

When  $a = 3$ ,  $a + b = 9$  gives  $b = 6$ . When  $a = 6$ ,  $a + b = 9$  gives  $b = 3$ .

Hence, the equations of the line are  $\frac{x}{3} + \frac{y}{6} = 1$  and,  $\frac{x}{6} + \frac{y}{3} = 1$  or,  $2x + y = 6$  and  $x + 2y = 6$ .

### 22.6.5 NORMAL FORM OR PERPENDICULAR FORM OF A LINE

**THEOREM** The equation of the straight line upon which the length of the perpendicular from the origin is  $p$  and this perpendicular makes an angle  $\alpha$  with  $x$ -axis is  $x \cos \alpha + y \sin \alpha = p$ .

**PROOF** Let the line  $AB$  be such that the length of the perpendicular  $OQ$  from the origin  $O$  to the line be  $p$  and  $\angle XOQ = \alpha$ . Let  $P(x, y)$  be any point on the line. Draw  $PL \perp OX$ ,  $LM \perp OQ$  and  $PN \perp LM$ . Then,  $OL = x$  and  $LP = y$ .

In  $\triangle OLM$ , we obtain

$$\cos \alpha = \frac{OM}{OL} \Rightarrow OM = OL \cos \alpha = x \cos \alpha.$$

In  $\triangle PNL$ , we obtain

$$\sin \alpha = \frac{PN}{PL} \Rightarrow PN = PL \sin \alpha = y \sin \alpha$$

$$\therefore MQ = PN = y \sin \alpha$$

$$\text{Now, } p = OQ = OM + MQ = x \cos \alpha + y \sin \alpha.$$

Hence, the equation of the required line is  $x \cos \alpha + y \sin \alpha = p$ .

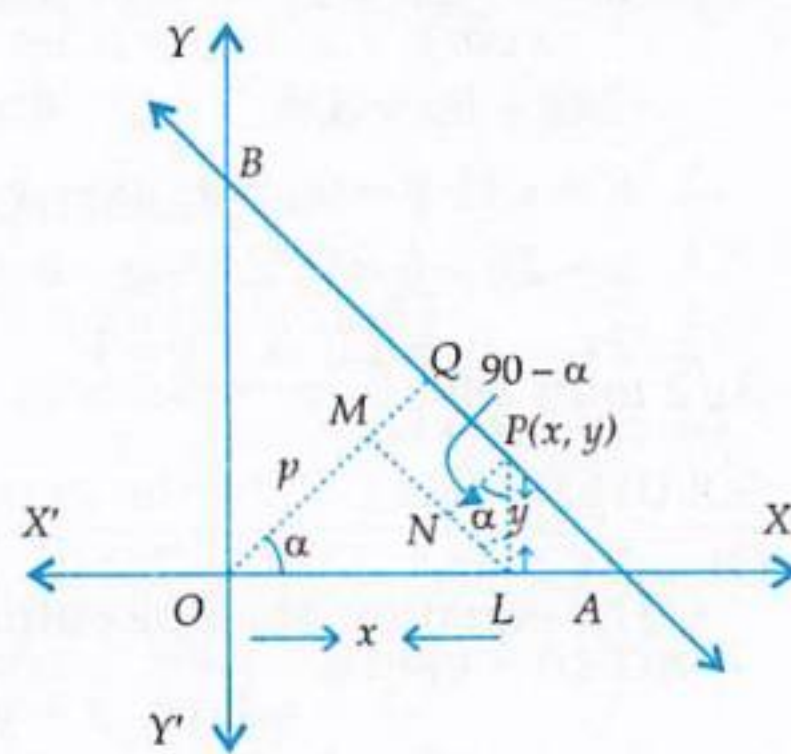


Fig. 22.41

### ILLUSTRATIVE EXAMPLES

#### BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** Find the equation of the line which is at a distance 3 from the origin and the perpendicular from the origin to the line makes an angle of  $30^\circ$  with the positive direction of the  $x$ -axis.

**SOLUTION** Here,  $p = 3$ ,  $\alpha = 30^\circ$ . The equation of the line in the normal form is



$$x \cos 30^\circ + y \sin 30^\circ = 3 \Rightarrow x \frac{\sqrt{3}}{2} + \frac{y}{2} = 3 \Rightarrow \sqrt{3}x + y = 6.$$

**EXAMPLE 2** Find the equation of the straight line on which the length of the perpendicular from the origin is 4 units and the line makes an angle of  $120^\circ$  with positive direction of  $x$ -axis.

[NCERT EXEMPLAR]

**SOLUTION** It is given that  $\angle XAB = 120^\circ$ . Therefore,  $\angle AOP = 30^\circ$ . Thus, we have  $p = 4$  and  $\alpha = 30^\circ$  (Fig. 22.42). So, the equation of the line is  $x \cos \alpha + y \sin \alpha = p$  or,  $x \cos 30^\circ + y \sin 30^\circ = 4$  or,  $\sqrt{3}x + y = 8$ .

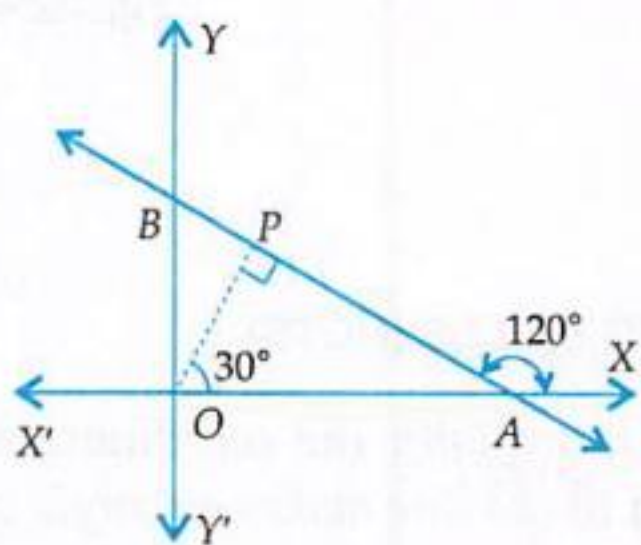


Fig. 22.42

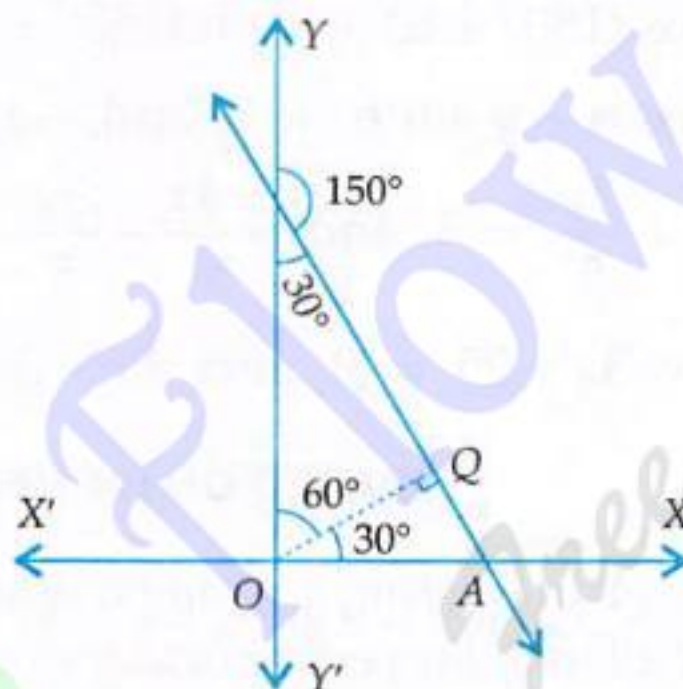


Fig. 22.43

**EXAMPLE 3** The length of the perpendicular from the origin to a line is 7 and the line makes an angle of  $150^\circ$  with the positive direction of  $y$ -axis. Find the equation of the line.

**SOLUTION** It is evident from the Figure 22.43 that the perpendicular  $OQ$  from the origin on the line makes  $30^\circ$  angle with  $x$ -axis. Therefore,  $\alpha = 30^\circ$ . It is given that  $OQ = 7$ . Therefore,  $p = 7$ . So, the equation of the required line is

$$x \cos \alpha + y \sin \alpha = p \text{ or, } x \cos 30^\circ + y \sin 30^\circ = 7 \Rightarrow \frac{\sqrt{3}x}{2} + \frac{y}{2} = 7 \Rightarrow \sqrt{3}x + y = 14.$$

#### BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 4** Find the equation of the straight line upon which the length of perpendicular from origin is  $3\sqrt{2}$  units and this perpendicular makes an angle of  $75^\circ$  with the positive direction of  $x$ -axis.

**SOLUTION** Let  $OL$  be the perpendicular from the origin on the required line. It is given that  $OL = 3\sqrt{2}$  and  $\angle XOL = 75^\circ$  i.e.  $p = 3\sqrt{2}$  and  $\alpha = 75^\circ$ . So, the equation of the line is

$$x \cos \alpha + y \sin \alpha = p \text{ or, } x \cos 75^\circ + y \sin 75^\circ = 3\sqrt{2} \quad \dots(i)$$

Putting  $\cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$  and  $\sin 75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$ , we obtain

$$\left( \frac{\sqrt{3}-1}{2\sqrt{2}} \right) x + \left( \frac{\sqrt{3}+1}{2\sqrt{2}} \right) y = 3\sqrt{2}$$

or,  $(\sqrt{3}-1)x + (\sqrt{3}+1)y = 12$ , which is the required equation.

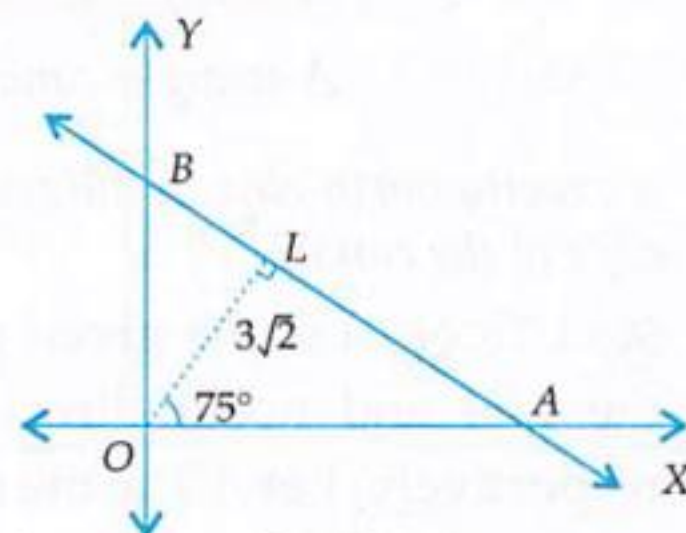


Fig. 22.44

**EXAMPLE 5** Find the equation of the straight line upon which the length of the perpendicular from the origin is 5 and the slope of this perpendicular is  $\frac{3}{4}$ .

**SOLUTION** Suppose the perpendicular  $OL$  drawn from the origin  $O$  on the given line makes acute angle  $\alpha$  with  $x$ -axis. Then, the slope of  $OL$  is  $\tan \alpha$ . But, it is given that the slope of  $OL$  is  $\frac{3}{4}$ .



$$\therefore \tan \alpha = \frac{3}{4}$$

[Given]

Since  $\tan(180^\circ + \alpha) = \tan \alpha$ . So, there are two possible lines  $AB$  and  $CD$  on which the perpendicular drawn from the origin has slope  $\frac{3}{4}$ .

Now,  $\tan \alpha = \frac{3}{4} \Rightarrow \sin \alpha = \frac{3}{5}$  and  $\cos \alpha = \frac{4}{5}$ . Here,  $p = 5$ .

So, the equations of the required lines are

$$x \cos \alpha + y \sin \alpha = p$$

and,  $x \cos(180^\circ + \alpha) + y \sin(180^\circ + \alpha) = p$

or,  $x \cos \alpha + y \sin \alpha = p$  and,  $-x \cos \alpha - y \sin \alpha = p$

or,  $\frac{4x}{5} + \frac{3y}{5} = 5$  and  $\frac{-4x}{5} - \frac{3y}{5} = 5$

or,  $4x + 3y - 25 = 0$  and  $4x + 3y + 25 = 0$

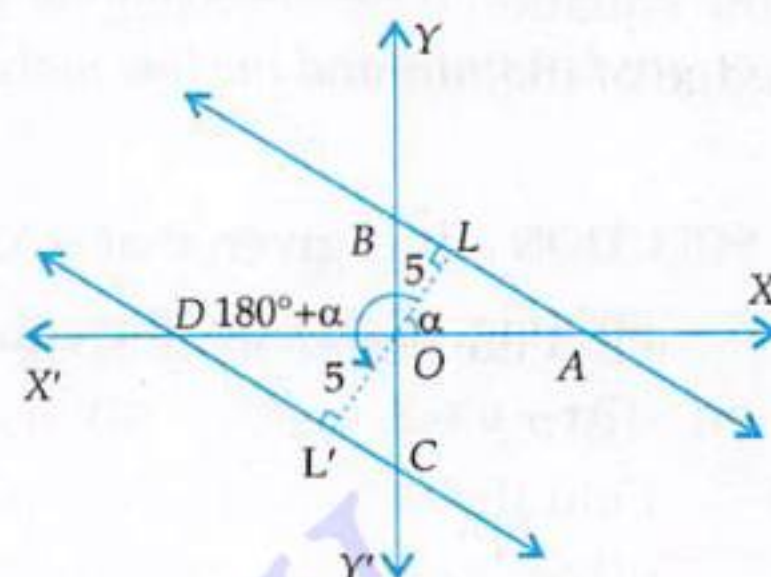


Fig. 22.45

### BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

**EXAMPLE 6** A line forms a triangle of area  $54\sqrt{3}$  square units with the coordinate axes. Find the equation of the line if the perpendicular drawn from the origin to the line makes an angle of  $60^\circ$  with the X-axis.

**SOLUTION** Let  $AB$  be the given line and  $OL = p$  be the perpendicular drawn from the origin on the line.

It is given that the perpendicular  $OL$  makes  $60^\circ$  angle with  $x$ -axis.

Therefore,  $\alpha = 60^\circ$ . Thus, the equation of the line  $AB$  is

$$x \cos \alpha + y \sin \alpha = p \text{ or, } x \cos 60^\circ + y \sin 60^\circ = p$$

or,  $x + \sqrt{3}y = 2p$  or,  $\frac{x}{2p} + \frac{\sqrt{3}y}{2p} = 1$  ... (i)

This, cuts the coordinates axes at  $A$  and  $B$  such that

$OA = 2p$  and  $OB = \frac{2p}{\sqrt{3}}$ . It is given that area of  $\triangle OAB$  is  $54\sqrt{3}$  sq. units.

$$\therefore \frac{1}{2} \times OA \times OB = 54\sqrt{3} \Rightarrow \frac{1}{2} \times 2p \times \frac{2p}{\sqrt{3}} = 54\sqrt{3} \Rightarrow p^2 = 81 \Rightarrow p = 9$$

Substituting  $p = 9$  in (i), we get  $x + \sqrt{3}y = 18$  as the equation of the required line.

**EXAMPLE 7** A straight canal is  $4\frac{1}{2}$  miles from a place and the shortest route from this place to the canal is exactly north-east. A village is 3 miles north and four miles east from the place. Does it lie by the nearest edge of the canal?

**SOLUTION** Let the given place be  $O$ . Take this as the origin and the east and north directions through  $O$  as the  $x$  and  $y$  axes respectively. Let  $AB$  be the nearest edge of the canal. It is given that the canal is  $4\frac{1}{2}$  miles from  $O$ . This means that the perpendicular

distance of  $AB$  from  $O$  is  $4\frac{1}{2}$  miles i.e.  $OL \perp AB$  and  $OL = 4\frac{1}{2}$ . It is

also given that  $OL$  is exactly north-east. Therefore,  $\angle LOA = 45^\circ$ . So, the equation of the edge  $AB$  of the canal is

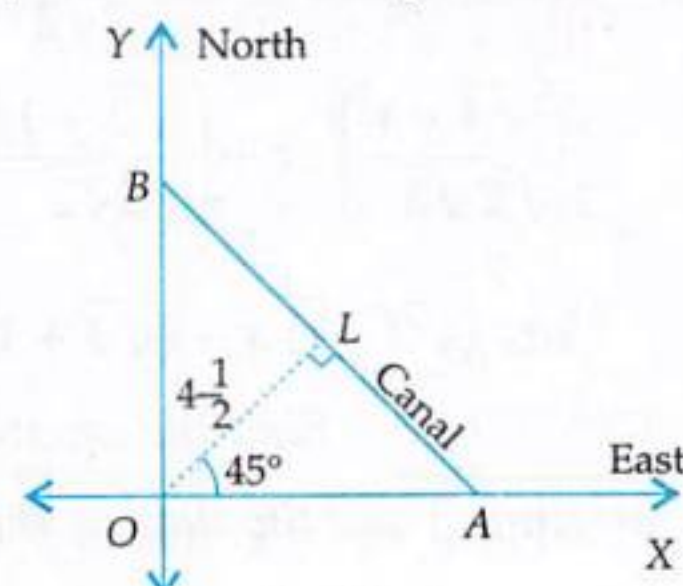


Fig. 22.47



$$x \cos 45^\circ + y \sin 45^\circ = 4 \frac{1}{2} \quad \text{or, } \sqrt{2} (x + y) = 9 \quad \dots(i)$$

The position of the village is (4, 3). The village will lie on the edge of the canal, if (4, 3) satisfies the equation (i). Clearly, (4, 3) does not satisfy (i). Hence, the village does not lie by the nearer edge of the canal.

## EXERCISE 22.7

## BASIC

- Find the equation of a line for which  
(i)  $p = 5$ ,  $\alpha = 60^\circ$  (ii)  $p = 4$ ,  $\alpha = 150^\circ$  (iii)  $p = 8$ ,  $\alpha = 225^\circ$  (iv)  $p = 8$ ,  $\alpha = 300^\circ$
- Find the equation of the line on which the length of the perpendicular segment from the origin to the line is 4 and the inclination of the perpendicular segment with the positive direction of  $x$ -axis is  $30^\circ$ . [NCERT EXEMPLAR]
- Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with the positive direction of  $x$ -axis is  $15^\circ$ . [NCERT]
- Find the equation of the straight line at a distance of 3 units from the origin such that the perpendicular from the origin to the line makes an angle  $\alpha$  given by  $\tan \alpha = \frac{5}{12}$  with the positive direction of  $x$ -axis.
- Find the equation of the straight line on which the length of the perpendicular from the origin is 2 and the perpendicular makes an angle  $\alpha$  with  $x$ -axis such that  $\sin \alpha = \frac{1}{3}$ .

## BASED ON LOTS

- Find the equation of the straight line upon which the length of the perpendicular from the origin is 2 and the slope of this perpendicular is  $\frac{5}{12}$ .
- The length of the perpendicular from the origin to a line is 7 and the line makes an angle of  $150^\circ$  with the positive direction of  $y$ -axis. Find the equation of the line.
- Find the value of  $\theta$  and  $p$ , if the equation  $x \cos \theta + y \sin \theta = p$  is the normal form of the line  $\sqrt{3}x + y + 2 = 0$ . [NCERT]

## BASED ON HOTS

- Find the equation of the straight line which makes a triangle of area  $96\sqrt{3}$  with the axes and perpendicular from the origin to it makes an angle of  $30^\circ$  with  $y$ -axis.
- Find the equation of a straight line on which the perpendicular from the origin makes an angle of  $30^\circ$  with  $x$ -axis and which forms a triangle of area  $50/\sqrt{3}$  with the axes.

## ANSWERS

- (i)  $x + \sqrt{3}y = 10$  (ii)  $-\sqrt{3}x + y = 8$  (iii)  $x + y + 8\sqrt{2}$  (iv)  $x - \sqrt{3}y = 16$
- $\sqrt{3}x + y = 8$  3.  $(\sqrt{3} + 1)x + (\sqrt{3} - 1)y = 8\sqrt{2}$  4.  $12x + 5y = 39$
- $2\sqrt{2}x + y = 6$  6.  $12x + 5y \pm 26 = 0$  7.  $\sqrt{3}x + y = 14$
- $\theta = \frac{7\pi}{6}$ ,  $p = 1$  9.  $x + \sqrt{3}y = 24$  10.  $\sqrt{3}x + y = \pm 10$

## HINTS TO SELECTED PROBLEMS

- Here,  $\alpha = 15^\circ$  and  $p = 4$ . So, the equation of the line is

$$x \cos 15^\circ + y \sin 15^\circ = 4 \quad \text{or, } (\sqrt{3} + 1)x + (\sqrt{3} - 1)y = 8\sqrt{2}.$$



8. We have  $\sqrt{3}x + y + 2 = 0 \Rightarrow -\sqrt{3}x - y = 2 \Rightarrow \left(\frac{-\sqrt{3}}{2}\right)x + \left(-\frac{1}{2}\right)y = 1$

This is same as  $x \cos \theta + y \sin \theta = p$ .

$\therefore \cos \theta = -\frac{\sqrt{3}}{2}, \sin \theta = -\frac{1}{2}$  and  $p = 1 \Rightarrow \theta = \frac{7\pi}{6}$  and  $p = 1$

### 22.6.6 DISTANCE FORM OF A LINE

**THEOREM** The equation of the straight line passing through  $(x_1, y_1)$  and making an angle  $\theta$  with the positive direction of  $x$ -axis is  $\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r$ , where  $r$  is the distance of the point  $(x, y)$  on the line from the point  $(x_1, y_1)$ .

**PROOF** Let the given line meets  $x$ -axis at  $A$ ,  $y$ -axis at  $B$  and passes through the point  $Q(x_1, y_1)$ . Let  $P(x, y)$  be any point on the line at a distance  $r$  from  $Q(x_1, y_1)$  i.e.  $PQ = r$ . Draw  $PL \perp OX$ ,  $QM \perp OX$  and  $QN \perp PL$ . Then,

$$QN = ML = OL - OM = x - x_1$$

and,  $PN = PL - NL = PL - QM = y - y_1$ .

In  $\triangle PQN$ , we have

$$\cos \theta = \frac{QN}{PQ} \Rightarrow \cos \theta = \frac{x - x_1}{r} \quad \dots(i)$$

and,  $\sin \theta = \frac{PN}{PQ} \Rightarrow \sin \theta = \frac{y - y_1}{r} \quad \dots(ii)$

From (i) and (ii), we get:  $\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r$

This is the required equation of the line in the distance form.

**Q.E.D.**

**NOTE 1** The equation of the line is  $\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r$

$$\Rightarrow x - x_1 = r \cos \theta \text{ and } y - y_1 = r \sin \theta \Rightarrow x = x_1 + r \cos \theta \text{ and } y = y_1 + r \sin \theta.$$

Thus, the coordinates of any point on the line at a distance  $r$  from the given point  $(x_1, y_1)$  are  $(x_1 + r \cos \theta, y_1 + r \sin \theta)$ . If  $P$  is on the right side of  $(x_1, y_1)$ , then  $r$  is positive and if  $P$  is on the left side of  $(x_1, y_1)$ , then  $r$  is negative. Since different values of  $r$  determine different points on the line, therefore the above form of the line is also called parametric form or symmetric form of a line.

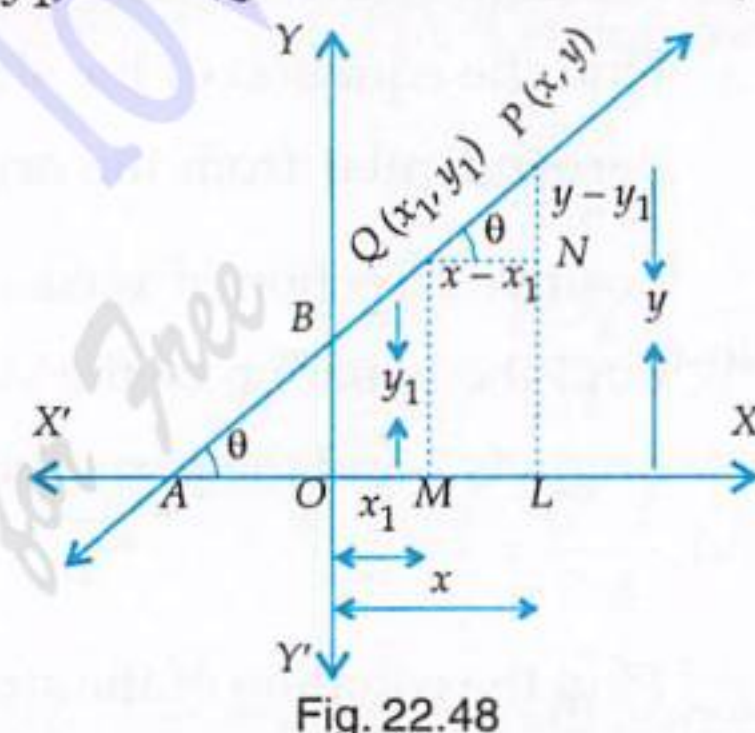
**NOTE 2** In the above form one can determine the coordinates of any point on the line at a given distance from the given point through which it passes. At a given distance  $r$  from the point  $(x_1, y_1)$  on the line  $\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta}$  there are two points viz.  $(x_1 + r \cos \theta, y_1 + r \sin \theta)$  and  $(x_1 - r \cos \theta, y_1 - r \sin \theta)$ .

## ILLUSTRATIVE EXAMPLES

### BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** A straight line is drawn through the point  $P(2, 3)$  and is inclined at an angle of  $30^\circ$  with the  $x$ -axis. Find the coordinates of two points on it at a distance 4 from  $P$  on either side of  $P$ .

**SOLUTION** Here,  $(x_1, y_1) = (2, 3)$ ,  $\theta = 30^\circ$ . So, the equation of the line is





$$\frac{x-2}{\cos 30^\circ} = \frac{y-3}{\sin 30^\circ} \text{ or, } \frac{x-2}{\frac{\sqrt{3}}{2}} = \frac{y-3}{\frac{1}{2}} \text{ or, } x-2 = \sqrt{3}(y-3) \text{ or, } x - \sqrt{3}y = 2 - 3\sqrt{3}.$$

Points on the line at a distance 4 from  $P(2, 3)$  are

$$(x_1 \pm r \cos \theta, y_1 \pm r \sin \theta) \text{ or, } (2 \pm 4 \cos 30^\circ, 3 \pm 4 \sin 30^\circ) \text{ or, } (2 \pm 2\sqrt{3}, 3 \pm 2)$$

or,  $(2 + 2\sqrt{3}, 5)$  and  $(2 - 2\sqrt{3}, 1)$ .

**EXAMPLE 2** The slope of a straight line through  $A(3, 2)$  is  $3/4$ . Find the coordinates of the points on the line that are 5 units away from  $A$ . **[NCERT EXEMPLAR]**

**SOLUTION** Suppose the given line makes an angle  $\theta$  with  $x$ -axis. It is given that its slope is  $3/4$ .

$$\therefore \tan \theta = \frac{3}{4} \Rightarrow \sin \theta = \frac{3}{5} \text{ and } \cos \theta = \frac{4}{5}$$

The equation of the line in distance form is  $\frac{x-3}{\cos \theta} = \frac{y-2}{\sin \theta}$  or,  $\frac{x-3}{4/5} = \frac{y-2}{3/5}$  and the coordinates of two points  $P$  and  $Q$  at a distance of 5 units from  $A$  are given by

$$\frac{x-3}{4/5} = \frac{y-2}{3/5} = \pm 5$$

$$\text{Now, } \frac{x-3}{4/5} = \frac{y-2}{3/5} = 5 \Rightarrow x-3 = \frac{4}{5} \times 5 \text{ and, } y-2 = \frac{3}{5} \times 5 \Rightarrow x=7, y=5$$

$$\text{and, } \frac{x-3}{4/5} = \frac{y-2}{3/5} = -5 \Rightarrow x-3 = \frac{4}{5} \times -5 \text{ and, } y-2 = \frac{3}{5} \times -5 \Rightarrow x=-1, y=-1$$

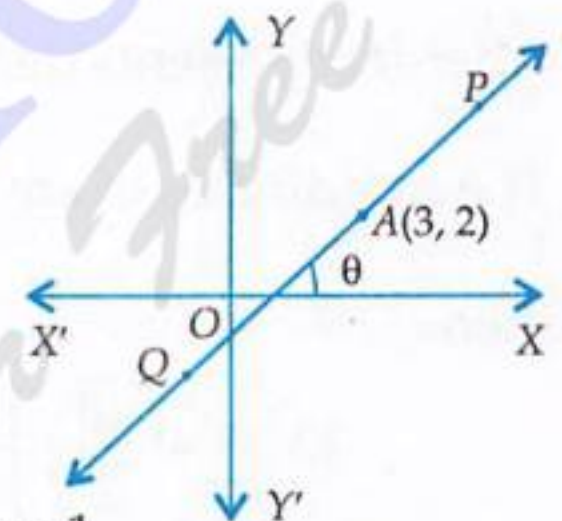


Fig. 22.49

Hence, the coordinates of  $P$  and  $Q$  are  $(7, 5)$  and  $(-1, -1)$  respectively.

**REMARK** The coordinates of  $P$  and  $Q$  are  $(3 \pm 5 \cos \theta, 2 \pm 5 \sin \theta)$ , where  $\cos \theta = \frac{4}{5}$  and  $\sin \theta = \frac{3}{5}$ .

#### BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 3** Find the equation of the line through the point  $A(2, 3)$  and making an angle of  $45^\circ$  with the  $x$ -axis. Also, determine the length of intercept on it between  $A$  and the line  $x + y + 1 = 0$ .

**SOLUTION** The equation of a line through  $A$  and making an angle of  $45^\circ$  with the  $x$ -axis is

$$\frac{x-2}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ} \Rightarrow \frac{x-2}{\frac{1}{\sqrt{2}}} = \frac{y-3}{\frac{1}{\sqrt{2}}} \Rightarrow x - y + 1 = 0$$

Suppose this line meets the line  $x + y + 1 = 0$  at  $P$  such that  $AP = r$ . Then, the coordinates of  $P$  are given by

$$\frac{x-2}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ} = r$$

$$\Rightarrow x = 2 + r \cos 45^\circ, y = 3 + r \sin 45^\circ$$

$$\Rightarrow x = 2 + \frac{r}{\sqrt{2}}, y = 3 + \frac{r}{\sqrt{2}}$$

Thus, the coordinates of  $P$  are  $\left(2 + \frac{r}{\sqrt{2}}, 3 + \frac{r}{\sqrt{2}}\right)$ . Since  $P$  lies on

$$x + y + 1 = 0.$$

$$\therefore 2 + \frac{r}{\sqrt{2}} + 3 + \frac{r}{\sqrt{2}} + 1 = 0 \Rightarrow \sqrt{2}r = -6 \Rightarrow r = -3\sqrt{2}$$

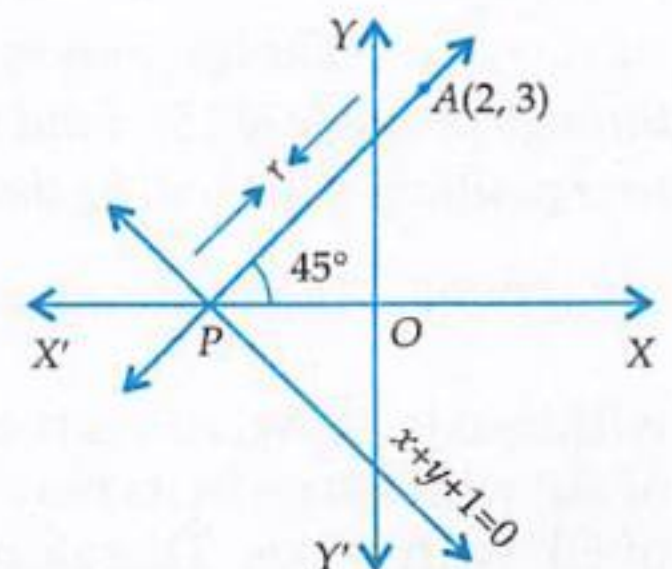


Fig. 22.50



$\therefore$  Length  $AP = |r| = 3\sqrt{2}$ . Thus, the length of the intercept is  $3\sqrt{2}$ .

**ALITER** The equation of the line through  $A(2, 3)$  and making an angle of  $45^\circ$  with  $x$ -axis is  $y - 3 = \tan 45^\circ (x - 2)$  or,  $x - y + 1 = 0$ .

This line intersects  $x + y + 1 = 0$  at  $P(-1, 0)$ .

$$\therefore AP = \sqrt{(2+1)^2 + (3-0)^2} = \sqrt{18} = 3\sqrt{2}$$

**EXAMPLE 4** If the straight line through the point  $P(3, 4)$  makes an angle  $\frac{\pi}{6}$  with  $x$ -axis and meets the line  $12x + 5y + 10 = 0$  at  $Q$ , find the length of  $PQ$ .

**SOLUTION** The equation of a line passing through  $P(3, 4)$  and making an angle  $\frac{\pi}{6}$  with  $x$ -axis is

$$\frac{x-3}{\cos \frac{\pi}{6}} = \frac{y-4}{\sin \frac{\pi}{6}} = r \text{ or, } \frac{x-3}{\frac{\sqrt{3}}{2}} = \frac{y-4}{\frac{1}{2}} = r$$

where  $r$  represents the distance of any point on this line from the given point  $P(3, 4)$ .

The coordinates of any point  $Q$  on this line are  $\left(3 + \frac{\sqrt{3}}{2}r, 4 + \frac{r}{2}\right)$ . If  $Q$  lies on  $12x + 5y + 10 = 0$ , then

$$12\left(3 + \frac{\sqrt{3}}{2}r\right) + 5\left(4 + \frac{r}{2}\right) + 10 = 0 \Rightarrow r = \frac{-132}{12\sqrt{3} + 5}$$

$$\text{Hence, length } PQ = \frac{132}{12\sqrt{3} + 5}.$$

**ALITER** The equation of the line through the point  $P(3, 4)$  and making an angle of  $\frac{\pi}{6}$  with  $x$ -axis is

$$y - 4 = \tan \frac{\pi}{6} (x - 3) \text{ or, } x - \sqrt{3}y + 4\sqrt{3} - 3 = 0$$

This intersects the line  $12x + 5y + 10 = 0$  at  $Q\left(\frac{15 - 30\sqrt{3}}{5 + 12\sqrt{3}}, \frac{48\sqrt{3} - 46}{5 + 12\sqrt{3}}\right)$ .

$$\therefore PQ = \sqrt{\left(\frac{15 - 30\sqrt{3}}{5 + 12\sqrt{3}} - 3\right)^2 + \left(\frac{48\sqrt{3} - 46}{5 + 12\sqrt{3}} - 4\right)^2} = \frac{132}{5 + 12\sqrt{3}}$$

#### BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

**EXAMPLE 5** The line joining two points  $A(2, 0)$ ,  $B(3, 1)$  is rotated about  $A$  in anti-clockwise direction through an angle of  $15^\circ$ . Find the equation of the line in the new position. If point  $B$  goes to point  $C$  in the new position, what will be the coordinates of  $C$ ?

**SOLUTION** The slope  $m$  of the line  $AB$  is given by  $m = \frac{1-0}{3-2} = 1$ . So,  $AB$  makes an angle of  $45^\circ$

with  $x$ -axis. Now,  $AB$  is rotated through  $15^\circ$  in anticlockwise direction and so it makes an angle of  $60^\circ$  with  $x$ -axis in its new position  $AC$ . Clearly,  $AC$  passes through  $A(2, 0)$  and makes an angle of  $60^\circ$  with  $x$ -axis. Therefore, the equation of  $AC$  in distance form is

$$\frac{x-2}{\cos 60^\circ} = \frac{y-0}{\sin 60^\circ} \text{ or, } \frac{x-2}{\frac{1}{2}} = \frac{y-0}{\frac{\sqrt{3}}{2}}$$



Clearly,  $AB = \sqrt{(3-2)^2 + (1-0)^2} = \sqrt{2}$ .

The point C is at a distance  $\sqrt{2}$  from A. So, the coordinates of C are given by

$$\frac{x-2}{\frac{1}{2}} = \frac{y-0}{\frac{\sqrt{3}}{2}} = \sqrt{2} \Rightarrow x = 2 + \frac{1}{2}\sqrt{2} = 2 + \frac{1}{\sqrt{2}} \text{ and } y = \frac{\sqrt{3}}{2} \times \sqrt{2} = \frac{\sqrt{6}}{2}.$$

Hence, the coordinates of C are  $\left(2 + \frac{1}{\sqrt{2}}, \frac{\sqrt{6}}{2}\right)$ .

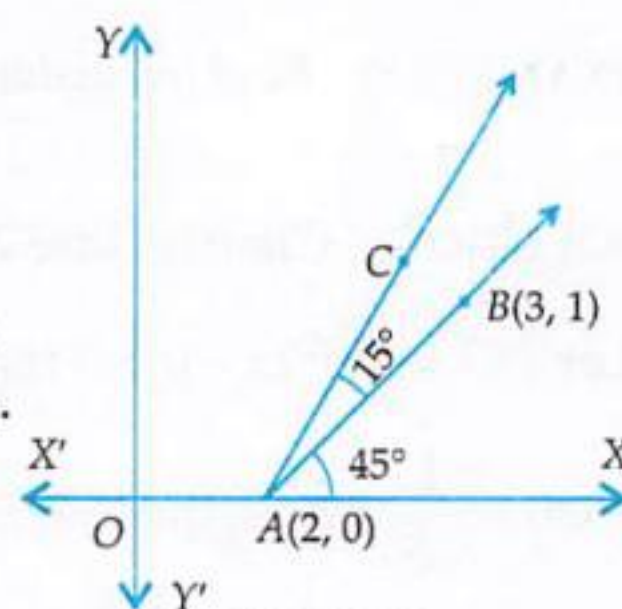


Fig. 22.51

**EXAMPLE 6** Find the distance of the line  $4x - y = 0$  from the point  $P(4, 1)$  measured along the line making an angle of  $135^\circ$  with the positive  $x$ -axis. [NCERT]

**SOLUTION** The equation in distance form of the line passing through  $P(4, 1)$  and making an angle of  $135^\circ$  with the positive  $x$ -axis is

$$\frac{x-4}{\cos 135^\circ} = \frac{y-1}{\sin 135^\circ}$$

Suppose it cuts  $4x - y = 0$  at  $Q$  such that  $PQ = r$ . Then, the coordinates of  $Q$  are given by

$$\frac{x-4}{\cos 135^\circ} = \frac{y-1}{\sin 135^\circ} = r$$

$$\Rightarrow \frac{x-4}{-1/\sqrt{2}} = \frac{y-1}{1/\sqrt{2}} = r \Rightarrow x = 4 - \frac{r}{\sqrt{2}}, y = 1 + \frac{r}{\sqrt{2}}$$

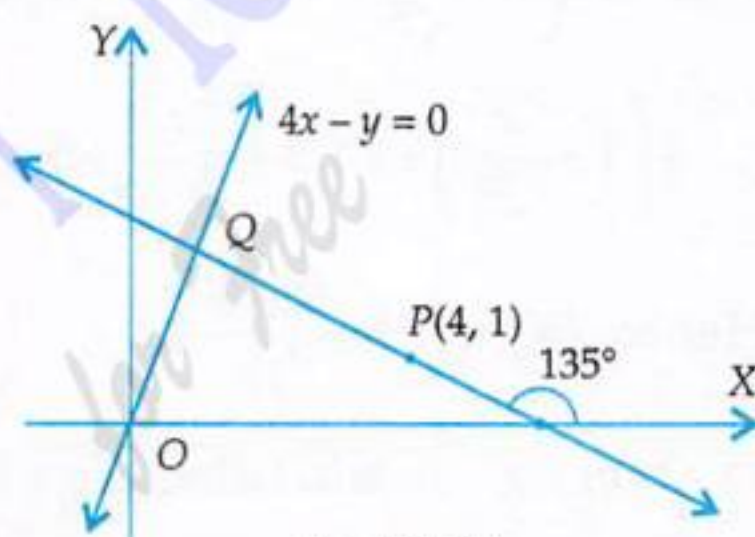


Fig. 22.52

So, the coordinates of  $Q$  are  $\left(4 - \frac{r}{\sqrt{2}}, 1 + \frac{r}{\sqrt{2}}\right)$ . Clearly,  $Q$  lies on  $4x - y = 0$ .

$$\therefore 16 - \frac{4r}{\sqrt{2}} - 1 - \frac{r}{\sqrt{2}} = 0 \Rightarrow \frac{5r}{\sqrt{2}} = 15 \Rightarrow r = 3\sqrt{2}. \text{ Hence, required distance is } 3\sqrt{2} \text{ units.}$$

**EXAMPLE 7** Find the distance of the point  $(2, 3)$  from the line  $2x - 3y + 9 = 0$  measured along a line  $x - y + 1 = 0$ .

**SOLUTION** The slope of the line  $x - y + 1 = 0$  is 1. So it makes an angle of  $45^\circ$  with  $x$ -axis.

The equation of a line, in distance form, passing through  $P(2, 3)$  and making an angle of  $45^\circ$  with  $x$ -axis is

$$\frac{x-2}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ}$$

The coordinates of any point  $Q$  on this line are given by

$$\frac{x-2}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ} = r \left[ \text{Using: } \frac{x-x_1}{\cos \theta} = \frac{y-y_1}{\sin \theta} = r \right]$$

So, the coordinates of  $Q$  are

$$(2 + r \cos 45^\circ, 3 + r \sin 45^\circ) \text{ or, } \left(2 + \frac{r}{\sqrt{2}}, 3 + \frac{r}{\sqrt{2}}\right)$$

If this point lies on the line  $2x - 3y + 9 = 0$ , then

$$4 + r\sqrt{2} - 9 - \frac{3r}{\sqrt{2}} + 9 = 0 \Rightarrow r = 4\sqrt{2} \Rightarrow PQ = 4\sqrt{2}$$

Hence, the required distance is  $4\sqrt{2}$  units.

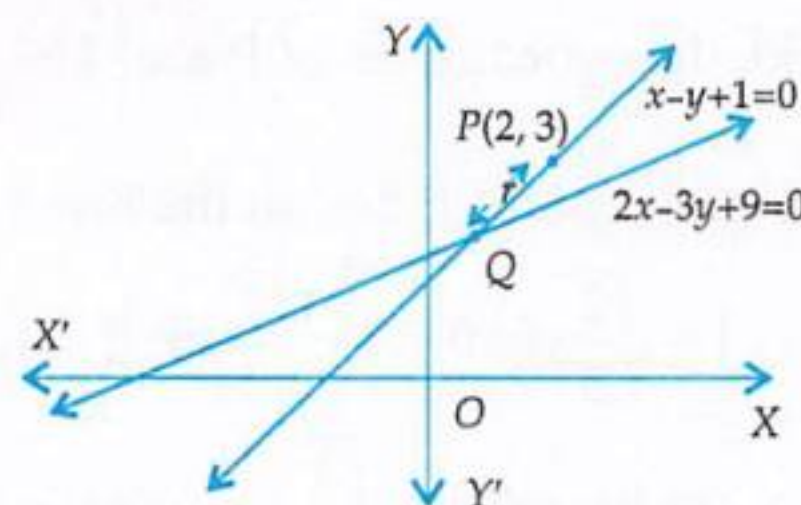


Fig. 22.53



**EXAMPLE 8** Find the distance of the line  $4x + 7y + 5 = 0$  from the point  $(1, 2)$  along the line  $2x - y = 0$ . [NCERT]

**SOLUTION** Clearly, line  $2x - y = 0$  passes through  $P(1, 2)$  and intersects  $4x + 7y + 5 = 0$  at  $Q$ .

Let  $PQ = r$ . If  $2x - y = 0$  makes an angle  $\theta$  with  $x$ -axis. Then,  $\tan \theta = 2$  and hence  $\sin \theta = \frac{2}{\sqrt{5}}$  and  $\cos \theta = \frac{1}{\sqrt{5}}$ . Thus, the line  $2x - y = 0$  passes through  $P(1, 2)$  and makes an angle  $\theta$  with  $x$ -axis such

that  $\cos \theta = \frac{1}{\sqrt{5}}$  and  $\sin \theta = \frac{2}{\sqrt{5}}$ . So, its equation in distance form is  $\frac{x-1}{1/\sqrt{5}} = \frac{y-2}{2/\sqrt{5}} = r$  and the

coordinates of  $Q$  are given by  $\frac{x-1}{1/\sqrt{5}} = \frac{y-2}{2/\sqrt{5}} = r$ .

Now,  $\frac{x-1}{1/\sqrt{5}} = \frac{y-2}{2/\sqrt{5}} = r \Rightarrow x = 1 + \frac{r}{\sqrt{5}}, y = 2 + \frac{2r}{\sqrt{5}}$ . So, the coordinates

of  $Q$  are  $\left(1 + \frac{r}{\sqrt{5}}, 2 + \frac{2r}{\sqrt{5}}\right)$ . As  $Q$  lies on  $4x + 7y + 5 = 0$ .

$$\therefore 4\left(1 + \frac{r}{\sqrt{5}}\right) + 7\left(2 + \frac{2r}{\sqrt{5}}\right) + 5 = 0 \Rightarrow \frac{18r}{\sqrt{5}} = -23 \Rightarrow r = -\frac{23\sqrt{5}}{18}$$

Hence,  $PQ = |r| = \frac{23\sqrt{5}}{18}$ .

