CHAPTER 28

LIMITS

28.1 INFORMAL APPROACH TO LIMIT

Consider the function $f(x) = \dfrac{x^2 - 4}{x - 2}$. Clearly, this function is defined for all x except at $x = 2$ as it assumes the form $\dfrac{0}{0}$ (known as an indeterminate form) at $x = 2$. However, if $x \neq 2$, then

$$f(x) = \frac{(x-2)(x+2)}{x-2} = x + 2$$

The following table exhibits the values of $f(x)$ at points which are close to 2 on its two sides *viz.* left and right on the real line.

x	1.4	1.5	1.6	1.7	1.8	1.9	1.99	2	2.01	2.1	2.2	2.3	2.4	2.5	2.6
$f(x)$	3.4	3.5	3.6	3.7	3.8	3.9	3.99	$\frac{0}{0}$	4.01	4.1	4.2	4.3	4.4	4.5	4.6

The graph of this function is shown in Fig. 28.1. It is evident from the above table and the graph of $f(x)$ that as x increases and comes closer to 2 from left hand side of 2, the values of $f(x)$ increase and come closer to 4. This is interpreted as:

When x approaches to 2 from its left hand side, the function f(x) tends to the limit 4.

It we use the notation '$x \to 2^-$' to denote 'x tends to 2 from left hand side', the above statement can be restated as:

as $x \to 2^-$, $f(x) \to 4$

or, $\lim\limits_{x \to 2^-} f(x) = 4$ or, Left hand limit of $f(x)$ *at* $x = 2$ is 4.

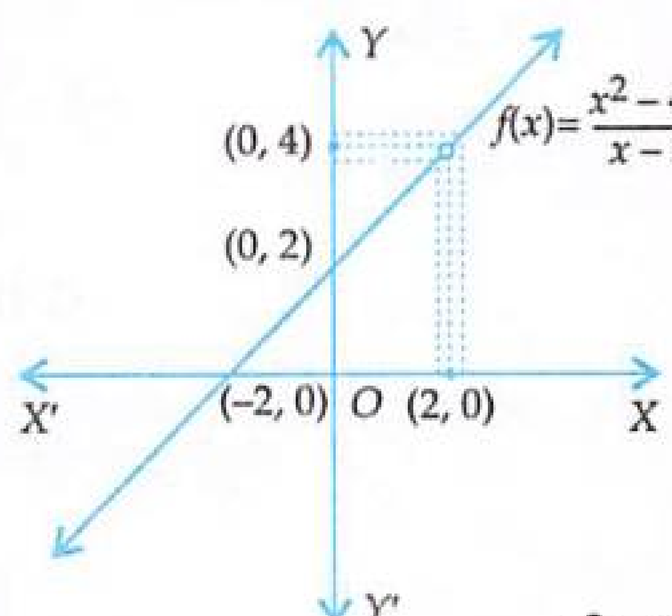

Fig. 28 .1 Graph of $f(x) = \dfrac{x^2 - 4}{x - 2}$

Thus, $\lim\limits_{x \to 2^-} f(x) = 4$ means that as x tends to 2 from left hand side, the values of $f(x)$ are tending to 4.

From the above table as well as the graph of $f(x)$, shown in Fig. 28.1, we observe that as x decreases and comes closer to 2 from right hand side, the values of $f(x)$ decrease and come closer to 4. This is interpreted as:

When x approaches to 2 from its right hand side, the function f(x) tends to the limit 4.

Using the notation '$x \to 2^+$' to denote 'x tends to 2 from right hand side', the above statement can be re-stated as:

as $x \to 2^+$, $f(x) \to 4$ or, $\lim\limits_{x \to 2^+} f(x) = 4$ or, Right hand limit of $f(x)$ at $x = 2$ is 4.

Thus, $\lim\limits_{x \to 2^+} f(x) = 4$ means that as x tends to 2 from right hand side, the values of $f(x)$ are tending to 4.

It follows from the above discussion that for the function $f(x)$ given by $f(x) = \dfrac{x^2 - 4}{x - 2}$:

(i) $\lim\limits_{x \to 2^-} f(x) = 4$ (ii) $\lim\limits_{x \to 2^+} f(x) = 4$ (iii) $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x)$

(iv) $f(2)$ does not exist i.e. $f(x)$ is not defined at $x = 2$.

Now, consider the function $f(x) = \dfrac{|x-3|}{x-3}$. This function is defined for all x except $x = 3$, as it assumes the form $\dfrac{0}{0}$ (an indeterminate form) at $x = 3$. The graph of $f(x)$ is shown in Fig. 28.2.

The following table shows the values of $f(x)$ at points which are close to 3 and are on its two sides.

x	2.3	2.4	2.5	2.6	2.7	2.8	2.9	2.99	3	3.01	3.1	3.2	3.3	3.4	3.5	3.6
$f(x)$	-1	-1	-1	-1	-1	-1	-1	-1	$\frac{0}{0}$	1	1	1	1	1	1	1

It is evident from the table and the graph of $f(x)$ that as $x \to 3$ from its left hand side the values of $f(x)$ are everywhere -1.

i.e. $\lim\limits_{x \to 3^-} f(x) = -1$ or, Left hand limit (LHL) of $f(x)$ at $x = 3$ is -1.

We also observe that at every point on the right hand side of 3, the function assumes value 1.