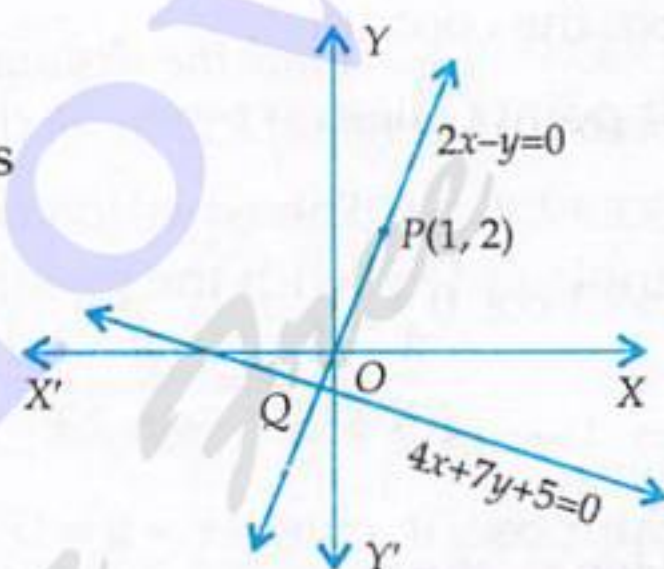


Fig. 22.54

**EXAMPLE 9** In what direction a line be drawn through the point  $(1, 2)$  that its point of intersection with the line  $x + y = 4$  is at a distance  $\frac{\sqrt{6}}{3}$  from the given point. [NCERT EXEMPLAR]

**SOLUTION** Let the line drawn through  $A(1, 2)$  makes an angle  $\theta$  with the positive direction of  $x$ -axis and intersects the line  $x + y = 4$  at  $P$  such that  $AP = \frac{\sqrt{6}}{3}$ . Then, the coordinates of  $P$  are given by

$$\frac{x-1}{\cos \theta} = \frac{y-2}{\sin \theta} = \frac{\sqrt{6}}{3} \Rightarrow x = 1 + \sqrt{\frac{2}{3}} \cos \theta, y = 2 + \sqrt{\frac{2}{3}} \sin \theta$$

So, the coordinates of  $P$  are  $\left(1 + \sqrt{\frac{2}{3}} \cos \theta, 2 + \sqrt{\frac{2}{3}} \sin \theta\right)$ .

Clearly, point  $P$  lies on the line  $x + y = 4$ .

$$\therefore 1 + \sqrt{\frac{2}{3}} \cos \theta + 2 + \sqrt{\frac{2}{3}} \sin \theta = 4$$

$$\Rightarrow \cos \theta + \sin \theta = \sqrt{\frac{3}{2}} \Rightarrow (\cos \theta + \sin \theta)^2 = \frac{3}{2}$$

$$\Rightarrow 1 + \sin 2\theta = \frac{3}{2} \Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6} \text{ or } 2\theta = \frac{5\pi}{6} \Rightarrow \frac{\pi}{12}, \frac{5\pi}{12}$$

Hence, the line drawn makes an angle whose measure is either  $\frac{\pi}{12}$  or  $\frac{5\pi}{12}$  with the  $x$ -axis.

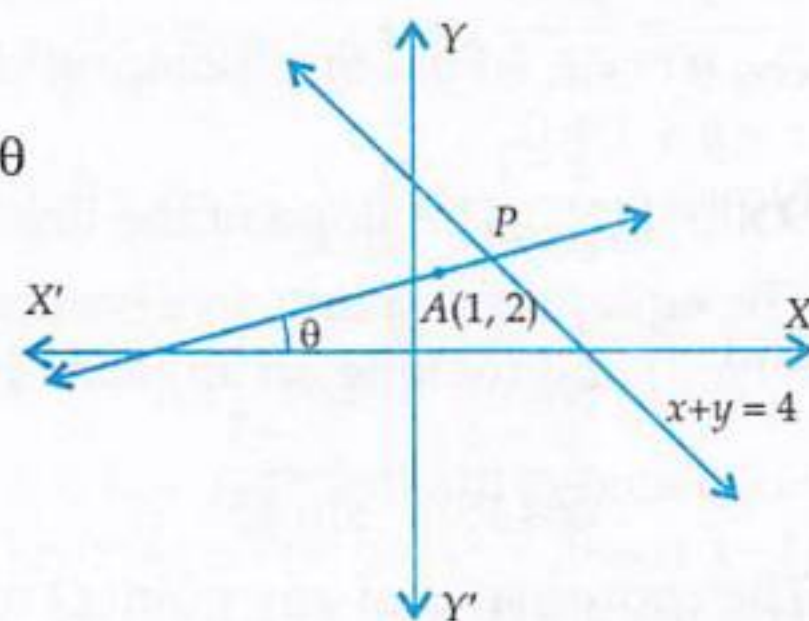


Fig. 22.55

**EXAMPLE 10** Find the direction in which a straight line must be drawn through the point  $(-1, 2)$  so that its point of intersection with the line  $x + y = 4$  may be at a distance of 3 units from the point. [NCERT]



**SOLUTION** Suppose the required line makes an angle  $\theta$  with  $x$ -axis. It passes through the point  $P(-1, 2)$ . So, its equation is

$$\frac{x - (-1)}{\cos \theta} = \frac{y - 2}{\sin \theta} \text{ or, } \frac{x + 1}{\cos \theta} = \frac{y - 2}{\sin \theta}$$

The coordinates of a point  $Q$  on this line at a distance of 3 units from  $P(-1, 2)$  are given by

$$\frac{x + 1}{\cos \theta} = \frac{y - 2}{\sin \theta} = 3$$

$$\Rightarrow x = -1 + 3 \cos \theta, y = 2 + 3 \sin \theta$$

So, the coordinates of  $Q$  are:  $(-1 + 3 \cos \theta, 2 + 3 \sin \theta)$ .

If point  $Q$  lies on  $x + y = 4$ , then

$$-1 + 3 \cos \theta + 2 + 3 \sin \theta = 4$$

$$\Rightarrow 3 \cos \theta + 3 \sin \theta = 3 \Rightarrow \cos \theta + \sin \theta = 1 \Rightarrow (\cos \theta + \sin \theta)^2 = 1$$

$$\Rightarrow 1 + \sin 2\theta = 1 \Rightarrow \sin 2\theta = 0 \Rightarrow 2\theta = 0 \text{ or } 2\theta = \pi \Rightarrow \theta = 0 \text{ or } \theta = \frac{\pi}{2}$$

Hence, the required line must be either parallel to  $x$ -axis or to  $y$ -axis.

**EXAMPLE 11** A line is such that its segment between the lines  $5x - y + 4 = 0$  and  $3x + 4y - 4 = 0$  is bisected at the point  $(1, 5)$ . Obtain its equation. [NCERT]

**SOLUTION** Let  $AB$  be the line making angle  $\theta$  with  $x$ -axis such that its intercept  $AB$  between the lines  $5x - y + 4 = 0$  and  $3x + 4y - 4 = 0$  is bisected at  $P(1, 5)$ . Then, the equation of the line is

$$\frac{x - 1}{\cos \theta} = \frac{y - 5}{\sin \theta} \text{ or, } y - 5 = \tan \theta (x - 1) \quad \dots(i)$$

Let  $AP = BP = r$ . Then, the coordinates of  $A$  and  $B$  are given by  $\frac{x - 1}{\cos \theta} = \frac{y - 5}{\sin \theta} = r$  and

$$\frac{x - 1}{\cos \theta} = \frac{y - 5}{\sin \theta} = -r \text{ respectively.}$$

$$\text{Now, } \frac{x - 1}{\cos \theta} = \frac{y - 5}{\sin \theta} = r \Rightarrow x = 1 + r \cos \theta, y = 5 + r \sin \theta$$

$$\text{and, } \frac{x - 1}{\cos \theta} = \frac{y - 5}{\sin \theta} = -r \Rightarrow x = 1 - r \cos \theta, y = 5 - r \sin \theta$$

So, the co-ordinates of  $A$  and  $B$  are  $(1 + r \cos \theta, 5 + r \sin \theta)$  and  $(1 - r \cos \theta, 5 - r \sin \theta)$  respectively. Points  $A$  and  $B$  lie on lines  $5x - y + 4 = 0$  and  $3x + 4y - 4 = 0$  respectively.

$$\therefore 5(1 + r \cos \theta) - (5 + r \sin \theta) + 4 = 0$$

$$\text{and, } 3(1 - r \cos \theta) + 4(5 - r \sin \theta) - 4 = 0$$

$$\Rightarrow r(5 \cos \theta - \sin \theta) = -4 \text{ and } r(3 \cos \theta + 4 \sin \theta) = 19$$

$$\Rightarrow r = \frac{-4}{5 \cos \theta - \sin \theta} \text{ and } r = \frac{19}{3 \cos \theta + 4 \sin \theta}$$

$$\Rightarrow \frac{-4}{5 \cos \theta - \sin \theta} = \frac{19}{3 \cos \theta + 4 \sin \theta}$$

$$\Rightarrow -12 \cos \theta - 16 \sin \theta = 95 \cos \theta - 19 \sin \theta \Rightarrow 107 \cos \theta = 3 \sin \theta \Rightarrow \tan \theta = \frac{107}{3}$$

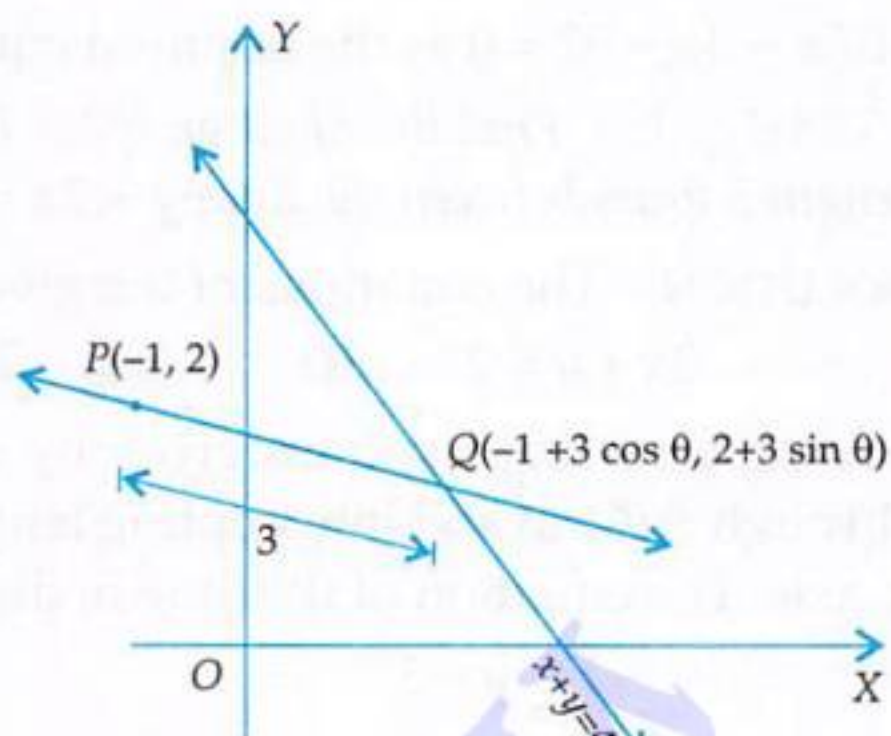


Fig. 22.56

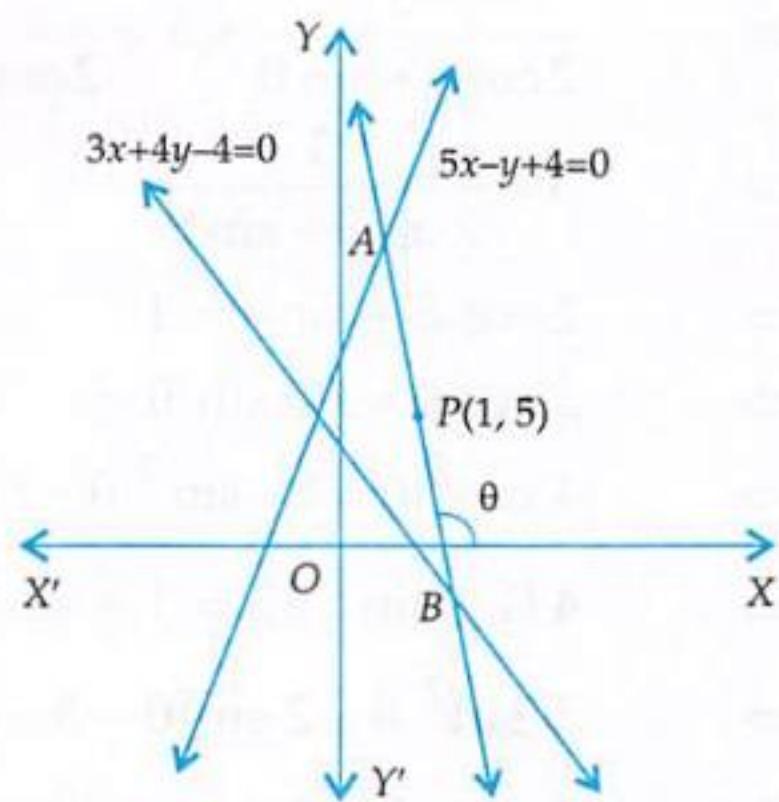


Fig. 22.57



Putting the value of  $\tan \theta$  in (i), we obtain  $y - 5 = \tan \theta (x - 1)$  or,  $y - 5 = \frac{107}{3} (x - 1)$  or,  $107x - 3y - 92 = 0$  as the required equation of the line.

**EXAMPLE 12** Find the equation of the line passing through the point  $(2, 3)$  and making an intercept of length 3 units between the lines  $y + 2x = 2$  and  $y + 2x = 5$ .

**SOLUTION** The equations of the given lines are

$$2x + y = 2 \quad \dots(i) \quad \text{and,} \quad 2x + y = 5 \quad \dots(ii)$$

We observe that the lines given by equations (i) and (ii) are parallel. Suppose a line passing through  $A(2, 3)$  and intercepting length  $BC = 3$  between lines (i) and (ii) makes an angle  $\theta$  with  $x$ -axis. The equation of this line in distance form is

$$\frac{x-2}{\cos \theta} = \frac{y-3}{\sin \theta} \quad \dots(iii)$$

Let  $AB = r$ . Then,  $AC = AB + BC = (r + 3)$ . Clearly,  $B$  and  $C$  are points on line (iii) at distances  $r$  and  $r + 3$  respectively from  $A$ . So, the coordinates of  $B$  and  $C$  are given by

$$\frac{x-2}{\cos \theta} = \frac{y-3}{\sin \theta} = r \quad \text{and} \quad \frac{x-2}{\cos \theta} = \frac{y-3}{\sin \theta} = r + 3 \quad \text{respectively.}$$

Now,

$$\frac{x-2}{\cos \theta} = \frac{y-3}{\sin \theta} = r \Rightarrow x = 2 + r \cos \theta, y = 3 + r \sin \theta$$

$$\text{and, } \frac{x-2}{\cos \theta} = \frac{y-3}{\sin \theta} = r + 3 \Rightarrow x = 2 + (r + 3) \cos \theta, y = 3 + (r + 3) \sin \theta$$

So, the coordinates of  $B$  and  $C$  are

$$(2 + r \cos \theta, 3 + r \sin \theta) \text{ and } (2 + (r + 3) \cos \theta, 3 + (r + 3) \sin \theta) \text{ respectively.}$$

We observe that points  $B$  and  $C$  lie on lines (i) and (ii) respectively.

$$\therefore 2(2 + r \cos \theta) + (3 + r \sin \theta) = 2 \text{ and, } 2[2 + (r + 3) \cos \theta] + [3 + (r + 3) \sin \theta] = 5$$

$$\Rightarrow r(2 \cos \theta + \sin \theta) = -5 \text{ and, } (r + 3)(2 \cos \theta + \sin \theta) = -2$$

$$\Rightarrow r = \frac{-5}{2 \cos \theta + \sin \theta} \text{ and } r + 3 = \frac{-2}{2 \cos \theta + \sin \theta}$$

$$\Rightarrow \frac{-5}{2 \cos \theta + \sin \theta} + 3 = \frac{-2}{2 \cos \theta + \sin \theta} \quad \text{[On eliminating } r]$$

$$\Rightarrow 3 = \frac{3}{2 \cos \theta + \sin \theta} \quad \dots(iv)$$

$$\Rightarrow 2 \cos \theta + \sin \theta = 1$$

$$\Rightarrow 2 \cos \theta = 1 - \sin \theta \quad \dots(v)$$

$$\Rightarrow 4 \cos^2 \theta = 1 + \sin^2 \theta - 2 \sin \theta$$

$$\Rightarrow 4(1 - \sin^2 \theta) = 1 + \sin^2 \theta - 2 \sin \theta$$

$$\Rightarrow 5 \sin^2 \theta - 2 \sin \theta - 3 = 0 \Rightarrow (\sin \theta - 1)(5 \sin \theta + 3) = 0 \Rightarrow \sin \theta = 1 \text{ or } \sin \theta = -3/5$$

Putting  $\sin \theta = 1$  in (v), we obtain  $\cos \theta = 0$ . Putting  $\sin \theta = -3/5$  in (v), we obtain  $\cos \theta = 4/5$ .

Substituting the values of  $\sin \theta$  and  $\cos \theta$  in (iii), we obtain that the equations of the required lines are

$$\frac{x-2}{0} = \frac{y-3}{1} \text{ and } \frac{x-2}{4/5} = \frac{y-3}{-3/5} \text{ or, } x-2=0 \text{ and } 3x+4y=18.$$

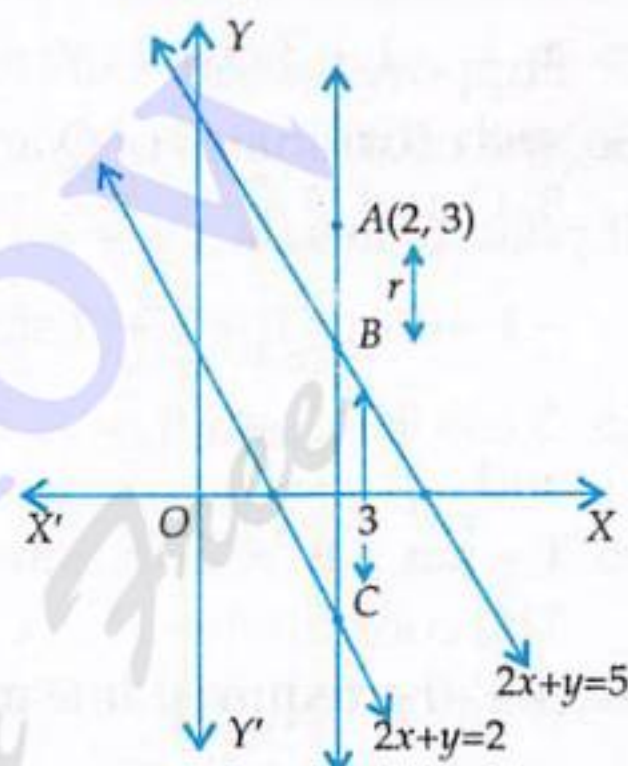


Fig. 22.58



**EXAMPLE 13** A line through  $A(-5, -4)$  meets the lines  $x + 3y + 2 = 0$ ,  $2x + y + 4 = 0$  and  $x - y - 5 = 0$  at the points  $B$ ,  $C$  and  $D$  respectively, if  $\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$  find the equation of the line.

**SOLUTION** The equations of the given lines are

$$x + 3y + 2 = 0 \quad \dots(i), \quad 2x + y + 4 = 0 \quad \dots(ii) \quad \text{and} \quad x - y - 5 = 0 \quad \dots(iii)$$

The equation of a line passing through  $A(-5, -4)$  and making an angle  $\theta$  with  $x$ -axis is

$$\frac{x + 5}{\cos \theta} = \frac{y + 4}{\sin \theta} \quad \dots(iv)$$

Suppose this line cuts lines (i), (ii) and (iii) at  $B$ ,  $C$  and  $D$  respectively such that  $AB = r_1$ ,  $AC = r_2$  and  $AD = r_3$ . Then, the coordinates of  $B$ ,  $C$  and  $D$  are given by

$$\frac{x + 5}{\cos \theta} = \frac{y + 4}{\sin \theta} = r_1, \quad \frac{x + 5}{\cos \theta} = \frac{y + 4}{\sin \theta} = r_2$$

and  $\frac{x + 5}{\cos \theta} = \frac{y + 4}{\sin \theta} = r_3$  respectively.

The coordinates of  $B$ ,  $C$  and  $D$  are  $(-5 + r_1 \cos \theta, -4 + r_1 \sin \theta)$ ,  $(-5 + r_2 \cos \theta, -4 + r_2 \sin \theta)$ , and  $(-5 + r_3 \cos \theta, -4 + r_3 \sin \theta)$  respectively.

Points  $B$ ,  $C$  and  $D$  lie on lines (i), (ii) and (iii) respectively.

$$\therefore (-5 + r_1 \cos \theta) + 3(-4 + r_1 \sin \theta) + 2 = 0, \quad 2(-5 + r_2 \cos \theta) + (-4 + r_2 \sin \theta) + 4 = 0$$

$$\text{and} \quad (-5 + r_3 \cos \theta) - (-4 + r_3 \sin \theta) - 5 = 0$$

$$\Rightarrow r_1 = \frac{15}{\cos \theta + 3 \sin \theta}, \quad r_2 = \frac{10}{2 \cos \theta + \sin \theta} \quad \text{and} \quad r_3 = \frac{6}{\cos \theta - \sin \theta}$$

$$\Rightarrow \frac{15}{AB} = \cos \theta + 3 \sin \theta, \quad \frac{10}{AC} = 2 \cos \theta + \sin \theta \quad \text{and} \quad \frac{6}{AD} = \cos \theta - \sin \theta$$

Substituting these values in  $\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$ , we obtain

$$(\cos \theta + 3 \sin \theta)^2 + (2 \cos \theta + \sin \theta)^2 = (\cos \theta - \sin \theta)^2$$

$$\Rightarrow 4 \cos^2 \theta + 12 \sin \theta \cos \theta + 9 \sin^2 \theta = 0 \Rightarrow (2 \cos \theta + 3 \sin \theta)^2 = 0$$

$$\Rightarrow 2 \cos \theta + 3 \sin \theta = 0 \Rightarrow \tan \theta = -2/3$$

Putting  $\tan \theta = -2/3$  in (iv), we obtain

$$y + 4 = -(2/3)(x + 5) \quad \text{or,} \quad 2x + 3y + 22 = 0 \quad \text{as the required equation of the line.}$$

**EXAMPLE 14** The sides  $AB$  and  $AC$  of a triangle  $ABC$  are respectively  $2x + 3y = 29$  and  $x + 2y = 16$  respectively. If the mid-point of  $BC$  is  $(5, 6)$  then find the equation of  $BC$ .

**SOLUTION** Suppose  $BC$  makes an angle  $\theta$  with the  $x$ -axis. Then, its equation is

$$\frac{x - 5}{\cos \theta} = \frac{y - 6}{\sin \theta} \quad \dots(i)$$

Let  $BD = CD = r$ . Then, the coordinates  $B$  and  $C$  are given by

$$\frac{x - 5}{\cos \theta} = \frac{y - 6}{\sin \theta} = -r \quad \text{and} \quad \frac{x - 5}{\cos \theta} = \frac{y - 6}{\sin \theta} = r \quad \text{respectively.}$$

$$\text{Now,} \quad \frac{x - 5}{\cos \theta} = \frac{y - 6}{\sin \theta} = -r \Rightarrow x = 5 - r \cos \theta, \quad y = 6 - r \sin \theta$$

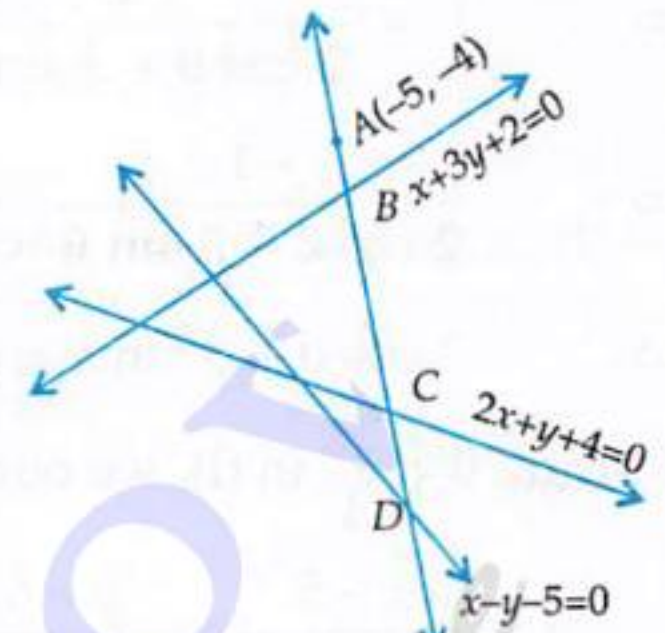


Fig. 22.59

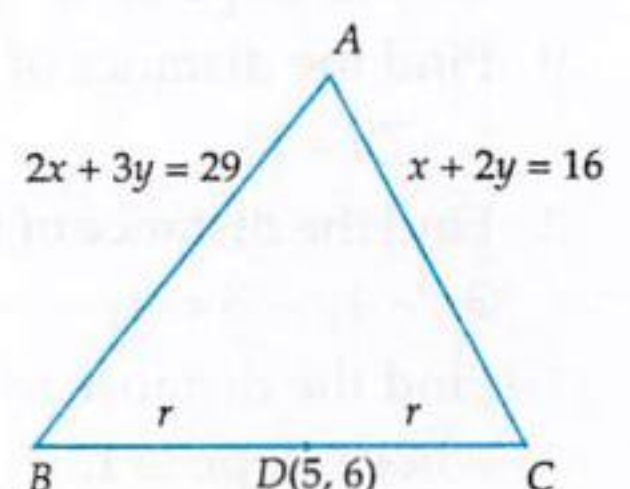


Fig. 22.60



and,  $\frac{x-5}{\cos \theta} = \frac{y-6}{\sin \theta} = r \Rightarrow x = 5 + r \cos \theta, y = 6 + r \sin \theta$

Thus, the coordinates of  $B$  and  $C$  are  $B(5 - r \cos \theta, 6 - r \sin \theta)$  and  $C(5 + r \cos \theta, 6 + r \sin \theta)$  respectively which lie on lines  $2x + 3y = 29$  and  $x + 2y = 16$  respectively.

$\therefore 2(5 - r \cos \theta) + 3(6 - r \sin \theta) = 29$  and  $(5 + r \cos \theta) + 2(6 + r \sin \theta) = 16$

$\Rightarrow r = \frac{-1}{2 \cos \theta + 3 \sin \theta}$  and  $r = \frac{-1}{\cos \theta + 2 \sin \theta}$

$\Rightarrow \frac{-1}{2 \cos \theta + 3 \sin \theta} = \frac{-1}{\cos \theta + 2 \sin \theta}$  [On eliminating  $r$ ]

$\Rightarrow 2 \cos \theta + 3 \sin \theta = \cos \theta + 2 \sin \theta \Rightarrow \sin \theta = -\cos \theta \Rightarrow \tan \theta = -1 \Rightarrow \theta = 3\pi/4$

Putting  $\theta = \frac{3\pi}{4}$  in (i), we obtain

$\frac{x-5}{\cos 3\pi/4} = \frac{y-6}{\sin 3\pi/4}$  or,  $x + y - 11 = 0$  as the required equation of the line.

### EXERCISE 22.8

#### BASIC

1. A line passes through a point  $A(1, 2)$  and makes an angle of  $60^\circ$  with the  $x$ -axis and intersects the line  $x + y = 6$  at the point  $P$ . Find  $AP$ .
2. If the straight line through the point  $P(3, 4)$  makes an angle  $\pi/6$  with the  $x$ -axis and meets the line  $12x + 5y + 10 = 0$  at  $Q$ , find the length  $PQ$ .
3. A straight line drawn through the point  $A(2, 1)$  making an angle  $\pi/4$  with positive  $x$ -axis intersects another line  $x + 2y + 1 = 0$  in the point  $B$ . Find length  $AB$ .

#### BASED ON LOTS

4. A line is drawn through  $A(4, -1)$  parallel to the line  $3x - 4y + 1 = 0$ . Find the coordinates of the two points on this line which are at a distance of 5 units from  $A$ .
5. The straight line through  $P(x_1, y_1)$  inclined at an angle  $\theta$  with the  $x$ -axis meets the line  $ax + by + c = 0$  in  $Q$ . Find the length of  $PQ$ .

#### BASED ON HOTS

6. Find the distance of the point  $(2, 3)$  from the line  $2x - 3y + 9 = 0$  measured along a line making an angle of  $45^\circ$  with the  $x$ -axis.
7. Find the distance of the point  $(3, 5)$  from the line  $2x + 3y = 14$  measured parallel to a line having slope  $1/2$ .
8. Find the distance of the point  $(2, 5)$  from the line  $3x + y + 4 = 0$  measured parallel to a line having slope  $3/4$ .
9. Find the distance of the point  $(3, 5)$  from the line  $2x + 3y = 14$  measured parallel to the line  $x - 2y = 1$ .
10. Find the distance of the point  $(2, 5)$  from the line  $3x + y + 4 = 0$  measured parallel to the line  $3x - 4y + 8 = 0$ .
11. Find the distance of the line  $2x + y = 3$  from the point  $(-1, -3)$  in the direction of the line whose slope is 1.



12. A line is such that its segment between the straight lines  $5x - y - 4 = 0$  and  $3x + 4y - 4 = 0$  is bisected at the point  $(1, 5)$ . Obtain its equation.
13. Find the equation of straight line passing through  $(-2, -7)$  and having an intercept of length 3 between the straight lines  $4x + 3y = 12$  and  $4x + 3y = 3$ .
14. In what direction should a line be drawn through the point  $(1, 2)$  so that its point of intersection with the line  $x + y = 4$  is at a distance  $\frac{\sqrt{6}}{3}$  from the given point.

[NCERT EXEMPLAR]

ANSWERS

1.  $3(\sqrt{3} - 1)$       2.  $\frac{132}{5 + 12\sqrt{3}}$       3.  $\frac{5\sqrt{2}}{3}$       4.  $(8, 2), (0, -4)$
5.  $\left| \frac{ax_1 + by_1 + c}{a \cos \theta + b \sin \theta} \right|$       6.  $4\sqrt{2}$       7.  $\sqrt{5}$       8. 5 units
9.  $\sqrt{5}$       10. 5      11.  $\frac{8\sqrt{2}}{3}$       12.  $83x - 35y + 92 = 0$
13.  $x + 2 = 0, 7x + 24y + 182 = 0$       14.  $\frac{\pi}{12}$  or  $\frac{5\pi}{12}$

## 22.7 TRANSFORMATION OF GENERAL EQUATION IN DIFFERENT STANDARD FORMS

The general equation of a straight line is  $Ax + By + C = 0$  which can be transformed to various standard forms as discussed below.

(i) Transformation of  $Ax + By + C = 0$  in the slope intercept form ( $y = mx + c$ ):

The equation of the line is

$$Ax + By + C = 0 \Rightarrow By = -Ax - C \Rightarrow y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)$$

This is of the form  $y = mx + c$ , where  $m = -\frac{A}{B}$  and,  $c = -\frac{C}{B}$ .

Thus, for the straight line  $Ax + By + C = 0$ , we obtain:

$$m = \text{Slope} = -\frac{A}{B} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y} \text{ and, Intercept on } y\text{-axis} = -\frac{C}{B} = -\frac{\text{Constant term}}{\text{Coefficient of } y}$$

**NOTE** To determine the slope of a line by the formula  $m = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y}$ , we must first transfer all terms in the equation on one side.

(ii) Transformation of  $Ax + By + C = 0$  in intercept form  $\frac{x}{a} + \frac{y}{b} = 1$ : The equation of the line is

$$Ax + By + C = 0 \Rightarrow Ax + By = -C \Rightarrow \frac{Ax}{-C} + \frac{By}{-C} = 1 \Rightarrow \frac{x}{\left(-\frac{C}{A}\right)} + \frac{y}{\left(-\frac{C}{B}\right)} = 1$$

This is of the form  $\frac{x}{a} + \frac{y}{b} = 1$ . Thus, for the straight line  $Ax + By + C = 0$ , we obtain

$$\text{Intercept on } x\text{-axis} = -\frac{C}{A} = -\frac{\text{Constant term}}{\text{Coefficient of } x}, \text{ Intercept on } y\text{-axis} = -\frac{C}{B} = -\frac{\text{Constant term}}{\text{Coefficient of } y}$$

**NOTE** As discussed above the intercepts made by a line with the coordinate axes can be determined by reducing its equation to intercept form. We may also use the following method to determine the intercepts on the coordinate axes:



For intercept on x-axis : Put  $y = 0$  in the equation of the line and find the value of  $x$ . Similarly to find y-intercept, put  $x = 0$  in the equation of the line and find the value of  $y$ .

(iii) Transformation of  $Ax + By + C = 0$  in the normal form  $x \cos \alpha + y \sin \alpha = p$ :

We have,  $Ax + By + C = 0$  ... (i)

Let  $x \cos \alpha + y \sin \alpha - p = 0$  ... (ii)

be the normal form of  $Ax + By + C = 0$ .

Then, (i) and (ii) represent the same straight line.

$$\therefore \frac{A}{\cos \alpha} = \frac{B}{\sin \alpha} = \frac{C}{-p} \Rightarrow \cos \alpha = -\frac{Ap}{C} \text{ and, } \sin \alpha = -\frac{Bp}{C} \quad \dots (iii)$$

$$\Rightarrow \cos^2 \alpha + \sin^2 \alpha = \frac{A^2 p^2}{C^2} + \frac{B^2 p^2}{C^2} \Rightarrow 1 = \frac{p^2}{C^2} (A^2 + B^2) \Rightarrow p = \pm \frac{C}{\sqrt{A^2 + B^2}}$$

But,  $p$  denotes the length of the perpendicular from the origin to the line and is always positive.

$$\therefore p = \frac{|C|}{\sqrt{A^2 + B^2}}$$

Putting the value of  $p$  in (iii), we obtain:  $\cos \alpha = -\frac{A}{\sqrt{A^2 + B^2}}$  and,  $\sin \alpha = -\frac{B}{\sqrt{A^2 + B^2}}$

Thus, the equation (ii) takes the form

$$-\frac{A}{\sqrt{A^2 + B^2}} x - \frac{B}{\sqrt{A^2 + B^2}} y - \frac{C}{\sqrt{A^2 + B^2}} = 0 \text{ or, } -\frac{A}{\sqrt{A^2 + B^2}} x - \frac{B}{\sqrt{A^2 + B^2}} y = \frac{C}{\sqrt{A^2 + B^2}}$$

This is the required normal form of the line  $Ax + By + C = 0$ .

In order to transform the general equation of a line to the normal form, we use the following steps :

Step I Shift the constant term on the RHS and make it positive

Step II Divide both sides by  $\sqrt{(\text{Coefficient of } x)^2 + (\text{Coefficient of } y)^2}$

The equation so obtained is in the normal form.

### ILLUSTRATIVE EXAMPLES

#### BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** Transform the equation of the line  $\sqrt{3}x + y - 8 = 0$  to (i) slope intercept form and find its slope and y-intercept (ii) intercept form and find intercepts on the coordinate axes (iii) normal form and find the inclination of the perpendicular segment from the origin on the line with the axis and its length.

**SOLUTION** (i) Given equation is

$$\sqrt{3}x + y - 8 = 0 \Rightarrow y = -\sqrt{3}x + 8, \text{ which is the slope intercept form of the given line.}$$

$$\therefore \text{Slope} = -\sqrt{3}, \text{ and } y\text{-intercept} = 8$$

(ii) Given equation is

$$\sqrt{3}x + y - 8 = 0 \Rightarrow \frac{x}{8/\sqrt{3}} + \frac{y}{8} = 1, \text{ which is the intercept form of the given line.}$$

$$\text{So, } x\text{-intercept} = \frac{8}{\sqrt{3}} \text{ and, } y\text{-intercept} = 8$$

(iii) We have,  $\sqrt{3}x + y - 8 = 0$  or,  $\sqrt{3}x + y = 8$ . Dividing throughout by

$\sqrt{(\text{Coefficient of } x)^2 + (\text{Coefficient of } y)^2}$ , we obtain



$$\Rightarrow \frac{\frac{\sqrt{3}}{\sqrt{(\sqrt{3})^2 + 1^2}} x + \frac{1}{\sqrt{(\sqrt{3})^2 + 1^2}} y = \frac{8}{\sqrt{(\sqrt{3})^2 + 1^2}} \Rightarrow \frac{\sqrt{3}}{2} x + \frac{1}{2} y = 4,$$

which is the normal form of the given line.

Comparing this equation with  $x \cos \alpha + y \sin \alpha = p$ , we obtain  $\cos \alpha = \frac{\sqrt{3}}{2}$ ,  $\sin \alpha = \frac{1}{2}$  and  $p = 4$ .

As  $\sin \alpha$  and  $\cos \alpha$  both are positive, therefore  $\alpha$  is in first quadrant and is equal to  $\pi/6$ . Hence, for the given line, we have  $\alpha = \pi/6$  and  $p = 4$ .

**EXAMPLE 2** Reduce the lines  $3x - 4y + 4 = 0$  and  $4x - 3y + 12 = 0$  to the normal form and hence determine which line is nearer to the origin.

**SOLUTION** The equation of the first line is  $3x - 4y + 4 = 0$  or,  $-3x + 4y = 4$

Dividing throughout by  $\sqrt{(-3)^2 + (4)^2}$ , we obtain

$$-\frac{3x}{\sqrt{(-3)^2 + 4^2}} + \frac{4y}{\sqrt{(-3)^2 + 4^2}} = \frac{4}{\sqrt{(-3)^2 + 4^2}} \text{ or, } -\frac{3}{5}x + \frac{4}{5}y = \frac{4}{5}$$

This is the normal form of  $3x - 4y + 4 = 0$  from which we find that the length of the perpendicular from the origin to it is given by  $p_1 = 4/5$ .

The equation of the second line is  $4x - 3y + 12 = 0$  or,  $-4x + 3y = 12$

Dividing throughout by  $\sqrt{(\text{Coefficient of } x)^2 + (\text{Coefficient of } y)^2}$ , we obtain

$$-\frac{4x}{\sqrt{(-4)^2 + 3^2}} + \frac{3y}{\sqrt{(-4)^2 + 3^2}} = \frac{12}{\sqrt{(-4)^2 + 3^2}} \text{ or, } -\frac{4}{5}x + \frac{3}{5}y = \frac{12}{5}$$

This is the normal form of  $4x - 3y + 12 = 0$  from which we find that the length of the perpendicular from the origin is given by  $p_2 = \frac{12}{5}$ . Clearly,  $p_2 > p_1$ . Therefore, the line  $3x - 4y + 4 = 0$  is nearer to the origin.

**EXAMPLE 3** Find the values of  $k$  for which the line  $(k-3)x - (4-k^2)y + k^2 - 7k + 6 = 0$  is

- (i) parallel to the  $x$ -axis. (ii) parallel to the  $y$ -axis. (iii) passing through the origin.

[NCERT]

**SOLUTION** Let  $m$  be the slope of the line  $(k-3)x - (4-k^2)y + k^2 - 7k + 6 = 0$  ... (i)

$$\text{Then, } m = -\frac{(k-3)}{-(4-k^2)} = \frac{k-3}{4-k^2}$$

(i) If the line is parallel to  $x$ -axis, then

$$\text{Slope} = 0 \Rightarrow \frac{k-3}{4-k^2} = 0 \Rightarrow k-3 = 0 \Rightarrow k = 3$$

(ii) If the line is parallel to  $y$ -axis, then

$$\frac{1}{m} = 0 \Rightarrow \frac{4-k^2}{k-3} = 0 \Rightarrow 4-k^2 = 0 \Rightarrow k = \pm 2$$

(iii) If the line passes through the origin, then  $(0, 0)$  must satisfy the equation (i).

$$\therefore (k-3) \times 0 - (4-k^2) \times 0 + k^2 - 7k + 6 = 0 \Rightarrow (k-1)(k-6) = 0 \Rightarrow k = 1, 6.$$

#### BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 4** Find the equation of a line with slope 2 and the length of the perpendicular from the origin equal to  $\sqrt{5}$ .

**SOLUTION** Let the  $y$ -intercept of the required line be  $c$ . Then, its equation is



$$y = 2x + c \text{ or, } -2x + y = c \quad \dots(i)$$

Dividing throughout by  $\sqrt{(\text{Coefficient of } x)^2 + (\text{Coefficient of } y)^2}$ , we obtain

$$-\frac{2}{\sqrt{(-2)^2 + 1^2}}x + \frac{y}{\sqrt{(-2)^2 + 1^2}} = \frac{c}{\sqrt{(-2)^2 + 1^2}} \text{ or, } -\frac{2}{\sqrt{5}}x + \frac{y}{\sqrt{5}} = \frac{c}{\sqrt{5}}$$

This is the normal form of line (i). Therefore, the absolute value of the RHS represents the length of the perpendicular from the origin. But, the length of the perpendicular from the origin is given to be  $\sqrt{5}$ .

$$\therefore \left| \frac{c}{\sqrt{5}} \right| = \sqrt{5} \Rightarrow |c| = 5 \Rightarrow c = \pm 5$$

Putting  $c = \pm 5$  in (i), we obtain the equations of the required lines as  $y = 2x \pm 5$ .

### BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

**EXAMPLE 5** Prove that the slope of a line is invariant under the translation of the axes.

**SOLUTION** Let the equation of a straight line referred to a system of coordinate axes be

$$ax + by + c = 0 \quad \dots(i)$$

The slope of this line is  $m = -\frac{a}{b}$ .

Now, let the origin be shifted to the point  $(h, k)$  under some translation of the axes. Then, any point  $(X, Y)$  with respect to the new system of coordinate axes is given by the relation

$$x = X + h \text{ and } y = Y + k,$$

where  $(x, y)$  are the coordinates of the point in the old system of coordinate axes.

The equation of line (i) in the new system of axes is given by

$$a(X + h) + b(Y + k) + c = 0 \text{ or, } aX + bY + (ah + bk + c) = 0 \quad \dots(ii)$$

Let  $m'$  be the slope of this line. Then,  $m' = -\frac{a}{b}$ . Clearly,  $m = m'$ .

Hence, the slope of a straight line is invariant under the translation of coordinate axes.

**EXAMPLE 6** The line  $2x - y = 5$  turns about the point on it, whose ordinate and abscissae are equal, through an angle of  $45^\circ$  in the anti-clockwise direction. Find the equation of the line in the new position.

**SOLUTION** If the line  $2x - y = 5$  makes an angle  $\theta$  with  $x$ -axis. Then,  $\tan \theta = 2$ . Let  $P(\alpha, \alpha)$  be a point on the line  $2x - y = 5$ . Then,

$$2\alpha - \alpha = 5 \Rightarrow \alpha = 5$$

So, the coordinates of  $P$  are  $(5, 5)$ .

If the line  $2x - y - 5 = 0$  is rotated about point  $P$  through  $45^\circ$  in anti-clockwise direction, then the line in its new position makes angle  $\theta + 45^\circ$  with  $x$ -axis. Let  $m'$  be the slope of the line in its new position. Then,

$$m' = \tan(\theta + 45^\circ) = \frac{\tan \theta + \tan 45^\circ}{1 - \tan \theta \tan 45^\circ} = \frac{2 + 1}{1 - 2 \times 1} = -3$$

Thus, the line in its new position passes through  $P(5, 5)$  and has slope  $m' = -3$ . So, its equation is

$$y - 5 = m'(x - 5) \text{ or, } y - 5 = -3(x - 5) \text{ or, } 3x + y - 20 = 0$$

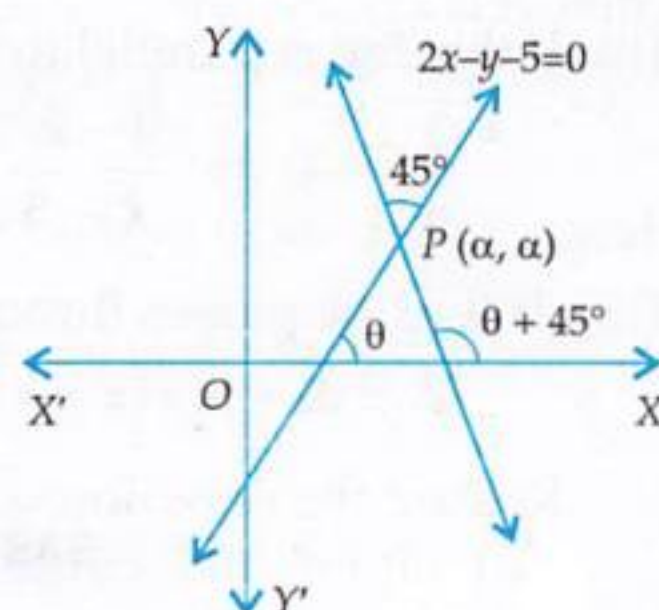


Fig. 22.61



**EXAMPLE 7** Find the coordinates of one vertex of an equilateral triangle with centroid at the origin and the opposite side  $x + y - 2 = 0$ . [NCERT EXEMPLAR]

**SOLUTION** Let  $ABC$  be an equilateral triangle having  $x + y - 2 = 0$  as the equation of side  $BC$  and opposite vertex  $A$ . Let the coordinates of its vertices be  $A(x_1, y_1)$ ,  $B(x_2, y_2)$  and  $C(x_3, y_3)$ . It is given that the centroid of  $\triangle ABC$  is at the origin.

$$\therefore \frac{x_1 + x_2 + x_3}{3} = 0 \text{ and } \frac{y_1 + y_2 + y_3}{3} = 0$$

$$\Rightarrow x_1 + x_2 + x_3 = 0 \text{ and } y_1 + y_2 + y_3 = 0$$

$$\Rightarrow x_2 + x_3 = -x_1 \text{ and } y_2 + y_3 = -y_1 \quad \dots(i)$$

It is given that  $\triangle ABC$  is an equilateral triangle. Therefore, median  $AD \perp BC$  and so  $OA \perp BC$ .

$$\therefore \text{Slope of } OA \times \text{Slope of } BC = -1$$

$$\Rightarrow \frac{y_1 - 0}{x_1 - 0} \times -1 = 1 \Rightarrow y_1 = x_1 \quad \dots(ii)$$

Clearly,  $D$  is the mid-point of  $BC$ . So, the coordinates of  $D$  are

$$\left( \frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2} \right) = \left( -\frac{x_1}{2}, -\frac{y_1}{2} \right)$$

[Using (i)]

But from (ii), we obtain  $y_1 = x_1$ . So, the coordinates of  $D$  are  $\left( -\frac{x_1}{2}, -\frac{x_1}{2} \right)$ . Point  $D$  lies on  $BC$  whose equation is  $x + y - 2 = 0$ .

$$\therefore -\frac{x_1}{2} - \frac{x_1}{2} - 2 = 0 \Rightarrow x_1 = -2 \Rightarrow y_1 = -2 \quad [\because y_1 = x_1]$$

Hence, the coordinates of  $A$  are  $(-2, -2)$ .