$\therefore \quad \lim\limits_{x \to 3^+} f(x) = 1$

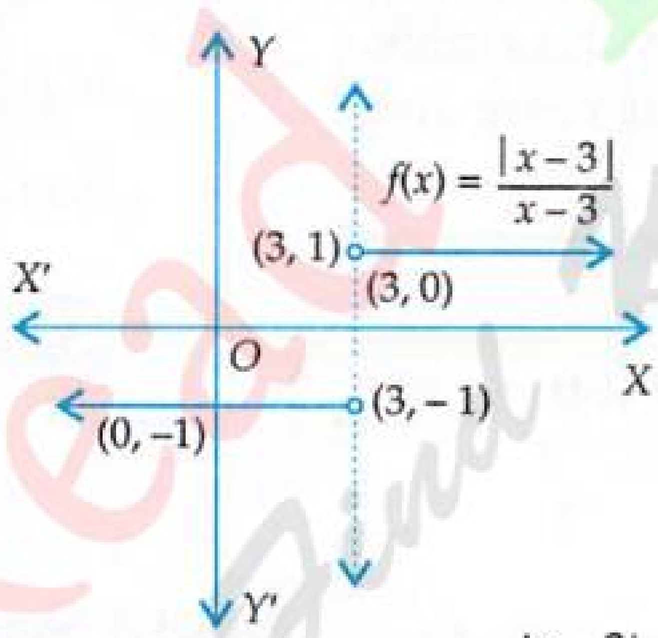

Fig. 28.2 Graph of $f(x) = \dfrac{|x-3|}{x-3}$

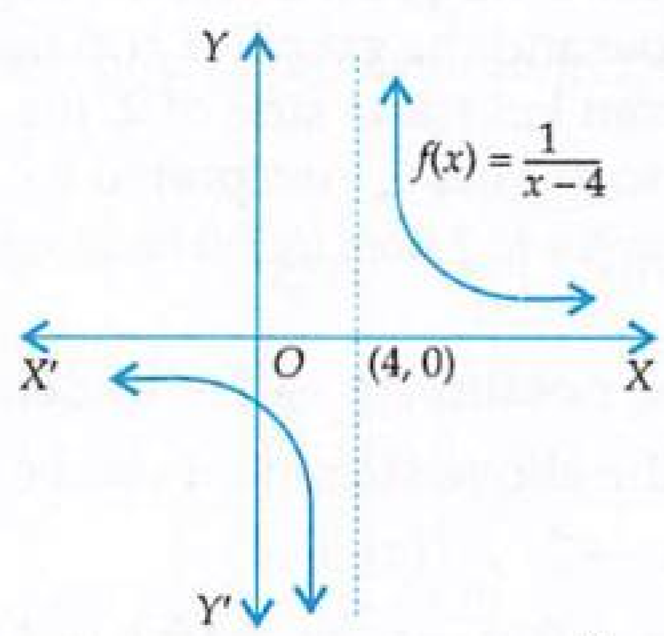

Fig. 28.3 Graph of $f(x) = \dfrac{1}{x-4}$

Let us now consider the function $f(x) = \dfrac{1}{x-4}$, $x \neq 4$. Here also the function is undefined at $x = 4$ as $f(4)$ assumes the form $\dfrac{1}{0}$. In this case it is evident from the graph shown in Fig. 28.3 that as x approaches to 4 from the left hand side, $f(x)$ decreases to $-\infty$.

i.e. $\lim\limits_{x \to 4^-} f(x) = -\infty$

Also, we observe that $f(x)$ increases to $+\infty$ as x approaches to 4 from the right

i.e. $\lim\limits_{x \to 4^+} f(x) = +\infty$

So, we say that $\lim\limits_{x \to 4^-} f(x)$ and $\lim\limits_{x \to 4^+} f(x)$ both do not exist.

It follows from the above discussion that as we can approach to a given number 'a' (say) on the real line either from its left hand side by increasing numbers which are less than 'a' or from right hand side by decreasing numbers which are greater than 'a'. So, there are two types of limits viz. (i) left hand limit and, (ii) right hand limit. We also observe that for some functions at a given point 'a' (say) left hand and right hand limits are equal whereas for some functions these two limits are not equal and even sometimes either left hand limit or right hand limit or both do not exist.

If $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$ i.e. (LHL at $x = a$) = (RHL at $x = a$), then we say that $\lim\limits_{x \to a} f(x)$ exists.

Otherwise, $\lim\limits_{x \to a} f(x)$ does not exist.

28.2 EVALUATION OF LEFT HAND AND RIGHT HAND LIMITS

In the previous sections, we have learnt that a real number l_1 is the left hand limit of function $f(x)$ at $x = a$ if the values of $f(x)$ can be made as close as desired to the number l_1 at points closed to a and on the left of a. In such a case, we write $\lim\limits_{x \to a^-} f(x) = l_1$. Also, a real number l_2 is the right hand limit of $f(x)$ at $x = a$ i.e. $\lim\limits_{x \to a^+} f(x) = l_2$, if the values of $f(x)$ can be made as close as desired to the number l_2 at points close to 'a' on the right of 'a'.

In this section, we shall discuss methods of evaluation of left hand and right hand limits of a function at a given point.

As discussed earlier that statement $x \to a^-$ means that x is tending to a from the left hand side i.e. x is a number less than a but very very close to a. Therefore, $x \to a^-$ is equivalent to $x = a - h$ where $h > 0$ such that $h \to 0$.

Similarly, $x \to a^+$ is equivalent to $x = a + h$ where $h \to 0$. Thus, we have the following algorithms for finding left hand and right hand limits at $x = a$.

ALGORITHM

Step I *Write* $\lim\limits_{x \to a^-} f(x)$

Step II *Put* $x = a - h$ *and replace* $x \to a^-$ by $h \to 0$ *to obtain* $\lim\limits_{x \to a^-} = f(x) = \lim\limits_{h \to 0} f(a-h)$.

Step III *Simplify* $\lim\limits_{h \to 0} f(a-h)$ *by using the formula for the given function.*

Step IV *The value obtain in step III is the LHL of* $f(x)$ *at* $x = a$.