**ALITER** It is given that  $ABC$  is an equilateral triangle with centroid at the origin  $O$ . Therefore,  $OA \perp BC$ . Consequently,

$$\text{Slope of } OA \times \text{Slope of } BC = -1 \Rightarrow \text{Slope of } OA \times -1 = -1 \Rightarrow \text{Slope of } OA = 1$$

Thus,  $OA$  makes an angle of  $45^\circ$  with  $x$ -axis. So, the equation  $OA$  in distance form is

$$\frac{x-0}{\cos 45^\circ} = \frac{y-0}{\sin 45^\circ} \text{ or, } \frac{x-0}{1/\sqrt{2}} = \frac{y-0}{1/\sqrt{2}}$$

The equation of  $BC$  in normal form is  $\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = \sqrt{2}$ . Therefore,  $OD = \sqrt{2}$ .

Since  $O$  is the centroid of  $\triangle ABC$ . Therefore,  $OA = 2(OD) = 2\sqrt{2}$ .

Thus,  $A$  is a point on  $OA$  at a distance  $2\sqrt{2}$  from the origin  $O$ . So, the coordinates of  $A$  are given by

$$\frac{x-0}{1/\sqrt{2}} = \frac{y-0}{1/\sqrt{2}} = -2\sqrt{2} \Rightarrow x = -2 \text{ and } y = -2$$

Hence, the co-ordinates of  $A$  are  $(-2, -2)$ .

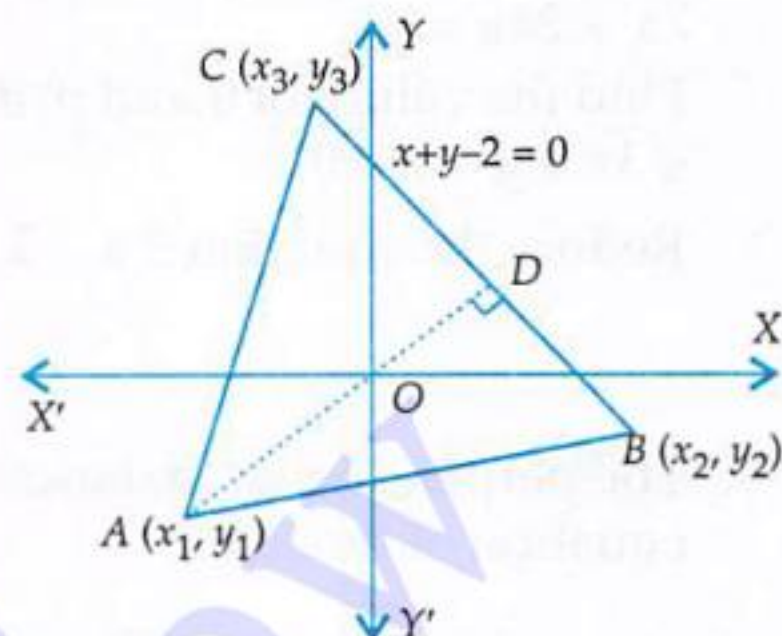


Fig 22.62

## EXERCISE 22.9

### BASIC

- Reduce the equation  $\sqrt{3}x + y + 2 = 0$  to:
  - slope-intercept form and find slope and  $y$ -intercept;
  - intercept form and find intercept on the axes;
  - the normal form and find  $p$  and  $\alpha$ .
- Reduce the following equations to the normal form and find  $p$  and  $\alpha$  in each case:
  - $x + \sqrt{3}y - 4 = 0$
  - $x + y + \sqrt{2} = 0$
  - $x - y + 2\sqrt{2} = 0$
  - $x - 3 = 0$
  - $y - 2 = 0$ .



3. Put the equation  $\frac{x}{a} + \frac{y}{b} = 1$  to the slope intercept form and find its slope and  $y$ -intercept.
4. Reduce the lines  $3x - 4y + 4 = 0$  and  $2x + 4y - 5 = 0$  to the normal form and hence find which line is nearer to the origin.
5. Show that the origin is equidistant from the lines  $4x + 3y + 10 = 0$ ;  $5x - 12y + 26 = 0$  and  $7x + 24y = 50$ .
6. Find the values of  $\theta$  and  $p$ , if the equation  $x \cos \theta + y \sin \theta = p$  is the normal form of the line  $\sqrt{3}x + y + 2 = 0$ . [NCERT]
7. Reduce the equation  $3x - 2y + 6 = 0$  to the intercept form and find the  $x$  and  $y$  intercepts.

#### BASED ON HOTS

8. The perpendicular distance of a line from the origin is 5 units and its slope is  $-1$ . Find the equation of the line.

#### ANSWERS

1. (i) Slope  $= -\sqrt{3}$ ,  $y$ -intercept  $= -2$  (ii)  $x$ -intercept  $= -\frac{2}{\sqrt{3}}$ ,  $y$ -intercept  $= -2$   
(iii)  $p = 1$ ,  $\alpha = 210^\circ$
2. (i)  $p = 2$ ,  $\alpha = \frac{\pi}{3}$  (ii)  $p = 1$ ,  $\alpha = 225^\circ$  (iii)  $p = 2$ ,  $\alpha = 135^\circ$  (iv)  $p = 3$ ,  $\alpha = 0$   
(v)  $p = 2$ ,  $\alpha = \frac{\pi}{2}$
3. Slope  $= -\frac{b}{a}$ ,  $y$ -intercept  $= b$
4.  $3x - 4y + 4 = 0$
6.  $\alpha = 210^\circ$ ,  $p = 1$
7.  $x$ -intercept  $= -2$ ,  $y$ -intercept  $= 3$
8.  $x + y - 5\sqrt{2} = 0$

### 22.8 POINT OF INTERSECTION OF TWO LINES

Let the equations of two lines be

$$a_1 x + b_1 y + c_1 = 0 \quad \dots(i)$$

and,  $a_2 x + b_2 y + c_2 = 0 \quad \dots(ii)$

Suppose these two lines intersect at a point  $P(x_1, y_1)$ . Then,  $(x_1, y_1)$  satisfies each of the given equations.

$$\therefore a_1 x_1 + b_1 y_1 + c_1 = 0 \text{ and } a_2 x_1 + b_2 y_1 + c_2 = 0$$

Solving these two equations by cross-multiplication, we get

$$\frac{x_1}{b_1 c_2 - b_2 c_1} = \frac{y_1}{c_1 a_2 - c_2 a_1} = \frac{1}{a_1 b_2 - a_2 b_1} \Rightarrow x_1 = \frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, y_1 = \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}$$

Hence, the coordinates of the point of intersection of lines (i) and (ii) are:

$$\left( \frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1} \right)$$

**NOTE** To find the coordinates of the point of intersection of two non-parallel lines, we solve the given equations simultaneously and the values of  $x$  and  $y$  so obtained determine the coordinates of the point of intersection.

#### ILLUSTRATIVE EXAMPLES

##### BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** Find the coordinates of the point of intersection of the lines  $2x - y + 3 = 0$  and  $x + 2y - 4 = 0$ .

**SOLUTION** Solving the equations  $2x - y + 3 = 0$  and  $x + 2y - 4 = 0$ , simultaneously we obtain

$$\frac{x}{4-6} = \frac{y}{3+8} = \frac{1}{4+1} \Rightarrow \frac{x}{-2} = \frac{y}{11} = \frac{1}{5} \Rightarrow x = -\frac{2}{5}, y = \frac{11}{5}$$

Hence,  $(-2/5, 11/5)$  is the required point of intersection.



**EXAMPLE 2** Find the area of the triangle formed by the lines  $y = x$ ,  $y = 2x$  and  $y = 3x + 4$ .

**SOLUTION** The given equations are

$$y = x \quad \dots(i) \quad y = 2x \quad \dots(ii) \quad \text{and} \quad y = 3x + 4 \quad \dots(iii)$$

Suppose the equations (i), (ii) and (iii) represent the sides  $AB$ ,  $BC$  and  $CA$  respectively of a triangle  $ABC$ .

Solving (i) and (ii), we get:  $x = 0$  and  $y = 0$ . Thus,  $AB$  and  $BC$  intersect at  $B(0, 0)$ .

Solving (ii) and (iii), we obtain:  $x = -4$ ,  $y = -8$ . Thus,  $BC$  and  $CA$  intersect at  $C(-4, -8)$ .

Solving (iii) and (i), we get:  $x = -2$  and  $y = -2$ . So,  $CA$  and  $AB$  intersect at  $A(-2, -2)$ .

Thus, the coordinates of the vertices of the triangle  $ABC$  are:  $A(-2, -2)$ ,  $B(0, 0)$  and  $C(-4, -8)$ .

$$\therefore \text{Area of } \Delta ABC = \frac{1}{2} \begin{vmatrix} -2 & -2 & 1 \\ 0 & 0 & 1 \\ -4 & -8 & 1 \end{vmatrix} = 4 \text{ sq. units.}$$

### BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 3** Find the equations of the medians of a triangle formed by the lines  $x + y - 6 = 0$ ,  $x - 3y - 2 = 0$  and  $5x - 3y + 2 = 0$ .

**SOLUTION** The given equations are :

$$x + y - 6 = 0 \quad \dots(i) \quad x - 3y - 2 = 0 \quad \dots(ii) \quad \text{and} \quad 5x - 3y + 2 = 0 \quad \dots(iii)$$

Suppose equations (i), (ii) and (iii) represent the sides,  $AB$ ,  $BC$  and  $CA$  respectively of triangle  $ABC$ .

Solving (i) and (ii), we get:  $x = 5$  and  $y = 1$ . Thus,  $AB$  and  $BC$  intersect at  $B(5, 1)$ .

Solving (ii) and (iii), we get:  $x = -1$  and  $y = -1$ . Thus,  $BC$  and  $CA$  intersect at  $C(-1, -1)$ .

Solving (i) and (iii), we get:  $x = 2$  and  $y = 4$ . Thus,  $AB$  and  $CA$  intersect at  $A(2, 4)$ .

Thus, the coordinates of the vertices  $A$ ,  $B$  and  $C$  of triangle  $ABC$  are  $(2, 4)$ ,  $(5, 1)$  and  $(-1, -1)$  respectively. Let  $D$ ,  $E$  and  $F$  be the mid-points of sides  $BC$ ,  $CA$  and  $AB$  respectively. Then, the coordinates of  $D$ ,  $E$  and  $F$  are

$$D\left(\frac{5-1}{2}, \frac{1-1}{2}\right) = (2, 0); E\left(\frac{2-1}{2}, \frac{4-1}{2}\right) = \left(\frac{1}{2}, \frac{3}{2}\right) \text{ and } F\left(\frac{2+5}{2}, \frac{4+1}{2}\right) = \left(\frac{7}{2}, \frac{5}{2}\right)$$

The median  $AD$  passes through  $A(2, 4)$  and  $D(2, 0)$ . So, its equation is

$$y - 4 = \frac{0-4}{2-2}(x-2) \Rightarrow x - 2 = \frac{2-2}{0-4}(y-4) \Rightarrow x - 2 = 0 \Rightarrow x = 2$$

The median  $BE$  passes through points  $B(5, 1)$  and  $E(1/2, 3/2)$ . So, its equation is

$$y - 1 = \frac{\frac{3}{2} - 1}{\frac{1}{2} - 5}(x - 5) \Rightarrow y - 1 = -\frac{1}{9}(x - 5) \Rightarrow x + 9y - 14 = 0$$

The median  $CF$  passes through points  $C(-1, -1)$  and  $F(7/2, 5/2)$ . So, its equation is

$$y + 1 = \frac{\frac{5}{2} + 1}{\frac{7}{2} + 1}(x + 1) \Rightarrow y + 1 = \frac{7}{9}(x + 1) \Rightarrow 7x - 9y - 2 = 0$$

Hence, the equations of the medians of the triangle are  $x = 2$ ,  $x + 9y - 14 = 0$  and  $7x - 9y - 2 = 0$ .

**EXAMPLE 4** Find the value of  $m$  for which the lines  $mx + (2m + 3)y + m + 6 = 0$  and  $(2m + 1)x + (m - 1)y + m - 9 = 0$  intersect at a point on  $y$ -axis.

**SOLUTION** The equations of the lines are

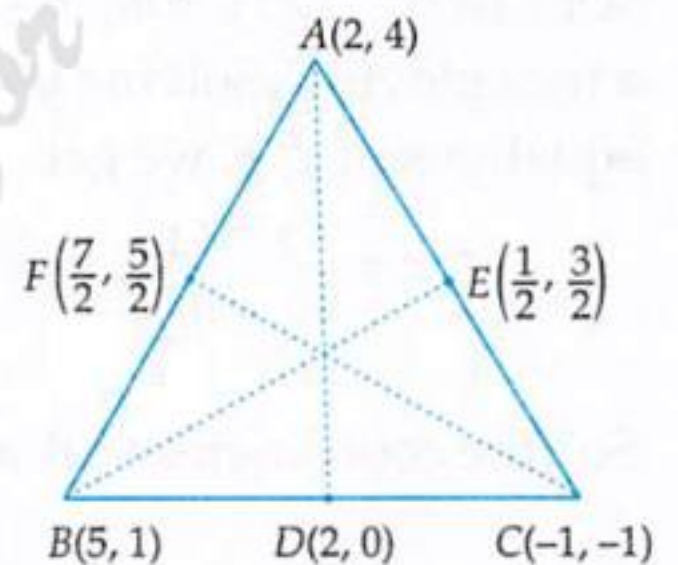


Fig 22.63



$$mx + (2m + 3)y + m + 6 = 0 \quad \dots(i)$$

$$(2m + 1)x + (m - 1)y + m - 9 = 0 \quad \dots(ii)$$

Solving these two equations by cross-multiplication, we obtain

$$\begin{aligned} \frac{x}{(2m+3)(m-9) - (m-1)(m+6)} &= \frac{y}{(2m+1)(m+6) - m(m-9)} = \frac{1}{m(m-1) - (2m+1)(2m+3)} \\ \Rightarrow \frac{x}{m^2 - 20m - 21} &= \frac{y}{m^2 + 22m + 6} = \frac{1}{-3(m^2 + 3m + 1)} \\ \Rightarrow x &= \frac{m^2 - 20m - 21}{-3(m^2 + 3m + 1)} \text{ and } y = \frac{m^2 + 22m + 6}{-3(m^2 + 3m + 1)} \end{aligned}$$

So, given lines intersect at the point  $\left( \frac{m^2 - 20m - 21}{-3(m^2 + 3m + 1)}, \frac{m^2 + 22m + 6}{-3(m^2 + 3m + 1)} \right)$ .

If it lies on y-axis, then its x-coordinate is zero.

$$\therefore \frac{m^2 - 20m - 21}{-3(m^2 + 3m + 1)} = 0 \Rightarrow m^2 - 20m - 21 = 0 \Rightarrow (m - 21)(m + 1) = 0 \Rightarrow m = -1, 21$$

**EXAMPLE 5** Find the area of the triangle formed by the lines  $y = m_1 x + c_1$ ,  $y = m_2 x + c_2$  and  $x = 0$ .

[NCERT]

**SOLUTION** Let  $y = m_1 x + c_1$ ,  $y = m_2 x + c_2$  and  $x = 0$  be the sides AB, BC and CA respectively of a triangle ABC. Solving  $y = m_1 x + c_1$  and  $y = m_2 x + c_2$  as linear equations in  $x, y$ , we get

$$x = \frac{c_2 - c_1}{m_1 - m_2}, \quad y = \frac{m_1 c_2 - m_2 c_1}{m_1 - m_2}$$

So, the coordinates of B are  $\left( \frac{c_2 - c_1}{m_1 - m_2}, \frac{m_1 c_2 - m_2 c_1}{m_1 - m_2} \right)$ .

Solving  $y = m_2 x + c_2$  and  $x = 0$ , we get:  $x = 0, y = c_2$ .  
So, coordinates of C are  $(0, c_2)$ .

Similarly, by solving  $x = 0$  and  $y = m_1 x + c_1$ , we get the coordinates of A as  $(0, c_1)$ .

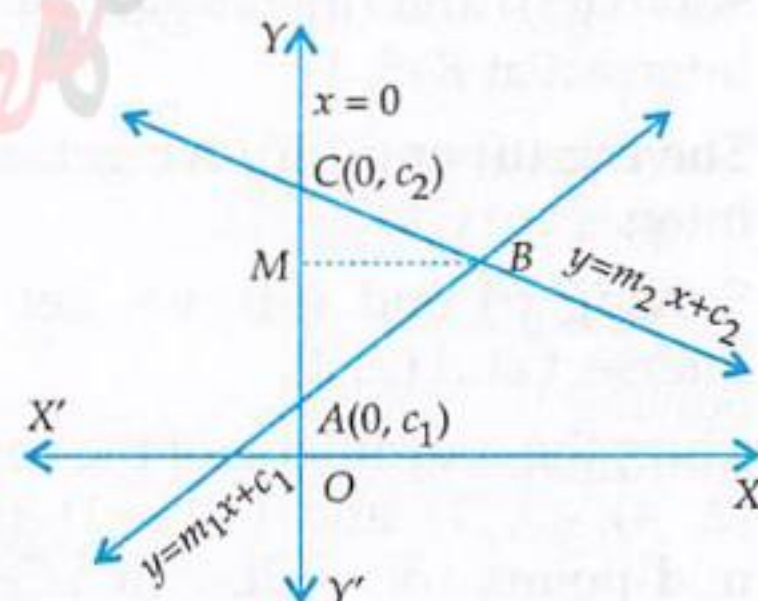


Fig. 22.64

$$\begin{aligned} \therefore \text{Area of } \triangle ABC &= \text{Absolute value of } \frac{1}{2} \begin{vmatrix} 0 & c_1 & 1 \\ 0 & c_2 & 1 \\ \frac{c_2 - c_1}{m_1 - m_2} & \frac{m_1 c_2 - m_2 c_1}{m_1 - m_2} & 1 \end{vmatrix} = \frac{1}{2} \left| \left( \frac{c_2 - c_1}{m_1 - m_2} \right) (c_1 - c_2) \right| \\ &= \frac{1}{2} \frac{(c_1 - c_2)^2}{|m_1 - m_2|} \end{aligned}$$

**ALITER** The equations of the given lines are

$$y = m_1 x + c_1 \quad \dots (i) \quad y = m_2 x + c_2 \quad \dots (ii) \quad x = 0 \quad \dots (iii)$$

Lines (i) and (ii) intersect line (iii) at A  $(0, c_1)$  and C  $(0, c_2)$  respectively. Solving (i) and (ii), we

obtain the coordinates of B as  $\left( \frac{c_2 - c_1}{m_1 - m_2}, \frac{m_1 c_2 - m_2 c_1}{m_1 - m_2} \right)$ .

$$\therefore BM = x\text{-coordinate of } B = \frac{c_2 - c_1}{m_1 - m_2}$$



Clearly,  $AC = |c_2 - c_1|$

$$\therefore \text{Area of } \triangle ABC = \frac{1}{2} (AC \times BM) = \frac{1}{2} |c_2 - c_1| \times \left| \frac{c_2 - c_1}{m_1 - m_2} \right| = \frac{1}{2} \frac{(c_2 - c_1)^2}{|m_1 - m_2|}$$

**EXAMPLE 6** Find the equation of the line parallel to  $y$ -axis and drawn through the point of intersection of the lines  $x - 7y + 5 = 0$  and  $3x + y = 0$ .

**SOLUTION** On solving the equations  $x - 7y + 5 = 0$  and  $3x + y = 0$ , we obtain:

$$x = -\frac{5}{22} \text{ and } y = \frac{15}{22}.$$

So, the given lines intersect at the point whose coordinates are :  $(-5/22, 15/22)$ .

We know that, the equation of a line parallel to  $y$ -axis is of the form  $x = \text{constant}$ . So, let the equation of the required line be

$$x = \lambda \quad \dots(i)$$

It passes through  $(-5/22, 15/22)$ . Therefore,  $\frac{-5}{22} = \lambda$ .

Substituting the value of  $\lambda$  in (i), we obtain:  $x = -5/22$  or,  $22x + 5 = 0$  as the equation of the required line.

#### BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

**EXAMPLE 7** Show that the lines  $4x + y - 9 = 0$ ,  $x - 2y + 3 = 0$ ,  $5x - y - 6 = 0$  make equal intercepts on any line of gradient 2.

**SOLUTION** The equation of any line of gradient 2 is

$$y = 2x + c \quad \dots(i)$$

The equations of given lines are

$$4x + y - 9 = 0 \quad \dots(ii) \quad x - 2y + 3 = 0 \quad \dots(iii)$$

$$5x - y - 6 = 0 \quad \dots(iv)$$

Solving (i) with (ii), (iii) and (iv) respectively, we obtain the coordinates of  $P$ ,  $Q$  and  $R$  as

$$P\left(\frac{3}{2} - \frac{c}{6}, 3 + \frac{2c}{3}\right), Q\left(1 - \frac{2c}{3}, 2 - \frac{c}{3}\right) \text{ and } R\left(2 + \frac{c}{3}, 4 + \frac{5c}{3}\right)$$

Clearly,  $P$  is the mid-point of  $QR$ . Therefore  $PQ = PR$ .

Hence, lines (ii), (iii) and (iv) make equal intercepts on any line of gradient 2.

**EXAMPLE 8** Two vertices of a triangle are  $(3, -1)$  and  $(-2, 3)$  and its orthocentre is at the origin. Find the coordinates of the third vertex.

**SOLUTION** Clearly,  $A$  is the intersection of sides  $AB$  and  $AC$  of  $\triangle ABC$ . Side  $AB$  passes through  $B(3, -1)$  and is perpendicular to  $OC$  whose slope is  $-3/2$ . So, equation of side  $AB$  is

$$y + 1 = \frac{2}{3} (x - 3) \text{ or, } 2x - 3y - 9 = 0 \quad \dots(i)$$

Similarly, side  $AC$  passes through  $C(-2, 3)$  and is perpendicular to  $OB$  whose slope is  $-1/3$ . So, equation of side  $AC$  is

$$y - 3 = 3(x + 2) \text{ or, } 3x - y + 9 = 0 \quad \dots(ii)$$

Solving (i) and (ii), we get:  $x = -36/7$ ,  $y = -45/7$ .

Hence, coordinates of  $A$  are  $(-36/7, -45/7)$ .

**EXAMPLE 9** Two consecutive sides of a parallelogram are  $4x + 5y = 0$  and  $7x + 2y = 0$ . If the equation of one diagonal is  $11x + 7y = 9$ , find the equation of the other diagonal.

**SOLUTION** Let  $AB$  and  $AD$  be consecutive sides of parallelogram  $ABCD$ . Let the equations of  $AB$  and  $AD$  be  $4x + 5y = 0$  and  $7x + 2y = 0$  respectively. Clearly, these two lines intersect at  $A(0, 0)$ .

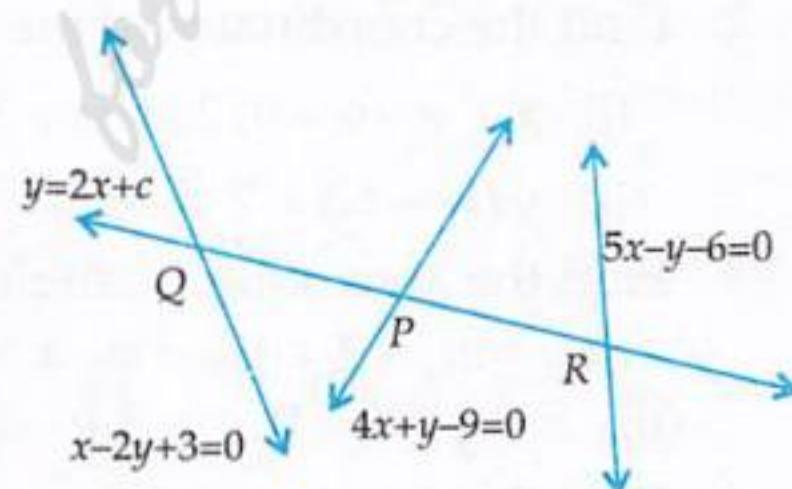


Fig. 22.65

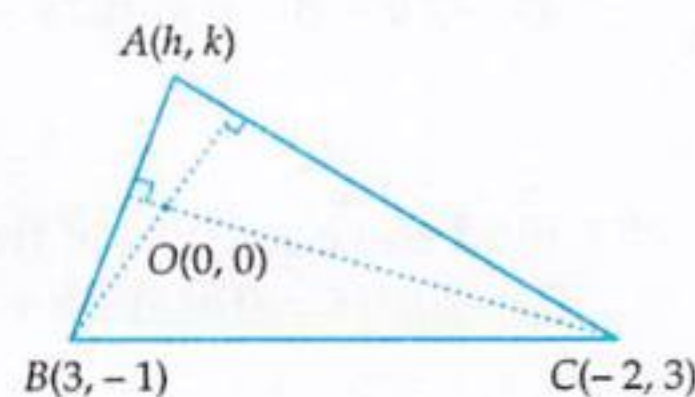


Fig. 22.66



Solving  $11x + 7y = 9$  and  $4x + 5y = 0$ , we get:  $x = 5/3$  and  $y = -4/3$ . So, the coordinates of  $B$  are  $(5/3, -4/3)$ . Similarly, by solving  $11x + 7y = 9$  and  $7x + 2y = 0$ , we obtain that the coordinates of  $D$  are  $(-2/3, 7/3)$ . The diagonals of a parallelogram bisect each other. So,  $P$  is the mid-point of  $BD$  and hence its coordinates

are  $\left(\frac{\frac{5}{3} - \frac{2}{3}}{2}, \frac{-\frac{4}{3} + \frac{7}{3}}{2}\right)$  or,  $\left(\frac{1}{2}, \frac{1}{2}\right)$ . Clearly,  $AC$  passes through

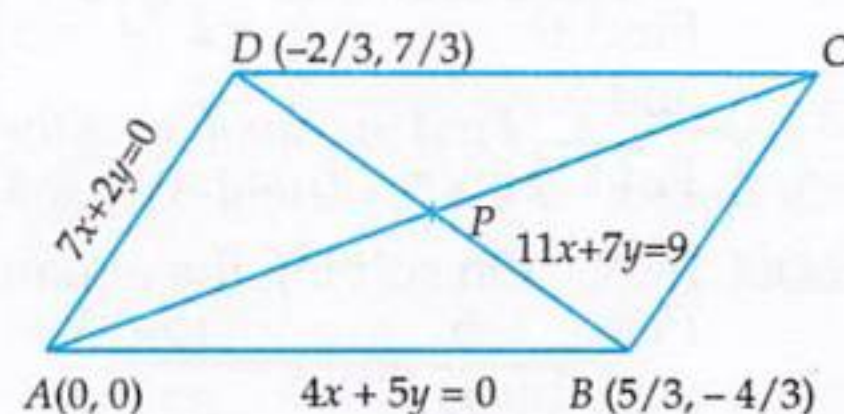


Fig. 22.67

$A(0,0)$  and  $C(1/2, 1/2)$ . Hence, equation of  $AC$  is

$$y - 0 = \frac{\frac{1}{2} - 0}{\frac{1}{2} - 0} (x - 0) \text{ or, } y = x.$$

## EXERCISE 22.10

## BASIC

- Find the point of intersection of the following pairs of lines :
  - $2x - y + 3 = 0$  and  $x + y - 5 = 0$
  - $bx + ay = ab$  and  $ax + by = ab$ .
  - $y = m_1 x + \frac{a}{m_1}$  and  $y = m_2 x + \frac{a}{m_2}$ .
- Find the coordinates of the vertices of a triangle, the equations of whose sides are:
  - $x + y - 4 = 0$ ,  $2x - y + 3 = 0$  and  $x - 3y + 2 = 0$
  - $y(t_1 + t_2) = 2x + 2at_1t_2$ ,  $y(t_2 + t_3) = 2x + 2at_2t_3$  and,  $y(t_3 + t_1) = 2x + 2at_1t_3$ .
- Find the area of the triangle formed by the lines
  - $y = m_1 x + c_1$ ,  $y = m_2 x + c_2$  and  $x = 0$
  - $y = 0$ ,  $x = 2$  and  $x + 2y = 3$ .
  - $x + y - 6 = 0$ ,  $x - 3y - 2 = 0$  and  $5x - 3y + 2 = 0$
- Find the equations of the medians of a triangle, the equations of whose sides are:
  $3x + 2y + 6 = 0$ ,  $2x - 5y + 4 = 0$  and  $x - 3y - 6 = 0$
- Prove that the lines  $y = \sqrt{3}x + 1$ ,  $y = 4$  and  $y = -\sqrt{3}x + 2$  form an equilateral triangle.
- Classify the following pairs of lines as coincident, parallel or intersecting:
  - $2x + y - 1 = 0$  and  $3x + 2y + 5 = 0$
  - $x - y = 0$  and  $3x - 3y + 5 = 0$
  - $3x + 2y - 4 = 0$  and  $6x + 4y - 8 = 0$ .
- Find the equation of the line joining the point  $(3, 5)$  to the point of intersection of the lines  $4x + y - 1 = 0$  and  $7x - 3y - 35 = 0$ .
- Find the equation of the line passing through the point of intersection of the lines  $4x - 7y - 3 = 0$  and  $2x - 3y + 1 = 0$  that has equal intercepts on the axes. [NCERT]

## BASED ON LOTS

- Find the equation of the straight line passing through the point of intersection of the lines  $5x - 6y - 1 = 0$  and  $3x + 2y + 5 = 0$  and perpendicular to the line  $3x - 5y + 11 = 0$ . [NCERT EXEMPLAR]
- Find the equation of the line passing through the intersection of the lines  $2x + y = 5$  and  $x + 3y + 8 = 0$  and parallel to the line  $3x + 4y = 7$ .

## BASED ON HOTS

- Find the orthocentre of the triangle the equations of whose sides are  $x + y = 1$ ,  $2x + 3y = 6$  and  $4x - y + 4 = 0$ .



12. Three sides  $AB$ ,  $BC$  and  $CA$  of a triangle  $ABC$  are  $5x - 3y + 2 = 0$ ,  $x - 3y - 2 = 0$  and  $x + y - 6 = 0$  respectively. Find the equation of the altitude through the vertex  $A$ .
13. Find the coordinates of the orthocentre of the triangle whose vertices are  $(-1, 3)$ ,  $(2, -1)$  and  $(0, 0)$ .
14. Find the coordinates of the incentre and centroid of the triangle whose sides have the equations  $3x - 4y = 0$ ,  $12y + 5x = 0$  and  $y - 15 = 0$ .
15. Prove that the lines  $\sqrt{3}x + y = 0$ ,  $\sqrt{3}y + x = 0$ ,  $\sqrt{3}x + y = 1$  and  $\sqrt{3}y + x = 1$  form a rhombus.
16. If the straight line  $\frac{x}{a} + \frac{y}{b} = 1$  passes through the point of intersection of the lines  $x + y = 3$  and  $2x - 3y = 1$  and is parallel to  $x - y - 6 = 0$ , find  $a$  and  $b$ .
17. Show that the area of the triangle formed by the lines  $y = m_1 x$ ,  $y = m_2 x$  and  $y = c$  is equal to  $\frac{c^2}{4} (\sqrt{33} + \sqrt{11})$ , where  $m_1, m_2$  are the roots of the equation  $x^2 + (\sqrt{3} + 2)x + \sqrt{3} - 1 = 0$ .

## ANSWERS

1. (i)  $\left(\frac{2}{3}, \frac{13}{3}\right)$  (ii)  $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$  (iii)  $\left(\frac{a}{m_1 m_2}, a\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\right)$
2. (i)  $\left(\frac{1}{3}, \frac{11}{3}\right), \left(-\frac{7}{5}, \frac{1}{5}\right), \left(\frac{5}{2}, \frac{3}{2}\right)$  (ii)  $(at_1^2, 2at_1), (at_2^2, 2at_2), (at_3^2, 2at_3)$
3. (i)  $\frac{(c_1 - c_2)^2}{2(m_1 - m_2)}$  (ii) 0 (iii) 12. sq. units
4.  $41x - 112y - 70 = 0$ ,  $16x - 59y - 120 = 0$  and  $25x - 53y + 50 = 0$
6. (i) intersecting (ii) parallel (iii) coincident
7.  $12x - y - 31 = 0$  8.  $x + y + 13 = 0$  9.  $5x + 3y + 8 = 0$  10.  $3x + 4y + 3 = 0$
11.  $\left(\frac{3}{7}, \frac{22}{7}\right)$  12.  $3x + y - 10 = 0$  13.  $(-4, -3)$  14.  $(-1, 8), \left(-\frac{16}{3}, 10\right)$
16.  $a = 1, b = -1$

## HINTS TO SELECTED PROBLEMS

8. Given lines intersect at  $(-8, -5)$ . The equation of a line making equal intercepts on the coordinates axes is  $\frac{x}{a} + \frac{y}{a} = 1$  or,  $x + y = a$ . It passes through  $(-8, -5)$ .  
 $\therefore -8 - 5 + a = 0 \Rightarrow a = 13$   
Hence, the equation of the line is  $x + y = 13$ .
9. The lines  $5x - 6y - 1 = 0$  and  $3x + 2y + 5 = 0$  intersect at the point  $(-1, -1)$ . The slope of the line  $3x - 5y + 11 = 0$  is  $3/5$ . So, the slope of a line perpendicular to it is  $-5/3$ . Hence, the equation of the required line is  
 $y + 1 = -5/3(x + 1)$  or,  $5x + 3y + 8 = 0$ .

## 22.9 CONDITION OF CONCURRENCY OF THREE LINES

Three lines are said to be concurrent if they pass through a common point i.e. they meet at a point.

Thus, if three lines are concurrent the point of intersection of two lines lies on the third line. Let

$$a_1 x + b_1 y + c_1 = 0 \dots(i) \quad a_2 x + b_2 y + c_2 = 0 \dots(ii) \quad a_3 x + b_3 y + c_3 = 0 \dots(iii)$$



be three concurrent lines. Then the point of intersection of (i) and (ii) must lie on the third.

The coordinates of the point of intersection of (i) and (ii) are:

$$\left( \frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1} \right) \quad [\text{See section 22.8}]$$

This point must lie on line (iii).

$$\therefore a_3 \left( \frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1} \right) + b_3 \left( \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1} \right) + c_3 = 0$$

$$\Rightarrow a_3 (b_1 c_2 - b_2 c_1) + b_3 (c_1 a_2 - c_2 a_1) + c_3 (a_1 b_2 - a_2 b_1) = 0 \Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

This is the required condition of concurrency of three lines.

#### ANOTHER CONDITION

$$L_1 \equiv a_1 x + b_1 y + c_1 = 0; \quad L_2 \equiv a_2 x + b_2 y + c_2 = 0; \quad L_3 \equiv a_3 x + b_3 y + c_3 = 0$$

are concurrent iff there exist constants  $\lambda_1, \lambda_2, \lambda_3$  not all zero such that

$$\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0$$

$$\text{i.e.} \quad \lambda_1 (a_1 x + b_1 y + c_1) + \lambda_2 (a_2 x + b_2 y + c_2) + \lambda_3 (a_3 x + b_3 y + c_3) = 0.$$

### ILLUSTRATIVE EXAMPLES

#### BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** Prove that the lines  $3x + y - 14 = 0$ ,  $x - 2y = 0$  and  $3x - 8y + 4 = 0$  are concurrent.

**SOLUTION** Given lines are  $3x + y - 14 = 0$ ,  $x - 2y + 0 = 0$  and  $3x - 8y + 4 = 0$ .

$$\text{We have, } \begin{vmatrix} 3 & 1 & -14 \\ 1 & -2 & 0 \\ 3 & -8 & 4 \end{vmatrix} = 3(-8 + 0) - 1(4 - 0) - 14(-8 + 6) = -24 - 4 + 28 = 0.$$

So, the given lines are concurrent.

**EXAMPLE 2** Show that the lines  $x - y - 6 = 0$ ,  $4x - 3y - 20 = 0$  and  $6x + 5y + 8 = 0$  are concurrent. Also, find their common point of intersection.

**SOLUTION** The given lines are

$$x - y - 6 = 0 \quad \dots(i) \quad 4x - 3y - 20 = 0 \quad \dots(ii) \quad 6x + 5y + 8 = 0 \quad \dots(iii)$$

Solving (i) and (ii) by cross-multiplication, we get

$$\frac{x}{20 - 18} = \frac{y}{-24 + 20} = \frac{1}{-3 + 4} \Rightarrow x = 2 \text{ and } y = -4.$$

Thus, the first two lines intersect at the point  $(2, -4)$ . Putting  $x = 2$  and  $y = -4$  in (iii), we get

$$6 \times 2 + 5 \times -4 + 8 = 0$$

Thus, the point  $(2, -4)$  lies on line (iii). Hence, the given lines are concurrent and their common point of intersection is  $(2, -4)$ .

**EXAMPLE 3** Find the value of  $\lambda$ , if the lines  $3x - 4y - 13 = 0$ ,  $8x - 11y - 33 = 0$  and  $2x - 3y + \lambda = 0$  are concurrent.

**SOLUTION** The given lines are concurrent, if



$$\begin{vmatrix} 3 & -4 & -13 \\ 8 & -11 & -33 \\ 2 & -3 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow 3(-11\lambda - 99) + 4(8\lambda + 66) - 13(-24 + 22) = 0 \Rightarrow -\lambda - 7 = 0 \Rightarrow \lambda = -7$$

**ALITER** The given equations are

$$3x - 4y - 13 = 0 \quad \dots(i) \quad 8x - 11y - 33 = 0 \quad \dots(ii) \quad \text{and} \quad 2x - 3y + \lambda = 0 \quad \dots(iii)$$

Solving equations (i) and (ii), we get  $x = 11$  and  $y = 5$ . Thus,  $(11, 5)$  is the point of intersection of lines (i) and (ii). The given lines will be concurrent if they pass through the common point i.e. the point of intersection of any two lies on the third. Therefore, the point  $(11, 5)$  must lie on the line (iii).

$$\therefore 2 \times 11 - 3 \times 5 + \lambda = 0 \Rightarrow \lambda = -7.$$

#### BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 4** If the lines  $a_1x + b_1y + 1 = 0$ ,  $a_2x + b_2y + 1 = 0$  and  $a_3x + b_3y + 1 = 0$  are concurrent, show that the points  $(a_1, b_1)$ ,  $(a_2, b_2)$  and  $(a_3, b_3)$  are collinear.

**SOLUTION** The given lines are

$$a_1x + b_1y + 1 = 0 \quad \dots(i) \quad a_2x + b_2y + 1 = 0 \quad \dots(ii) \quad \text{and} \quad a_3x + b_3y + 1 = 0 \quad \dots(iii).$$

If these lines are concurrent, we must have

$$\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0, \text{ which is the condition of collinearity of three points } (a_1, b_1), (a_2, b_2) \text{ and } (a_3, b_3).$$

Hence, if the given lines are concurrent, the given points are collinear.

#### BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

**EXAMPLE 5** If the lines  $ax + y + 1 = 0$ ,  $x + by + 1 = 0$  and  $x + y + c = 0$  are concurrent ( $a \neq b \neq c \neq 1$ ), prove that  $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$ .

**SOLUTION** The equations of the given lines are

$$ax + y + 1 = 0 \quad \dots(i) \quad x + by + 1 = 0 \quad \dots(ii) \quad x + y + c = 0 \quad \dots(iii)$$

If these lines are concurrent, then

$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0 \Rightarrow a \begin{vmatrix} b & 1 \\ 1 & c \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & c \end{vmatrix} + 1 \begin{vmatrix} 1 & b \\ 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow a(bc - 1) - (c - 1) + (1 - b) = 0 \Rightarrow abc - a - c + 1 + 1 - b = 0 \Rightarrow abc = a + b + c - 2 \quad \dots(i)$$

$$\therefore \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = \frac{(1-b)(1-c) + (1-a)(1-c) + (1-a)(1-b)}{(1-a)(1-b)(1-c)}$$

$$= \frac{3 - 2(a+b+c) + ab + bc + ca}{1 - (a+b+c) + ab + bc + ca - abc}$$

$$= \frac{3 - 2(a+b+c) + ab + bc + ca}{1 - (a+b+c) + (ab + bc + ca) - (a+b+c-2)} \quad \text{[Using (i)]}$$

$$= \frac{3 - 2(a+b+c) + ab + bc + ca}{3 - 2(a+b+c) + ab + bc + ca} = 1$$



**EXAMPLE 6** Show that the following lines are concurrent:

$$L_1 \equiv (a-b)x + (b-c)y + (c-a) = 0$$

$$L_2 \equiv (b-c)x + (c-a)y + (a-b) = 0$$

and,  $L_3 \equiv (c-a)x + (a-b)y + (b-c) = 0.$

**SOLUTION** We observe that:  $\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0$ , where  $\lambda_1 = \lambda_2 = \lambda_3 =$  any non-zero number. Hence, the given lines are concurrent.

**EXAMPLE 7** Show that the altitudes of a triangle are concurrent.

**SOLUTION** Let  $ABC$  be a triangle such that the coordinates of its vertices are  $A(x_1, y_1), B(x_2, y_2)$  and  $C(x_3, y_3)$ . Let  $AD, BE$  and  $CF$  be the altitudes drawn from the vertices  $A, B, C$  respectively to the opposite sides  $BC, CA$  and,  $AB$  respectively. Then,

$$\text{Slope of } BC = \frac{y_2 - y_3}{x_2 - x_3}, \text{ Slope of } CA = \frac{y_3 - y_1}{x_3 - x_1}$$

and,  $\text{Slope of } AB = \frac{y_2 - y_1}{x_2 - x_1}$

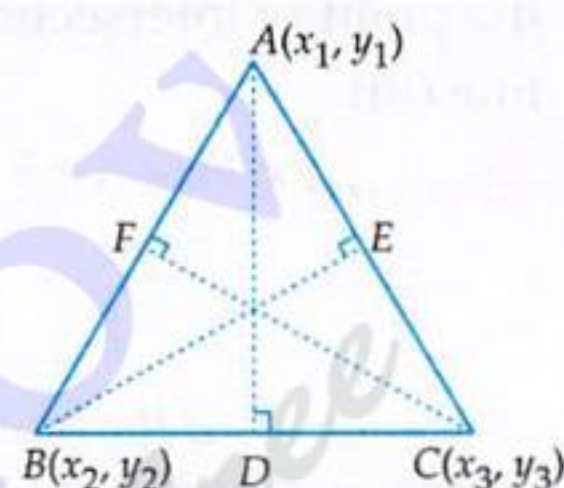


Fig. 22.68

Since  $AD \perp BC, BE \perp CA$  and  $CF \perp AB$ . Therefore,

$$\text{Slope of } AD = -\frac{x_2 - x_3}{y_2 - y_3}, \text{ Slope of } BE = -\frac{x_3 - x_1}{y_3 - y_1} \text{ and, Slope of } CF = -\frac{x_2 - x_1}{y_2 - y_1}.$$

The equation of altitude  $AD$  is

$$y - y_1 = -\frac{x_2 - x_3}{y_2 - y_3} (x - x_1)$$

i.e.  $L_1 \equiv x(x_2 - x_3) + y(y_2 - y_3) - x_1(x_2 - x_3) - y_1(y_2 - y_3) = 0$  ... (i)

Similarly, equations of altitudes  $BE$  and  $CF$  are

$$L_2 \equiv x(x_3 - x_1) + y(y_3 - y_1) - x_2(x_3 - x_1) - y_2(y_3 - y_1) = 0 \quad \dots (ii)$$

and,  $L_3 \equiv x(x_1 - x_2) + y(y_1 - y_2) - x_3(x_1 - x_2) - y_3(y_1 - y_2) = 0 \quad \dots (iii)$

Clearly,  $1 \cdot L_1 + 1 \cdot L_2 + 1 \cdot L_3 = 0$ . Hence, lines  $L_1 = 0, L_2 = 0$  and  $L_3 = 0$  i.e. the altitudes  $AD, BE$  and  $CF$  are concurrent.

**EXAMPLE 8** Prove analytically that the medians of a triangle are concurrent.

**SOLUTION** Let  $ABC$  be a triangle the coordinates of whose vertices are  $A(x_1, y_1), B(x_2, y_2)$  and  $C(x_3, y_3)$ . Let  $D, E$  and  $F$  be the mid-points of sides  $BC, CA$  and  $AB$  respectively.

The coordinates of  $D, E$  and  $F$  are  $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right), E\left(\frac{x_3 + x_1}{2}, \frac{y_3 + y_1}{2}\right)$  and

$F\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$  respectively.

Equation of median  $AD$  is

$$y - y_1 = \frac{y_1 - \frac{y_2 + y_3}{2}}{x_1 - \frac{x_2 + x_3}{2}} (x - x_1)$$

or,  $y - y_1 = \frac{2y_1 - y_2 - y_3}{2x_1 - x_2 - x_3} (x - x_1)$

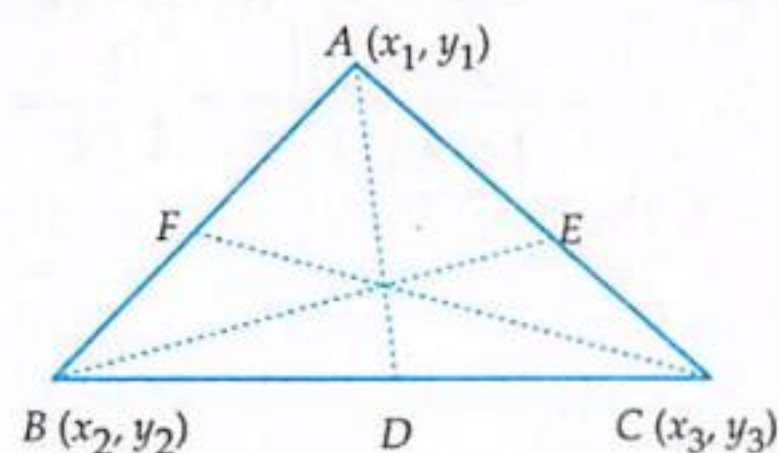


Fig. 22.69



$$\text{or, } (2y_1 - y_2 - y_3)x - (2x_1 - x_2 - x_3)y - x_1(2y_1 - y_2 - y_3) + y_1(2x_1 - x_2 - x_3) = 0$$

$$\text{or, } L_1 \equiv (2y_1 - y_2 - y_3)x - (2x_1 - x_2 - x_3)y + x_1(y_2 + y_3) - y_1(x_2 + x_3) = 0 \quad \dots(i)$$

Similarly, equations of medians  $BE$  and  $CF$  are respectively

$$L_2 \equiv (2y_2 - y_1 - y_3)x - (2x_2 - x_1 - x_3)y + x_2(y_1 + y_3) - y_2(x_1 + x_3) = 0 \quad \dots(ii)$$

$$L_3 \equiv (2y_3 - y_1 - y_2)x - (2x_3 - x_1 - x_2)y + x_3(y_1 + y_2) - y_3(x_1 + x_2) = 0 \quad \dots(iii)$$

We observe that  $1 \cdot L_1 + 1 \cdot L_2 + 1 \cdot L_3 = 0$  (identically). Hence, medians  $AD$ ,  $BE$  and  $CF$  are concurrent.

### EXERCISE 22.11

#### BASIC

1. Prove that the following sets of three lines are concurrent:

(i)  $15x - 18y + 1 = 0$ ,  $12x + 10y - 3 = 0$  and  $6x + 66y - 11 = 0$

(ii)  $3x - 5y - 11 = 0$ ,  $5x + 3y - 7 = 0$  and  $x + 2y = 0$

(iii)  $\frac{x}{a} + \frac{y}{b} = 1$ ,  $\frac{x}{b} + \frac{y}{a} = 1$  and  $y = x$ .

2. For what value of  $\lambda$  are the three lines  $2x - 5y + 3 = 0$ ,  $5x - 9y + \lambda = 0$  and  $x - 2y + 1 = 0$  concurrent?

#### BASED ON LOTS

3. Find the conditions that the straight lines  $y = m_1x + c_1$ ,  $y = m_2x + c_2$  and  $y = m_3x + c_3$  may meet in a point. [NCERT]

4. If the lines  $p_1x + q_1y = 1$ ,  $p_2x + q_2y = 1$  and  $p_3x + q_3y = 1$  be concurrent, show that the points  $(p_1, q_1)$ ,  $(p_2, q_2)$  and  $(p_3, q_3)$  are collinear.

#### BASED ON HOTS

5. Show that the straight lines  $L_1 = (b + c)x + ay + 1 = 0$ ,  $L_2 = (c + a)x + by + 1 = 0$  and  $L_3 = (a + b)x + cy + 1 = 0$  are concurrent.

6. If the three lines  $ax + a^2y + 1 = 0$ ,  $bx + b^2y + 1 = 0$  and  $cx + c^2y + 1 = 0$  are concurrent, show that at least two of three constants  $a, b, c$  are equal.

7. If  $a, b, c$  are in A.P., prove that the straight lines  $ax + 2y + 1 = 0$ ,  $bx + 3y + 1 = 0$  and  $cx + 4y + 1 = 0$  are concurrent.

8. Show that the perpendicular bisectors of the sides of a triangle are concurrent.

### ANSWERS

2.  $\lambda = 4$

3.  $m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$

### HINTS TO SELECTED PROBLEMS

3. If the lines  $y = m_1x + c_1$  and  $y = m_2x + c_2$  and  $y = m_3x + c_3$  are concurrent, then

$$\begin{vmatrix} m_1 & -1 & c_1 \\ m_2 & -1 & c_2 \\ m_3 & -1 & c_3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} m_1 & c_1 & 1 \\ m_2 & c_2 & 1 \\ m_3 & c_3 & 1 \end{vmatrix} = 0 \quad \text{[Interchanging second and third column]}$$

$$\Rightarrow m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$$



## 22.10 LINES PARALLEL AND PERPENDICULAR TO A GIVEN LINE

### LINE PARALLEL TO A GIVEN LINE

**THEOREM 1** Prove that the equation of a line parallel to a given line  $ax + by + c = 0$  is  $ax + by + \lambda = 0$ , where  $\lambda$  is a constant.

**PROOF** Let  $m$  be the slope of the line  $ax + by + c = 0$ . Then,

$$m = -\frac{a}{b}$$

$$\left[ \text{Using : } m = -\frac{\text{Coeff. of } x}{\text{Coeff. of } y} \right]$$

The required line is parallel to the given line. So, the slope of the required line is also  $m$ . Let  $c_1$  be the  $y$ -intercept of the required line. Then, its equation is

$$y = mx + c_1$$

$$\Rightarrow y = -\frac{a}{b}x + c_1 \Rightarrow ax + by - bc_1 = 0 \Rightarrow ax + by + \lambda = 0, \text{ where } \lambda = -bc_1 = \text{constant.}$$

**Q.E.D.**

**NOTE** To write a line parallel to a given line we keep the expression containing  $x$  and  $y$  same and simply replace the given constant by an unknown constant  $\lambda$ . The value of unknown  $\lambda$  can be determined by some given condition.

### LINE PERPENDICULAR TO A GIVEN LINE

**THEOREM 2** Prove that the equation of a line perpendicular to a given line  $ax + by + c = 0$  is  $bx - ay + \lambda = 0$ , where  $\lambda$  is a constant.

**PROOF** Let  $m_1$  be the slope of the given line and  $m_2$  be the slope of a line perpendicular to the given line. Then,  $m_1 = -\frac{a}{b}$ . As the lines are perpendicular.

$$\therefore m_1 m_2 = -1 \Rightarrow m_2 = -\frac{1}{m_1} = \frac{b}{a}$$

Let  $c_2$  be the  $y$ -intercept of the required line. Then, its equation is

$$y = m_2 x + c_2$$

$$\Rightarrow y = \frac{b}{a}x + c_2 \Rightarrow bx - ay + ac_2 = 0 \Rightarrow bx - ay + \lambda = 0, \text{ where } \lambda = ac_2 = \text{constant.}$$

**Q.E.D.**

To write a line perpendicular to a given line we may use the following algorithm.

### ALGORITHM

- Step I Interchange  $x$  and  $y$ .  
 Step II If the coefficients of  $x$  and  $y$  in the given equation are of the same sign make them of opposite signs and if the coefficients are of opposite signs make them of the same sign.  
 Step III Replace the given constant by a new constant  $\lambda$  which is determined by a given condition.

## ILLUSTRATIVE EXAMPLES

### BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** Find the equation of the line which is parallel to  $3x - 2y + 5 = 0$  and passes through the point  $(5, -6)$ .

**SOLUTION** The equation of any line parallel to the line  $3x - 2y + 5 = 0$  is

$$3x - 2y + \lambda = 0$$

...(i)

This line passes through  $(5, -6)$ . Therefore,  $3 \times 5 - 2 \times -6 + \lambda = 0 \Rightarrow \lambda = -27$ .

Putting  $\lambda = -27$  in (i), we obtain  $3x - 2y - 27 = 0$  as the required equation.



**ALITER** The slope of the given line is  $3/2$ . Therefore, the slope of the required line is also  $3/2$ . Since the required line passes through  $(5, -6)$ , so its equation is

$$y + 6 = \frac{3}{2}(x - 5) \quad \text{or,} \quad 3x - 2y - 27 = 0 \quad [\text{Using: } y - y_1 = m(x - x_1)]$$

**EXAMPLE 2** Find the equation of the straight line that passes through the point  $(3, 4)$  and perpendicular to the line  $3x + 2y + 5 = 0$ .

**SOLUTION** The equation of a line perpendicular to  $3x + 2y + 5 = 0$  is

$$2x - 3y + \lambda = 0 \quad \dots(i)$$

This passes through the point  $(3, 4)$ . Therefore,  $3 \times 2 - 3 \times 4 + \lambda = 0 \Rightarrow \lambda = 6$

Putting  $\lambda = 6$  in (i), we obtain  $2x - 3y + 6 = 0$  as the required equation.

**ALITER** The slope of the given line is  $-3/2$ . Since the required line is perpendicular to the given line. So, the slope of the required line is  $2/3$ . As it passes through  $(3, 4)$ . So, its equation is

$$y - 4 = \frac{2}{3}(x - 3) \quad \text{or,} \quad 2x - 3y + 6 = 0 \quad [\text{Using: } y - y_1 = m(x - x_1)].$$

**EXAMPLE 3** Find the equation of the line perpendicular to  $x - 7y + 5 = 0$  and having  $x$ -intercept 3.

**SOLUTION** The equation of a line perpendicular to  $x - 7y + 5 = 0$  is

$$7x + y + \lambda = 0 \quad \dots(i)$$

Its  $x$ -intercept is 3. This means that the line cuts  $x$ -axis at a distance of 3 units from the origin. Consequently, it passes through the point  $(3, 0)$  on  $x$ -axis.

$$\therefore 21 + 0 + \lambda = 0 \Rightarrow \lambda = -21$$

Putting  $\lambda = -21$  in (i), we obtain  $7x + y - 21 = 0$  as the equation of the required line.

#### BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 4** Find the coordinates of the foot of the perpendicular drawn from the point  $(1, -2)$  on the line  $y = 2x + 1$ .

**SOLUTION** Let  $M$  be the foot of the perpendicular drawn from  $P(1, -2)$  on the line  $y = 2x + 1$ . Then,  $M$  is the point of intersection of  $y = 2x + 1$  and a line passing through  $P(1, -2)$  and perpendicular to  $y = 2x + 1$ . The equation of a line perpendicular to  $y = 2x + 1$  or,  $2x - y + 1 = 0$  is

$$x + 2y + \lambda = 0 \quad \dots(i)$$

This passes through  $P(1, -2)$ .

$$\therefore 1 - 4 + \lambda = 0 \Rightarrow \lambda = 3$$

Putting  $\lambda = 3$  in (i), we get:  $x + 2y + 3 = 0$

Point  $M$  is the point of intersection of the lines

$$2x - y + 1 = 0 \quad \text{and} \quad x + 2y + 3 = 0.$$

Solving these equations by cross-multiplication, we obtain:

$$\frac{x}{-5} = \frac{y}{-5} = \frac{1}{5} \Rightarrow x = -1, \quad y = -1.$$

Hence, the coordinates of the foot of the perpendicular are  $(-1, -1)$ .

**EXAMPLE 5** Find the equation of a straight line parallel to  $2x + 3y + 11 = 0$  and which is such that the sum of its intercepts on the axes is 15.

**SOLUTION** The equation of a line parallel to  $2x + 3y + 11 = 0$  is

$$2x + 3y + \lambda = 0, \quad \lambda \text{ is a constant} \quad \dots(i)$$

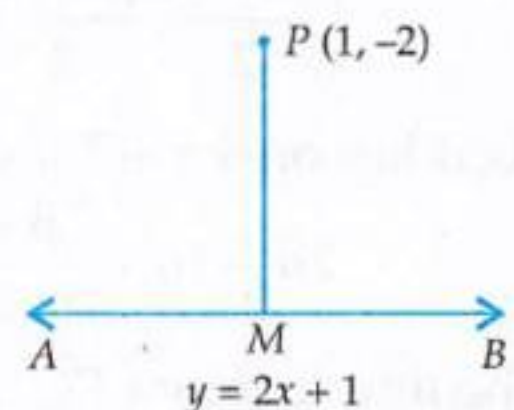


Fig. 22.70



To find  $x$ -intercept of this line, we put  $y = 0$  in its equation. Putting  $y = 0$  in (i), we get

$$\Rightarrow 2x + \lambda = 0 \Rightarrow x = -\lambda/2$$

So,  $x$ -intercept  $= -\lambda/2$ .

To find  $y$ -intercept of this line, we put  $x = 0$  in its equation. Putting  $x = 0$  in (i), we get

$$3y + \lambda = 0 \Rightarrow y = -\lambda/3$$

So,  $y$ -intercept  $= -\lambda/3$ .

It is given that the sum of the intercepts of the line (i) on the coordinate axes is 15.

$$\therefore \left(-\frac{\lambda}{2}\right) + \left(-\frac{\lambda}{3}\right) = 15 \Rightarrow -\frac{5\lambda}{6} = 15 \Rightarrow \lambda = -18$$

Putting  $\lambda = -18$  in (i), we get:  $2x + 3y - 18 = 0$ .

Hence, the equation of the required line is  $2x + 3y - 18 = 0$ .

**EXAMPLE 6** Show that the equation of a line passing through  $(a \cos^3 \theta, a \sin^3 \theta)$  and perpendicular to the line  $x \sec \theta + y \operatorname{cosec} \theta = a$  is  $x \cos \theta - y \sin \theta = a \cos 2 \theta$ .

**SOLUTION** The equation of a line perpendicular to the line  $x \sec \theta + y \operatorname{cosec} \theta = a$  is

$$x \operatorname{cosec} \theta - y \sec \theta + \lambda = 0, \lambda \text{ is a constant} \quad \dots(i)$$

This line passes through  $(a \cos^3 \theta, a \sin^3 \theta)$ .

$$\therefore a \cos^3 \theta \operatorname{cosec} \theta - a \sin^3 \theta \sec \theta + \lambda = 0 \Rightarrow \lambda = a(\sin^3 \theta \sec \theta - \cos^3 \theta \operatorname{cosec} \theta)$$

Putting the value of  $\lambda$  in (i), we get

$$x \operatorname{cosec} \theta - y \sec \theta + a(\sin^3 \theta \sec \theta - \cos^3 \theta \operatorname{cosec} \theta) = 0$$

$$\Rightarrow \frac{x}{\sin \theta} - \frac{y}{\cos \theta} + a \left( \frac{\sin^3 \theta}{\cos \theta} - \frac{\cos^3 \theta}{\sin \theta} \right) = 0$$

$$\Rightarrow x \cos \theta - y \sin \theta + a(\sin^4 \theta - \cos^4 \theta) = 0$$

$$\Rightarrow x \cos \theta - y \sin \theta + a(\sin^2 \theta + \cos^2 \theta)(\sin^2 \theta - \cos^2 \theta) = 0$$

$$\Rightarrow x \cos \theta - y \sin \theta - a(\cos^2 \theta - \sin^2 \theta) = 0$$

$$\Rightarrow x \cos \theta - y \sin \theta - a \cos 2 \theta = 0 \Rightarrow x \cos \theta - y \sin \theta = a \cos 2 \theta.$$

**EXAMPLE 7** Find the image of the point  $(-8, 12)$  with respect to the line mirror  $4x + 7y + 13 = 0$ .

**SOLUTION** Let the image of the point  $P(-8, 12)$  in the line mirror  $AB$  be  $Q(\alpha, \beta)$ . Then, the line segment  $PQ$  is perpendicularly bisected at  $R$ . So, the coordinates of  $R$  are

$$\left( \frac{\alpha - 8}{2}, \frac{\beta + 12}{2} \right).$$

As it lies on  $4x + 7y + 13 = 0$ .

$$\therefore 2\alpha - 16 + \frac{7\beta + 84}{2} + 13 = 0 \Rightarrow 4\alpha + 7\beta + 78 = 0 \quad \dots(i)$$

The line segment  $PQ$  is perpendicular to  $AB$ .

$$\therefore (\text{Slope of } AB) \times (\text{Slope of } PQ) = -1$$

$$\Rightarrow -\frac{4}{7} \times \frac{\beta - 12}{\alpha + 8} = -1 \Rightarrow 7\alpha - 4\beta + 104 = 0 \quad \dots(ii)$$

Solving (i) and (ii), we get:  $\alpha = -16, \beta = -2$ .

Hence, the image of the point  $(-8, 12)$  in the line mirror  $4x + 7y + 13 = 0$  is the point  $(-16, -2)$ .

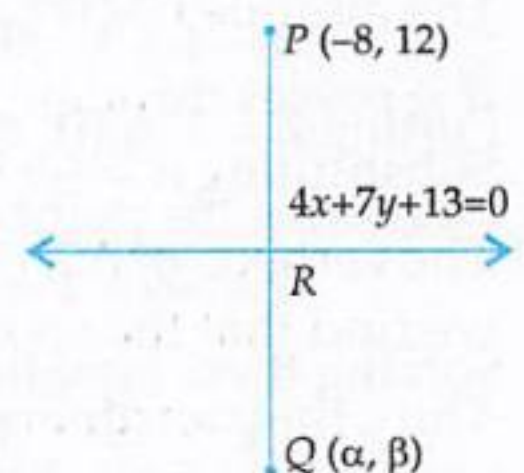


Fig. 22.71



## BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

**EXAMPLE 8** A person stranding at a junction (crossing) of two straight paths represented by the equations  $2x - 3y - 4 = 0$  and  $3x - 4y - 5 = 0$ , wants to reach the path whose equation is  $6x - 7y + 8 = 0$  in the least time. Find the equation of the path that he should follow. [NCERT]

**SOLUTION** The lines  $2x - 3y - 4 = 0$  and  $3x - 4y - 5 = 0$  intersect at  $(-1, -2)$ . In order to reach the path, represented by the equation  $6x - 7y + 8 = 0$ , in the least time, the person should move along the line passing through A and perpendicular to  $6x - 7y + 8 = 0$ .

Clearly, slope of the line  $6x - 7y + 8 = 0$  is  $\frac{6}{7}$ . Therefore, slope of a line perpendicular to it is  $-\frac{7}{6}$ .

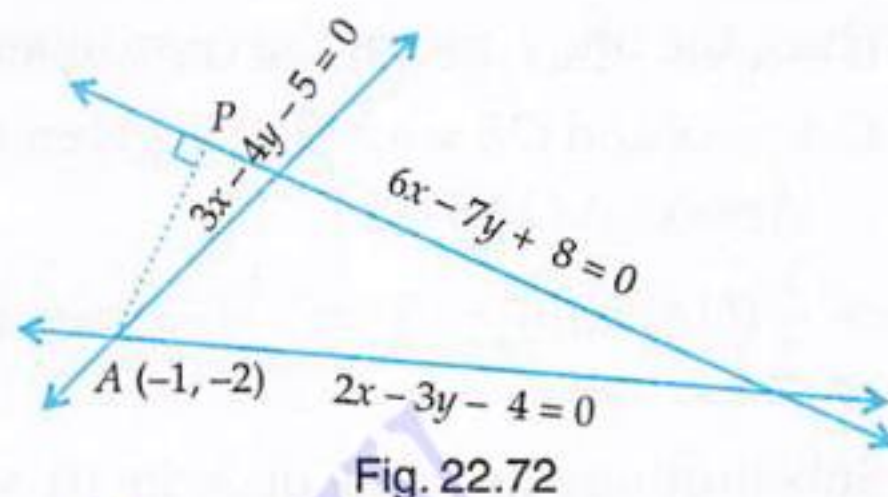


Fig. 22.72

Hence, the equation of the required path is  $y + 2 = -\frac{7}{6}(x + 1)$  or,  $7x + 6y + 19 = 0$

**EXAMPLE 9** The equations of two sides of a triangle are  $3x - 2y + 6 = 0$  and  $4x + 5y - 20 = 0$  and the orthocentre is  $(1, 1)$ . Find the equation of the third side. [NCERT]

**SOLUTION** Let the equations of sides AB and AC of triangle ABC be respectively

$$3x - 2y + 6 = 0 \quad \dots(i)$$

$$\text{and, } 4x + 5y - 20 = 0 \quad \dots(ii)$$

Let  $H(1, 1)$  be the orthocentre of triangle ABC where the altitudes AL, BM and CN intersect each other. Clearly, BM passes through  $H(1, 1)$  and is perpendicular to AC. The equation of a line perpendicular to AC is

$$5x - 4y + \lambda = 0 \quad \dots(iii)$$

If it passes through the orthocentre  $H(1, 1)$ , then

$$5 - 4 + \lambda = 0 \Rightarrow \lambda = -1$$

Substituting  $\lambda = -1$  in (iii), we get:

$$5x - 4y - 1 = 0 \quad \dots(iv)$$

This is the equation of altitude BM.

The vertex B of  $\triangle ABC$  is the intersection point of side AB and altitude BM. Solving their equations given by (i) and (iii), we get  $x = -13$  and  $y = -33/2$ . So, coordinates of B are  $(-13, -33/2)$ .

The altitude CN is perpendicular to AB. So, let its equation be  $2x + 3y + \mu = 0$  ...(v)

If it passes through the orthocentre  $H(1, 1)$ , then  $2 + 3 + \mu = 0 \Rightarrow \mu = -5$ .

Substituting  $\mu = -5$  in (v), we get:  $2x + 3y - 5 = 0$  as the equation of CN. ...(vi)

The vertex C of  $\triangle ABC$  is the point of intersection of side AC and altitude CN. Solving (ii) and (vi), we find that the coordinates of C are  $(35/2, -10)$ .

Thus, the coordinates of B and C are  $(-13, -33/2)$  and  $(35/2, -10)$  respectively. Hence, equation of side BC is

$$y + \frac{33}{2} = \frac{-10 + \frac{33}{2}}{\frac{35}{2} + 13} (x + 13) \text{ or, } \frac{2y + 33}{2} = \frac{13}{61} (x + 13) \text{ or, } 26x - 122y - 1675 = 0$$

**EXAMPLE 10** A straight line L is perpendicular to the line  $5x - y = 1$ . The area of the triangle formed by the line L and the coordinate axes is 5 square units. Find the equation of the line L.

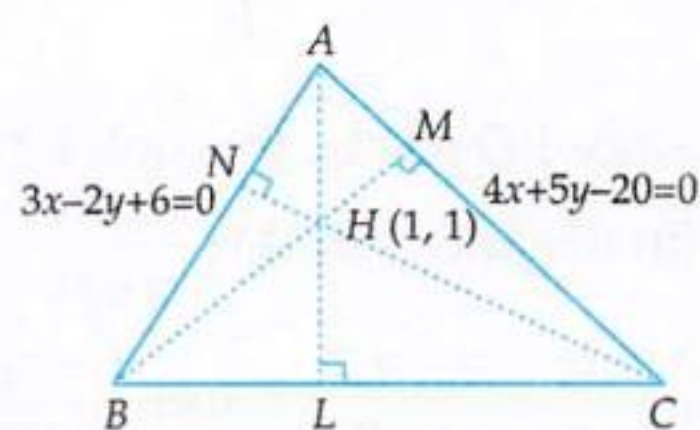


Fig. 22.73



**SOLUTION** The equation of a line  $L$  perpendicular to the line  $5x - y = 1$  is

$$x + 5y + \lambda = 0 \quad \dots(i)$$

This line meets  $x$ -axis at  $y = 0$ . Putting  $y = 0$ , we get  $x = -\lambda$ . So, the line  $L$  meets  $x$ -axis at  $A(-\lambda, 0)$ . The line  $L$  meets  $y$ -axis at  $x = 0$ . Putting  $x = 0$  in the equation  $x + 5y + \lambda = 0$ , we obtain  $y = -\lambda/5$ . So, the line  $L$  meets  $y$ -axis at  $B(0, -\lambda/5)$ .

Thus, the line  $L$  meets the coordinate axes at  $A$  and  $B$  such that

$OA = -\lambda$  and  $OB = -\lambda/5$ . It is given that

$$\text{Area of } \triangle OAB = 5$$

$$\Rightarrow \frac{1}{2}(OA)(OB) = 5 \Rightarrow \frac{1}{2}(-\lambda)\left(-\frac{\lambda}{5}\right) = 5 \Rightarrow \lambda^2 = 50 \Rightarrow \lambda = \pm 5\sqrt{2}$$

Substituting the value of  $\lambda$  in (i), we obtain  $x + 5y \pm 5\sqrt{2} = 0$  as the equations of the required line  $L$ .

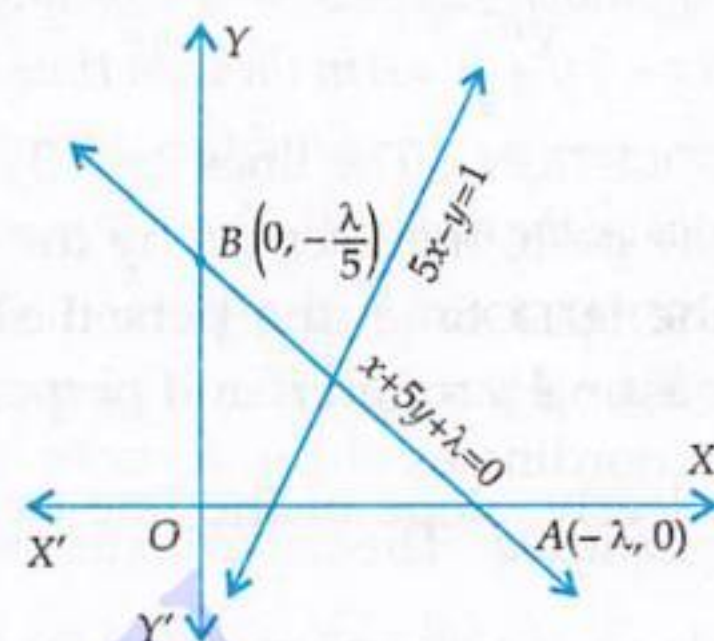


Fig. 22.74

**EXAMPLE 11** Let  $P(x_1, y_1)$  be a point and let  $ax + by + c = 0$  be a line. If  $L(h, k)$  is the foot of perpendicular drawn from  $P$  on this line and  $Q(\alpha, \beta)$  is the image of  $P$  in the given line, then prove that

$$(i) \quad \frac{h - x_1}{a} = \frac{k - y_1}{b} = -\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right) \quad (ii) \quad \frac{\alpha - x_1}{a} = \frac{\beta - y_1}{b} = -2\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right)$$

**SOLUTION** Suppose  $PQ$  makes an angle  $\theta$  with  $x$ -axis. Since  $PQ$  is perpendicular to  $ax + by + c = 0$ .

$$\therefore \text{Slope of } PQ \times (\text{Slope of } ax + by + c = 0) = -1$$

$$\Rightarrow \tan \theta \times -\frac{a}{b} = -1 \Rightarrow \tan \theta = \frac{b}{a} \Rightarrow \sin \theta = \frac{b}{\sqrt{a^2 + b^2}} \text{ and } \cos \theta = \frac{a}{\sqrt{a^2 + b^2}}$$

Since  $PQ$  passes through  $P(x_1, y_1)$  and makes an angle  $\theta$  with  $x$ -axis. Therefore, equation of  $PQ$  (in distance form) is  $\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta}$ . Let  $PL = LQ = r$ . Then, coordinates of  $L$  and  $Q$  are given by

$$\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r \text{ and } \frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = 2r \text{ respectively.}$$

$$\therefore \frac{h - x_1}{\cos \theta} = \frac{k - y_1}{\sin \theta} = r \dots(i) \text{ and } \frac{\alpha - x_1}{\cos \theta} = \frac{\beta - y_1}{\sin \theta} = 2r \dots(ii)$$

$$\text{Now, } \frac{h - x_1}{\cos \theta} = \frac{k - y_1}{\sin \theta} = r \Rightarrow h = x_1 + r \cos \theta, k = y_1 + r \sin \theta$$

So, the coordinates of  $L$  are  $(x_1 + r \cos \theta, y_1 + r \sin \theta)$ .

Point  $L$  lies on  $ax + by + c = 0$ .

$$\therefore a(x_1 + r \cos \theta) + b(y_1 + r \sin \theta) + c = 0$$

$$\Rightarrow r = -\frac{ax_1 + by_1 + c}{a \cos \theta + b \sin \theta} \Rightarrow r = -\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \left[ \because \cos \theta = \frac{a}{\sqrt{a^2 + b^2}}, \sin \theta = \frac{b}{\sqrt{a^2 + b^2}} \right]$$

(i) Substituting the values of  $\cos \theta$ ,  $\sin \theta$  and  $r$  in (i), we get

$$\frac{h - x_1}{\frac{a}{\sqrt{a^2 + b^2}}} = \frac{k - y_1}{\frac{b}{\sqrt{a^2 + b^2}}} = -\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \Rightarrow \frac{h - x_1}{a} = \frac{k - y_1}{b} = -\frac{ax_1 + by_1 + c}{a^2 + b^2}$$

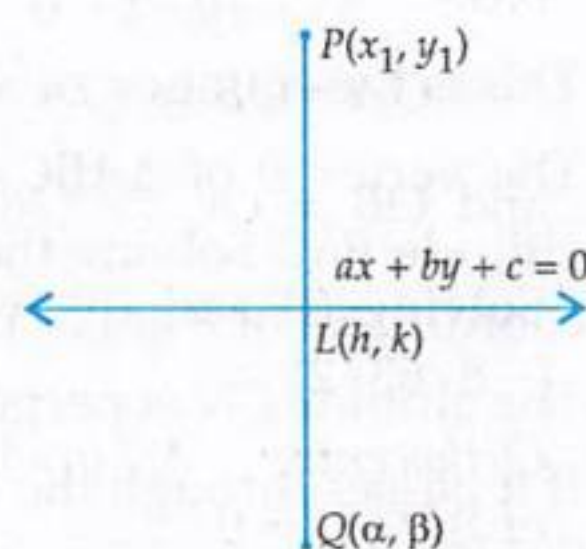


Fig. 22.75



(ii) Substituting the values of  $\cos \theta$ ,  $\sin \theta$  and  $r$  in (ii), we get

$$\frac{\frac{\alpha - x_1}{a}}{\sqrt{a^2 + b^2}} = \frac{\frac{\beta - y_1}{b}}{\sqrt{a^2 + b^2}} = -2 \left( \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right) \Rightarrow \frac{\alpha - x_1}{a} = \frac{\beta - y_1}{b} = -2 \left( \frac{ax_1 + by_1 + c}{a^2 + b^2} \right)$$

**EXAMPLE 12** Find the centroid, incentre circum-centre and orthocentre of the triangle whose sides have the equations  $3x - 4y = 0$ ,  $12y + 5x = 0$  and  $y - 15 = 0$

**SOLUTION** Let  $ABC$  be the triangle whose sides  $BC$ ,  $CA$  and  $AB$  have the equations  $y - 15 = 0$ ,  $3x - 4y = 0$  and  $5x + 12y = 0$  respectively. Solving these equations pair wise we can obtain the coordinates of the vertices  $A$ ,  $B$ ,  $C$  as  $A(0, 0)$ ,  $B(-36, 15)$  and  $C(20, 15)$  respectively.

**Centroid:** The coordinates of the centroid are

$$\left( \frac{0 - 36 + 20}{3}, \frac{0 + 15 + 15}{3} \right) = \left( -\frac{16}{3}, 10 \right) \quad \left[ \text{Using: } \left( \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right) \right]$$

**In centre:** We have,

$$a = BC = \sqrt{(-36 - 20)^2 + (15 - 15)^2} = 56,$$

$$b = CA = \sqrt{20^2 + 15^2} = 25,$$

$$\text{and, } c = AB = \sqrt{(-36 - 0)^2 + (15 - 0)^2} = 39.$$

Using  $\left( \frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c} \right)$  the coordinates of the incentre are

$$\left( \frac{56 \times 0 + 25 \times -36 + 39 \times 20}{56 + 25 + 39}, \frac{56 \times 0 + 25 \times 15 + 39 \times 15}{56 + 25 + 39} \right) = (-1, 8)$$

**Circum-centre:** Let  $(x, y)$  be the coordinates of the circum-centre  $O$  (say). Then,  $OA = OB = OC$ . Now,

$$OA = OB \Rightarrow OA^2 = OB^2 \Rightarrow x^2 + y^2 = (x + 36)^2 + (y - 15)^2 \Rightarrow 72x - 30y + 1521 = 0 \quad \dots(i)$$

$$\text{and, } OB = OC \Rightarrow OB^2 = OC^2 \Rightarrow (x + 36)^2 + (y - 15)^2 = (x - 20)^2 + (y - 15)^2 \Rightarrow x = -8 \quad \dots(ii)$$

Solving (i) and (ii), we get :  $x = -8$  and,  $y = 63/2$ . So, the coordinates of circumcentre are  $(-8, 63/2)$ .

**Orthocentre:**  $AD$  is a line passing through  $A(0, 0)$  and perpendicular to  $y - 15 = 0$ . So, equation of  $AD$  is  $x = 0$ .

The equation of any line perpendicular to side  $AC$  having equation  $3x - 4y = 0$  is  $4x + 3y + \lambda = 0$ . If it passes through  $B(-36, 15)$ , then

$$-144 + 45 + \lambda = 0 \Rightarrow \lambda = 99.$$

So, the equation of  $BE$  is  $4x + 3y + 99 = 0$ . Solving the equations of  $AD$  and  $BE$ , we obtain  $x = 0$  and  $y = -33$ . Hence, the coordinates of the orthocentre are  $(0, -33)$ .

**EXAMPLE 13** Find the circumcentre of the triangle whose sides are  $3x - y + 3 = 0$ ,  $3x + 4y + 3 = 0$  and  $x + 3y + 11 = 0$ .

**SOLUTION** Let  $ABC$  be the triangle whose sides  $AB$ ,  $BC$  and  $CA$  have the equations  $3x - y + 3 = 0$ ,  $3x + 4y + 3 = 0$  and  $x + 3y + 11 = 0$  respectively.

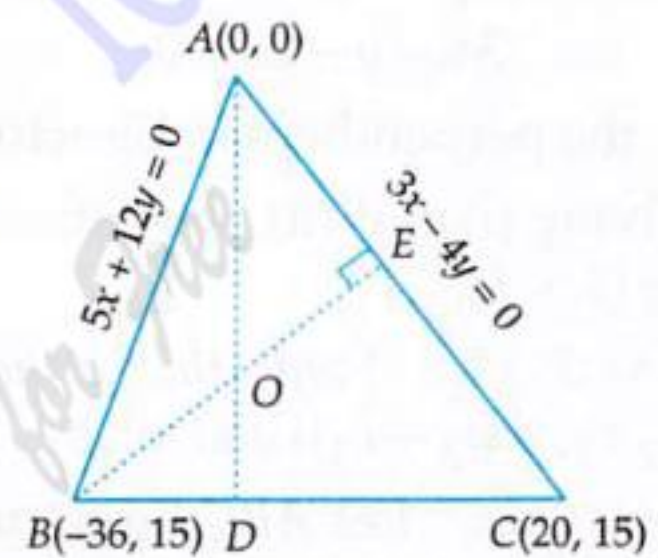


Fig. 22.76



The circumcentre of  $\Delta ABC$  is the point of concurrence of its perpendicular bisectors. So, let us first find the perpendicular bisector of sides  $BC$  and  $AC$ .

Solving the equations of  $AB$  and  $AC$ ;  $BC$  and  $AB$ ;  $AC$  and  $BC$  in pairs we obtain that the coordinates of the vertices  $A$ ,  $B$  and  $C$  are  $(-2, -3)$ ,  $(-1, 0)$  and  $(7, -6)$  respectively.

The equation of a line perpendicular to  $BC$  is  $4x - 3y + \lambda = 0$ .

This will pass through  $(3, -3)$ , the mid point of  $BC$ , if

$$12 + 9 + \lambda = 0 \Rightarrow \lambda = -21.$$

Putting  $\lambda = -21$  in  $4x - 3y + \lambda = 0$ , we get

$$4x - 3y - 21 = 0. \quad \dots(i)$$

as the equation of the perpendicular bisector of  $BC$ .

The equation of a line perpendicular to  $AC$  is  $3x - y + \lambda_1 = 0$ .

This will pass through  $(5/2, -9/2)$  i.e. the mid point of  $AC$ , if

$$\frac{15}{2} + \frac{9}{2} + \lambda_1 = 0 \Rightarrow \lambda_1 = -12.$$

Putting  $\lambda_1 = -12$  in  $3x - y + \lambda_1 = 0$ , we get

$$3x - y - 12 = 0. \quad \dots(ii)$$

as the perpendicular bisector of  $AC$ .

Solving (i) and (ii), we get:  $x = 3$ ,  $y = -3$ . Hence, the coordinates of the circumcentre of  $\Delta ABC$  are  $(3, -3)$ .

**EXAMPLE 14** Find the orthocentre of the triangle whose vertices are  $(at_1 t_2, a(t_1 + t_2))$ ,  $(at_2 t_3, a(t_2 + t_3))$  and  $(at_1 t_3, a(t_1 + t_3))$ .

**SOLUTION** Let  $ABC$  be a triangle whose vertices are  $A(at_1 t_2, a(t_1 + t_2))$ ,  $B(at_2 t_3, a(t_2 + t_3))$  and  $C(at_1 t_3, a(t_1 + t_3))$ . The orthocentre of  $\Delta ABC$  is the point of concurrence of its altitudes. So, let us find their equations.

Clearly,

$$\text{Slope of } BC = \frac{a(t_2 + t_3) - a(t_1 + t_3)}{at_2 t_3 - at_1 t_3} = \frac{1}{t_3} \text{ and, Slope of } AC = \frac{a(t_1 + t_3) - a(t_1 + t_2)}{at_1 t_3 - at_1 t_2} = \frac{1}{t_1}.$$

The equation of the line through  $A$  perpendicular to  $BC$  i.e. the altitude through vertex  $A$  is

$$y - a(t_1 + t_2) = -t_3(x - at_1 t_2) \quad \dots(i)$$

The equation of the line through  $B$  perpendicular to  $AC$  i.e. the altitude through vertex  $B$  is

$$y - a(t_2 + t_3) = -t_1(x - at_2 t_3) \quad \dots(ii)$$

The point of intersection of (i) and (ii) is the orthocentre.

Subtracting (ii) from (i), we get  $x = -a$ .

Putting  $x = -a$  in (i), we get  $y = a(t_1 + t_2 + t_3 + t_1 t_2 t_3)$ .

Hence, the coordinates of the orthocentre are  $(-a, a(t_1 + t_2 + t_3 + t_1 t_2 t_3))$ .

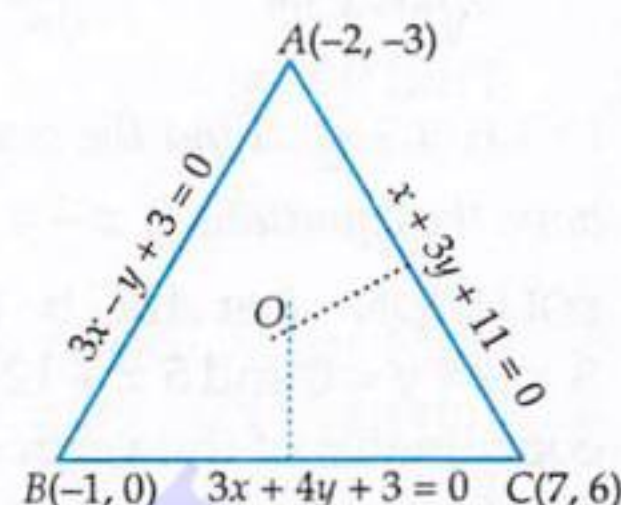


Fig. 22.77

## EXERCISE 22.12

### BASIC

- Find the equation of a line passing through the point  $(2, 3)$  and parallel to the line  $3x - 4y + 5 = 0$ .
- Find the equation of a line passing through  $(3, -2)$  and perpendicular to the line  $x - 3y + 5 = 0$ .
- Find the equation of the perpendicular bisector of the line joining the points  $(1, 3)$  and  $(3, 1)$ .
- Find the equations of the altitudes of a  $\Delta ABC$  whose vertices are  $A(1, 4)$ ,  $B(-3, 2)$  and  $C(-5, -3)$ .



5. Find the equation of a line which is perpendicular to the line  $\sqrt{3}x - y + 5 = 0$  and which cuts off an intercept of 4 units with the negative direction of  $y$ -axis.
6. If the image of the point  $(2, 1)$  with respect to a line mirror is  $(5, 2)$ , find the equation of the mirror.
7. Find the equation of the straight line through the point  $(\alpha, \beta)$  and perpendicular to the line  $lx + my + n = 0$ .

**BASED ON LOTS**

8. Find the equation of the straight line perpendicular to  $2x - 3y = 5$  and cutting off an intercept 1 on the positive direction of the  $x$ -axis.
9. Find the equation of the straight line perpendicular to  $5x - 2y = 8$  and which passes through the mid-point of the line segment joining  $(2, 3)$  and  $(4, 5)$ .
10. Find the equation of the straight line which has  $y$ -intercept equal to  $4/3$  and is perpendicular to  $3x - 4y + 11 = 0$ .
11. Find the equation of the right bisector of the line segment joining the points  $(a, b)$  and  $(a_1, b_1)$ .
12. Find the image of the point  $(2, 1)$  with respect to the line mirror  $x + y - 5 = 0$ .
13. If the image of the point  $(2, 1)$  with respect to the line mirror be  $(5, 2)$ , find the equation of the mirror.
14. Find the equation to the straight line parallel to  $3x - 4y + 6 = 0$  and passing through the middle point of the join of points  $(2, 3)$  and  $(4, -1)$ .
15. Prove that the lines  $2x - 3y + 1 = 0$ ,  $x + y = 3$ ,  $2x - 3y = 2$  and  $x + y = 4$  form a parallelogram.
16. Find the equation of a line drawn perpendicular to the line  $\frac{x}{4} + \frac{y}{6} = 1$  through the point where it meets the  $y$ -axis. [NCERT]
17. The perpendicular from the origin to the line  $y = mx + c$  meets it at the point  $(-1, 2)$ . Find the values of  $m$  and  $c$ . [NCERT]
18. Find the equation of the right bisector of the line segment joining the points  $(3, 4)$  and  $(-1, 2)$ .
19. The line through  $(h, 3)$  and  $(4, 1)$  intersects the line  $7x - 9y - 19 = 0$  at right angle. Find the value of  $h$ .
20. Find the image of the point  $(3, 8)$  with respect to the line  $x + 3y = 7$  assuming the line to be a plane mirror. [NCERT]
21. Find the coordinates of the foot of the perpendicular from the point  $(-1, 3)$  to the line  $3x - 4y - 16 = 0$ . [NCERT]
22. Find the projection of the point  $(1, 0)$  on the line joining the points  $(-1, 2)$  and  $(5, 4)$ .
23. Find the equation of a line perpendicular to the line  $\sqrt{3}x - y + 5 = 0$  and at a distance of 3 units from the origin.

**BASED ON HOTS**

24. The line  $2x + 3y = 12$  meets the  $x$ -axis at  $A$  and  $y$ -axis at  $B$ . The line through  $(5, 5)$  perpendicular to  $AB$  meets the  $x$ -axis and the line  $AB$  at  $C$  and  $E$  respectively. If  $O$  is the origin of coordinates, find the area of figure  $OCEB$ .
25. Find the equation of the straight line which cuts off intercepts on  $x$ -axis twice that on  $y$ -axis and is at a unit distance from the origin.
26. The equations of perpendicular bisectors of the sides  $AB$  and  $AC$  of a triangle  $ABC$  are  $x - y + 5 = 0$  and  $x + 2y = 0$  respectively. If the point  $A$  is  $(1, -2)$ , find the equation of the line  $BC$ .