ILLUSTRATION 1 *Evaluate the left hand limit of the function*

$$f(x) = \begin{cases} \dfrac{|x-4|}{x-4} & , x \neq 4 \\ 0 & , x = 4 \end{cases} \quad \text{at } x = 4.$$

SOLUTION (LHL of $f(x)$ at $x = 4$) (Step I)

$$= \lim_{x \to 4^-} f(x)$$

$$= \lim_{h \to 0} f(4-h) \qquad \text{(Step II)}$$

$$= \lim_{h \to 0} \frac{|4-h-4|}{4-h-4} = \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} -1 = -1. \qquad \text{(Step III)}$$

To evaluate RHL of $f(x)$ at $x = a$ i.e. $\lim\limits_{x \to a^+} f(x)$ we use the following algorithm.

ALGORITHM

Step I *Write* $\lim\limits_{x \to a^+} f(x)$

Step II *Put* $x = a + h$ *and replace* $x \to a^+$ by $h \to 0$ *to obtain* $\lim\limits_{x \to a^+} f(x) = \lim\limits_{h \to 0} f(a + h)$.

Step III *Simplify* $\lim\limits_{h \to 0} f(a + h)$ *by using the formula for the given function.*

Step IV *The value obtained in step III is the RHL of* $f(x)$ *at* $x = a$.

ILLUSTRATION 2 *Evaluate the right hand limit of the function*

$$f(x) = \begin{cases} \dfrac{|x-4|}{x-4} & , x \neq 4 \\ 0 & , x = 4 \end{cases} \quad \text{at } x = 4.$$

SOLUTION (RHL of $f(x)$ at $x = 4$) (Step I)

$$= \lim_{x \to 4^+} f(x)$$

$$= \lim_{h \to 0} f(4 + h) \qquad \text{(Step II)}$$

$$= \lim_{h \to 0} \frac{|4 + h - 4|}{4 + h - 4} = \lim_{h \to 0} \frac{|h|}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1 \qquad \text{(Step III)}$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate the left hand and right hand limits of the function defined by*

$$f(x) = \begin{cases} 1 + x^2, & \text{if } 0 \le x \le 1 \\ 2 - x, & \text{if } x > 1 \end{cases}$$ at $x = 1$. *Also, show that* $\lim\limits_{x \to 1} f(x)$ *does not exist.*

SOLUTION (LHL of $f(x)$ at $x = 1$)

$$= \lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1 - h) = \lim_{h \to 0} 1 + (1 - h)^2 = \lim_{h \to 0} 2 - 2h + h^2 = 2.$$

and, (RHL of $f(x)$ at $x = 1$)

$$= \lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1 + h) = \lim_{h \to 0} 2 - (1 + h) = \lim_{h \to 0} 1 - h = 1.$$

Clearly, $\lim\limits_{x \to 1^-} f(x) \neq \lim\limits_{x \to 1^+} f(x)$. Hence, $\lim\limits_{x \to 1} f(x)$ does not exist.

EXAMPLE 2 *If* $f(x) = \begin{cases} \dfrac{x - |x|}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$ *show that* $\lim\limits_{x \to 0} f(x)$ *does not exist.*

SOLUTION We find that:

(LHL of $f(x)$ at $x = 0$)

$$= \lim_{x \to 0^-} f(x)$$

$$= \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} \frac{-h - |-h|}{(-h)} = \lim_{h \to 0} \frac{-h - h}{-h} = \lim_{h \to 0} \frac{-2h}{-h} = \lim_{h \to 0} 2 = 2.$$

(RHL of $f(x)$ at $x = 0$)

$$= \lim_{x \to 0^+} f(x)$$

$$= \lim_{x \to 0} f(0+h) = \lim_{h \to 0} \frac{h-|h|}{h} = \lim_{h \to 0} \frac{h-h}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0.$$

Clearly, $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$. Hence, $\lim_{x \to 0} f(x)$ does not exist.