## ANSWERS

1.  $3x - 4y + 6 = 0$
2.  $3x + y - 7 = 0$
3.  $y = x$
4.  $2x + 5y - 22 = 0, 6x + 7y + 4 = 0, 2x + y + 13 = 0$
5.  $x + \sqrt{3}y + 4\sqrt{3} = 0$
6.  $3x + y - 12 = 0$
7.  $m(x - \alpha) = l(y - \beta)$
8.  $3x + 2y - 3 = 0$
9.  $2x + 5y - 26 = 0$
10.  $4x + 3y - 4 = 0$
11.  $2x(a_1 - a) + 2y(b_1 - b) + (a^2 + b^2) - (a_1^2 + b_1^2) = 0$
12.  $(4, 3)$
13.  $3x + y = 12$
14.  $3x - 4y = 5$
16.  $2x - 3y + 18 = 0$
17.  $m = 1/2, c = 5/2$
18.  $2x + y - 5 = 0$
19.  $22/9$
20.  $(-1, -4)$
21.  $\left(\frac{68}{25}, -\frac{49}{25}\right)$
22.  $\left(\frac{1}{5}, \frac{12}{5}\right)$
23.  $x + \sqrt{3}y \pm 6 = 0$
24.  $\frac{23}{3}$  sq. units
25.  $x + 2y \pm \sqrt{5} = 0$
26.  $14x + 23y - 40 = 0$
27.  $5x + 3y + 8 = 0$

## HINTS TO SELECTED PROBLEMS

16. The line  $\frac{x}{4} + \frac{y}{6} = 1$  cuts  $y$ -axis at  $(0, 6)$  and has slope is  $-\frac{3}{2}$ .  
Hence, equation of the required line is  $y - 6 = \frac{2}{3}(x - 0)$  or,  $2x - 3y + 18 = 0$
17. Clearly,  $(-1, 2)$  lies on  $y = mx + c$ .  
 $\therefore 2 = -m + c$  ... (i)  
The line joining the origin to  $(-1, 2)$  is perpendicular to  $y = mx + c$ .  
 $\therefore \frac{2 - 0}{-1 - 0} \times m = -1 \Rightarrow m = \frac{1}{2}$ . Putting  $m = \frac{1}{2}$  in (i), we get  $c = \frac{5}{2}$ .
20. Let the image of the point  $P(3, 8)$  in the line mirror  $x + 3y = 7$  be  $Q(\alpha, \beta)$ . Then,  $PQ$  is perpendicularly bisected at  $M$ . The coordinates of  $M$  are  $\left(\frac{\alpha + 3}{2}, \frac{\beta + 8}{2}\right)$ . Since  $M$  lies on  $x + 3y = 7$ .  
 $\therefore \frac{\alpha + 3}{2} + 3\left(\frac{\beta + 8}{2}\right) = 7 \Rightarrow \alpha + 3\beta + 13 = 0$  ... (i)  
Line segment  $PQ$  is perpendicular to  $x + 3y = 7$ .  
 $\therefore \frac{\beta - 8}{\alpha - 3} \times -\frac{1}{3} = -1$   
 $\Rightarrow 3\alpha - 9 = \beta - 8 \Rightarrow 3\alpha - \beta - 1 = 0$  ... (ii)  
Solving (i) and (ii), we get  $\alpha = -1$  and  $\beta = -4$ .

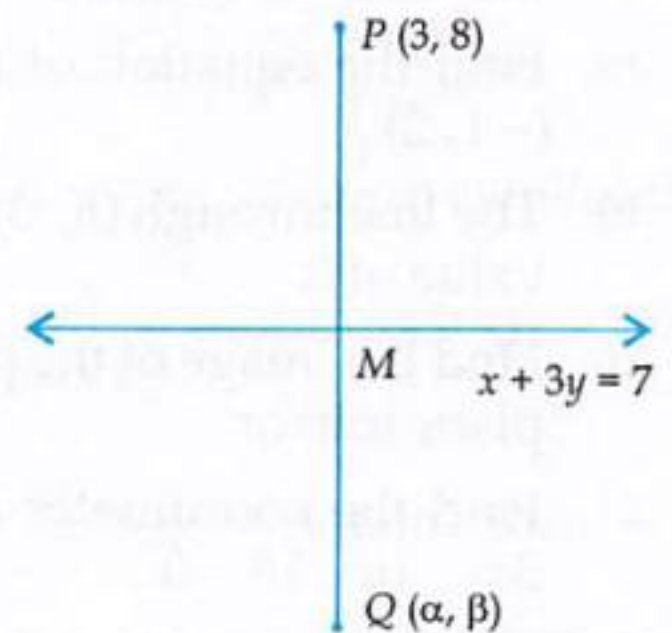


Fig. 22.78

- Hence, the coordinates of  $Q$  are  $(-1, -4)$ .
21. Let  $M(\alpha, \beta)$  be the foot of perpendicular from  $P(-1, 3)$  on the line  $3x - 4y - 16 = 0$ . Then,  
 $\frac{\beta - 3}{\alpha + 1} \times \frac{3}{4} = -1 \Rightarrow 4\alpha + 3\beta - 5 = 0$  ... (i)

Point  $M(\alpha, \beta)$  lies on  $3x - 4y - 16 = 0$ .

$$\therefore 3\alpha - 4\beta - 16 = 0 \quad \dots (ii)$$

Solving (i) and (ii), we get:  $\alpha = \frac{68}{25}$  and  $\beta = -\frac{49}{25}$ .

Hence, the coordinates of  $M$  are  $(68/25, -49/25)$ .

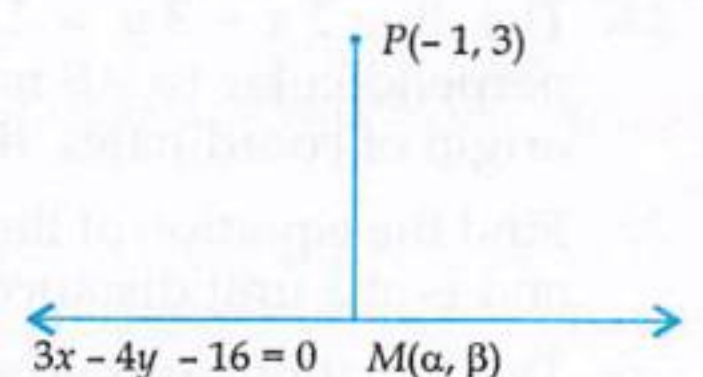


Fig. 22.79



### 22.11 ANGLE BETWEEN TWO STRAIGHT LINES WHEN THEIR EQUATIONS ARE GIVEN

**THEOREM** Prove that the acute angle  $\theta$  between the lines  $a_1 x + b_1 y + c_1 = 0$  and  $a_2 x + b_2 y + c_2 = 0$  is given by

$$\tan \theta = \left| \frac{a_2 b_1 - a_1 b_2}{a_1 a_2 + b_1 b_2} \right|$$

**PROOF** Let  $m_1$  and  $m_2$  be the slopes of the lines  $a_1 x + b_1 y + c_1 = 0$  and  $a_2 x + b_2 y + c_2 = 0$ . Then,

$$m_1 = -\frac{a_1}{b_1} \text{ and } m_2 = -\frac{a_2}{b_2}$$

$$\text{Now, } \tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$\Rightarrow \tan \theta = \left| \frac{-\frac{a_1}{b_1} + \frac{a_2}{b_2}}{1 + \left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right)} \right| \Rightarrow \tan \theta = \left| \frac{a_2 b_1 - a_1 b_2}{a_1 a_2 + b_1 b_2} \right| \Rightarrow \theta = \tan^{-1} \left| \frac{a_2 b_1 - a_1 b_2}{a_1 a_2 + b_1 b_2} \right|$$

**Q.E.D.**

**CONDITION FOR THE LINES TO BE PARALLEL** If the lines  $a_1 x + b_1 y + c_1 = 0$  and  $a_2 x + b_2 y + c_2 = 0$  are parallel, then

$$m_1 = m_2 \Rightarrow -\frac{a_1}{b_1} = -\frac{a_2}{b_2} \Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

**CONDITION FOR THE LINES TO BE PERPENDICULAR** If the lines  $a_1 x + b_1 y + c_1 = 0$  and  $a_2 x + b_2 y + c_2 = 0$  are perpendicular, then

$$m_1 m_2 = -1 \Rightarrow -\frac{a_1}{b_1} \times -\frac{a_2}{b_2} = -1 \Rightarrow a_1 a_2 + b_1 b_2 = 0$$

It follows from the above discussion that the lines  $a_1 x + b_1 y + c_1 = 0$  and  $a_2 x + b_2 y + c_2 = 0$  are:

- |                                                                          |                                                                            |
|--------------------------------------------------------------------------|----------------------------------------------------------------------------|
| (i) Coincident, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ | (ii) Parallel, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ |
| (iii) Intersecting, if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$            | (iv) Perpendicular, if $a_1 a_2 + b_1 b_2 = 0$                             |

To find the acute angle between two lines when their slopes are given, we may use the following algorithm.

#### ALGORITHM

Step I Obtain the equations of the lines.

Step II Obtain the slopes  $m_1$  and  $m_2$  of two lines by using the formula:  $\text{Slope} = -\frac{\text{Coeff. of } x}{\text{Coeff. of } y}$ .

Step II Use the formula :  $\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$  to find the acute angle  $\theta$  between the lines.



## ILLUSTRATIVE EXAMPLES

## BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** Find the angles between the pairs of straight lines

$$(i) \ x - \sqrt{3}y - 5 = 0 \text{ and } \sqrt{3}x + y - 7 = 0 \quad (ii) \ y = (2 - \sqrt{3})x + 5 \text{ and } y = (2 + \sqrt{3})x - 7.$$

**SOLUTION** (i) The equations of two straight lines are:

$$x - \sqrt{3}y - 5 = 0 \quad \dots(i) \quad \text{and} \quad \sqrt{3}x + y - 7 = 0 \quad \dots(ii)$$

Let  $m_1$  and  $m_2$  be the slopes of these two lines. Then,

$$m_1 = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}} \text{ and } m_2 = -\frac{\sqrt{3}}{1} = -\sqrt{3}$$

We observe that  $m_1 m_2 = -1$ . Thus, the two lines are at right angle.(ii) Let  $m_1$  and  $m_2$  be the slopes of the straight lines  $y = (2 - \sqrt{3})x + 5$  and  $y = (2 + \sqrt{3})x - 7$  respectively. Then,  $m_1 = 2 - \sqrt{3}$  and  $m_2 = 2 + \sqrt{3}$ .Let  $\theta$  be the angle between the lines. Then,

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| = \left| \frac{(2 - \sqrt{3}) - (2 + \sqrt{3})}{1 + (2 - \sqrt{3})(2 + \sqrt{3})} \right| = \left| \frac{-2\sqrt{3}}{1 + 4 - 3} \right| = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$$

Thus, the acute angle between the lines is of  $60^\circ$ .**EXAMPLE 2** Find the tangent of the angle between the lines whose intercepts on the axes are respectively  $a, -b$  and  $b, -a$ .**SOLUTION** The line which cuts off intercepts  $a$  and  $-b$  on the coordinate axes passes through points  $A(a, 0)$  and  $B(0, -b)$ .

$$\therefore \text{Slope of line } AB = m_1 = \frac{-b - 0}{0 - a} = \frac{b}{a}$$

The line which cuts off intercepts  $b$  and  $-a$  on the coordinate axes passes through points  $C(b, 0)$  and  $D(0, -a)$ .

$$\therefore \text{Slope of line } CD = m_2 = \frac{-a - 0}{0 - b} = \frac{a}{b}$$

Let  $\theta$  be the angle between  $AB$  and  $CD$ . Then,

$$\tan \theta = \pm \frac{m_1 - m_2}{1 + m_1 m_2} = \pm \frac{\frac{b}{a} - \frac{a}{b}}{1 + \frac{b}{a} \cdot \frac{a}{b}} = \pm \frac{b^2 - a^2}{2ab}$$

**EXAMPLE 3** Find the obtuse angle between the lines  $x - 2y + 3 = 0$  and  $3x + y - 1 = 0$ .**SOLUTION** Let  $m_1$  and  $m_2$  be the slopes of the straight lines  $x - 2y + 3 = 0$  and  $3x + y - 1 = 0$ . Then,

$$m_1 = -\frac{1}{-2} = \frac{1}{2} \text{ and } m_2 = -\frac{3}{1} = -3.$$

Let  $\theta$  be the angle between the given lines. Then,

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| = \left| \frac{\frac{1}{2} + 3}{1 - \frac{3}{2}} \right| = 7 \Rightarrow \theta = \tan^{-1}(7).$$

Thus, the acute angle between the lines is  $\tan^{-1}(7)$  and the obtuse angle is  $\pi - \tan^{-1}(7)$ .



**EXAMPLE 4** Find the value of  $k$  if the straight line  $2x + 3y + 4 + k(6x - y + 12) = 0$  is perpendicular to the line  $7x + 5y - 4 = 0$ .

**SOLUTION** The two lines are

$$x(2 + 6k) + y(3 - k) + 4 + 12k = 0 \quad \dots(i) \quad \text{and} \quad 7x + 5y - 4 = 0 \quad \dots(ii)$$

Let  $m_1$  and  $m_2$  be the slopes of (i) and (ii) respectively. Then,

$$m_1 = -\frac{2 + 6k}{3 - k}, \quad m_2 = -\frac{7}{5}$$

If lines (i) and (ii) are perpendicular. Then,

$$m_1 m_2 = -1 \Rightarrow \left(-\frac{2 + 6k}{3 - k}\right)\left(-\frac{7}{5}\right) = -1 \Rightarrow 14 + 42k = -15 + 5k \Rightarrow k = -\frac{29}{37}.$$

**EXAMPLE 5** A line passing through the points  $(a, 2a)$  and  $(-2, 3)$  is perpendicular to the line  $4x + 3y + 5 = 0$ , find the value of  $a$ .

**SOLUTION** Let  $m_1$  be the slope of the line joining  $A(a, 2a)$  and  $B(-2, 3)$ . Then,  $m_1 = \frac{2a - 3}{a + 2}$

Let  $m_2$  be the slope of the line  $4x + 3y + 5 = 0$ . Then,  $m_2 = -\frac{4}{3}$ .

Since given lines are perpendicular. Therefore,

$$m_1 m_2 = -1 \Rightarrow \frac{2a - 3}{a + 2} \times -\frac{4}{3} = -1 \Rightarrow 8a - 12 = 3a + 6 \Rightarrow a = 18/5.$$

**EXAMPLE 6** Classify the following pairs of lines as coincident, parallel or intersecting:

- (i)  $x + 2y - 3 = 0$  and  $-3x - 6y + 9 = 0$  (ii)  $x + 2y + 1 = 0$  and  $2x + 4y + 3 = 0$   
 (iii)  $3x - 2y + 5 = 0$  and  $2x + y - 9 = 0$

**SOLUTION** (i) The given lines are  $a_1x + b_1y + c_1 = 0$  and  $a_2x + b_2y + c_2 = 0$ , where  $a_1 = 1, b_1 = 2, c_1 = -3, a_2 = -3, b_2 = -6$  and  $c_2 = 9$ .

Clearly,  $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = -\frac{1}{3}$ . So, the given lines are coincident.

(ii) The given lines are  $a_1x + b_1y + c_1 = 0$  and  $a_2x + b_2y + c_2 = 0$ , where  $a_1 = 1, b_1 = 2, c_1 = 1, a_2 = 2, b_2 = 4$ , and  $c_2 = 3$ .

Clearly,  $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ . So, the given lines are parallel.

(iii) The given lines are  $a_1x + b_1y + c_1 = 0$  and  $a_2x + b_2y + c_2 = 0$ , where  $a_1 = 3, b_1 = -2, c_1 = 5, a_2 = 2, b_2 = 1$ , and  $c_2 = -9$ .

Clearly,  $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ . So, the given lines are intersecting.

#### BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 7** The hypotenuse of a right isosceles triangle has its ends at the points  $(1, 3)$  and  $(-4, 1)$ . Find the equations of the legs (perpendicular sides) of the triangle. [NCERT]



**SOLUTION** Let  $ABC$  be the right triangle with diagonal  $AC$ . Let  $m$  be the slope of a line making  $45^\circ$  angle with  $AC$ . Clearly, Slope of  $AC = \frac{1-3}{-4-1} = \frac{2}{5}$  and  $\theta$  is the angle between  $AC$  and a line of slope  $m$ .

$$\therefore \tan 45^\circ = \left| \frac{m - \frac{2}{5}}{1 + \frac{2m}{5}} \right|$$

$$\Rightarrow 2m + 5 = \pm (5m - 2)$$

$$\Rightarrow 2m + 5 = 5m - 2 \text{ or, } 2m + 5 = -(5m - 2) \Rightarrow m = \frac{7}{3} \text{ or, } m = -\frac{3}{7}$$

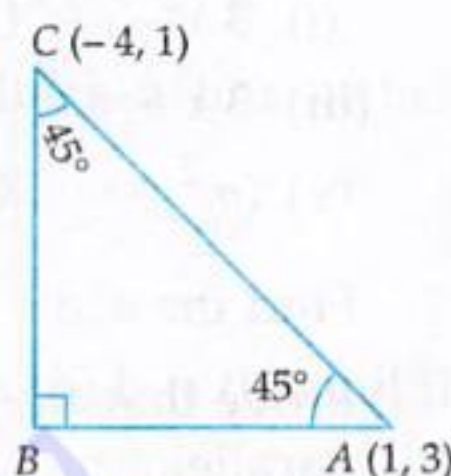


Fig. 22.80

Thus, the lines making  $45^\circ$  angle with  $AC$  having slopes  $\frac{7}{3}$  or  $-\frac{3}{7}$ . So, the possible equations of  $AB$  are

$$y - 3 = \frac{7}{3}(x - 1) \text{ and } y - 3 = -\frac{3}{7}(x - 1) \Rightarrow 7x - 3y + 2 = 0 \text{ and } 3x + 7y - 24 = 0$$

The possible equations of  $BC$  are

$$y - 1 = \frac{7}{3}(x + 4) \text{ and } y - 1 = -\frac{3}{7}(x + 4) \Rightarrow 7x - 3y + 31 = 0 \text{ and } 3x + 7y + 5 = 0$$

Hence, the equations of the sides are:

$$7x - 3y + 2 = 0 \text{ and } 3x + 7y + 5 = 0 \text{ or, } 7x - 3y + 31 = 0 \text{ and } 3x + 7y - 24 = 0$$

**EXAMPLE 8** If the lines  $y = 3x + 1$  and  $2y = x + 3$  are equally inclined to the line  $y = mx + 4$ . Find the values of  $m$ . [NCERT]

**SOLUTION** Let  $\theta$  be the acute angle which the line  $y = mx + 4$  makes with the lines  $y = 3x + 1$  and  $2y = x + 3$ . Then,

$$\tan \theta = \left| \frac{m - 3}{1 + 3m} \right| \text{ and, } \tan \theta = \left| \frac{m - \frac{1}{2}}{1 + \frac{m}{2}} \right|$$

$$\Rightarrow \left| \frac{m - 3}{1 + 3m} \right| = \left| \frac{2m - 1}{m + 2} \right| \Rightarrow \frac{m - 3}{3m + 1} = \pm \frac{2m - 1}{m + 2} \Rightarrow m^2 - m - 6 = \pm (6m^2 - m - 1)$$

$$\Rightarrow 5m^2 + 5 = 0 \text{ or } 7m^2 - 2m - 7 = 0 \Rightarrow 7m^2 - 2m - 7 = 0 \Rightarrow m = \frac{1 \pm 5\sqrt{2}}{7}$$

**EXAMPLE 9** Find the slope of the lines which make an angle of  $45^\circ$  with the line  $3x - y + 5 = 0$ .

**SOLUTION** Let  $m$  be the slope of the line which make an angle of  $45^\circ$  with the line  $3x - y + 5 = 0$ . Then,

$$\tan 45^\circ = \left| \frac{m - 3}{1 + 3m} \right|$$

[ $\because$  Slope of  $3x - y + 5 = 0$  is 3]

$$\Rightarrow 1 = \left| \frac{m - 3}{1 + 3m} \right|$$

$$\Rightarrow |1 + 3m| = |m - 3| \Rightarrow 1 + 3m = \pm (m - 3) \Rightarrow 1 + 3m = m - 3, 1 + 3m = -m + 3 \Rightarrow m = -2, \frac{1}{2}$$



## EXERCISE 22.13

## BASIC

- Find the angles between each of the following pairs of straight lines:
  - $3x + y + 12 = 0$  and  $x + 2y - 1 = 0$
  - $3x - y + 5 = 0$  and  $x - 3y + 1 = 0$
  - $3x + 4y - 7 = 0$  and  $4x - 3y + 5 = 0$
  - $x - 4y = 3$  and  $6x - y = 11$
  - $(m^2 - mn)y = (mn + n^2)x + n^3$  and  $(mn + m^2)y = (mn - n^2)x + m^3$ .
- Find the acute angle between the lines  $2x - y + 3 = 0$  and  $x + y + 2 = 0$ .
- Prove that the points  $(2, -1)$ ,  $(0, 2)$ ,  $(2, 3)$  and  $(4, 0)$  are the coordinates of the vertices of a parallelogram and find the angle between its diagonals.
- Find the angle between the line joining the points  $(2, 0)$ ,  $(0, 3)$  and the line  $x + y = 1$ .

## BASED ON LOTS

- If  $\theta$  is the angle which the straight line joining the points  $(x_1, y_1)$  and  $(x_2, y_2)$  subtends at the origin, prove that  $\tan \theta = \frac{x_2 y_1 - x_1 y_2}{x_1 x_2 + y_1 y_2}$  and  $\cos \theta = \frac{x_1 x_2 + y_1 y_2}{\sqrt{x_1^2 + y_1^2} \sqrt{x_2^2 + y_2^2}}$ .
- Prove that the straight lines  $(a + b)x + (a - b)y = 2ab$ ,  $(a - b)x + (a + b)y = 2ab$  and  $x + y = 0$  form an isosceles triangle whose vertical angle is  $2 \tan^{-1} \left( \frac{a}{b} \right)$ .
- Find the angle between the lines  $x = a$  and  $by + c = 0$ .
- Find the tangent of the angle between the lines which have intercepts 3, 4 and 1, 8 on the axes respectively.
- Show that the line  $a^2x + ay + 1 = 0$  is perpendicular to the line  $x - ay = 1$  for all non-zero real values of  $a$ .
- Show that the tangent of an angle between the lines  $\frac{x}{a} + \frac{y}{b} = 1$  and  $\frac{x}{a} - \frac{y}{b} = 1$  is  $\frac{2ab}{a^2 - b^2}$ .

[NCERT EXEMPLAR]

## ANSWERS

- (i)  $45^\circ$  (ii)  $\tan^{-1} \left( \frac{4}{3} \right)$  (iii)  $90^\circ$  (iv)  $\tan^{-1} \left( \frac{23}{10} \right)$  (v)  $\tan^{-1} \left( \frac{4m^2n^2}{m^4 - n^4} \right)$
- $\tan^{-1} 3$
- $-\tan^{-1} \frac{1}{2} - \frac{\pi}{2}$
- $\tan^{-1} \left( \frac{1}{5} \right)$
- $90^\circ$
- $\frac{4}{7}$

## 22.12 POSITION OF TWO POINTS RELATIVE TO A LINE

In this section, we shall see how to check whether two given points are on the same side or opposite sides of a given line.

Let the equation of the given line be  $ax + by + c = 0$  ... (i)

and let the coordinates of the two given points be  $P(x_1, y_1)$  and  $Q(x_2, y_2)$ .

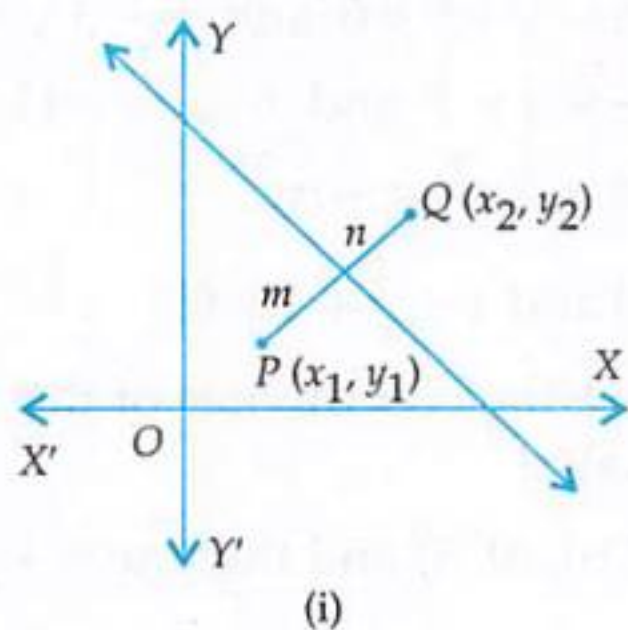
The coordinates of the point  $R$  which divides the line joining  $P$  and  $Q$  in the ratio  $m : n$  are

$$\left( \frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right) \quad \dots (ii)$$

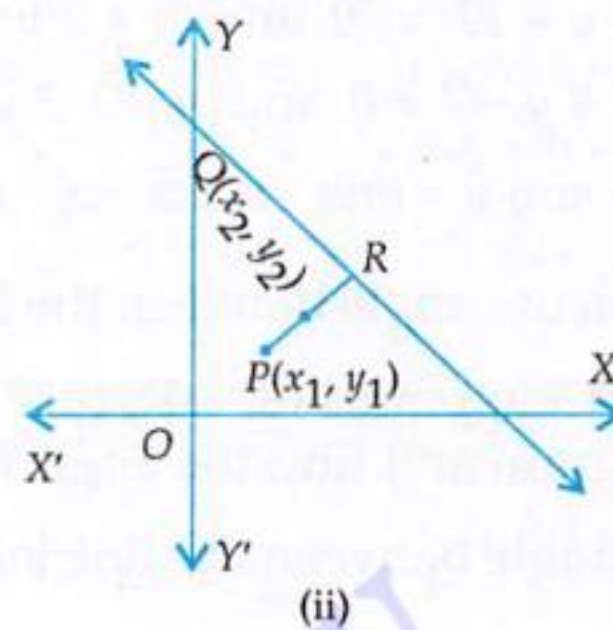


If this point lies on (i), then

$$a \left( \frac{m x_2 + n x_1}{m + n} \right) + b \left( \frac{m y_2 + n y_1}{m + n} \right) + c = 0$$



(i)



(ii)

Fig. 22.81

$$\Rightarrow m(ax_2 + by_2 + c) + n(ax_1 + by_1 + c) = 0 \Rightarrow \frac{m}{n} = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} \quad \dots(iii)$$

If the point R is between the points P and Q i.e. points P and Q are on the opposite sides of the given line, then the ratio  $m:n$  is positive.

$$\therefore -\left( \frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} \right) > 0 \Rightarrow \frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} < 0$$

$$\Rightarrow ax_1 + by_1 + c \text{ and } ax_2 + by_2 + c \text{ are of opposite signs}$$

If the point R is not between P and Q i.e. point P and Q are on the same side of the given line, then the ratio  $m:n$  is negative.

$$\therefore -\left( \frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} \right) < 0 \Rightarrow \frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} > 0 \quad [\text{From (iii)}]$$

$$\Rightarrow ax_1 + by_1 + c \text{ and } ax_2 + by_2 + c \text{ are of the same sign.}$$

Thus, the two points  $(x_1, y_1)$  and  $(x_2, y_2)$  are on the same (or opposite) sides of the straight line  $ax + by + c = 0$  according as the quantities  $ax_1 + by_1 + c$  and  $ax_2 + by_2 + c$  have the same (or opposite) signs.

**REMARK 1** A point  $(x_1, y_1)$  will lie on the side of the origin relative to a line  $ax + by + c = 0$ , if  $ax_1 + by_1 + c$  and  $c$  have the same sign.

**REMARK 2** A point  $(x_1, y_1)$  will lie on the opposite side of the origin relative to the line  $ax + by + c = 0$ , if  $ax_1 + by_1 + c$  and  $c$  have the opposite signs.

### ILLUSTRATIVE EXAMPLES

#### BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** Are the points  $(3, 4)$  and  $(2, -6)$  on the same or opposite sides of the line  $3x - 4y = 8$ ?

**SOLUTION** Let  $Z = 3x - 4y - 8$ . Then, the value of  $Z$  at  $(3, 4)$  is given by

$$Z_1 = 3 \times 3 - 4 \times 4 - 8 = 9 - 16 - 8 = -15 < 0$$

The value of  $Z$  at  $(2, -6)$  is given by

$$Z_2 = 3 \times 2 - 4 \times -6 - 8 = 6 + 24 - 8 = 22 > 0$$

Since  $Z_1$  and  $Z_2$  are of opposite signs, therefore the two points are on the opposite sides of the given line.



## BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

**EXAMPLE 2** If the points  $(4, 7)$  and  $(\cos \theta, \sin \theta)$ , where  $0 < \theta < \pi$ , lie on the same side of the line  $x + y - 1 = 0$ , then prove that  $\theta$  lies in the first quadrant.

**SOLUTION** If the points  $(4, 7)$  and  $(\cos \theta, \sin \theta)$  lie on the same side of  $x + y - 1 = 0$ , then  $4 + 7 - 1$  and  $\cos \theta + \sin \theta - 1$  must be of the same sign.

$$\therefore \cos \theta + \sin \theta - 1 > 0$$

$$\Rightarrow \frac{1}{\sqrt{2}} \cos \theta + \frac{1}{\sqrt{2}} \sin \theta > \frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos \theta \sin \frac{\pi}{4} + \sin \theta \cos \frac{\pi}{4} > \frac{1}{\sqrt{2}}$$

$$\Rightarrow \sin \left( \theta + \frac{\pi}{4} \right) > \frac{1}{\sqrt{2}} \Rightarrow \frac{\pi}{4} < \theta + \frac{\pi}{4} < \frac{3\pi}{4} \Rightarrow 0 < \theta < \frac{\pi}{2} \Rightarrow \theta \in \left( 0, \frac{\pi}{2} \right)$$

Hence,  $\theta$  lies in the first quadrant.

**EXAMPLE 3** Find the values of  $\beta$  so that the point  $(0, \beta)$  lies on or inside the triangle having the sides  $3x + y + 2 = 0$ ,  $2x - 3y + 5 = 0$  and  $x + 4y - 14 = 0$ .

**SOLUTION** Let  $ABC$  be the given triangle. The coordinates of the vertices of the triangle  $ABC$  are marked in Fig. 22.82. The point  $P(0, \beta)$  will lie inside or on the triangle  $ABC$ , if the following three conditions hold simultaneously:

- (i)  $A$  and  $P$  lie on the same side of  $BC$
- (ii)  $B$  and  $P$  lie on the same side of  $AC$ ,
- (iii)  $C$  and  $P$  lie on the same side of  $AB$ .

Now,

$A$  and  $P$  will lie on the same side of  $BC$ , if

$$(3 \times 2 + 3 + 2)(3 \times 0 + \beta + 2) > 0 \Rightarrow 11(\beta + 2) \geq 0 \Rightarrow \beta + 2 \geq 0 \Rightarrow \beta \geq -2 \quad \dots(i)$$

$B$  and  $P$  will lie on the same side of  $AC$ , if

$$(-2 \times 2 - 3 \times 4 + 5)(2 \times 0 - 3\beta + 5) \geq 0 \Rightarrow -11(-3\beta + 5) \geq 0 \Rightarrow 3\beta - 5 \geq 0 \Rightarrow \beta \geq \frac{5}{3} \quad \dots(ii)$$

$C$  and  $P$  will lie on the same side of  $AB$ , if

$$(-1 + 1 \times 4 - 14)(0 + 4\beta - 14) \geq 0 \Rightarrow -22(2\beta - 7) \geq 0 \Rightarrow 2\beta - 7 \leq 0 \Rightarrow \beta \leq \frac{7}{2} \quad \dots(iii)$$

From (i), (ii) and (iii), we obtain that:  $\frac{5}{3} \leq \beta \leq \frac{7}{2}$  i.e.,  $\beta \in [5/3, 7/2]$ .

**EXAMPLE 4** Determine all values of  $\alpha$  for which the point  $(\alpha, \alpha^2)$  lies inside the triangle formed by the lines  $2x + 3y - 1 = 0$ ,  $x + 2y - 3 = 0$  and  $5x - 6y - 1 = 0$ .

**SOLUTION** Let  $ABC$  be the triangle, the equations of whose sides  $AB$ ,  $BC$  and  $CA$  are respectively  $2x + 3y - 1 = 0$ ,  $x + 2y - 3 = 0$  and  $5x - 6y - 1 = 0$ . The coordinates of the vertices are  $A(1/3, 1/9)$ ,  $B(-7, 5)$  and  $C(5/4, 7/8)$ . If the point  $P(\alpha, \alpha^2)$  lies inside the  $\Delta ABC$ , then

- (i)  $A$  and  $P$  must lie on the same side of  $BC$
- (ii)  $B$  and  $P$  must lie on the same side of  $AC$
- (iii)  $C$  and  $P$  must lie on the same side of  $AB$ .

Now,

$A$  and  $P$  will lie on the same side of  $BC$ , if

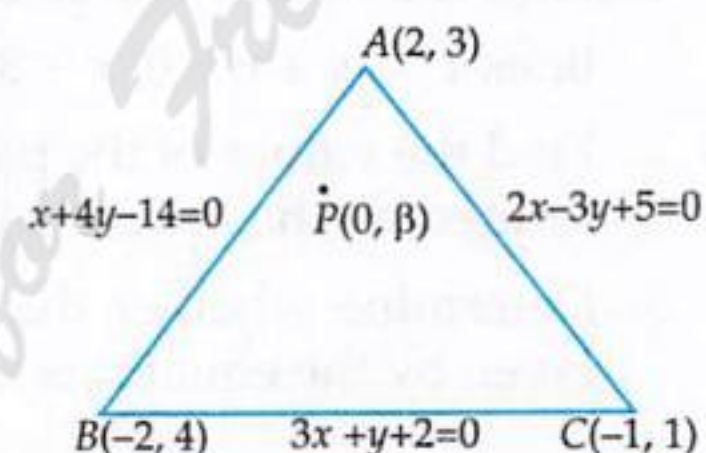


Fig. 22.82

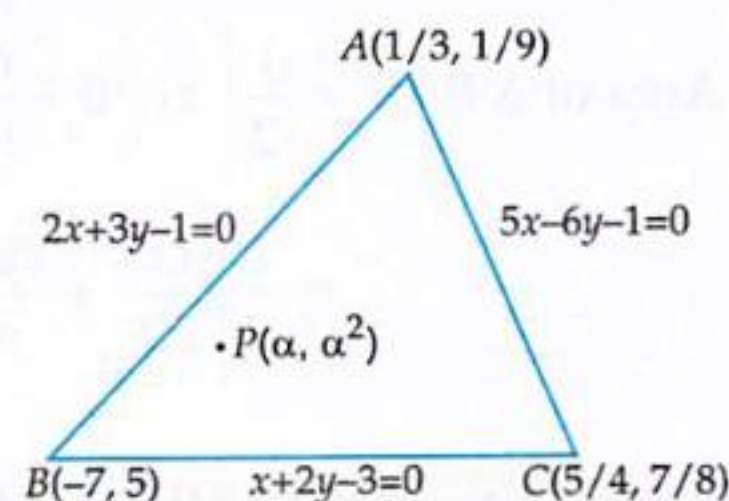


Fig. 22.83



$$\left(\frac{1}{3} + \frac{2}{9} - 3\right)(\alpha + 2\alpha^2 - 3) > 0$$

$$\Rightarrow \alpha + 2\alpha^2 - 3 < 0 \Rightarrow 2\alpha^2 + \alpha - 3 < 0 \Rightarrow (\alpha - 1)(2\alpha + 3) < 0 \Rightarrow \alpha \in (-3/2, 1) \quad \dots(i)$$

$B$  and  $P$  will lie on the same side of  $CA$ , if

$$(-35 - 30 - 1)(5\alpha - 6\alpha^2 - 1) > 0$$

$$\Rightarrow 6\alpha^2 - 5\alpha + 1 > 0 \Rightarrow (3\alpha - 1)(2\alpha - 1) > 0 \Rightarrow \alpha \in (-\infty, 1/3) \cup (1/2, \infty) \quad \dots(ii)$$

$C$  and  $P$  will lie on the same side of  $AB$ , if

$$\left(\frac{5}{2} + \frac{21}{8} - 3\right)(2\alpha + 3\alpha^2 - 1) > 0$$

$$\Rightarrow 3\alpha^2 + 2\alpha - 1 > 0 \Rightarrow (\alpha + 1)(3\alpha - 1) > 0 \Rightarrow \alpha \in (-\infty, -1) \cup (1/3, \infty) \quad \dots(iii)$$

From (i), (ii) and (iii), we obtain:  $\alpha \in (-3/2, -1) \cup (1/2, 1)$

### EXERCISE 22.14

#### BASED ON HOTS

- Find the values of  $\alpha$  so that the point  $P(\alpha^2, \alpha)$  lies inside or on the triangle formed by the lines  $x - 5y + 6 = 0$ ,  $x - 3y + 2 = 0$  and  $x - 2y - 3 = 0$ .
- Find the values of the parameter  $a$  so that the point  $(a, 2)$  is an interior point of the triangle formed by the lines  $x + y - 4 = 0$ ,  $3x - 7y - 8 = 0$  and  $4x - y - 31 = 0$ .
- Determine whether the point  $(-3, 2)$  lies inside or outside the triangle whose sides are given by the equations  $x + y - 4 = 0$ ,  $3x - 7y + 8 = 0$ ,  $4x - y - 31 = 0$ .

### ANSWERS

1.  $\alpha \in [2, 3]$

2.  $a \in (22/3, 33/4)$

3. Outside

### 22.13 DISTANCE OF A POINT FROM A LINE

**THEOREM** Prove that the length of the perpendicular from a point  $(x_1, y_1)$  to a line  $ax + by + c = 0$  is

$$\left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

**PROOF** The line  $ax + by + c = 0$  meets  $x$ -axis at  $y = 0$ . Therefore, putting  $y = 0$  in  $ax + by + c = 0$ , we get  $x = -\frac{c}{a}$ . Thus, the coordinates of the point  $A$  where the line

$ax + by + c = 0$  meets  $x$ -axis are  $(-c/a, 0)$ . Similarly, the coordinates of  $B$  where the line cuts  $y$ -axis are  $(0, -c/b)$ .

Let  $P(x_1, y_1)$  be the point. Draw  $PN \perp AB$ .

Now,

$$\begin{aligned} \text{Area of } \triangle PAB &= \frac{1}{2} \left| x_1 \left( 0 + \frac{c}{b} \right) - \frac{c}{a} \left( -\frac{c}{b} - y_1 \right) + 0(y_1 - 0) \right| \\ &= \frac{1}{2} \left| \frac{cx_1}{b} + \frac{cy_1}{a} + \frac{c^2}{ab} \right| = \left| (ax_1 + by_1 + c) \frac{c}{2ab} \right| \quad \dots(i) \end{aligned}$$

$$\text{Also, Area of } \triangle PAB = \frac{1}{2} AB \times PN = \frac{1}{2} \sqrt{\frac{c^2}{a^2} + \frac{c^2}{b^2}} \times PN = \frac{c}{2ab} \sqrt{a^2 + b^2} \times PN \quad \dots(ii)$$

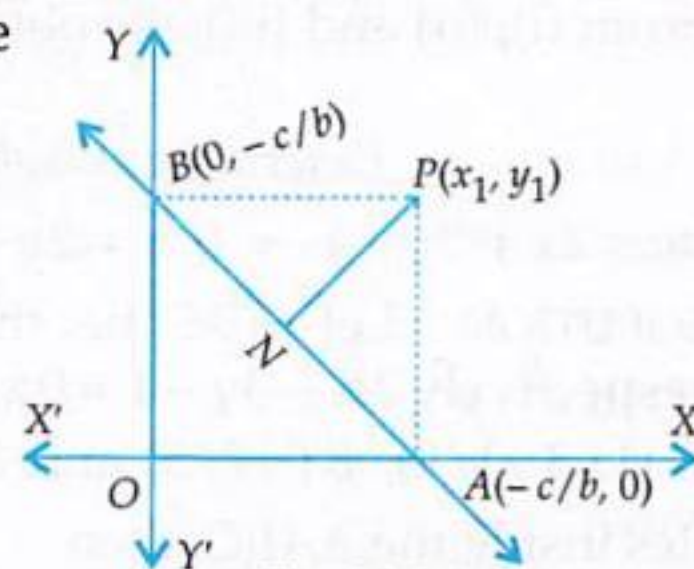


Fig. 22.84



From (i) and (ii), we get

$$\left| (ax_1 + by_1 + c) \frac{c}{2ab} \right| = \frac{c}{2ab} \sqrt{a^2 + b^2} \times PN \Rightarrow PN = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \quad \text{Q.E.D.}$$

**COROLLARY** The length of the perpendicular from the origin to the line  $ax + by + c = 0$  is  $\frac{|c|}{\sqrt{a^2 + b^2}}$ .

We may use the following algorithm for finding the length of the perpendicular from a point  $(x_1, y_1)$  to the line  $ax + by + c = 0$ .

#### ALGORITHM

- Step I Write the equation of the line in the form  $ax + by + c = 0$ .  
 Step II Substitute the coordinates  $x_1$  and  $y_1$  of the point in place of  $x$  and  $y$  respectively in the expression.  
 Step III Divide the result obtained in step II by the square root of the sum of the squares of the coefficients of  $x$  and  $y$ .  
 Step IV Take the modulus of the expression obtained in step III.

### ILLUSTRATIVE EXAMPLES

#### BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** Find the distance between the line  $12x - 5y + 9 = 0$  and the point  $(2, 1)$ .

**SOLUTION** Required distance  $= \frac{|12 \times 2 - 5 \times 1 + 9|}{\sqrt{12^2 + (-5)^2}} = \frac{|24 - 5 + 9|}{13} = \frac{28}{13}$

#### BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 2** If  $p$  is the length of the perpendicular from the origin to the line  $\frac{x}{a} + \frac{y}{b} = 1$ , then prove that

(i)  $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$  (ii)  $a^4 + b^4 = 0$ , if  $a^2, p^2, b^2$  are in A.P. [NCERT]

**SOLUTION** (i) The equation given line is  $bx + ay - ab = 0$  ... (i)

It is given that

$p$  = Length of the perpendicular from the origin to line (i)

$$\Rightarrow p = \frac{|b(0) + a(0) - ab|}{\sqrt{b^2 + a^2}} = \frac{ab}{\sqrt{a^2 + b^2}}$$

$$\Rightarrow p^2 = \frac{a^2 b^2}{a^2 + b^2} \Rightarrow \frac{1}{p^2} = \frac{a^2 + b^2}{a^2 b^2} \Rightarrow \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

(ii) If  $a^2, p^2, b^2$  are in A.P, then  $2p^2 = a^2 + b^2$ .

Now,  $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$  [from (i)]

$$\Rightarrow p^2(a^2 + b^2) = a^2 b^2 \Rightarrow \left( \frac{a^2 + b^2}{2} \right) (a^2 + b^2) = a^2 b^2 \quad [\text{Using } 2p^2 = a^2 + b^2]$$

$$\Rightarrow (a^2 + b^2)^2 = 2a^2 b^2 \Rightarrow a^4 + b^4 = 0.$$



**EXAMPLE 3** If  $p$  and  $p'$  be the perpendicular from the origin upon the straight lines  $x \sec \theta + y \operatorname{cosec} \theta = a$  and  $x \cos \theta - y \sin \theta = a \cos 2\theta$ . Prove that:  $4p^2 + p'^2 = a^2$ . [NCERT]

**SOLUTION** We have,

$p$  = Length of the perpendicular from  $(0, 0)$  to  $x \sec \theta + y \operatorname{cosec} \theta - a = 0$

$$\Rightarrow p = \left| \frac{0 \sec \theta + 0 \operatorname{cosec} \theta - a}{\sqrt{\sec^2 \theta + \operatorname{cosec}^2 \theta}} \right| = \frac{a \cos \theta \sin \theta}{\sqrt{\sin^2 \theta + \cos^2 \theta}} = a \sin \theta \cos \theta$$

and,  $p'$  = Length of the perpendicular from  $(0, 0)$  to  $x \cos \theta - y \sin \theta - a \cos 2\theta = 0$

$$\Rightarrow p' = \left| \frac{0 \cos \theta - 0 \sin \theta - a \cos 2\theta}{\sqrt{\cos^2 \theta + \sin^2 \theta}} \right| = a \cos 2\theta.$$

$$\text{Now, } 4p^2 + p'^2 = 4a^2 \sin^2 \theta \cos^2 \theta + a^2 \cos^2 2\theta = a^2 (2 \sin \theta \cos \theta)^2 + a^2 \cos^2 2\theta$$

$$\Rightarrow 4p^2 + p'^2 = a^2 (\sin^2 2\theta + \cos^2 2\theta) = a^2.$$

**EXAMPLE 4** What are the points on  $x$ -axis whose perpendicular distance from the line  $4x + 3y = 12$  is 4?

**SOLUTION** Let the required point be  $P(\alpha, 0)$ . Then, Length of the perpendicular from  $P(\alpha, 0)$  on  $4x + 3y - 12 = 0$  is 4.

$$\therefore \left| \frac{4\alpha + 3 \times 0 - 12}{\sqrt{4^2 + 3^2}} \right| = 4$$

$$\Rightarrow \left| \frac{4\alpha - 12}{5} \right| = 4 \Rightarrow |4\alpha - 12| = 20 \Rightarrow |\alpha - 3| = 5 \Rightarrow \alpha - 3 = \pm 5 \Rightarrow \alpha = 8, -2$$

Hence, the required points are  $(8, 0)$  and  $(-2, 0)$ .

**EXAMPLE 5** Find the points on  $y$ -axis whose perpendicular distance from the line  $4x - 3y - 12 = 0$  is 3.

**SOLUTION** Let the required point be  $P(0, \alpha)$ . It is given that the length of the perpendicular from  $P(0, \alpha)$  on  $4x - 3y - 12 = 0$  is 3.

$$\therefore \left| \frac{4 \times 0 - 3\alpha - 12}{\sqrt{4^2 + (-3)^2}} \right| = 3$$

$$\Rightarrow |3\alpha + 12| = 15 \Rightarrow |\alpha + 4| = 5 \Rightarrow \alpha + 4 = \pm 5 \Rightarrow \alpha = 1, -9$$

Hence, the required points are  $(0, 1)$  and  $(0, -9)$ .

**EXAMPLE 6** Find the equation of the straight line which cuts off intercept on  $X$ -axis which is twice that on  $Y$ -axis and is at a unit distance from the origin. [NCERT EXEMPLAR]

**SOLUTION** Let the equation of the straight line be  $\frac{x}{a} + \frac{y}{b} = 1$ . It is given that  $a = 2b$ .

Putting  $a = 2b$  in the above equation, we get

$$x + 2y - 2b = 0$$

...(i)

This line is at  $a$  unit distance from the origin.

$$\therefore \left| \frac{0 + 2 \times 0 - 2b}{\sqrt{1^2 + 2^2}} \right| = 1 \Rightarrow \frac{|2b|}{\sqrt{5}} = 1 \Rightarrow |2b| = \sqrt{5} \Rightarrow 2b = \pm \sqrt{5} \Rightarrow b = \pm \frac{\sqrt{5}}{2}$$

Substituting the value of  $b$  in (i), we obtain  $x + 2y \pm \sqrt{5} = 0$  as the equations of the required line.



**EXAMPLE 7** The equation of the base of an equilateral triangle is  $x + y - 2 = 0$  and the opposite vertex has coordinates  $(2, -1)$ . Find the area of the triangle. [NCERT EXEMPLAR]

**SOLUTION** Let  $p$  be the altitude of the given triangle and ' $a$ ' be the length of each side. Then,

$$p = \text{Length of perpendicular from } (2, -1) \text{ on } x + y - 2 = 0$$

$$\Rightarrow p = \left| \frac{2 - 1 - 2}{\sqrt{1^2 + 1^2}} \right| = \frac{1}{\sqrt{2}}$$

In  $\triangle ABD$ , we have

$$\sin 60^\circ = \frac{p}{a} \Rightarrow \frac{\sqrt{3}}{2} = \frac{p}{a} \Rightarrow a = \frac{2p}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{2}} = \sqrt{\frac{2}{3}}$$

$$\therefore \text{Area of the triangle} = \frac{\sqrt{3}a^2}{4} = \frac{\sqrt{3}}{4} \times \left( \sqrt{\frac{2}{3}} \right)^2 = \frac{1}{2\sqrt{3}} \text{ sq. units}$$

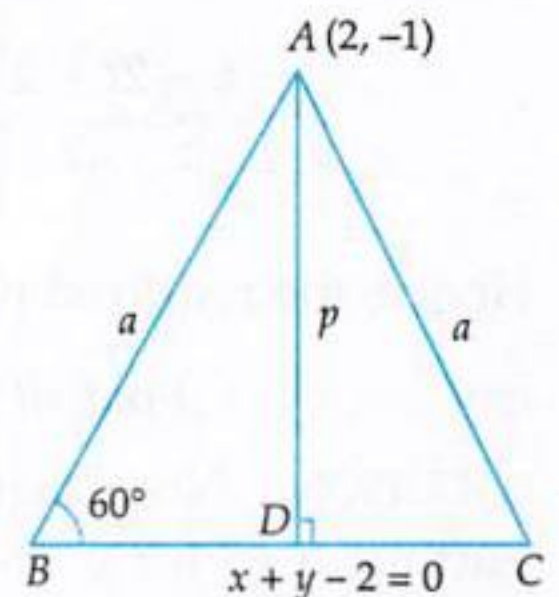


Fig. 22.85

**EXAMPLE 8** Prove that the length of perpendiculars from points  $P(m^2, 2m)$ ,  $Q(mn, m+n)$  and  $R(n^2, 2n)$  to the line  $x \cos^2 \theta + y \sin \theta \cos \theta + \sin^2 \theta = 0$  are in G.P.

**SOLUTION** Let  $a, b$  and  $c$  denote the lengths of perpendiculars drawn from  $P, Q$  and  $R$  respectively on the line  $x \cos^2 \theta + y \sin \theta \cos \theta + \sin^2 \theta = 0$ . Then,

$$a = \left| \frac{m^2 \cos^2 \theta + 2m \sin \theta \cos \theta + \sin^2 \theta}{\sqrt{\cos^4 \theta + \sin^2 \theta \cos^2 \theta}} \right| = \left| \frac{(m \cos \theta + \sin \theta)^2}{\cos \theta} \right| \quad \dots(i)$$

$$b = \left| \frac{mn \cos^2 \theta + (m+n) \sin \theta \cos \theta + \sin^2 \theta}{\sqrt{\cos^4 \theta + \sin^2 \theta \cos^2 \theta}} \right| = \left| \frac{(m \cos \theta + \sin \theta)(n \cos \theta + \sin \theta)}{\cos \theta} \right| \quad \dots(ii)$$

$$\text{and, } c = \left| \frac{n^2 \cos^2 \theta + 2n \sin \theta \cos \theta + \sin^2 \theta}{\sqrt{\cos^4 \theta + \sin^2 \theta \cos^2 \theta}} \right| = \left| \frac{(n \cos \theta + \sin \theta)^2}{\cos \theta} \right| \quad \dots(iii)$$

$$\therefore b^2 = \frac{(m \cos \theta + \sin \theta)^2 (n \cos \theta + \sin \theta)^2}{\cos^2 \theta} = \frac{(m \cos \theta + \sin \theta)^2}{\cos \theta} \times \frac{(n \cos \theta + \sin \theta)^2}{\cos \theta}$$

$$\Rightarrow b^2 = ac \Rightarrow a, b, c \text{ are in G.P.}$$

#### BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

**EXAMPLE 9** Find the coordinates of a point on  $x + y + 3 = 0$ , whose distance from  $x + 2y + 2 = 0$  is  $\sqrt{5}$ .

**SOLUTION** Let the required point be  $(x_1, y_1)$ . Since it lies on  $x + y + 3 = 0$ .

$$\therefore x_1 + y_1 + 3 = 0 \quad \dots(i)$$

Now,

Length of the perpendicular from  $(x_1, y_1)$  to  $x + 2y + 2 = 0$  is  $\sqrt{5}$ .

$$\Rightarrow \left| \frac{x_1 + 2y_1 + 2}{\sqrt{1^2 + 2^2}} \right| = \sqrt{5} \Rightarrow x_1 + 2y_1 + 2 = \pm 5 \quad \dots(ii)$$

Solving equation (i) and (ii), we get:  $x_1 = -9, y_1 = 6$  and  $x_1 = 1, y_1 = -4$ .



Hence, the required points are  $(-9, 6)$  and  $(1, -4)$ .

**ALITER** Putting  $x = t$  in  $x + y + 3 = 0$ , we get:  $y = -3 - t$ . So, let the required point be  $(t, -3 - t)$ . This point is at a distance of  $\sqrt{5}$  units from  $x + 2y + 2 = 0$ .

$$\therefore \left| \frac{t - 6 - 2t + 2}{\sqrt{1^2 + 2^2}} \right| = \sqrt{5} \Rightarrow \left| \frac{-t - 4}{\sqrt{5}} \right| = \sqrt{5} \Rightarrow t + 4 = \pm 5 \Rightarrow t = 1, -9$$

Hence, the required points are  $(1, -4)$  and  $(-9, 6)$ .

**EXAMPLE 10** Find all points on  $x + y = 4$  that lie at a unit distance from the line  $4x + 3y - 10 = 0$ .

**SOLUTION** Note that the coordinates of an arbitrary point on  $x + y = 4$  can be obtained by putting  $x = t$  (or  $y = t$ ) and then obtaining  $y$  (or  $x$ ) from the equation of the line, where  $t$  is a parameter. Putting  $x = t$  in the equation  $x + y = 4$  of the given line, we obtain  $y = 4 - t$ .

So, coordinates of an arbitrary point on the given line are  $P(t, 4 - t)$ .

Let  $P(t, 4 - t)$  be the required point. Then, distance of  $P$  from the line  $4x + 3y - 10 = 0$  is unity.

$$\therefore \left| \frac{4t + 3(4 - t) - 10}{\sqrt{4^2 + 3^2}} \right| = 1 \Rightarrow |t + 2| = 5 \Rightarrow t + 2 = \pm 5 \Rightarrow t = -7 \text{ or } t = 3$$

Hence, required points are  $(-7, 11)$  and  $(3, 1)$ .

**EXAMPLE 11** Find the equations of lines passing through the point  $(1, 0)$  and at a distance  $\frac{\sqrt{3}}{2}$  from the origin. **[NCERT EXEMPLAR]**

**SOLUTION** Let  $m$  be the slope of a line passing through  $(1, 0)$ . Then, its equation is

$$y - 0 = m(x - 1) \text{ or, } mx - y - m = 0 \quad \dots(i)$$

It is given that line (i) is at a distance  $\frac{\sqrt{3}}{2}$  from the origin.

$$\therefore \left| \frac{m \times 0 - 0 - m}{\sqrt{m^2 + (-1)^2}} \right| = \frac{\sqrt{3}}{2} \Rightarrow \frac{|m|}{\sqrt{m^2 + 1}} = \frac{\sqrt{3}}{2} \Rightarrow 4m^2 = 3m^2 + 3 \Rightarrow m^2 = 3 \Rightarrow m = \pm \sqrt{3}$$

Substituting the values of  $m$  in (i), we obtain  $\sqrt{3}x - y - \sqrt{3} = 0$  and  $\sqrt{3}x + y - \sqrt{3} = 0$  as the equations of the line.

**EXAMPLE 12** Find the locus of a point which moves in such a way that the square of its distance from the point  $(3, -2)$  is numerically equal to its distance from the line  $5x - 12y = 13$ . **[NCERT EXEMPLAR]**

**SOLUTION** Let  $P(h, k)$  be a variable point moving in such a way that the square of its distance from  $A(3, -2)$  is numerically equal to its distance from the line  $5x - 12y = 13$ .

$$\therefore (h - 3)^2 + (k + 2)^2 = \frac{|5h - 12k + 13|}{\sqrt{5^2 + (-12)^2}}$$

$$\Rightarrow 13 \left\{ (h - 3)^2 + (k + 2)^2 \right\} = \pm (5h - 12k + 13)$$

$$\Rightarrow 13(h^2 + k^2) - 83h + 64k + 182 = 0 \text{ or, } 13(h^2 + k^2) - 73h + 40k + 156 = 0$$

Hence, the locus of  $(h, k)$  is

$$13(x^2 + y^2) - 83x + 64y + 182 = 0 \text{ or, } 13(x^2 + y^2) - 73x + 40y + 156 = 0$$

**EXAMPLE 13** A point moves such that its distance from the point  $(4, 0)$  is half that of its distance from the line  $x = 16$ , find its locus. **[NCERT EXEMPLAR]**



**SOLUTION** Let  $P(h, k)$  be the variable point. It is given that the distance of  $P(h, k)$  from  $(4, 0)$  is half of its distance from  $x - 16 = 0$ .

$$\therefore \sqrt{(h-4)^2 + (k-0)^2} = \frac{1}{2} \left| \frac{h-16}{\sqrt{1^2 + 0^2}} \right| \Rightarrow 4 \left\{ (h-4)^2 + k^2 \right\} = (h-16)^2 \Rightarrow 3h^2 + 4k^2 = 192$$

Hence, the locus of  $(h, k)$  is  $3x^2 + 4y^2 = 192$ .

**EXERCISE 22.15****BASIC**

- Find the distance of the point  $(4, 5)$  from the straight line  $3x - 5y + 7 = 0$ .
- Find the perpendicular distance of the line joining the points  $(\cos \theta, \sin \theta)$  and  $(\cos \phi, \sin \phi)$  from the origin.
- Find the length of the perpendicular from the origin to the straight line joining the two points whose coordinates are  $(a \cos \alpha, a \sin \alpha)$  and  $(a \cos \beta, a \sin \beta)$ .
- Show that the perpendiculars let fall from any point on the straight line  $2x + 11y - 5 = 0$  upon the two straight lines  $24x + 7y = 20$  and  $4x - 3y - 2 = 0$  are equal to each other.
- Find the distance of the point of intersection of the lines  $2x + 3y = 21$  and  $3x - 4y + 11 = 0$  from the line  $8x + 6y + 5 = 0$ .
- Find the length of the perpendicular from the point  $(4, -7)$  to the line joining the origin and the point of intersection of the lines  $2x - 3y + 14 = 0$  and  $5x + 4y - 7 = 0$ .

**BASED ON LOTS**

- What are the points on X-axis whose perpendicular distance from the straight line  $\frac{x}{a} + \frac{y}{b} = 1$  is  $a$ ?
- Show that the product of perpendiculars on the line  $\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$  from the points  $(\pm \sqrt{a^2 - b^2}, 0)$  is  $b^2$ . [NCERT]
- Find the perpendicular distance from the origin of the perpendicular from the point  $(1, 2)$  upon the straight line  $x - \sqrt{3}y + 4 = 0$ .
- Find the distance of the point  $(1, 2)$  from the straight line with slope 5 and passing through the point of intersection of  $x + 2y = 5$  and  $x - 3y = 7$ . [NCERT]
- What are the points on y-axis whose distance from the line  $\frac{x}{3} + \frac{y}{4} = 1$  is 4 units?
- In the triangle  $ABC$  with vertices  $A(2, 3)$ ,  $B(4, -1)$  and  $C(1, 2)$ , find the equation and the length of the altitude from the vertex  $A$ . [NCERT]
- Show that the path of a moving point such that its distances from two lines  $3x - 2y = 5$  and  $3x + 2y = 5$  are equal is a straight line. [NCERT]
- If sum of perpendicular distances of a variable point  $P(x, y)$  from the lines  $x + y - 5 = 0$  and  $3x - 2y + 7 = 0$  is always 10. Show that  $P$  must move on a line. [NCERT]

**BASED ON HOTS**

- If the length of the perpendicular from the point  $(1, 1)$  to the line  $ax - by + c = 0$  be unity, show that  $\frac{1}{c} + \frac{1}{a} - \frac{1}{b} = \frac{c}{2ab}$ .

**ANSWERS**

- $\frac{6}{\sqrt{34}}$
- $\cos\left(\frac{\theta - \phi}{2}\right)$
- $a \cos\left(\frac{\alpha - \beta}{2}\right)$
- $\frac{59}{10}$



6. 1

7.  $\left\{ \frac{a}{b} (b \pm \sqrt{a^2 + b^2}), 0 \right\}$

9.  $\frac{1}{2} (2 + \sqrt{3})$

10.  $\frac{132}{\sqrt{650}}$

11.  $(0, 32/3), (0, -8/3)$

12.  $x - y + 1 = 0, \sqrt{2}$

**HINTS TO SELECTED PROBLEMS**2. The equation of the line joining  $(\cos \theta, \sin \theta)$  and  $(\cos \phi, \sin \phi)$  is

$$x \cos \left( \frac{\theta + \phi}{2} \right) + y \sin \left( \frac{\theta + \phi}{2} \right) = \cos \left( \frac{\theta - \phi}{2} \right)$$

$$\therefore \text{ Required distance } = \frac{\left| 0 \cos \left( \frac{\theta + \phi}{2} \right) + 0 \sin \left( \frac{\theta + \phi}{2} \right) - \cos \left( \frac{\theta - \phi}{2} \right) \right|}{\sqrt{\cos^2 \left( \frac{\theta + \phi}{2} \right) + \sin^2 \left( \frac{\theta + \phi}{2} \right)}} = \cos \left( \frac{\theta - \phi}{2} \right)$$

8. Let  $p_1$  and  $p_2$  be the lengths of perpendiculars drawn from points  $P(\sqrt{a^2 - b^2}, 0)$  and  $Q(-\sqrt{a^2 - b^2}, 0)$  on the line  $\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$ . Then,

$$p_1 = \frac{\left| \frac{\sqrt{a^2 - b^2}}{a} \cos \theta + \frac{0}{b} \sin \theta - 1 \right|}{\sqrt{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}}}, \quad p_2 = \frac{\left| -\frac{\sqrt{a^2 - b^2}}{a} \cos \theta + \frac{0}{b} \sin \theta - 1 \right|}{\sqrt{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}}}$$

$$\therefore p_1 p_2 = \frac{\left| \frac{\sqrt{a^2 - b^2}}{a} \cos \theta - 1 \right|}{\sqrt{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}}} \times \frac{\left| \frac{\sqrt{a^2 - b^2}}{a} \cos \theta + 1 \right|}{\sqrt{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}}}$$

$$\Rightarrow p_1 p_2 = \frac{\left( \frac{\sqrt{a^2 - b^2}}{a} \cos \theta - 1 \right) \left( \frac{\sqrt{a^2 - b^2}}{a} \cos \theta + 1 \right)}{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}}$$

$$\Rightarrow p_1 p_2 = \frac{\left| \frac{a^2 - b^2}{a^2} \cos^2 \theta - 1 \right|}{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}} = \frac{\left| (a^2 - b^2) \cos^2 \theta - a^2 \right|}{b^2 \cos^2 \theta + a^2 \sin^2 \theta} b^2 = \frac{\left| -b^2 \cos^2 \theta - a^2 \sin^2 \theta \right|}{b^2 \cos^2 \theta + a^2 \sin^2 \theta} b^2 = b^2$$

10. Given lines  $x + 2y = 5$  and  $x - 3y = 7$  intersect at  $\left( \frac{29}{5}, -\frac{2}{5} \right)$ . The equation of the line of slope5 passing through this point is  $y + \frac{2}{5} = 5 \left( x - \frac{29}{5} \right)$  or,  $25x - 5y - 147 = 0$ .The distance  $d$  of the point  $(1, 2)$  from this line is  $d = \frac{|25 - 10 - 147|}{\sqrt{625 + 25}} = \frac{132}{\sqrt{650}}$ .



11. Let the required point on  $y$ -axis be  $(0, a)$ . Its distance from  $\frac{x}{3} + \frac{y}{4} - 1 = 0$  or  $4x + 3y - 12 = 0$  is  $a$ .

$$\therefore \left| \frac{4 \times 0 + 3 \times a - 12}{\sqrt{4^2 + 3^2}} \right| = 4 \Rightarrow \left| \frac{3a - 12}{5} \right| = 4 \Rightarrow 3a - 12 = \pm 20 \Rightarrow 3a = 32, -8 \Rightarrow a = \frac{32}{3}, -\frac{8}{3}$$

12. The equation of  $BC$  is  $y + 1 = \frac{2+1}{1-4}(x-4)$  or,  $y + 1 = -x + 4$  or,  $x + y - 3 = 0$ .

$$\therefore AL = \left| \frac{2 + 3 - 3}{\sqrt{1+1}} \right| = \sqrt{2}$$

Clearly, slope of  $BC$  having equation  $x + y - 3 = 0$  is  $-1$ . So, Slope of  $AL$  is  $1$ . As it passes through  $A(2, 3)$ . So, its equation is  $y - 3 = 1(x - 2)$  or,  $x - y + 1 = 0$ .

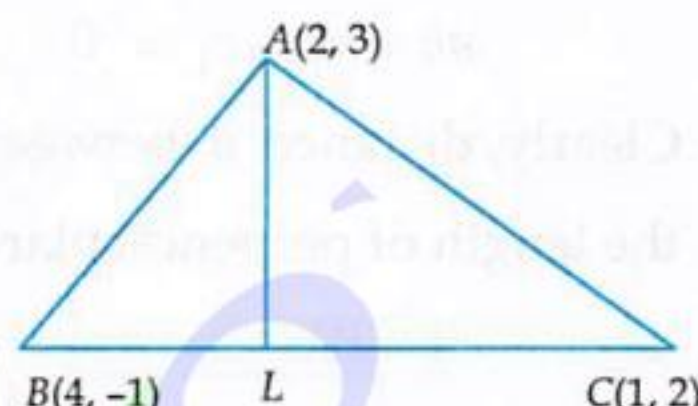


Fig. 22.86

13. Let  $P(h, k)$  be a moving point such that it is equidistant from the lines  $3x - 2y - 5 = 0$  and  $3x + 2y - 5 = 0$ . Then,

$$\left| \frac{3h - 2k - 5}{\sqrt{9 + 4}} \right| = \left| \frac{3h + 2k - 5}{\sqrt{9 + 4}} \right| \Rightarrow |3h - 2k - 5| = |3h + 2k - 5|$$

$$\Rightarrow 3h - 2k - 5 = \pm(3h + 2k - 5) \Rightarrow 4k = 0 \text{ or, } 6h - 10 = 0 \Rightarrow k = 0 \text{ or, } 3h = 5$$

Hence, the locus of  $(h, k)$  is  $y = 0$  or  $3x = 5$ , which are straight lines.

14. It is given that the sum of the perpendicular distances of a variable point  $P(x, y)$  from the lines  $x + y - 5 = 0$  and  $3x - 2y + 7 = 0$  is always 10.

$$\therefore \frac{(x + y - 5)}{\sqrt{2}} + \frac{(3x - 2y + 7)}{\sqrt{9 + 4}} = 10$$

$$\Rightarrow (3\sqrt{2} + \sqrt{13})x + (\sqrt{13} - 2\sqrt{2})y + (7\sqrt{2} - 5\sqrt{13} - 10\sqrt{26}) = 0$$

Clearly, it is a straight line.

## 22.14 DISTANCE BETWEEN PARALLEL LINES

If two lines are parallel, then they have the same distance between them throughout. Therefore to find the distance between two parallel lines choose an arbitrary point on one of them and find the length of the perpendicular on the other. In order to choose a point on a line, we give an arbitrary value to  $x$  or  $y$  and find the value of the other variable.

We may use the following algorithm to find the distance between two parallel lines.

### ALGORITHM

Let the two parallel lines be  $ax + by + c_1 = 0$  and  $ax + by + c_2 = 0$ . To find the distance between these two lines we proceed as follows:

- Step I Choose a point on any one of the two lines by giving a particular value to  $x$  or  $y$  of your choice.  
 Step II Find the length of the perpendicular from the chosen point in step I to the other line.  
 Step III The length obtained in step II is the required distance between the parallel lines.



**THEOREM** Prove that the distance between two parallel lines  $ax + by + c_1 = 0$  and  $ax + by + c_2 = 0$  is given by  $\frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$ .

**PROOF** Given lines are

$$ax + by + c_1 = 0 \quad \dots(i) \quad ax + by + c_2 = 0 \quad \dots(ii)$$

Let  $P(h, k)$  be a point on the line  $ax + by + c_1 = 0$ . Then,

$$ah + bk + c_1 = 0 \quad \dots(iii)$$

Clearly, distance ' $d$ ' between parallel lines (i) and (ii) is equal to the length of perpendicular from  $P$  on line (ii).

$$\therefore d = PM$$

$$\Rightarrow d = \left| \frac{ah + bk + c_2}{\sqrt{a^2 + b^2}} \right| = \left| \frac{-c_1 + c_2}{\sqrt{a^2 + b^2}} \right| \quad [\text{From (iii): } ah + bk = -c_1]$$

$$\Rightarrow d = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$

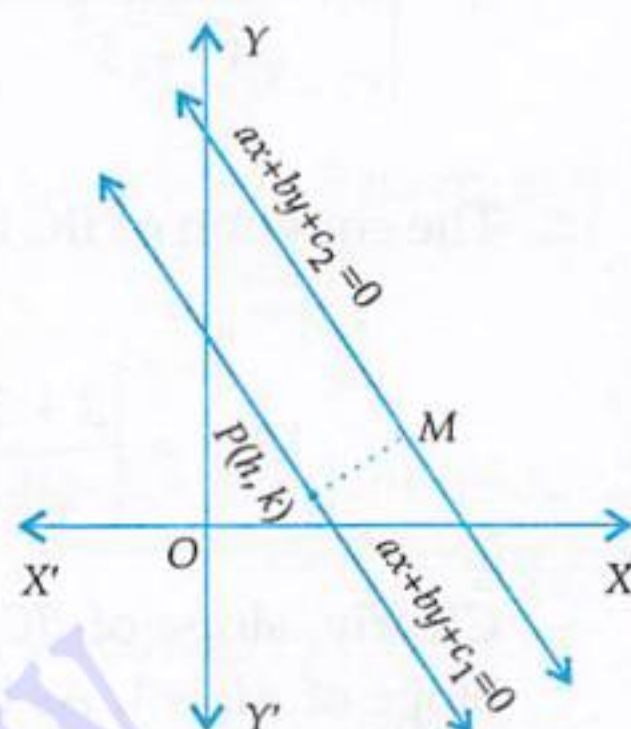


Fig. 22.87

**Q.E.D.**

### ILLUSTRATIVE EXAMPLES

#### BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** Find the distance between the parallel lines  $3x - 4y + 9 = 0$  and  $6x - 8y - 15 = 0$ .

**SOLUTION** Putting  $y = 0$  in  $3x - 4y + 9 = 0$ , we get  $x = -3$ . Thus,  $(-3, 0)$  is a point on the line  $3x - 4y + 9 = 0$ . Length of the perpendicular from  $(-3, 0)$  to  $6x - 8y - 15 = 0$  is given by

$$d = \frac{|-3 \times 6 - 8 \times 0 - 15|}{\sqrt{6^2 + (-8)^2}} = \frac{33}{10}$$

Hence, the distance between the given lines is  $\frac{33}{10}$  units.

**ALITER** Given lines are

$$3x - 4y + 9 = 0 \quad \dots(i) \quad \text{and,} \quad 6x - 8y - 15 = 0 \quad \text{or,} \quad 3x - 4y - \frac{15}{2} = 0 \quad \dots(ii)$$

$$\therefore \text{Required distance} = \frac{\left| 9 - \left( -\frac{15}{2} \right) \right|}{\sqrt{3^2 + (-4)^2}} = \frac{9 + \frac{15}{2}}{5} = \frac{33}{10}$$

#### BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 2** Find the equations of lines parallel to  $3x - 4y - 5 = 0$  at a unit distance from it.

**SOLUTION** Equation of any line parallel to  $3x - 4y - 5 = 0$  is

$$3x - 4y + \lambda = 0 \quad \dots(i)$$

Putting  $x = -1$  in  $3x - 4y - 5 = 0$ , we get  $y = -2$ . Therefore,  $(-1, -2)$  is a point on  $3x - 4y - 5 = 0$ .

Since the distance between the two lines is one unit. Therefore, the length of the perpendicular from  $(-1, -2)$  to  $3x - 4y + \lambda = 0$  is one unit.

$$\text{i.e. } \frac{|3 \times -1 - 4 \times -2 + \lambda|}{\sqrt{3^2 + (-4)^2}} = 1 \Rightarrow \frac{|5 + \lambda|}{5} = 1 \Rightarrow |5 + \lambda| = 5 \Rightarrow 5 + \lambda = \pm 5 \Rightarrow \lambda = 0 \text{ or } -10.$$



Substituting the values of  $\lambda$  in (i), we get

$3x - 4y = 0$  and,  $3x - 4y - 10 = 0$  as the equations of the required lines.

**ALITER** Let the equation of a line parallel to  $3x - 4y - 5 = 0$  be

$$3x - 4y + \lambda = 0 \quad \dots(i)$$

It is given that the distance between the line  $3x - 4y - 5 = 0$  and line (i) is 1 unit.

$$\therefore \frac{|\lambda - (-5)|}{\sqrt{3^2 + (-4)^2}} = 1 \Rightarrow \frac{|\lambda + 5|}{5} = 1 \Rightarrow |\lambda + 5| = 5 \Rightarrow \lambda + 5 = \pm 5 \Rightarrow \lambda = 0, -10.$$

Substituting the values of  $\lambda$  in (i), we get  $3x - 4y = 0$  and  $3x - 4y - 10 = 0$  as the equations of required lines.

**EXAMPLE 3** Two sides of a square lie on the lines  $x + y = 1$  and  $x + y + 2 = 0$ . What is its area?

**SOLUTION** clearly, the length of the side of the square is equal to the distance between the parallel lines

$$x + y - 1 = 0 \quad \dots(i) \quad \text{and} \quad x + y + 2 = 0 \quad \dots(ii)$$

Putting  $x = 0$  in (i), we get  $y = 1$ . So  $(0, 1)$  is a point on line (i).

$\therefore$  Distance between the parallel lines

$$= \{\text{Length of the perpendicular from } (0, 1) \text{ to } x + y + 2 = 0\} = \frac{|0 + 1 + 2|}{\sqrt{1^2 + 1^2}} = \frac{3}{\sqrt{2}}.$$

Thus, the length of the side of the square is  $\frac{3}{\sqrt{2}}$  and hence its area is  $\left(\frac{3}{\sqrt{2}}\right)^2 = \frac{9}{2}$  square units

**ALITER** The equations of parallel sides of the square are  $x + y - 1 = 0$  and  $x + y + 2 = 0$ .

$$\therefore \text{Length of the side of the square} = \text{Distance between parallel side} = \frac{|2 - (-1)|}{\sqrt{1^2 + 1^2}} = \frac{3}{\sqrt{2}}$$

$$\text{Hence, Area of the square} = (\text{Side})^2 = \left(\frac{3}{\sqrt{2}}\right)^2 = \frac{9}{2} \text{ sq. units.}$$

**EXAMPLE 4** Prove that the line  $5x - 2y - 1 = 0$  is mid-parallel to the lines  $5x - 2y - 9 = 0$  and  $5x - 2y + 7 = 0$ .

**SOLUTION** Clearly, the slope of each of the given lines is same equal to  $5/2$ . Hence, the line  $5x - 2y - 1 = 0$  is parallel to each of the given lines.

In order to prove that the line  $5x - 2y - 1 = 0$  is mid-parallel to the given lines it is sufficient to show that the line  $5x - 2y - 1 = 0$  is equidistant from the given lines.

Putting  $y = 0$  in  $5x - 2y - 1 = 0$ , we get  $x = 1/5$ . So, the coordinates of a point on  $5x - 2y - 1 = 0$  are  $(1/5, 0)$ .

The distance  $d_1$  between the lines  $5x - 2y - 1 = 0$  and  $5x - 2y - 9 = 0$  is given by

$$\begin{aligned} d_1 &= \text{Length of the perpendicular from } (1/5, 0) \text{ to } 5x - 2y - 9 = 0 \\ \Rightarrow d_1 &= \left| \frac{5 \times (1/5) - 2 \times 0 - 9}{\sqrt{5^2 + (-2)^2}} \right| = \frac{8}{\sqrt{29}} \end{aligned}$$

The distance  $d_2$  between the lines  $5x - 2y - 1 = 0$  and  $5x - 2y + 7 = 0$  is given by

$$\begin{aligned} d_2 &= \text{Length of the perpendicular } (1/5, 0) \text{ to } 5x - 2y + 7 = 0 \\ \Rightarrow d_2 &= \left| \frac{5 \times (1/5) - 2 \times 0 + 7}{\sqrt{5^2 + (-2)^2}} \right| = \frac{8}{\sqrt{29}} \end{aligned}$$



Clearly,  $d_1 = d_2$ . Consequently the line  $5x - 2y - 1 = 0$  is equidistant from the lines  $5x - 2y - 9 = 0$  and  $5x - 2y + 7 = 0$ . Hence, the result follows.

### BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

**EXAMPLE 5** Prove that the parallelogram formed by the lines  $\frac{x}{a} + \frac{y}{b} = 1$ ,  $\frac{x}{b} + \frac{y}{a} = 1$ ,  $\frac{x}{a} + \frac{y}{b} = 2$  and  $\frac{x}{b} + \frac{y}{a} = 2$  is a rhombus.

**SOLUTION** Let the given straight lines be  $AB, BC, CD$  and  $CA$  whose equations are respectively

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \dots(i) \quad \frac{x}{b} + \frac{y}{a} = 1 \quad \dots(ii) \quad \frac{x}{a} + \frac{y}{b} = 2 \quad \dots(iii) \quad \text{and} \quad \frac{x}{b} + \frac{y}{a} = 2 \quad \dots(iv)$$

Putting  $y = 0$  in (i) and (ii), we get  $x = a$  and  $x = b$  respectively. So, the coordinate points on lines (i) and (ii) are  $(a, 0)$  and  $(b, 0)$  respectively.

Now,  $d_1 =$  Distance between the parallel lines (i) and (iii)

$\Rightarrow d_1 =$  Length of the perpendicular drawn from  $(a, 0)$  upon the line (iii)

$$\Rightarrow d_1 = \frac{\left| \frac{a}{a} + \frac{0}{b} - 2 \right|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \frac{ab}{\sqrt{a^2 + b^2}}$$

and,  $d_2 =$  Distance between the parallel lines (ii) and (iv)

$\Rightarrow d_2 =$  Length of the perpendicular drawn from  $(b, 0)$  upon the line (iv)

$$\Rightarrow d_2 = \frac{\left| \frac{b}{b} + \frac{0}{a} - 2 \right|}{\sqrt{\frac{1}{b^2} + \frac{1}{a^2}}} = \frac{1}{\sqrt{\frac{1}{b^2} + \frac{1}{a^2}}} = \frac{ab}{\sqrt{a^2 + b^2}}$$

Clearly,  $d_1 = d_2$  i.e. the distances between the pairs of parallel lines are equal. Hence,  $ABCD$  is a rhombus.

**EXAMPLE 6** Find the equation of the line mid-way between the parallel lines  $9x + 6y - 7 = 0$  and  $3x + 2y + 6 = 0$ . [NCERT]

**SOLUTION** The equations of the lines are

$$3x + 2y - \frac{7}{3} = 0 \quad \dots(i) \quad 3x + 2y + 6 = 0 \quad \dots(ii)$$

Let the equation of the line mid-way between the parallel lines (i) and (ii) be

$$3x + 2y + \lambda = 0 \quad \dots(iii)$$

Then,

Distance between lines (i) and (iii) = Distance between lines (ii) and (iii)

$$\Rightarrow \frac{\left| \lambda + \frac{7}{3} \right|}{\sqrt{9 + 4}} = \frac{|\lambda - 6|}{\sqrt{9 + 4}}$$

$$\Rightarrow \left| \lambda + \frac{7}{3} \right| = |\lambda - 6| \Rightarrow \lambda + \frac{7}{3} = -\lambda + 6 \Rightarrow 2\lambda = 6 - \frac{7}{3} \Rightarrow 2\lambda = \frac{11}{3} \Rightarrow \lambda = \frac{11}{6}$$

Hence, the equation of the required line is  $3x + 2y + \frac{11}{6} = 0$ .



## EXERCISE 22.16

## BASIC

1. Determine the distance between the following pair of parallel lines :

- (i)  $4x - 3y - 9 = 0$  and  $4x - 3y - 24 = 0$
- (ii)  $8x + 15y - 34 = 0$  and  $8x + 15y + 31 = 0$
- (iii)  $y = mx + c$  and  $y = mx + d$
- (iv)  $4x + 3y - 11 = 0$  and  $8x + 6y = 15$

## BASED ON LOTS

- 2. The equations of two sides of a square are  $5x - 12y - 65 = 0$  and  $5x - 12y + 26 = 0$ . Find the area of the square.
- 3. Find the equation of two straight lines which are parallel to  $x + 7y + 2 = 0$  and at unit distance from the point  $(1, -1)$ .
- 4. Prove that the lines  $2x + 3y = 19$  and  $2x + 3y + 7 = 0$  are equidistant from the line  $2x + 3y = 6$ .
- 5. Find the equation of the line mid-way between the parallel lines  $9x + 6y - 7 = 0$  and  $3x + 2y + 6 = 0$ .

## BASED ON HOTS

- 6. Find the ratio in which the line  $3x + 4y + 2 = 0$  divides the distance between the lines  $3x + 4y + 5 = 0$  and  $3x + 4y - 5 = 0$ . [NCERT EXEMPLAR]

## ANSWERS

- 1. (i) 3 units      (ii)  $\frac{65}{17}$  units      (iii)  $\frac{|c-d|}{\sqrt{1+m^2}}$       (iv)  $\frac{7}{10}$  units
- 2. 49 sq. units      3.  $x + 7y + 6 \pm 5\sqrt{2} = 0$       5.  $18x + 12y + 11 = 0$       6. 3 : 7

## HINTS TO SELECTED PROBLEMS

- 6. Given lines are parallel. So, required ratio is the ratio of distances between first and second line and between first and third line.

## 22.15 AREA OF A PARALLELOGRAM

Let  $ABCD$  be a parallelogram the equations of whose sides  $AB, BC, CD$  and  $DA$  are  $a_1x + b_1y + c_1 = 0$ ,  $a_2x + b_2y + c_2 = 0$ ,  $a_1x + b_1y + d_1 = 0$  and  $a_2x + b_2y + d_2 = 0$ . Let  $p_1$  and  $p_2$  be the distances between the pairs of parallel sides of the parallelogram. In  $\Delta s ALD$  and  $AMB$ , we obtain

$$\sin \theta = \frac{p_1}{AD} \text{ and, } \sin \theta = \frac{p_2}{AB} \text{ respectively.}$$

$$\Rightarrow AD = \frac{p_1}{\sin \theta} \text{ and } AB = \frac{p_2}{\sin \theta}$$

Now,

$$\text{Area of parallelogram } ABCD = AB \times p_1 = \frac{p_1 p_2}{\sin \theta}$$

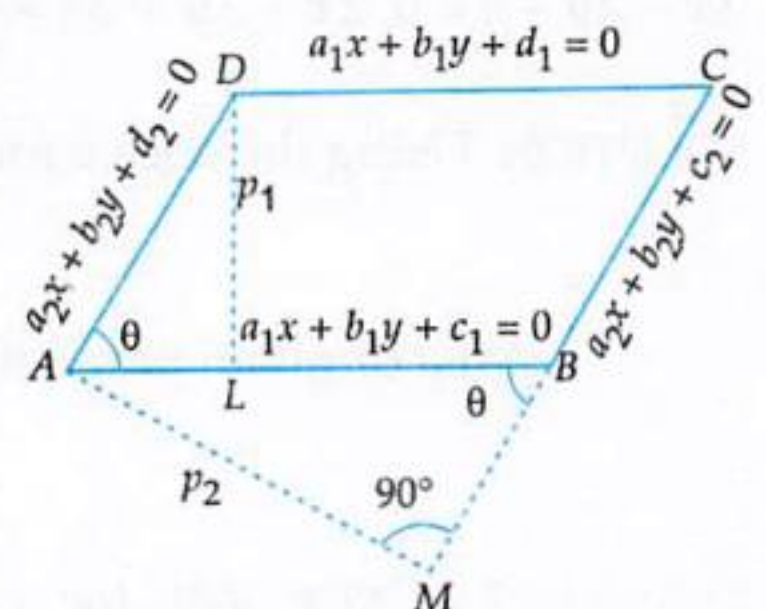


Fig. 22.88

$$\left[ \because AB = \frac{p_2}{\sin \theta} \right]$$



$$\text{Also, Area of parallelogram } ABCD = AD \times p_2 = \frac{p_1 p_2}{\sin \theta} \quad \left[ \because AD = \frac{p_1}{\sin \theta} \right]$$

Thus, area of a parallelogram is  $\frac{p_1 p_2}{\sin \theta}$ , where  $p_1$  and  $p_2$  are the distances between pairs of parallel sides and  $\theta$  is the angle between two adjacent sides.

Let  $m_1$  and  $m_2$  be the slopes of sides  $AB$  and  $AD$  respectively. Then,

$$m_1 = -\frac{a_1}{b_1} \text{ and, } m_2 = -\frac{a_2}{b_2}$$

The angle  $\theta$  between  $AB$  and  $AD$  is given by

$$\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-\frac{a_1}{b_1} + \frac{a_2}{b_2}}{1 + \frac{a_1 a_2}{b_1 b_2}} = \frac{a_2 b_1 - a_1 b_2}{a_1 a_2 + b_1 b_2} \Rightarrow \sin \theta = \frac{a_2 b_1 - a_1 b_2}{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}}$$

We have,

$$p_1 = \text{Distance between parallel sides } AB \text{ and } AD = \frac{|c_1 - d_1|}{\sqrt{a_1^2 + b_1^2}}$$

$$\text{and, } p_2 = \text{Distance between parallel sides } AD \text{ and } BC = \frac{|c_2 - d_2|}{\sqrt{a_2^2 + b_2^2}}$$

$$\therefore \text{Area of parallelogram } ABCD = \frac{p_1 p_2}{\sin \theta} = \frac{|c_1 - d_1| |c_2 - d_2|}{|a_2 b_1 - a_1 b_2|} = \left| \frac{(c_1 - d_1)(c_2 - d_2)}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \right|$$

### ILLUSTRATIVE EXAMPLES

#### BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 1** Show that the area of the parallelogram formed by the lines  $2x - 3y + a = 0$ ,  $3x - 2y - a = 0$ ,  $2x - 3y + 3a = 0$  and  $3x - 2y - 2a = 0$  is  $\frac{2a^2}{5}$  sq. units.

**SOLUTION** Using the above formula, we obtain

$$\text{Area of given parallelogram} = \left| \frac{(3a - a) \{-2a - (-a)\}}{\begin{vmatrix} 2 & -3 \\ 3 & -2 \end{vmatrix}} \right| = \frac{2a^2}{5} \text{ sq. units}$$

**EXAMPLE 2** Prove that the area of the parallelogram formed by the lines  $x \cos \alpha + y \sin \alpha = p$ ,  $x \cos \alpha + y \sin \alpha = q$ ,  $x \cos \beta + y \sin \beta = r$  and  $x \cos \beta + y \sin \beta = s$  is  $\pm (p - q)(r - s) \operatorname{cosec}(\alpha - \beta)$ .

**SOLUTION** The equations of the sides of the parallelogram are:

$$x \cos \alpha + y \sin \alpha - p = 0, \quad x \cos \alpha + y \sin \alpha - q = 0, \quad x \cos \beta + y \sin \beta - r = 0$$

$$\text{and, } x \cos \beta + y \sin \beta - s = 0$$



$$\begin{aligned} \therefore \text{Area of the parallelogram} &= \left| \frac{\{(-p) - (-q)\} \{(-r) - (-s)\}}{\begin{vmatrix} \cos \alpha & \sin \alpha \\ \cos \beta & \sin \beta \end{vmatrix}} \right| \\ &= \left| \frac{(q-p)(s-r)}{(\cos \alpha \sin \beta - \sin \alpha \cos \beta)} \right| = \pm \frac{(p-q)(r-s)}{\sin(\alpha - \beta)} \end{aligned}$$

### BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

**EXAMPLE 3** Prove that the four straight lines  $\frac{x}{a} + \frac{y}{b} = 1$ ,  $\frac{x}{b} + \frac{y}{a} = 1$ ,  $\frac{x}{a} + \frac{y}{b} = 2$  and  $\frac{x}{b} + \frac{y}{a} = 2$  form a rhombus. Find its area.

**SOLUTION** The equations of the four sides are

$$\begin{aligned} \frac{x}{a} + \frac{y}{b} &= 1 & \dots(i) & \quad \frac{x}{b} + \frac{y}{a} = 1 & \dots(ii) \\ \frac{x}{a} + \frac{y}{b} &= 2 & \dots(iii) & \quad \frac{x}{b} + \frac{y}{a} = 2 & \dots(iv) \end{aligned}$$

Clearly, (i), (iii) and (ii), (iv) form two sets of parallel lines. So, the four lines form a parallelogram.

Let  $p_1$  be the distance between parallel lines (i) and (iii) and  $p_2$  be the distance between (ii) and (iv). Then,

$$p_1 = \left| \frac{2-1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} \right| = \frac{ab}{\sqrt{a^2 + b^2}} \text{ and, } p_2 = \left| \frac{2-1}{\sqrt{\frac{1}{b^2} + \frac{1}{a^2}}} \right| = \frac{ab}{\sqrt{a^2 + b^2}} \left[ \text{Using : } d = \left| \frac{c_1 - c_2}{\sqrt{a^2 + b^2}} \right| \right]$$

Clearly,  $p_1 = p_2$ . So, the given lines form a rhombus.

$$\therefore \text{Area of the rhombus} = \left| \frac{(2-1)(2-1)}{\begin{vmatrix} 1/a & 1/b \\ 1/b & 1/a \end{vmatrix}} \right| = \left| \frac{1}{(1/a^2 - 1/b^2)} \right| = \frac{a^2 b^2}{|b^2 - a^2|}$$

**EXAMPLE 4** Show that the four lines  $ax \pm by \pm c = 0$  enclose a rhombus whose area is  $\frac{2c^2}{ab}$ .

**SOLUTION** The four lines are

$$\begin{aligned} ax + by + c &= 0 & \dots(i) & \quad ax + by - c = 0 & \dots(ii) \\ ax - by + c &= 0 & \dots(iii) & \quad ax - by - c = 0 & \dots(iv) \end{aligned}$$

Clearly, (i), (ii) and (iii), (iv) are pairs of parallel lines. Solving (i) with (iii) and (ii) with (iv), we obtain the coordinates of C and A as  $(-c/a, 0)$  and  $(c/a, 0)$  respectively. Solving (ii) with (iii) and (i) with (iv), we obtain the coordinates of B and D as  $(0, c/b)$  and  $(0, -c/b)$  respectively.