EXAMPLE 3 *If* $f(x) = \begin{cases} 5x-4, & 0 < x \le 1 \\ 4x^3 - 3x, & 1 < x < 2 \end{cases}$, *show that* $\lim_{x \to 1} f(x)$ *exists.*

SOLUTION We find that

(LHL of $f(x)$ at $x = 1$) $= \lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} 5(1-h) - 4 = \lim_{h \to 0} 1 - 5h = 1.$

(RHL of $f(x)$ at $x = 1$) $= \lim_{x \to 1^+} f(x)$

$$= \lim_{h \to 0} f(1+h) = \lim_{h \to 0} 4(1+h)^3 - 3(1+h) = 4(1)^3 - 3(1) = 1$$

Clearly, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$. Hence, $\lim_{x \to 1} f(x)$ exists and is equal to 1.

EXAMPLE 4 *Discuss the existence of each of the following limits:*

(i) $\lim_{x \to 0} \frac{1}{x}$ (ii) $\lim_{x \to 0} \frac{1}{|x|}$

SOLUTION (i) The graph of $f(x) = \frac{1}{x}$ is as shown in Fig. 28.4. We observe that as x approaches to 0 from the LHS i.e. x is negative and very close to zero, then the values of $\frac{1}{x}$ are negative and very large in magnitude.

$\therefore \quad \lim_{x \to 0^-} \frac{1}{x} \to -\infty.$

Similarly, when x approaches to 0 from the right i.e. x is positive and very close to 0, then the values of $\frac{1}{x}$ are very large and positive.

$\therefore \quad \lim_{x \to 0^+} \frac{1}{x} \to \infty.$

Thus we obtain, $\lim_{x \to 0^-} \frac{1}{x} \neq \lim_{x \to 0^+} \frac{1}{x}$. Hence, $\lim_{x \to 0} \frac{1}{x}$ does not exist.