Thus, the vertices of the parallelogram ABCD are

$$A(c/a, 0), B(0, c/b), C(-c/a, 0) \text{ and } D(0, -c/b)$$

It shows that the vertices of the parallelogram are on the

coordinate axes such that one diagonal is along x-axis and other along y-axis. Since, the diagonals are at right angles. Hence, ABCD is a rhombus.

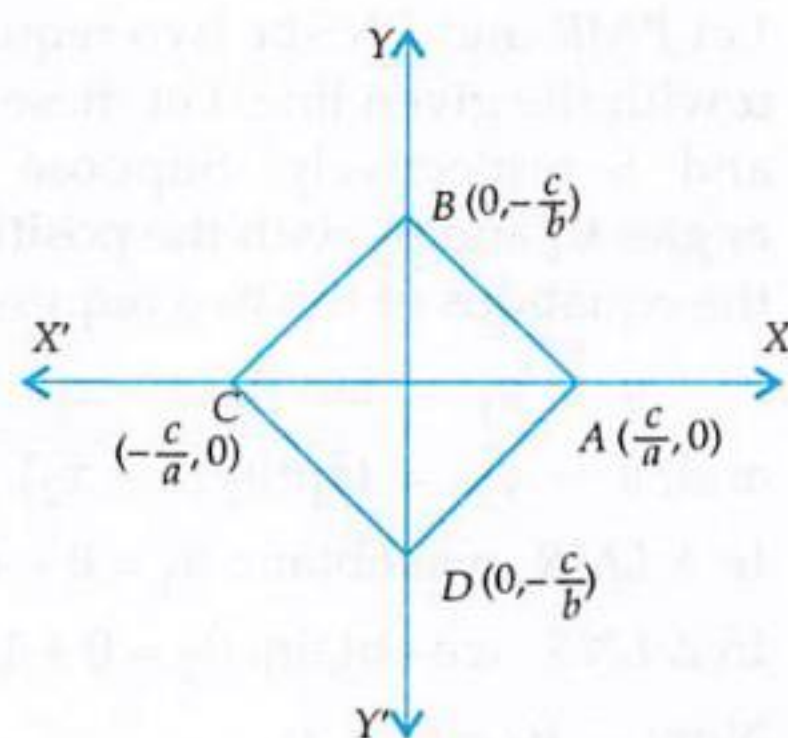


Fig. 22.89



$$\text{Area of the rhombus} = \frac{1}{2} AC \times BD = \frac{1}{2} \left( \frac{2c}{a} \times \frac{2c}{b} \right) = \frac{2c^2}{ab}.$$

**EXERCISE 22.17****BASED ON LOTS**

1. Prove that the area of the parallelogram formed by the lines  $a_1 x + b_1 y + c_1 = 0$ ,  $a_1 x + b_1 y + d_1 = 0$ ,  $a_2 x + b_2 y + c_2 = 0$ ,  $a_2 x + b_2 y + d_2 = 0$  is  $\left| \frac{(d_1 - c_1)(d_2 - c_2)}{a_1 b_2 - a_2 b_1} \right|$  sq. units.

Deduce the condition for these lines to form a rhombus.

2. Prove that the area of the parallelogram formed by the lines  $3x - 4y + a = 0$ ,  $3x - 4y + 3a = 0$ ,  $4x - 3y - a = 0$  and  $4x - 3y - 2a = 0$  is  $\frac{2a^2}{7}$  sq. units.

**BASED ON HOTS**

3. Show that the diagonals of the parallelogram whose sides are  $lx + my + n = 0$ ,  $lx + my + n' = 0$ ,  $mx + ly + n = 0$  and  $mx + ly + n' = 0$  include an angle  $\pi/2$ .

**HINTS TO SELECTED PROBLEMS**

1. Area =  $\frac{p_1 p_2}{\sin \theta}$ , where  $p_1, p_2$  are the distance between the pairs of parallel lines and  $\theta$  is the angle between two adjacent sides. For, rhombus use  $p_1 = p_2$ .
3. Use:  $p_1 = p_2$ .

**22.16 EQUATIONS OF LINES PASSING THROUGH A GIVEN POINT AND MAKING A GIVEN ANGLE WITH A LINE**

**THEOREM** Prove the equations of the straight lines which pass through a given point  $(x_1, y_1)$  and make a given angle  $\alpha$  with the given straight line  $y = mx + c$  are

$$y - y_1 = \frac{m \pm \tan \alpha}{1 \mp m \tan \alpha} (x - x_1)$$

**PROOF** Let  $P(x_1, y_1)$  be the given point and let the given line be  $LMN$ , making an angle  $\theta$  with the axis of  $x$ . Then,  $m = \tan \theta$ .

Let  $PMR$  and  $PNS$  be two required lines which make angle  $\alpha$  with the given line. Let these lines meet the axis of  $X$  at  $R$  and  $S$  respectively. Suppose lines  $PMR$  and  $PNS$  make angles  $\theta_1$  and  $\theta_2$  with the positive direction of  $X$ -axis. Then, the equations of the two required lines are

$$y - y_1 = \tan \theta_1 (x - x_1) \quad \dots(i)$$

$$\text{and, } y - y_2 = \tan \theta_2 (x - x_2) \quad \dots(ii)$$

In  $\triangle LMR$ , we obtain:  $\theta_1 = \theta + \alpha$ .

In  $\triangle LNS$ , we obtain:  $\theta_2 = \theta + 180 - \alpha$ .

Now,  $\theta_1 = \theta + \alpha$

$$\Rightarrow \tan \theta_1 = \tan (\theta + \alpha) = \frac{\tan \theta + \tan \alpha}{1 - \tan \theta \tan \alpha} = \frac{m + \tan \alpha}{1 - m \tan \alpha}$$

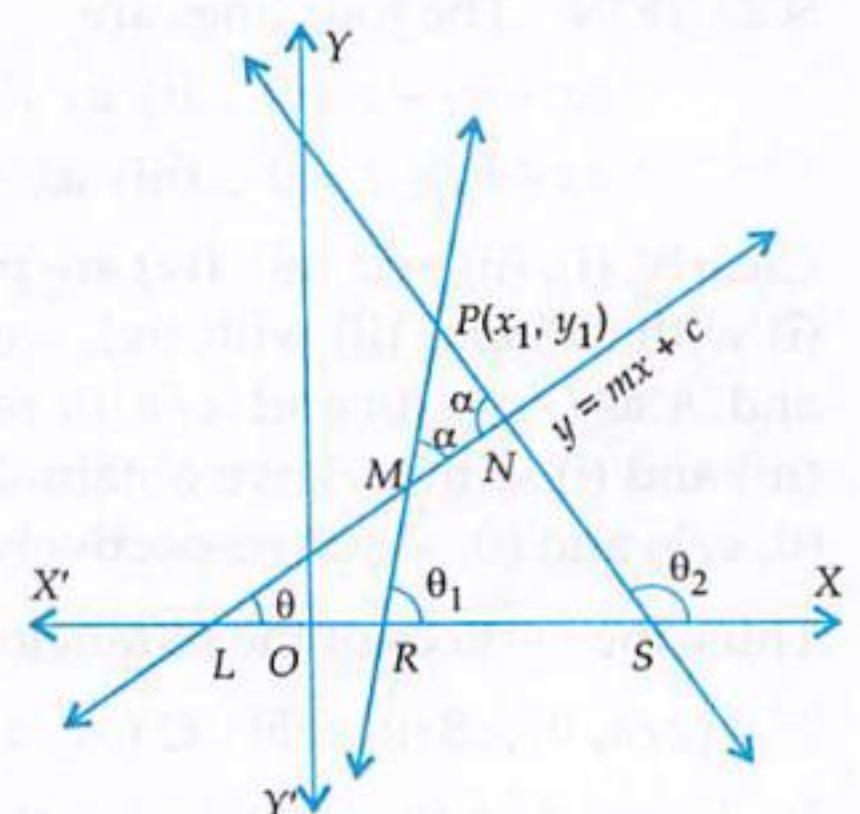


Fig. 22.90



and,  $\theta_2 = \theta + 180^\circ - \alpha$

$$\Rightarrow \tan \theta_2 = \tan (180 + \theta - \alpha) = \tan (\theta - \alpha) = \frac{\tan \theta - \tan \alpha}{1 + \tan \theta \tan \alpha} = \frac{m - \tan \alpha}{1 + m \tan \alpha}$$

On substituting the values of  $\tan \theta_1$  and  $\tan \theta_2$  in (i) and (ii), we get

$$y - y_1 = \frac{m + \tan \alpha}{1 - m \tan \alpha} (x - x_1) \quad \text{and} \quad y - y_1 = \frac{m - \tan \alpha}{1 + m \tan \alpha} (x - x_1)$$

These are the equations of the required lines.

**Q.E.D.**

### ILLUSTRATIVE EXAMPLES

#### BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 1** Find the equations of the two straight lines through  $(7, 9)$  and making an angle of  $60^\circ$  with the line  $x - \sqrt{3}y - 2\sqrt{3} = 0$ .

**SOLUTION** We know that the equations of two straight lines which pass through a point  $(x_1, y_1)$  and make a given angle  $\alpha$  with the line  $y = mx + c$  are

$$y - y_1 = \frac{m \pm \tan \alpha}{1 \mp m \tan \alpha} (x - x_1)$$

Here,  $x_1 = 7, y_1 = 9, \alpha = 60^\circ$  and  $m = (\text{Slope of the line } x - \sqrt{3}y - 2\sqrt{3} = 0) = \frac{1}{\sqrt{3}}$

So, equations of required lines are

$$y - 9 = \frac{\frac{1}{\sqrt{3}} + \tan 60^\circ}{1 - \frac{1}{\sqrt{3}} \tan 60^\circ} (x - 7) \quad \text{and} \quad y - 9 = \frac{\frac{1}{\sqrt{3}} - \tan 60^\circ}{1 + \frac{1}{\sqrt{3}} \tan 60^\circ} (x - 7)$$

$$\text{or, } (y - 9) \left( 1 - \frac{1}{\sqrt{3}} \tan 60^\circ \right) = \left( \frac{1}{\sqrt{3}} + \tan 60^\circ \right) (x - 7)$$

$$\text{and, } (y - 9) \left( 1 + \frac{1}{\sqrt{3}} \tan 60^\circ \right) = \left( \frac{1}{\sqrt{3}} - \tan 60^\circ \right) (x - 7)$$

$$\text{or, } 0 = \left( \frac{1}{\sqrt{3}} + \sqrt{3} \right) (x - 7), \text{ and } (y - 9)(2) = \left( \frac{1}{\sqrt{3}} - \sqrt{3} \right) (x - 7)$$

$$\text{or, } x - 7 = 0 \text{ and, } x + \sqrt{3}y = 7 + 9\sqrt{3}.$$

Hence, the required lines are  $x = 7$  and  $x + \sqrt{3}y = 7 + 9\sqrt{3}$ .

**EXAMPLE 2** Show that the equations of the straight lines passing through the point  $(3, -2)$  and inclined at  $60^\circ$  to the line  $\sqrt{3}x + y = 1$  are  $y + 2 = 0$  and  $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$ . **[NCERT EXEMPLAR]**

**SOLUTION** The equations of two straight lines passing through a point  $(x_1, y_1)$  and making an angle  $\alpha$  with  $y = mx + c$  are

$$y - y_1 = \frac{m \pm \tan \alpha}{1 \mp m \tan \alpha} (x - x_1)$$

Here,  $x_1 = 3, y_1 = -2, \alpha = 60^\circ$  and  $m = (\text{Slope of } \sqrt{3}x + y = 1) = -\sqrt{3}$ .

So, the equations of the required lines are

$$y + 2 = \frac{-\sqrt{3} - \tan 60^\circ}{1 - \sqrt{3} \tan 60^\circ} (x - 3) \quad \text{and} \quad y + 2 = \frac{-\sqrt{3} + \tan 60^\circ}{1 + \sqrt{3} \tan 60^\circ} (x - 3)$$

$$\text{or, } y + 2 = \sqrt{3}(x - 3) \quad \text{and} \quad y + 2 = 0$$



or,  $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$  and  $y + 2 = 0$ .

**EXAMPLE 3** Find the equations of the straight lines through  $(3, 2)$  which make acute angle of  $45^\circ$  with the line  $x - 2y - 3 = 0$ . **[NCERT]**

**SOLUTION** Here,  $x_1 = 3, y_1 = 2, \alpha = 45^\circ$  and,  $m = (\text{Slope of the line } x - 2y - 3 = 0) = \frac{1}{2}$

So, equations of required lines are

$$y - 2 = \frac{\frac{1}{2} - \tan 45^\circ}{1 + \frac{1}{2} \tan 45^\circ} (x - 3) \text{ and, } y - 2 = \frac{\frac{1}{2} + \tan 45^\circ}{1 - \frac{1}{2} \tan 45^\circ} (x - 3)$$

$$\Rightarrow y - 2 = -\frac{1}{3} (x - 3) \quad \text{and} \quad y - 2 = 3 (x - 3)$$

$$\Rightarrow x + 3y = 9 \text{ and } 3x - y = 7.$$

**ALITER** The equation of any line through  $(3, 2)$  is

$$y - 2 = m(x - 3) \quad \dots(i)$$

where  $m$  is the slope of the line and is to be determined.

It is given that the line (i) makes an acute angle of  $45^\circ$  with the line  $x - 2y - 3 = 0$ . Therefore,

$$\tan 45^\circ = \pm \frac{m - \frac{1}{2}}{1 + \frac{1}{2}m} \quad \left[ \text{Using: } \tan \theta = \pm \frac{m_1 - m_2}{1 + m_1 m_2} \right]$$

$$\Rightarrow 1 = \pm \left( \frac{2m - 1}{2 + m} \right) \Rightarrow 2 + m = \pm (2m - 1) \Rightarrow m = 3, -\frac{1}{3}$$

Putting the values of  $m$  in (i), equations of required lines are

$$y - 2 = 3(x - 3) \text{ and } y - 2 = -\frac{1}{3}(x - 3) \text{ or, } 3x - y = 7 \text{ and } x + 3y = 9.$$

#### BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 4** A vertex of an equilateral triangle is  $(2, 3)$  and the opposite side is  $x + y = 2$ . Find the equations of other sides. **[NCERT EXEMPLAR]**

**SOLUTION** Let  $A(2, 3)$  be one vertex and  $x + y = 2$  be the opposite side of an equilateral triangle. Clearly, remaining two sides pass through the point  $A(2, 3)$  and make an angle of  $60^\circ$  with  $x + y = 2$ .

Let  $m$  be the slope of  $x + y = 2$ . Then,  $m = -1$ . Fig. 22.91 the equations of the other two sides are

$$y - 3 = \frac{-1 - \tan 60^\circ}{1 - \tan 60^\circ} (x - 2)$$

$$\text{and } y - 3 = \frac{-1 + \tan 60^\circ}{1 + \tan 60^\circ} (x - 2)$$

$$\Rightarrow y - 3 = \frac{-(1 + \sqrt{3})}{1 - \sqrt{3}} (x - 2) \text{ and } y - 3 = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} (x - 2)$$

$$\Rightarrow y - 3 = (2 + \sqrt{3})(x - 2) \text{ and, } (y - 3)(2 + \sqrt{3}) = x - 2$$

$$\Rightarrow (2 + \sqrt{3})x - y = 1 + 2\sqrt{3} \text{ and, } (2 - \sqrt{3})x - y = 1 - 2\sqrt{3}.$$

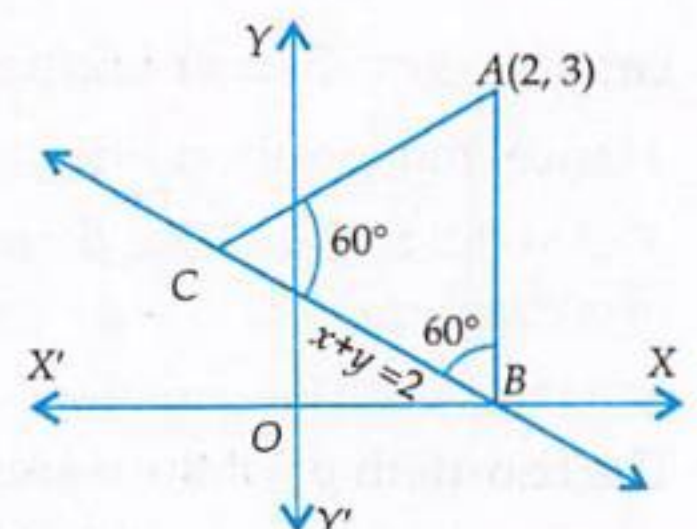


Fig. 22.91



**EXAMPLE 5** Show that the equation of the straight line through the origin making angle  $\phi$  with the line  $y = mx + b$  is  $\frac{y}{x} = \frac{m \pm \tan \phi}{1 \pm m \tan \phi}$ . [NCERT]

**SOLUTION** Let  $m_1$  be the slope of the desired line. Then, its equation is

$$(y - 0) = m_1(x - 0)$$

$$[\text{Using : } y - y_1 = m(x - x_1)]$$

$$\Rightarrow y = m_1 x$$

...(i)

If  $y = mx + b$  makes angle  $\phi$  with line (i), then

$$\tan \phi = \pm \frac{m - m_1}{1 + mm_1} \Rightarrow m_1 = \frac{m \pm \tan \phi}{1 \pm m \tan \phi}$$

Putting the values of  $m_1$  in (i), we get:  $y = \frac{m \pm \tan \phi}{1 \pm m \tan \phi} x$  as the required equations of two lines.

**EXAMPLE 6** The opposite angular points of a square are  $(3, 4)$  and  $(1, -1)$ . Find the coordinates of the other two vertices.

**SOLUTION** We have, Slope of  $AC = \frac{-1 - 4}{1 - 3} = \frac{5}{2}$

Clearly,  $AB$  and  $AD$  pass through  $A(3, 4)$  and make angle of  $45^\circ$  with  $AC$  whose slope is  $5/2$ . Therefore, equations of  $AB$  and  $AD$  are given by

$$y - 4 = \frac{\frac{5}{2} \pm \tan 45^\circ}{1 \pm \frac{5}{2} \tan 45^\circ} (x - 3) \Rightarrow y - 4 = \frac{5 \pm 2}{2 \pm 5} (x - 3)$$

$$\Rightarrow y - 4 = \frac{3}{7} (x - 3) \text{ and } y - 4 = -\frac{7}{3} (x - 3) \Rightarrow 3x - 7y + 19 = 0 \text{ and } 7x + 3y - 33 = 0.$$

Thus, equations of  $AB$  and  $AD$  are  $3x - 7y + 19 = 0$  and  $7x + 3y - 33 = 0$  respectively. Since  $BC$  is a line parallel to  $AD$ . Therefore, equation of  $BC$  is  $7x + 3y + \lambda = 0$ . This passes through  $(1, -1)$ .

$$\therefore 7 - 3 + \lambda = 0 \Rightarrow \lambda = -4.$$

So, the equation of  $BC$  is  $7x + 3y - 4 = 0$ . Since  $B$  is the point of intersection of  $AB$  and  $BC$ . Therefore, solving equations of  $AB$  and  $BC$  by cross-multiplication, we get

$$\frac{x}{-29} = \frac{y}{155} = \frac{1}{58} \Rightarrow x = -\frac{1}{2}, y = \frac{5}{2}$$

So, the coordinates of  $B$  are  $(-1/2, 5/2)$ .

Let the coordinates of  $D$  be  $(x_2, y_2)$ . Then, the coordinates of the mid-point of  $BD$  are

$$\left( \frac{x_2 - \frac{1}{2}}{2}, \frac{y_2 + \frac{5}{2}}{2} \right)$$

The coordinates of the mid-point of  $AC$  are  $(2, 3/2)$ . Since the diagonals  $AC$  and  $BD$  bisect each other.

$$\therefore \frac{x_2 - \frac{1}{2}}{2} = 2 \text{ and } \frac{y_2 + \frac{5}{2}}{2} = \frac{3}{2} \Rightarrow x_2 = \frac{9}{2} \text{ and } y_2 = \frac{1}{2}$$

So, the coordinates of  $D$  are  $(9/2, 1/2)$ . Hence, the other two vertices are  $(-1/2, 5/2)$  and  $(9/2, 1/2)$ .

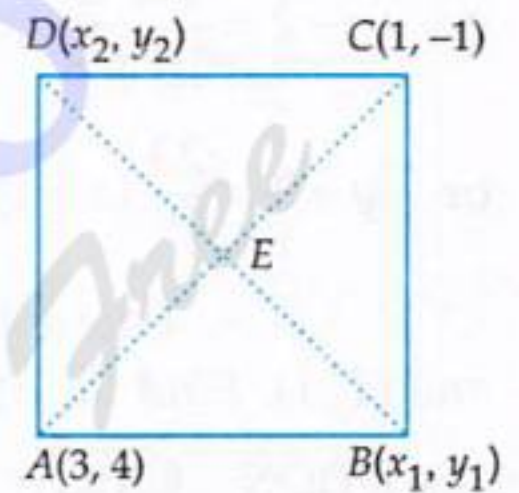


Fig. 22.92



### BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

**EXAMPLE 7** If one diagonal of a square is along the line  $8x - 15y = 0$  and one of its vertex is at  $(1, 2)$ , then find the equations of sides of the square passing through this vertex. [NCERT EXEMPLAR]

**SOLUTION** Let  $ABCD$  be the given square whose one vertex is at  $A(1, 2)$  and the diagonal  $BD$  is along the line  $8x - 15y = 0$ .

We observe that the sides  $AB$  and  $AD$  pass through the vertex  $A(1, 2)$  and make  $45^\circ$  angle with the diagonal  $BD$  of slope  $m = 8/15$ . Therefore, equations of  $AB$  and  $AD$  are given by

$$y - 2 = \frac{\frac{8}{15} \pm \tan 45^\circ}{1 \mp \frac{8}{15} \tan 45^\circ} (x - 1)$$

$$\text{or, } y - 2 = \frac{8 \pm 15}{15 \mp 8} (x - 1)$$

$$\text{or, } y - 2 = \frac{23}{7} (x - 1) \text{ and } y - 2 = -\frac{7}{23} (x - 1) \text{ or, } 23x - 7y - 9 = 0 \text{ and } 7x + 23y - 53 = 0$$

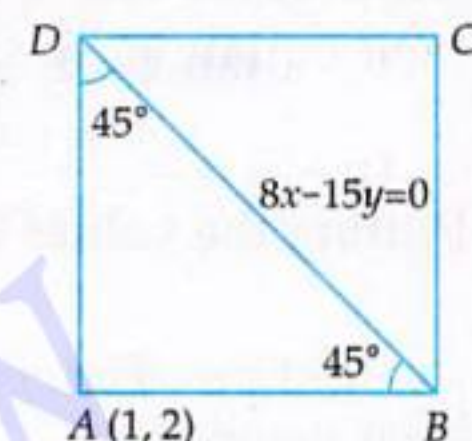


Fig. 22.93

**EXAMPLE 8** One side of a rectangle lies along the line  $4x + 7y + 5 = 0$ . Two of its vertices are  $(-3, 1)$  and  $(1, 1)$ . Find the equations of the other three sides.

**SOLUTION** Clearly, the point  $(-3, 1)$  lies on the line  $4x + 7y + 5 = 0$ . So let  $A(-3, 1)$  and  $C(1, 1)$  be two vertices of the rectangle  $ABCD$  and let the equation of the side  $AB$  be  $4x + 7y + 5 = 0$ .

Clearly,  $DC$  is a line parallel to  $AB$  passing through  $C(1, 1)$ .

Let the equation of  $DC$  be  $4x + 7y + \lambda = 0$  ... (i)

This passes through  $(1, 1)$ .

$$\therefore 4 + 7 + \lambda = 0 \Rightarrow \lambda = -11.$$

Putting the value of  $\lambda$  in (i), we get

$$4x + 7y - 11 = 0.$$

This is the equation of the side  $DC$ .

Since  $AD$  is a line through  $A$  perpendicular to  $AB$ , therefore equation of  $AD$  is

$$7x - 4y + \lambda_1 = 0 \quad \dots (ii)$$

This will pass through  $(-3, 1)$ , if

$$-21 - 4 + \lambda_1 = 0 \Rightarrow \lambda_1 = 25.$$

Putting the value of  $\lambda_1$  in the equation of  $AD$  is  $7x - 4y + 25 = 0$ .

Clearly,  $BC$  is a line perpendicular to  $AB$  passing through  $C(1, 1)$ . Let the equation of  $BC$  be

$$7x - 4y + \mu = 0 \quad \dots (iii)$$

This will pass through  $(1, 1)$  if

$$7 - 4 + \mu = 0 \Rightarrow \mu = -3.$$

Putting the value of  $\mu$  in (iii), we get  $7x - 4y - 3 = 0$ . This is the equation of  $BC$ .

**EXAMPLE 9** A line  $4x + y = 1$  through the point  $A(2, -7)$  meets the line  $BC$  whose equation is  $3x - 4y + 1 = 0$  at the point  $B$ . Find the equation to the line  $AC$  so that  $AB = AC$ .

**SOLUTION** The lines  $AB$  and  $BC$  meet at a point  $B$ . Let  $\alpha$  be the angle between them. Then,

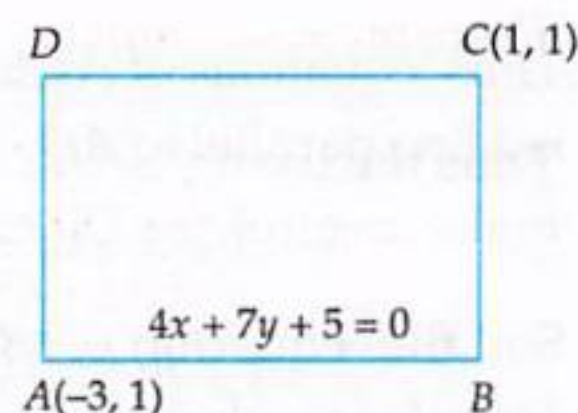


Fig. 22.94



$$\tan \alpha = \frac{m_1 - m_2}{1 + m_1 m_2}, \text{ where, } m_1 = \text{Slope of } AB = -4, m_2 = \text{Slope of } BC = \frac{3}{4}$$

$$\Rightarrow \tan \alpha = \frac{-4 - 3/4}{1 + (-4) \times 3/4} = \frac{19}{8}$$

It is given that  $AB = AC$ . Therefore, triangle  $ABC$  is an isosceles triangle.

Clearly,  $AB$  and  $AC$  both pass through  $A(2, -7)$  and are equally inclined to  $3x - 4y + 1 = 0$ . So, their equations are given by

$$(y+7) = \frac{m \pm \tan \alpha}{1 \mp m \tan \alpha} (x-2), \text{ where, } m = \text{Slope of } BC = \frac{3}{4} \text{ and } \tan \alpha = \frac{19}{8}$$

$$\text{or, } (y+7) = \frac{\frac{3}{4} + \frac{19}{8}}{1 - \frac{3}{4} \times \frac{19}{8}} (x-2) \text{ and } y+7 = \frac{\frac{3}{4} - \frac{19}{8}}{1 + \frac{3}{4} \times \frac{19}{8}} (x-2)$$

$$\text{or, } y+7 = -4(x-2) \text{ and } y+7 = -\frac{52}{89}(x-2)$$

$$\Rightarrow 4x + y = 1 \text{ and } 52x + 89y + 519 = 0$$

Clearly,  $4x + y = 1$  is the equation of  $AB$ . So, equation of  $AC$  is  $52x + 89y + 519 = 0$ .

**EXAMPLE 10** The straight lines  $3x + 4y = 5$  and  $4x - 3y = 15$  intersect at the point  $A$ . On these lines, the points  $B$  and  $C$  are chosen such that  $AB = AC$ . Find the possible equations of the line  $BC$  passing through the point  $(1, 2)$ .

**SOLUTION** Let  $m_1$  and  $m_2$  be the slopes of the lines  $3x + 4y = 5$  and  $4x - 3y = 15$  respectively.

Then,  $m_1 = -\frac{3}{4}$  and  $m_2 = \frac{4}{3}$ . Clearly,  $m_1 \cdot m_2 = -1$ . So, lines  $AB$  and  $AC$  are at right angle.

Thus the triangle  $ABC$  is a right angled isosceles triangle. Hence, the line  $BC$  through  $(1, 2)$  will make an angle of  $45^\circ$  with the given lines. So, possible equations of  $BC$  are

$$(y-2) = \frac{m \pm \tan 45^\circ}{1 \mp m \tan 45^\circ} (x-1), \text{ where } m = \text{slope of } AB = -\frac{3}{4}$$

$$\Rightarrow y-2 = \frac{-3/4 \pm 1}{1 \mp (-3/4)} (x-1)$$

$$\Rightarrow (y-2) = \frac{-3 \pm 4}{4 \pm 3} (x-1)$$

$$\Rightarrow y-2 = \frac{1}{7} (x-1) \text{ and } y-2 = -7 (x-1)$$

$$\Rightarrow x-7y+13 = 0 \text{ and } 7x+y-9 = 0$$

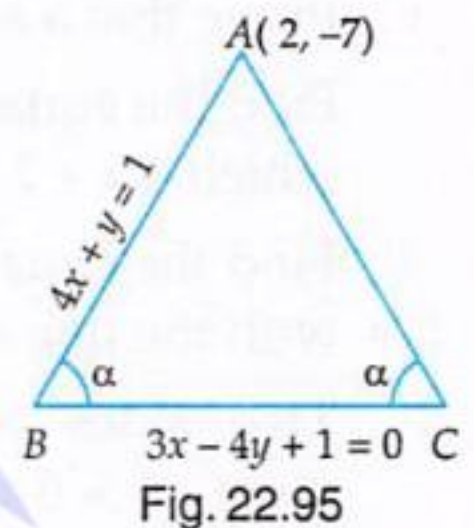


Fig. 22.95

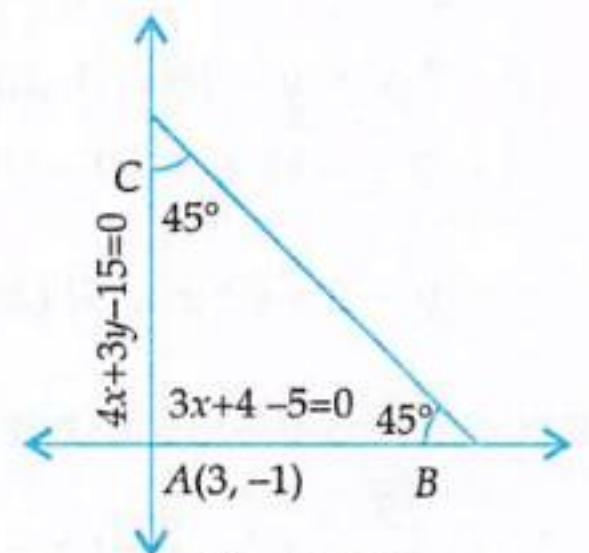


Fig. 22.96

### EXERCISE 22.18

#### BASED ON LOTS

- Find the equation of the straight lines passing through the origin and making an angle of  $45^\circ$  with the straight line  $\sqrt{3}x + y = 11$ .
- Find the equations to the straight lines which pass through the origin and are inclined at an angle of  $75^\circ$  to the straight line  $x + y + \sqrt{3}(y - x) = a$ .
- Find the equations of the straight lines passing through  $(2, -1)$  and making an angle of  $45^\circ$  with the line  $6x + 5y - 8 = 0$ .
- Find the equations to the straight lines which pass through the point  $(h, k)$  and are inclined at angle  $\tan^{-1} m$  to the straight line  $y = mx + c$ .



5. Find the equations to the straight lines passing through the point (2, 3) and inclined at an angle of  $45^\circ$  to the line  $3x + y - 5 = 0$ .
6. Find the equations to the sides of an isosceles right angled triangle the equation of whose hypotenuse is  $3x + 4y = 4$  and the opposite vertex is the point (2, 2).

[NCERT EXEMPLAR]

## BASED ON HOTS

7. The equation of one side of an equilateral triangle is  $x - y = 0$  and one vertex is  $(2 + \sqrt{3}, 5)$ . Prove that a second side is  $y + (2 - \sqrt{3})x = 6$  and find the equation of the third side.
8. Find the equations of the two straight lines through (1, 2) forming two sides of a square of which  $4x + 7y = 12$  is one diagonal.
9. Find the equations of two straight lines passing through (1, 2) and making an angle of  $60^\circ$  with the line  $x + y = 0$ . Find also the area of the triangle formed by the three lines.
10. Two sides of an isosceles triangle are given by the equations  $7x - y + 3 = 0$  and  $x + y - 3 = 0$  and its third side passes through the point (1, -10). Determine the equation of the third side.
11. Show that the point (3, -5) lies between the parallel lines  $2x + 3y - 7 = 0$  and  $2x + 3y + 12 = 0$  and find the equation of lines through (3, -5) cutting the above lines at an angle of  $45^\circ$ .
12. The equation of the base of an equilateral triangle is  $x + y = 2$  and its vertex is (2, -1). Find the length and equations of its sides.
13. If two opposite vertices of a square are (1, 2) and (5, 8), find the coordinates of its other two vertices and the equations of its sides.

## ANSWERS

1.  $y = (\sqrt{3} \pm 2)x$       2.  $x = 0, y + \sqrt{3}x = 0$       3.  $11x - y - 23 = 0$
4.  $y = k, (1 - m^2)(y - k) = 2m(x - h)$       5.  $x + 2y - 8 = 0$  and  $2x - y - 1 = 0$
6.  $7x + y - 16 = 0$  and  $x - 7y + 12 = 0$       7.  $y + (2 + \sqrt{3})x = 12 + 4\sqrt{3}$
8.  $3x - 11y + 19 = 0$  and  $11x + 3y - 17 = 0$
9.  $y - 2 = (2 \pm \sqrt{3})(x - 1)$ , Area =  $\frac{3\sqrt{3}}{2}$  sq. units
10.  $x - 3y - 31 = 0$  or  $3x + y + 7 = 0$       11.  $x - 5y - 28 = 0$  or  $5x + y - 10 = 0$
12.  $\sqrt{\frac{2}{3}}, (2 - \sqrt{3})x - y - 5 + 2\sqrt{3} = 0, (2 + \sqrt{3})x - y - 5 - 2\sqrt{3} = 0$
13. (6, 3), (0, 7),  $x - 5y + 9 = 0$ ;  $5x + y - 7 = 0$ ;  $5x + y - 33 = 0$ ;  $x - 5y + 35 = 0$

## HINTS TO SELECTED PROBLEMS

6. The two sides pass through (2, 2) and make an angle of  $45^\circ$  with the line  $3x + 4y = 4$ .

So their equations are given by  $y - 2 = \frac{-\frac{3}{4} \pm \tan \frac{\pi}{4}}{1 \pm \left(-\frac{3}{4}\right) \tan \frac{\pi}{4}} (x - 2)$

8. The two sides pass through (1, 2) and make  $45^\circ$  angle with the diagonal having slope  $m = -4/7$ .



10. Any line through  $(1, -10)$  is  $y + 10 = m(x - 1)$ . Since it makes equal angles, say  $\theta$ , with the given lines. Therefore,

$$\tan \theta = \frac{m-7}{1+7m} = -\frac{m-(-1)}{1+m(-1)} \Rightarrow m = -3 \text{ or } \frac{1}{3}.$$

12. Proceed as in example 4 on page 22.100.

## 22.17 FAMILY OF LINES THROUGH THE INTERSECTION OF TWO GIVEN LINES

**THEOREM** Prove that the equation of the family of lines passing through the intersection of the lines  $a_1 x + b_1 y + c_1 = 0$  and  $a_2 x + b_2 y + c_2 = 0$  is  $(a_1 x + b_1 y + c_1) + \lambda(a_2 x + b_2 y + c_2) = 0$ , where  $\lambda$  is a parameter.

**PROOF** The equations of the lines are

$$a_1 x + b_1 y + c_1 = 0 \quad \dots(i) \quad \text{and, } a_2 x + b_2 y + c_2 = 0 \quad \dots(ii)$$

Let  $(\alpha, \beta)$  be the point of intersection of the lines (i) and (ii). Then,

$$a_1 \alpha + b_1 \beta + c_1 = 0 \quad \dots(iii) \quad \text{and, } a_2 \alpha + b_2 \beta + c_2 = 0 \quad \dots(iv)$$

Now, consider the equation

$$(a_1 x + b_1 y + c_1) + \lambda(a_2 x + b_2 y + c_2) = 0 \quad \dots(v)$$

Clearly, this is a first degree equation in  $x$  and  $y$ . So it represents a straight line.

We find that

$$(a_1 \alpha + b_1 \beta + c_1) + \lambda(a_2 \alpha + b_2 \beta + c_2) = 0 + \lambda 0 = 0 \quad [\text{Using (iii) and (iv)}]$$

So,  $(\alpha, \beta)$  lies on the line given in equation (v). Hence, equation (v) represents family lines through the point of intersection of lines (i) and (ii).

Thus, the family of straight lines through the intersection of lines  $L_1 = a_1 x + b_1 y + c_1 = 0$  and  $L_2 = a_2 x + b_2 y + c_2 = 0$  is

$$(a_1 x + b_1 y + c_1) + \lambda(a_2 x + b_2 y + c_2) = 0 \text{ i.e. } L_1 + \lambda L_2 = 0$$

**Q.E.D.**

**REMARK** The equation  $L_1 + \lambda L_2 = 0$  represents a line passing through the intersection of the lines  $L_1 = 0$  and  $L_2 = 0$  which is a fixed point. Hence,  $L_1 + \lambda L_2 = 0$  represents a family of straight lines, for different values of  $\lambda$ , which pass through a fixed-point.

## ILLUSTRATIVE EXAMPLES

### BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** Find the equation of the straight line which passes through the point  $(2, -3)$  and the point of intersection of the lines  $x + y + 4 = 0$  and  $3x - y - 8 = 0$ .

**SOLUTION** Any line through the intersection of the lines  $x + y + 4 = 0$  and  $3x - y - 8 = 0$  has the equation

$$(x + y + 4) + \lambda(3x - y - 8) = 0 \quad \dots(i)$$

This will pass through  $(2, -3)$ , if

$$(2 - 3 + 4) + \lambda(6 + 3 - 8) = 0 \Rightarrow 3 + \lambda = 0 \Rightarrow \lambda = -3.$$

Putting the value of  $\lambda$  in (i), the equation of the required line is  $2x - y - 7 = 0$ .

**ALITER** Solving the equations  $x + y + 4 = 0$  and  $3x - y - 8 = 0$  by cross-multiplication, we get  $x = 1, y = -5$ . So, the two lines intersect at the point  $(1, -5)$ . Hence, the required line passes through points  $(2, -3)$  and  $(1, -5)$  and so its equation is

$$y + 3 = -\frac{5 + 3}{1 - 2}(x - 2) \Rightarrow 2x - y - 7 = 0.$$



**EXAMPLE 2** Find the equation of the straight line which passes through the intersection of the lines  $x - y - 1 = 0$  and  $2x - 3y + 1 = 0$  and is parallel to (i)  $x$ -axis (ii)  $y$ -axis (iii)  $3x + 4y = 14$ .

**SOLUTION** The equation of any line through the intersection of the lines  $x - y - 1 = 0$  and  $2x - 3y + 1 = 0$  is

$$(x - y - 1) + \lambda(2x - 3y + 1) = 0 \text{ or, } (2\lambda + 1)x - y(3\lambda + 1) + \lambda - 1 = 0 \quad \dots(i)$$

(i) The line in (i) will be parallel to  $x$ -axis if it is of the form  $y = \text{constant}$ .

$\therefore$  Coefficient of  $x$  in (i)  $= 0 \Rightarrow 2\lambda + 1 = 0 \Rightarrow \lambda = -1/2$ .

Putting  $\lambda = -1/2$  in (i), we get  $y = 3$  as is the equation of the required line.

(ii) The line in (i) will be parallel to  $y$ -axis if it is of the form  $x = \lambda$ .

$\therefore$  Coefficient of  $y$  in (i)  $= 0 \Rightarrow 3\lambda + 1 = 0 \Rightarrow \lambda = -1/3$ .

Putting  $\lambda = -1/3$  in (i), we get  $x = 4$  as the equation of the required line.

(iii) The line in (i) is parallel to the line  $3x + 4y - 14 = 0$ . Therefore, their slopes are equal

So, slope of line in equation (i) is same as that of the line  $3x + 4y - 14 = 0$ .

$$\text{i.e. } \frac{2\lambda + 1}{3\lambda + 1} = -\frac{3}{4} \Rightarrow \lambda = -\frac{7}{17}.$$

Putting this value of  $\lambda$  in (i) we get the equation of the required line as  $3x + 4y = 24$ .

**EXAMPLE 3** Find the equation of the straight line which passes through the point of intersection of the straight lines  $x + 2y = 5$  and  $3x + 7y = 17$  and is perpendicular to the straight line  $3x + 4y = 10$ .

**SOLUTION** The equation of any line through the intersection of the lines  $x + 2y - 5 = 0$  and  $3x + 7y - 17 = 0$  is

$$(x + 2y - 5) + \lambda(3x + 7y - 17) = 0 \text{ or, } x(3\lambda + 1) + y(7\lambda + 2) - (17\lambda + 5) = 0 \quad \dots(i)$$

This is perpendicular to the line  $3x + 4y = 10$ .

$$\therefore \text{Product of their slopes} = -1 \Rightarrow -\left(\frac{3\lambda + 1}{7\lambda + 2}\right)\left(-\frac{3}{4}\right) = -1 \Rightarrow \lambda = -\frac{11}{37}$$

Putting this value of  $\lambda$  in (i), the equation of the required line is  $4x - 3y + 2 = 0$ .

#### BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 4** Obtain the equations of the lines passing through the intersection of lines  $4x - 3y - 1 = 0$  and  $2x - 5y + 3 = 0$  and equally inclined to the axes.

**SOLUTION** The equation of any line through the intersection of the given lines is

$$(4x - 3y - 1) + \lambda(2x - 5y + 3) = 0 \text{ or, } x(2\lambda + 4) - y(5\lambda + 3) + 3\lambda - 1 = 0 \quad \dots(i)$$

Let  $m$  be the slope of this line. Then,  $m = \frac{2\lambda + 4}{5\lambda + 3}$ . As the line is equally inclined with the axes.

$$\therefore m = \pm 1 \Rightarrow \frac{2\lambda + 4}{5\lambda + 3} = \pm 1 \Rightarrow \lambda = -1 \text{ or, } 1/3$$

Putting the values of  $\lambda$  in (i), we obtain  $x + y - 2 = 0$  and  $x = y$  as the equations of the required lines.

#### BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

**EXAMPLE 5** If  $t_1$  and  $t_2$  are roots of the equation  $t^2 + \lambda t + 1 = 0$ , where  $\lambda$  is an arbitrary constant. Then prove that the line joining the points  $(at_1^2, 2at_1)$  and  $(at_2^2, 2at_2)$  always passes through a fixed point. Also, find that point.

**SOLUTION** Since  $t_1$  and  $t_2$  are roots of the equation  $t^2 + \lambda t + 1 = 0$ .

$$\therefore t_1 + t_2 = -\lambda \text{ and, } t_1 t_2 = 1 \quad \dots(i)$$

The equation of the line joining the points  $(at_1^2, 2at_1)$  and  $(at_2^2, 2at_2)$  is



$$y - 2at_1 = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2} (x - at_1^2) \quad \text{or, } y(t_2 + t_1) = 2x + 2at_1t_2 \quad \dots(ii)$$

Putting  $t_1 + t_2 = -\lambda$  and  $t_1t_2 = 1$  in (ii), we get

$$2x + 2a = -\lambda y \Rightarrow (x + a) + \left(-\frac{\lambda}{2}\right)y = 0$$

This is a line passing through the intersection of the lines  $x + a = 0$  and  $y = 0$  which is a fixed point having coordinates  $(-a, 0)$ .

**EXAMPLE 6** Show that the straight lines given by  $x(a + 2b) + y(a + 3b) = a + b$  for different values of  $a$  and  $b$  pass through a fixed point.

**SOLUTION** The given equation can be written as

$$a(x + y - 1) + b(2x + 3y - 1) = 0 \quad \text{or, } (x + y - 1) + \lambda(2x + 3y - 1) = 0, \text{ where } \lambda = b/a$$

This is of the form  $L_1 + \lambda L_2 = 0$ . So it represents a line passing through the intersection of  $x + y - 1 = 0$  and  $2x + 3y - 1 = 0$ . Solving these two equations, we get the point  $(2, -1)$ , which is the fixed point.

**EXAMPLE 7** If  $a, b, c$  are variables such that  $3a + 2b + 4c = 0$ , then show that the family of lines given by  $ax + by + c = 0$  pass through a fixed point. Also, find that point.

**SOLUTION** We have,  $3a + 2b + 4c = 0 \Rightarrow c = -\frac{3}{4}a - \frac{1}{2}b$ . Substituting this value of  $c$  in  $ax + by + c = 0$ , we get

$$ax + by - \frac{3}{4}a - \frac{1}{2}b = 0 \Rightarrow a\left(x - \frac{3}{4}\right) + b\left(y - \frac{1}{2}\right) = 0 \Rightarrow \left(x - \frac{3}{4}\right) + \lambda\left(y - \frac{1}{2}\right) = 0, \text{ where } \lambda = \frac{b}{a}.$$

This equation is of the form  $L_1 + \lambda L_2 = 0$  which represents a straight line through the intersection of the line  $L_1 = 0$  and  $L_2 = 0$  i.e.  $x - \frac{3}{4} = 0$  and  $y - \frac{1}{2} = 0$ . Solving these two equations, we get the point  $(3/4, 1/2)$ , which is a fixed point.

**ALITER** We have

$$3a + 2b + 4c = 0 \Rightarrow \left(\frac{3}{4}\right)a + \left(\frac{2}{4}\right)b + c = 0 \Rightarrow (3/4, 1/2) \text{ lies on the line } ax + by + c = 0$$

Hence, the given family of lines pass through the point  $(3/4, 1/2)$ .

**EXAMPLE 8** Let  $a, b, c$  be parameters. Then, the equation  $ax + by + c = 0$  will represent a family of straight lines passing through a fixed-point iff there exists a linear relation between  $a, b$  and  $c$ .

**SOLUTION** First, let the equation  $ax + by + c = 0$  represent a family of straight lines passing through a fixed-point  $(\alpha, \beta)$  (say) for different values of  $a, b, c$ . Then, we have to prove that there is a linear relation between  $a, b$  and  $c$ . Since,  $ax + by + c = 0$  represents a family of lines passing through  $(\alpha, \beta)$ . Therefore,

$$a\alpha + b\beta + c = 0, \text{ which is the required linear relation between } a, b \text{ and } c.$$

Conversely, let there be a linear relation between the parameters  $a, b, c$ . Then, we have to prove that the equation  $ax + by + c = 0$  represents a family of lines passing through a fixed-point.

Let the linear relation be

$$la + mb + nc = 0, \text{ where } l, m, n \text{ are constants.}$$

$$\Rightarrow \left(\frac{l}{n}\right)a + \left(\frac{m}{n}\right)b + c = 0 \Rightarrow ax + by + c = 0 \text{ passes through the fixed-point } (l/n, m/n)$$



**EXAMPLE 9** If the algebraic sum of the perpendiculars from the points  $(2, 0)$ ,  $(0, 2)$ ,  $(1, 1)$  to a variable line be zero, then prove that the line passes through a fixed-point whose coordinates are  $(1, 1)$ .

[NCERT EXEMPLAR]

**SOLUTION** Let the variable line be  $ax + by = 1$ . It is given that the algebraic sum of the perpendiculars from the points  $(2, 0)$ ,  $(0, 2)$  and  $(1, 1)$  to this line is zero. Therefore,

$$\frac{2a + 0 \times b - 1}{\sqrt{a^2 + b^2}} + \frac{0 \times a + 2b - 1}{\sqrt{a^2 + b^2}} + \frac{a + b - 1}{\sqrt{a^2 + b^2}} = 0 \Rightarrow 3a + 3b - 3 = 0 \Rightarrow a + b - 1 = 0 \Rightarrow a + b = 1.$$

This is a linear relation between the parameters  $a$  and  $b$ . So, the equation  $ax + by = 1$  represents a family of straight lines passing through a fixed-point. Comparing  $ax + by = 1$  and  $a + b = 1$ , we obtain that the coordinates of the fixed-point are  $(1, 1)$ .

**EXAMPLE 10** A ray of light is sent along the line  $x - 2y - 3 = 0$  upon reaching the line  $3x - 2y - 5 = 0$ , the ray is reflected from it. Find the equation of the line containing the reflected ray.

**SOLUTION** The point of intersection of the lines  $x - 2y - 3 = 0$  and  $3x - 2y - 5 = 0$  is  $B(1, -1)$ .  $BP$  is the normal at  $P$ . Clearly,  $BP$  passes through  $B(1, -1)$  and is perpendicular to  $3x - 2y - 5 = 0$ . So, equation of  $BP$  is

$$y + 1 = -(2/3)(x - 1) \text{ or, } 2x + 3y + 1 = 0$$

Since the reflected ray passes through the intersection of  $x - 2y - 3 = 0$  and the normal  $2x + 3y + 1 = 0$ . Therefore, equation of the reflected ray is

$$x - 2y - 3 + \lambda(2x + 3y + 1) = 0 \quad \dots(i)$$

$$\text{or, } x(1 + 2\lambda) + y(3\lambda - 2) + (\lambda - 3) = 0 \quad \dots(ii)$$

Let  $P(x_1, y_1)$  be an arbitrary point on the normal at  $P$ . Then,  $P$  is equidistant from the incident ray and the reflected ray.

$$\therefore \left| \frac{x_1 - 2y_1 - 3}{\sqrt{1 + 4}} \right| = \left| \frac{(x_1 - 2y_1 - 3) + \lambda(2x_1 + 3y_1 + 1)}{\sqrt{(1 + 2\lambda)^2 + (3\lambda - 2)^2}} \right|$$

$$\Rightarrow \left| \frac{x_1 - 2y_1 - 3}{\sqrt{5}} \right| = \left| \frac{(x_1 - 2y_1 - 3) + \lambda \times 0}{\sqrt{(1 + 2\lambda)^2 + (3\lambda - 2)^2}} \right| \quad \left[ \begin{array}{l} \because (x_1, y_1) \text{ lies on } 2x + 3y + 1 = 0 \\ \therefore 2x_1 + 3y_1 + 1 = 0 \end{array} \right]$$

$$\Rightarrow \left| \frac{1}{\sqrt{5}} \right| = \left| \frac{1}{\sqrt{(2\lambda + 1)^2 + (3\lambda - 2)^2}} \right|$$

$$\Rightarrow 5 = (2\lambda + 1)^2 + (3\lambda - 2)^2 \Rightarrow 13\lambda^2 - 8\lambda = 0 \Rightarrow \lambda = 0 \text{ or, } \lambda = \frac{8}{13}$$

Since  $\lambda = 0$  is not possible. So,  $\lambda = \frac{8}{13}$ . Putting the value of  $\lambda$  in (ii), we get  $29x - 2y - 31 = 0$  as the equation of the line containing the reflected ray.

**EXAMPLE 11** Lines  $L_1 \equiv ax + by + c = 0$  and  $L_2 \equiv lx + my + n = 0$  intersect at a point  $P$  and make an angle  $\theta$  with each other. Find the equation of the line  $L$  different from  $L_2$  which passes through  $P$  and makes the same angle with  $L_1$ .

**SOLUTION** Since the required line  $L$  passes through the intersection of  $L_1 = 0$  and  $L_2 = 0$ . So, equation of the required line  $L$  is

$$L_1 + \lambda L_2 = 0 \text{ i.e. } (ax + by + c) + \lambda(lx + my + n) = 0 \quad \dots(i)$$

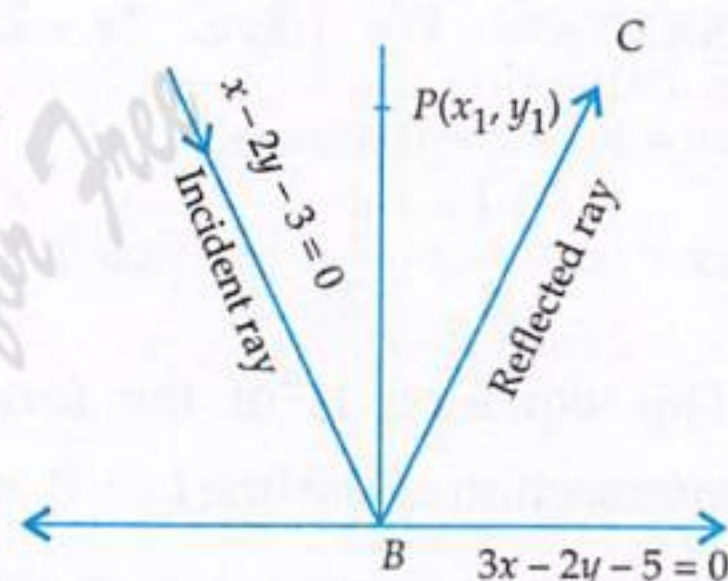


Fig. 22.97



where  $\lambda$  is a parameter.

Since  $L_1$  is the angle bisector of  $L = 0$  and  $L_2 = 0$ . Therefore any point  $A(x_1, y_1)$  on  $L_1$  is equidistant from  $L = 0$  and  $L_2 = 0$ .

$$\Rightarrow \frac{|lx_1 + my_1 + n|}{\sqrt{l^2 + m^2}} = \frac{|(ax_1 + by_1 + c) + \lambda(lx_1 + my_1 + n)|}{\sqrt{(a + \lambda l)^2 + (b + \lambda m)^2}} \quad \dots(ii)$$

But,  $A(x_1, y_1)$  lies on  $L_1$ . So, it must satisfy the equation of  $L_1$  i.e.  $ax + by + c = 0$ .

$$\therefore ax_1 + by_1 + c = 0$$

Substituting  $ax_1 + by_1 + c = 0$  in (ii), we get

$$\begin{aligned} \frac{|lx_1 + my_1 + n|}{\sqrt{l^2 + m^2}} &= \frac{|0 + \lambda(lx_1 + my_1 + n)|}{\sqrt{(a + \lambda l)^2 + (b + \lambda m)^2}} \\ \Rightarrow \lambda^2(l^2 + m^2) &= (a + \lambda l)^2 + (b + \lambda m)^2 \\ \Rightarrow \lambda &= -\frac{a^2 + b^2}{2al + 2bm} \end{aligned}$$

Substituting the value of  $\lambda$  in (i), we get

$$(ax + by + c) - \frac{(a^2 + b^2)}{2al + 2bm}(lx + my + n) = 0$$

or,  $2(al + bm)(ax + by + c) - (a^2 + b^2)(lx + my + n) = 0$  as the equation of the required line  $L$ .

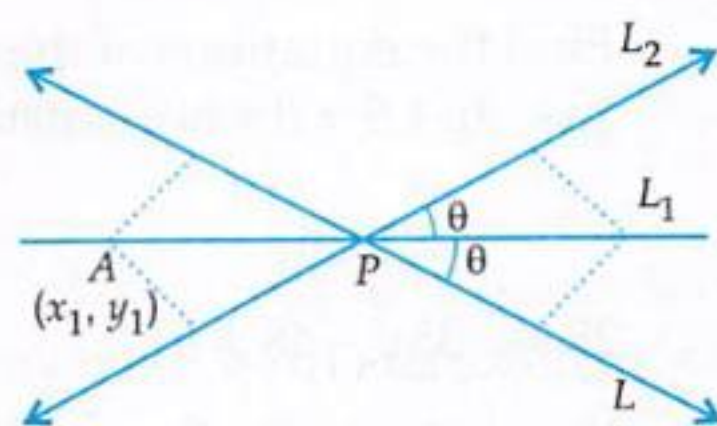


Fig. 22.98

### EXERCISE 22.19

#### BASIC

- Find the equation of a straight line through the point of intersection of the lines  $4x - 3y = 0$  and  $2x - 5y + 3 = 0$  and parallel to  $4x + 5y + 6 = 0$ .
- Find the equation of a straight line passing through the point of intersection of  $x + 2y + 3 = 0$  and  $3x + 4y + 7 = 0$  and perpendicular to the straight line  $x - y + 9 = 0$ .
- Find the equation of the line passing through the point of intersection of  $2x - 7y + 11 = 0$  and  $x + 3y - 8 = 0$  and is parallel to (i)  $x$ -axis (ii)  $y$ -axis.

#### BASED ON LOTS

- Find the equation of the straight line passing through the point of intersection of  $2x + 3y + 1 = 0$  and  $3x - 5y - 5 = 0$  and equally inclined to the axes.
- Find the equation of the straight line drawn through the point of intersection of the lines  $x + y = 4$  and  $2x - 3y = 1$  and perpendicular to the line cutting off intercepts 5, 6 on the axes.

#### BASED ON HOTS

- Prove that the family of lines represented by  $x(1 + \lambda) + y(2 - \lambda) + 5 = 0$ ,  $\lambda$  being arbitrary, pass through a fixed point. Also, find the fixed point.
- Show that the straight lines given by  $(2 + k)x + (1 + k)y = 5 + 7k$  for different values of  $k$  pass through a fixed point. Also, find that point.
- Find the equation of the straight line passing through the point of intersection of  $2x + y - 1 = 0$  and  $x + 3y - 2 = 0$  and making with the coordinate axes a triangle of area  $3/8$  sq. units.
- Find the equation of the straight line which passes through the point of intersection of the lines  $3x - y = 5$  and  $x + 3y = 1$  and makes equal and positive intercepts on the axes.
- Find the equations of the lines through the point of intersection of the lines  $x - 3y + 1 = 0$  and  $2x + 5y - 9 = 0$  and whose distance from the origin is  $\sqrt{5}$ .



11. Find the equations of the lines through the point of intersection of the lines  $x - y + 1 = 0$  and  $2x - 3y + 5 = 0$  whose distance from the point  $(3, 2)$  is  $7/5$ . [NCERT EXEMPLAR]

## ANSWERS

1.  $28x + 35y - 48 = 0$       2.  $x + y + 2 = 0$       3.  $13y = 27, 13x = 23$   
 4.  $19x + 19y + 3 = 0$  or,  $19x - 19y - 23 = 0$       5.  $25x - 30y - 23 = 0$   
 6.  $(-5/3, -5/3)$       7.  $(-2, 9)$   
 8.  $3x + 4y - 3 = 0$  or,  $12x + y - 3 = 0$       9.  $5x + 5y = 7$   
 10.  $2x + y - 5 = 0$       11.  $3x - 4y + 6 = 0, 4x - 3y + 1 = 0$

## HINTS TO SELECTED PROBLEMS

11. The equation of the family of lines through the intersection of the lines  $x - y + 1 = 0$  and  $2x - 3y + 5 = 0$  is

$$(x - y + 1) + \lambda(2x - 3y + 5) = 0 \text{ or, } x(2\lambda + 1) + y(-3\lambda - 1) + 5\lambda + 1 = 0 \quad \dots(i)$$

This is at a distance of  $\frac{7}{5}$  units from the point  $(3, 2)$ .

$$\therefore \frac{|3(2\lambda + 1) + 2(-3\lambda - 1) + 5\lambda + 1|}{\sqrt{(2\lambda + 1)^2 + (-3\lambda - 1)^2}} = \frac{7}{5}$$

$$\Rightarrow \frac{|5\lambda + 2|}{\sqrt{13\lambda^2 + 10\lambda + 2}} = \frac{7}{5} \Rightarrow 25(5\lambda + 2)^2 = 49(13\lambda^2 + 10\lambda + 2) \Rightarrow 6\lambda^2 - 5\lambda - 1 = 0 \Rightarrow \lambda = 1, -\frac{1}{6}$$

Substituting the values of  $\lambda$  in (i), we obtain

$3x - 4y + 6 = 0$  and  $4x - 3y + 1 = 0$  as the required equations of the line.

## FILL IN THE BLANKS TYPES QUESTIONS (FBQs)

- If  $2a + 3b + 4c = 0$ , then the straight lines  $ax + by + c = 0$  will always pass through the point .....
- If  $a, b, c$  are in A.P., then the straight lines  $ax + by + c = 0$  will always pass through the point.....
- The line which cuts off equal intercepts from the axes and pass through the point  $(1, -2)$  is.....
- The locus of the mid-points of the portion of the line  $x \sin \theta + y \cos \theta = p$  intercepted between the axes is .....
- The points  $(3, 4)$  and  $(2, -6)$  are situated on the ..... of the line  $3x - 4y - 8 = 0$ .
- If the points  $(-2, 1)$  and  $(-1, -2)$  are on the opposite sides of the line  $2x + y + a = 0$ , then the set of values of  $a$  is .....
- Equations of the lines through the point  $(3, 2)$  and making an angle of  $45^\circ$  with line  $x - 2y - 3 = 0$  are.....
- A point moves so that the square of its distance from the point  $(3, -2)$  is numerically equal to its distance from the line  $5x - 12y = 3$ . The equation of its locus is .....
- If the vertices of a triangle have integral coordinates, then the triangle cannot be.....
- If the lines  $ax + 2y + 1 = 0, bx + y + 1 = 0$  and  $cx + 4y + 1 = 0$  are concurrent, then  $a, b, c$  are in .....



11. If the line  $(2x + 3y + 4) + \lambda(6x - y + 12) = 0$  is perpendicular to the line  $7x + y - 4 = 0$ , then  $\lambda = \dots\dots\dots$ .
12. The equation of a straight line passing through  $(-3, 2)$  and cutting intercepts from the axes whose sum is zero, is  $\dots\dots\dots$ .
13. The number of points on the line  $x + y = 4$  which are at a unit distance from the line  $2x + 2y = 11$  is  $\dots\dots\dots$ .
14. If the lines  $2x - 3y + k = 0$ ,  $3x - 4y - 13 = 0$  and  $8x - 11y - 33 = 0$  are concurrent, then  $k = \dots\dots\dots$ .
15. If  $a + b + c = 0$ , then the family of lines  $4ax + 3by + c = 0$  are concurrent at  $\dots\dots\dots$ .
16. If  $2x + y - 5 = 0$  and  $4x + 2y - 15 = 0$  are two parallel sides of a square, then its area is  $\dots\dots\dots$ .
17. If the lines  $3x - 4y + 4 = 0$  and  $6x - 8y - 7 = 0$  are two parallel tangents to a circle, then the length of its diameter is  $\dots\dots\dots$ .

## ANSWERS

- |                                            |                   |                                     |                                                    |
|--------------------------------------------|-------------------|-------------------------------------|----------------------------------------------------|
| 1. $\left(\frac{1}{2}, \frac{3}{4}\right)$ | 2. $(1, -2)$      | 3. $x + y + 1 = 0$                  | 4. $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$ |
| 5. opposite                                | 6. $(3, 4)$       | 7. $3x - y - 7 = 0, x + 3y - 9 = 0$ |                                                    |
| 8. $13(x^2 + y^2) - 83x + 64y + 182 = 0$   | 9. Equilateral    | 10. AP                              | 11. $-\frac{17}{41}$                               |
| 12. $x - y + 5 = 0$                        | 13. Infinite      | 14. $-7$                            | 15. $\left(\frac{1}{4}, \frac{3}{4}\right)$        |
| 16. $\frac{5}{4}$ sq. units                | 17. $\frac{3}{2}$ |                                     |                                                    |

## VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write an equation representing a pair of lines through the point  $(a, b)$  and parallel to the coordinate axes.
2. Write the coordinates of the orthocentre of the triangle formed by the lines  $x^2 - y^2 = 0$  and  $x + 6y = 18$ .
3. If the centroid of a triangle formed by the points  $(0, 0)$ ,  $(\cos \theta, \sin \theta)$  and  $(\sin \theta, -\cos \theta)$  lies on the line  $y = 2x$ , then write the value of  $\tan \theta$ .
4. Write the value of  $\theta \in \left(0, \frac{\pi}{2}\right)$  for which area of the triangle formed by points  $O(0, 0)$ ,  $A(a \cos \theta, b \sin \theta)$  and  $B(a \cos \theta, -b \sin \theta)$  is maximum.
5. Write the distance between the lines  $4x + 3y - 11 = 0$  and  $8x + 6y - 15 = 0$ .
6. Write the coordinates of the orthocentre of the triangle formed by the lines  $xy = 0$  and  $x + y = 1$ .
7. If the lines  $x + ay + a = 0$ ,  $bx + y + b = 0$  and  $cx + cy + 1 = 0$  are concurrent, then write the value of  $2abc - ab - bc - ca$ .
8. Write the area of the triangle formed by the coordinate axes and the line  $(\sec \theta - \tan \theta)x + (\sec \theta + \tan \theta)y = 2$ .



9. If the diagonals of the quadrilateral formed by the lines  $l_1 x + m_1 y + n_1 = 0$ ,  $l_2 x + m_2 y + n_2 = 0$ ,  $l_1 x + m_1 y + n_1' = 0$  and  $l_2 x + m_2 y + n_2' = 0$  are perpendicular, then write the value of  $l_1^2 - l_2^2 + m_1^2 - m_2^2$ .
10. Write the coordinates of the image of the point  $(3, 8)$  in the line  $x + 3y - 7 = 0$ .
11. Write the integral values of  $m$  for which the  $x$ -coordinate of the point of intersection of the lines  $y = mx + 1$  and  $3x + 4y = 9$  is an integer.
12. If  $a \neq b \neq c$ , write the condition for which the equations  $(b - c)x + (c - a)y + (a - b) = 0$  and  $(b^3 - c^3)x + (c^3 - a^3)y + (a^3 - b^3) = 0$  represent the same line.
13. If  $a, b, c$  are in G.P. write the area of the triangle formed by the line  $ax + by + c = 0$  with the coordinates axes.
14. Write the area of the figure formed by the lines  $a|x| + b|y| + c = 0$ .
15. Write the locus of a point the sum of whose distances from the coordinates axes is unity.
16. If  $a, b, c$  are in A.P., then the line  $ax + by + c = 0$  passes through a fixed point. Write the coordinates of that point.
17. Write the equation of the line passing through the point  $(1, -2)$  and cutting off equal intercepts from the axes.
18. Find the locus of the mid-points of the portion of the line  $x \sin \theta + y \cos \theta = p$  intercepted between the axes.

**ANSWER**

- |                             |                                                     |              |                     |
|-----------------------------|-----------------------------------------------------|--------------|---------------------|
| 1. $(x - a)(y - b) = 0$     | 2. $(0, 0)$                                         | 3. $-3$      | 4. $\pi/4$          |
| 5. $7/10$ units             | 6. $(0, 0)$                                         | 7. $-1$      | 8. 2 sq. units      |
| 9. 0                        | 10. $(-1, -4)$                                      | 11. $-1, -2$ | 12. $a + b + c = 0$ |
| 13. $\frac{1}{2}$ Sq. units | 14. $\frac{2c^2}{ ab }$ Sq. units                   | 15. A square | 16. $(1, -2)$       |
| 17. $x + y + 1 = 0$         | 18. $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$ |              |                     |

**MULTIPLE CHOICE QUESTIONS (MCQs)**

Answers each of the following questions in one word or one sentence or as per exact requirement of the question:

1.  $L$  is a variable line such that the algebraic sum of the distances of the points  $(1, 1)$ ,  $(2, 0)$  and  $(0, 2)$  from the line is equal to zero. The line  $L$  will always pass through  
 (a)  $(1, 1)$                       (b)  $(2, 1)$                       (c)  $(1, 2)$                       (d) none of these
2. The acute angle between the medians drawn from the acute angles of a right angled isosceles triangle is  
 (a)  $\cos^{-1}\left(\frac{2}{3}\right)$                       (b)  $\cos^{-1}\left(\frac{3}{4}\right)$                       (c)  $\cos^{-1}\left(\frac{4}{5}\right)$                       (d)  $\cos^{-1}\left(\frac{5}{6}\right)$
3. The distance between the orthocentre and circumcentre of the triangle with vertices  $(1, 2)$ ,  $(2, 1)$  and  $\left(\frac{3 + \sqrt{3}}{2}, \frac{3 + \sqrt{3}}{2}\right)$  is



- (a) 0                      (b)  $\sqrt{2}$                       (c)  $3 + \sqrt{3}$                       (d) none of these
4. The equation of the straight line which passes through the point  $(-4, 3)$  such that the portion of the line between the axes is divided internally by the point in the ratio  $5 : 3$  is  
 (a)  $9x - 20y + 96 = 0$                       (b)  $9x + 20y = 24$   
 (c)  $20x + 9y + 53 = 0$                       (d) none of these
5. The point which divides the join of  $(1, 2)$  and  $(3, 4)$  externally in the ratio  $1 : 1$   
 (a) lies in the III quadrant                      (b) lies in the II quadrant  
 (c) lies in the I quadrant                      (d) cannot be found
6. A line passes through the point  $(2, 2)$  and is perpendicular to the line  $3x + y = 3$ . Its  $y$ -intercept is [NCERT EXEMPLAR]  
 (a)  $1/3$                       (b)  $2/3$                       (c) 1                      (d)  $4/3$
7. If the lines  $ax + 12y + 1 = 0$ ,  $bx + 13y + 1 = 0$  and  $cx + 14y + 1 = 0$  are concurrent, then  $a, b, c$  are in  
 (a) H.P.                      (b) G.P.                      (c) A.P.                      (d) none of these
8. The number of real values of  $\lambda$  for which the lines  $x - 2y + 3 = 0$ ,  $\lambda x + 3y + 1 = 0$  and  $4x - \lambda y + 2 = 0$  are concurrent is  
 (a) 0                      (b) 1                      (c) 2                      (d) Infinite
9. The equations of the sides  $AB, BC$  and  $CA$  of  $\triangle ABC$  are  $y - x = 2$ ,  $x + 2y = 1$  and  $3x + y + 5 = 0$  respectively. The equation of the altitude through  $B$  is  
 (a)  $x - 3y + 1 = 0$  (b)  $x - 3y + 4 = 0$  (c)  $3x - y + 2 = 0$  (d) none of these
10. If  $p_1$  and  $p_2$  are the lengths of the perpendiculars from the origin upon the lines  $x \sec \theta + y \csc \theta = a$  and  $x \cos \theta - y \sin \theta = a \cos 2\theta$  respectively, then  
 (a)  $4p_1^2 + p_2^2 = a^2$                       (b)  $p_1^2 + 4p_2^2 = a^2$   
 (c)  $p_1^2 + p_2^2 = a^2$                       (d) none of these
11. Area of the triangle formed by the points  $((a + 3)(a + 4), a + 3)$ ,  $((a + 2)(a + 3), (a + 2))$  and  $((a + 1)(a + 2), (a + 1))$  is  
 (a)  $25a^2$                       (b)  $5a^2$                       (c)  $24a^2$                       (d) none of these
12. If  $a + b + c = 0$ , then the family of lines  $3ax + by + 2c = 0$  pass through fixed point  
 (a)  $(2, 2/3)$                       (b)  $(2/3, 2)$                       (c)  $(-2, 2/3)$                       (d) none of these
13. The line segment joining the points  $(-3, -4)$  and  $(1, -2)$  is divided by  $y$ -axis in the ratio  
 (a)  $1 : 3$                       (b)  $2 : 3$                       (c)  $3 : 1$                       (d)  $3 : 2$
14. The area of a triangle with vertices at  $(-4, -1)$ ,  $(1, 2)$  and  $(4, -3)$  is  
 (a) 17                      (b) 16                      (c) 15                      (d) none of these
15. The line segment joining the points  $(1, 2)$  and  $(-2, 1)$  is divided by the line  $3x + 4y = 7$  in the ratio  
 (a)  $3 : 4$                       (b)  $4 : 3$                       (c)  $9 : 4$                       (d)  $4 : 9$
16. If the point  $(5, 2)$  bisects the intercept of a line between the axes, then its equation is  
 (a)  $5x + 2y = 20$                       (b)  $2x + 5y = 20$                       (c)  $5x - 2y = 20$                       (d)  $2x - 5y = 20$



17.  $A(6, 3)$ ,  $B(-3, 5)$ ,  $C(4, -2)$  and  $D(x, 3x)$  are four points. If  $\Delta DBC : \Delta ABC = 1 : 2$ , then  $x$  is equal to  
(a)  $11/8$  (b)  $8/11$  (c)  $3$  (d) none of these
18. If  $p$  be the length of the perpendicular from the origin on the line  $x/a + y/b = 1$ , then  
(a)  $p^2 = a^2 + b^2$  (b)  $p^2 = \frac{1}{a^2} + \frac{1}{b^2}$  (c)  $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$  (d) none of these
19. The equation of the line passing through  $(1, 5)$  and perpendicular to the line  $3x - 5y + 7 = 0$  is  
(a)  $5x + 3y - 20 = 0$  (b)  $3x - 5y + 7 = 0$  (c)  $3x - 5y + 6 = 0$  (d)  $5x + 3y + 7 = 0$
20. The figure formed by the lines  $ax \pm by \pm c = 0$  is  
(a) a rectangle (b) a square (c) a rhombus (d) none of these
21. Two vertices of a triangle are  $(-2, -1)$  and  $(3, 2)$  and third vertex lies on the line  $x + y = 5$ . If the area of the triangle is 4 square units, then the third vertex is  
(a)  $(0, 5)$  or,  $(4, 1)$  (b)  $(2, 3)$  or,  $(1, 4)$  (c)  $(5, 0)$  or,  $(4, 1)$  (d)  $(0, 5)$  or,  $(1, 4)$
22. The inclination of the straight line passing through the point  $(-3, 6)$  and the mid-point of the line joining the point  $(4, -5)$  and  $(-2, 9)$  is  
(a)  $\pi/4$  (b)  $\pi/6$  (c)  $\pi/3$  (d)  $3\pi/4$
23. Distance between the lines  $5x + 3y - 7 = 0$  and  $15x + 9y + 14 = 0$  is  
(a)  $\frac{35}{\sqrt{34}}$  (b)  $\frac{1}{3\sqrt{34}}$  (c)  $\frac{35}{3\sqrt{34}}$  (d)  $\frac{35}{2\sqrt{34}}$
24. The angle between the lines  $2x - y + 3 = 0$  and  $x + 2y + 3 = 0$  is  
(a)  $90^\circ$  (b)  $60^\circ$  (c)  $45^\circ$  (d)  $30^\circ$
25. The value of  $\lambda$  for which the lines  $3x + 4y = 5$ ,  $5x + 4y = 4$  and  $\lambda x + 4y = 6$  meet at a point is  
(a) 2 (b) 1 (c) 4 (d) 3
26. Three vertices of a parallelogram taken in order are  $(-1, -6)$ ,  $(2, -5)$  and  $(7, 2)$ . The fourth vertex is  
(a)  $(1, 4)$  (b)  $(4, 1)$  (c)  $(1, 1)$  (d)  $(4, 4)$
27. The centroid of a triangle is  $(2, 7)$  and two of its vertices are  $(4, 8)$  and  $(-2, 6)$ . The third vertex is  
(a)  $(0, 0)$  (b)  $(4, 7)$  (c)  $(7, 4)$  (d)  $(7, 7)$
28. If the lines  $x + q = 0$ ,  $y - 2 = 0$  and  $3x + 2y + 5 = 0$  are concurrent, then the value of  $q$  will be  
(a) 1 (b) 2 (c) 3 (d) 5
29. The medians  $AD$  and  $BE$  of a triangle with vertices  $A(0, b)$ ,  $B(0, 0)$  and  $C(a, 0)$  are perpendicular to each other, if  
(a)  $a = \frac{b}{2}$  (b)  $b = \frac{a}{2}$  (c)  $ab = 1$  (d)  $a = \pm \sqrt{2} b$
30. The equation of the line with slope  $-3/2$  and which is concurrent with the lines  $4x + 3y - 7 = 0$  and  $8x + 5y - 1 = 0$  is  
(a)  $3x + 2y - 63 = 0$  (b)  $3x + 2y - 2 = 0$   
(c)  $2y - 3x - 2 = 0$  (d) none of these



31. The vertices of a triangle are  $(6, 0)$ ,  $(0, 6)$  and  $(6, 6)$ . The distance between its circumcentre and centroid is  
 (a)  $2\sqrt{2}$  (b) 2 (c)  $\sqrt{2}$  (d) 1
32. A point equidistant from the line  $4x + 3y + 10 = 0$ ,  $5x - 12y + 26 = 0$  and  $7x + 24y - 50 = 0$  is  
 [NCERT EXEMPLAR]  
 (a)  $(1, -1)$  (b)  $(1, 1)$  (c)  $(0, 0)$  (d)  $(0, 1)$
33. The ratio in which the line  $3x + 4y + 2 = 0$  divides the distance between the lines  $3x + 4y + 5 = 0$  and  $3x + 4y - 5 = 0$  is  
 [NCERT EXEMPLAR]  
 (a) 1 : 2 (b) 3 : 7 (c) 2 : 3 (d) 2 : 5
34. The coordinates of the foot of the perpendicular from the point  $(2, 3)$  on the line  $x + y - 11 = 0$  are  
 [NCERT EXEMPLAR]  
 (a)  $(-6, 5)$  (b)  $(5, 6)$  (c)  $(-5, 6)$  (d)  $(6, 5)$
35. The reflection of the point  $(4, -13)$  about the line  $5x + y + 6 = 0$  is  
 [NCERT EXEMPLAR]  
 (a)  $(-1, -14)$  (b)  $(3, 4)$  (c)  $(0, 0)$  (d)  $(1, 2)$
36. The inclination of the line  $x - y + 3 = 0$  with positive direction of  $x$ -axis, is  
 (a)  $45^\circ$  (b)  $135^\circ$  (c)  $-45^\circ$  (d)  $-135^\circ$   
 [NCERT EXEMPLAR]
37. The two lines  $a_1x + b_1y = c_1$  and  $a_2x + b_2y = c_2$  are perpendicular if  
 [NCERT EXEMPLAR]  
 (a)  $a_1a_2 + b_1b_2 = 0$  (b)  $a_1b_2 = a_2b_1$  (c)  $a_1b_1 + a_2b_2 = 0$  (d)  $a_1b_2 + a_2b_1 = 0$
38. The coordinates of the foot of the perpendicular from the point  $(2, 3)$  on the line  $x + y - 11 = 0$  are  
 [NCERT EXEMPLAR]  
 (a)  $(-6, 5)$  (b)  $(5, 6)$  (c)  $(-5, 6)$  (d)  $(6, 5)$
39. The coordinates of the image of the point  $(2, 3)$  in the line mirror  $x + y - 11 = 0$  are  
 (a)  $(5, 6)$  (b)  $(9, 8)$  (c)  $(8, 9)$  (d)  $(-8, -9)$
40. The intercept cut-off by a line from  $y$ -axis is twice than from  $x$ -axis and the line passes through the point  $(1, 2)$ . The equation of the line is  
 [NCERT EXEMPLAR]  
 (a)  $2x + y = 4$  (b)  $2x + y + 4 = 0$  (c)  $2x - y = 4$  (d)  $2x - y + 4 = 0$
41. A line passes through the point  $P(1, 2)$  such that its intercept between the axes is bisected at  $P$ . The equation of the line is  
 [NCERT EXEMPLAR]  
 (a)  $x + 2y = 5$  (b)  $x - y + 1 = 0$  (c)  $x + y - 3 = 0$  (d)  $2x + y - 4 = 0$
42. The reflection of the point  $(4, -13)$  about the line  $5x + y + 6 = 0$  is  
 [NCERT EXEMPLAR]  
 (a)  $(-1, -14)$  (b)  $(3, 4)$  (c)  $(0, 0)$  (d)  $(1, 2)$
43. A point moves such that its distance from the point  $(4, 0)$  is half that of its distance from the line  $x = 16$ . the locus of the point is  
 [NCERT EXEMPLAR]  
 (a)  $3x^2 + 4y^2 = 192$  (b)  $4x^2 + 3y^2 = 192$  (c)  $x^2 + y^2 = 12$  (d) none of these
44. One vertex of the equilateral triangle with centroid at the origin and one side as  $x + y - 2 = 0$  is  
 [NCERT EXEMPLAR]  
 (a)  $(-1, -1)$  (b)  $(2, 2)$  (c)  $(-2, -2)$  (d)  $(2, -2)$
45. The coordinates of the foot of perpendicular from the point  $(2, 3)$  on the line  $y = 3x + 4$  are given by  
 [NCERT EXEMPLAR]  
 (a)  $\left(\frac{37}{10}, -\frac{1}{10}\right)$  (b)  $\left(-\frac{1}{10}, \frac{37}{10}\right)$  (c)  $\left(\frac{10}{37}, -10\right)$  (d)  $\left(\frac{2}{3}, -\frac{1}{3}\right)$
46. If the coordinates of the middle point of the portion of a line intercepted between the coordinate axes are  $(3, 2)$ , then the equation of the line will be  
 [NCERT EXEMPLAR]  
 (a)  $2x + 3y = 12$  (b)  $3x + 2y = 12$  (c)  $4x - 3y = 6$  (d)  $5x - 2y = 10$
47. Equations of diagonals of the square formed by the lines  $x = 0$ ,  $y = 0$ ,  $x = 1$  and  $y = 1$  are  
 [NCERT EXEMPLAR]



- (a)  $y = x, y = x + 1$  (b)  $y = x, x + y = 2$  (c)  $2y = x, x + y = \frac{1}{3}$  (d)  $y = 2x, y = 2x + 1$

48. For specifying a straight line, how many geometrical parameters should be known? [NCERT EXEMPLAR]

- (a) 1 (b) 2 (c) 3 (d) 4

49. The point (4, 1) undergoes the following two successive transformations:

(i) Reflection about the line  $y = x$

(ii) Translation through a distance of 2 units along the positive  $x$ -axis. Then the coordinates of the point are [NCERT EXEMPLAR]

- (a) (4, 3) (b) (3, 4) (c) (1, 4) (d)  $(7/2, 7/2)$

50. The tangent of the angle between the lines whose intercepts on the axes are  $a, -b$  and  $b, -a$  respectively, is [NCERT EXEMPLAR]

- (a)  $\frac{a^2 - b^2}{ab}$  (b)  $\frac{b^2 - a^2}{2}$  (c)  $\frac{b^2 - a^2}{2ab}$  (d) none of these

51. If the line  $\frac{x}{a} + \frac{y}{b} = 1$  passes through the points (2, -3) and (4, -5), then  $(a, b) =$

- (a) (1, 1) (b) (-1, 1) (c) (1, -1) (d) (-1, -1)

[NCERT EXEMPLAR]

52. The distance between the lines  $y = mx + c_1$  and  $y = mx + c_2$ , is [NCERT EXEMPLAR]

- (a)  $\frac{c_1 - c_2}{\sqrt{1 + m^2}}$  (b)  $\frac{|c_1 - c_2|}{\sqrt{1 + m^2}}$  (c)  $\frac{c_2 - c_1}{\sqrt{1 + m^2}}$  (d)  $|c_1 - c_2|$

53. The equations of the lines passing through the point (1, 0) and at a distance  $\frac{\sqrt{3}}{2}$  from the origin are

- (a)  $\sqrt{3}x \pm y - \sqrt{3} = 0$  (b)  $\sqrt{3}x \pm y + \sqrt{3} = 0$   
(c)  $x \pm \sqrt{3}y - \sqrt{3} = 0$  (d) none of these [NCERT EXEMPLAR]

54. The equations of the lines which pass through the point (3, -2) and are inclined at  $60^\circ$  to the line  $\sqrt{3}x + y = 1$  are [NCERT EXEMPLAR]

- (a)  $y + 2 = 0, \sqrt{3}x - y - 2 - 3\sqrt{3} = 0$  (b)  $x - 2 = 0, \sqrt{3}x - y + 2 + 3\sqrt{3} = 0$   
(c)  $\sqrt{3}x - y - 2 \pm 3\sqrt{3} = 0$  (d) none of these

55. The distance of the point of intersection of the lines  $2x - 3y + 5 = 0$  and  $3x + 4y = 0$  from the line  $5x - 2y = 0$ , is [NCERT EXEMPLAR]

- (a)  $\frac{130}{17\sqrt{29}}$  (b)  $\frac{13}{7\sqrt{29}}$  (c)  $\frac{130}{7}$  (d) none of these

56. Slopes of lines which cut off intercepts of equal lengths on the coordinate axes are

- (a) 0 (b)  $\pm \frac{1}{\sqrt{3}}$  (c)  $\pm 1$  (d)  $\pm \sqrt{3}$

[NCERT EXEMPLAR]

57. A line cutting off intercept -3 from the  $y$ -axis and the tangent of angle to the  $x$ -axis is  $\frac{3}{5}$ , its equation is

- (a)  $5y - 3x + 15 = 0$  (b)  $3y - 5x + 15 = 0$  (c)  $5y - 3x - 15 = 0$  (d) none of these

[NCERT EXEMPLAR]

58. The equation of the straight line passing through the points (3, 2) and perpendicular to the line  $y = x$  is [NCERT EXEMPLAR]

- (a)  $x - y = 5$  (b)  $x + y = 5$  (c)  $x + y = 1$  (d)  $x - y = 1$



## ANSWERS

- |         |         |         |         |         |         |         |         |
|---------|---------|---------|---------|---------|---------|---------|---------|
| 1. (a)  | 2. (c)  | 3. (a)  | 4. (a)  | 5. (d)  | 6. (d)  | 7. (c)  | 8. (a)  |
| 9. (b)  | 10. (a) | 11. (d) | 12. (b) | 13. (c) | 14. (a) | 15. (d) | 16. (b) |
| 17. (a) | 18. (c) | 19. (a) | 20. (c) | 21. (b) | 22. (d) | 23. (c) | 24. (a) |
| 25. (b) | 26. (b) | 27. (b) | 28. (c) | 29. (d) | 30. (b) | 31. (c) | 32. (c) |
| 33. (b) | 34. (b) | 35. (a) | 36. (a) | 37. (a) | 38. (b) | 39. (c) | 40. (a) |
| 41. (d) | 42. (a) | 43. (a) | 44. (b) | 45. (b) | 46. (a) | 47. (a) | 48. (b) |
| 49. (b) | 50. (c) | 51. (d) | 52. (b) | 53. (a) | 54. (a) | 55. (a) | 56. (c) |
| 57. (a) | 58. (b) |         |         |         |         |         |         |

## SUMMARY

- Every first degree equation in  $x, y$  represents a straight line.
- The trigonometrical tangent of the angle that a non-vertical line makes with the positive direction of the  $x$ -axis in anticlockwise sense is called the slope or gradient of the line.
- The slope  $m$  of a non-vertical line passing through the points  $(x_1, y_1)$  and  $(x_2, y_2)$  is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Difference of ordinates}}{\text{Difference of abscissae}}$$

- Slope of a horizontal line is zero and slope of a vertical line is undefined.
- An acute angle  $\theta$  between the lines having slopes  $m_1$  and  $m_2$  is given by

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|, 1 + m_1 m_2 \neq 0$$

- Two lines are parallel if and only if their slopes are equal.
- Two lines are perpendicular if and only if the product of their slopes is  $-1$ .
- Three points  $P, Q$  and  $R$  are collinear if and only if  
Slope of  $PQ$  = Slope of  $QR$
- If a straight line cuts  $x$ -axis at  $A$  and the  $y$ -axis at  $B$ , then  $OA$  and  $OB$  are known as the intercepts of the line on  $x$ -axis and  $y$ -axis respectively.
- The equation of a line parallel to  $x$ -axis at a distance  $a$  from it is  $y = a$  or  $y = -a$  according as it is above or below  $x$ -axis.
- The equation of a line parallel to  $y$ -axis at a distance  $b$  from it is  $x = b$  or  $x = -b$  according as it is on the right or on left side of  $y$ -axis.
- The equation of  $x$ -axis is  $y = 0$ .
- The equation of  $y$ -axis is  $x = 0$ .
- The equation of a line with slope  $m$  and making an intercept  $c$  on  $y$ -axis is  $y = mx + c$ .
- The equation of a line with slope  $m$  and passing through the origin is  $y = mx$ .
- The equation of the line which passes through the point  $(x_1, y_1)$  and has slope  $m$  is  
 $y - y_1 = m(x - x_1)$
- The equation of the line passing through the points  $(x_1, y_1)$  and  $(x_2, y_2)$  is



$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

18. The equation of the line making intercepts  $a$  and  $b$  on  $x$  and  $y$ -axis respectively is  $\frac{x}{a} + \frac{y}{b} = 1$ .
19. The equation of the straight line upon which the length of the perpendicular from the origin is  $p$  and the angle between this perpendicular and positive  $x$ -axis is  $\alpha$  is given by  $x \cos \alpha + y \sin \alpha = p$ .
20. The equation of the straight line passing through  $(x_1, y_1)$  and making an angle  $\theta$  with the positive direction of  $x$ -axis is

$$\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r, \text{ where } r \text{ is the distance of the point } (x, y) \text{ on the line from the point } (x_1, y_1).$$

The coordinates of any point on the line at a distance  $r$  from the point  $(x_1, y_1)$  are

$$(x_1 \pm r \cos \theta, y_1 \pm r \sin \theta)$$

21. The slope of the line  $ax + by + c = 0$  is

$$-\frac{a}{b} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y}$$

22. Three lines  $L_1 \equiv a_1 x + b_1 y + c_1 = 0$ ,  $L_2 \equiv a_2 x + b_2 y + c_2 = 0$  and  $L_3 \equiv a_3 x + b_3 y + c_3 = 0$  are concurrent, if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Also, these lines are concurrent iff there exist scalars  $\lambda_1, \lambda_2, \lambda_3$  such that

$$\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0$$

23. The equation of a line parallel to the line  $ax + by + c = 0$  is  $ax + by + \lambda = 0$ , where  $\lambda$  is a constant.
24. The equation of a line perpendicular to the line  $ax + by + c = 0$  is  $bx - ay + \lambda = 0$ , where  $\lambda$  is a constant.
25. The perpendicular distance ( $d$ ) of a line  $ax + by + c = 0$  from a point  $(x_1, y_1)$  is given by

$$d = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

26. The distance ( $d$ ) between the parallel lines  $ax + by + c_1 = 0$  and  $ax + by + c_2 = 0$  is given by  $d = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$ .

27. The equations of the lines passing through  $(x_1, y_1)$  and making an angle  $\alpha$  with the line  $y = mx + c$  are given by

$$y - y_1 = \frac{m \pm \tan \alpha}{1 \pm m \tan \alpha} (x - x_1).$$