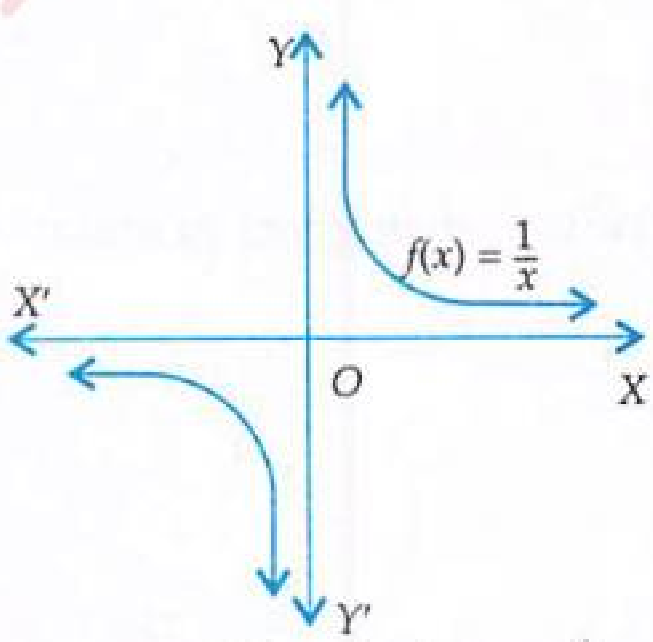

Fig. 28.4 Graph of $f(x) = \frac{1}{x}$

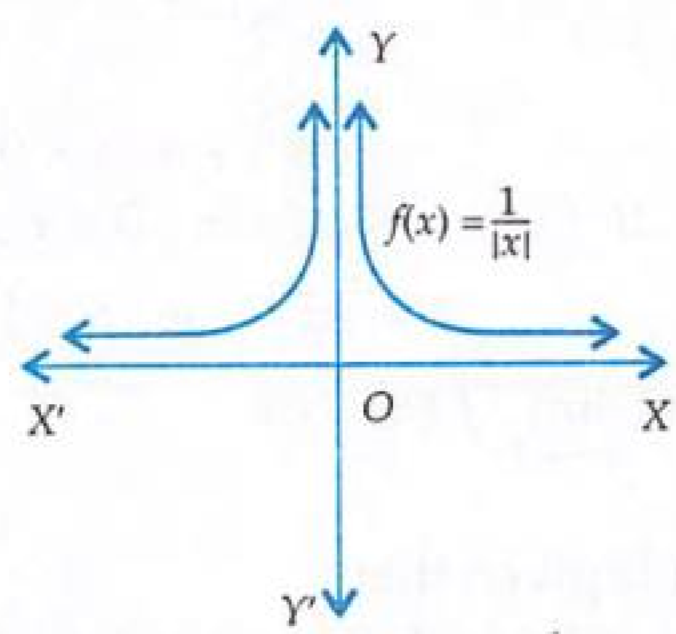

Fig. 28.5 Graph of $f(x) = \frac{1}{|x|}$

(ii) The graph of $f(x) = \frac{1}{|x|}$ is shown in Fig. 28.5. We observe that as x approaches to 0 from

LHS i.e. x is negative and close to 0, then $|x|$ is close to zero and is positive. Consequently, $\frac{1}{|x|}$ is large and positive.

$$\therefore \quad \lim_{x \to 0^-} \frac{1}{|x|} \to \infty$$

Also, if x approaches to 0 from RHS i.e. x is positive and close to 0, then $|x|$ is close to zero and is positive. Consequently, $\frac{1}{|x|}$ is large and positive.

$$\therefore \quad \lim_{x \to 0^+} \frac{1}{|x|} \to \infty$$

Thus, we obtain: $\lim_{x \to 0^-} \frac{1}{|x|} = \lim_{x \to 0^+} \frac{1}{|x|}$. Hence, $\lim_{x \to 0} \frac{1}{|x|}$ exists and it tends to infinity.

EXAMPLE 5 *Let* $f(x) = \begin{cases} \cos x, & \text{if } x > 0 \\ x + k, & \text{if } x < 0 \end{cases}$. *Find the value of constant k, given that* $\lim_{x \to 0} f(x)$ *exists.*

SOLUTION It is given that

$$\lim_{x \to 0} f(x) \text{ exists}$$

$$\Rightarrow \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$$

$$\Rightarrow \quad \lim_{x \to 0} x + k = \lim_{x \to 0} \cos x \Rightarrow 0 + k = \cos 0 \Rightarrow k = 1 \qquad \text{[Using definition of } f(x)\text{]}$$

EXAMPLE 6 *Let* $f(x)$ *be a function defined by* $f(x) = \begin{cases} 4x - 5, & \text{if } x \le 2 \\ x - \lambda, & \text{if } x > 2 \end{cases}$. *Find* λ, *if* $\lim_{x \to 2} f(x)$ *exists.*

SOLUTION We have, $f(x) = \begin{cases} 4x - 5, & \text{if } x \le 2 \\ x - \lambda, & \text{if } x > 2 \end{cases}$

$$\therefore \quad \lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2 - h) = \lim_{h \to 0} 4(2 - h) - 5 = \lim_{h \to 0} 3 - 4h = 3$$

and, $$\lim_{x \to 2^+} f(x) = \lim_{h \to 0} f(2 + h) = \lim_{h \to 0} 2 + h - \lambda = 2 - \lambda$$

If $\lim_{x \to 2} f(x)$ exists, then

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) \Rightarrow 3 = 2 - \lambda \Rightarrow \lambda = -1.$$

EXAMPLE 7 If $f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \le x \le 1 \\ nx^3 + m, & x > 1 \end{cases}$. *For what values of integers m and n does the limits* $\lim_{x \to 0} f(x)$ *and* $\lim_{x \to 1} f(x)$ *exist.* **[NCERT]**

SOLUTION It is given that

$$\lim_{x \to 0} f(x) \text{ and } \lim_{x \to 1} f(x) \text{ both exist}$$

$$\Leftrightarrow \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \text{ and, } \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$$

$$\Leftrightarrow \quad \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(0+h) \text{ and, } \lim_{h \to 0} f(1-h) = \lim_{h \to 0} f(1+h)$$

$$\Leftrightarrow \quad \lim_{h \to 0} m(-h)^2 + n = \lim_{h \to 0} n(h) + m \text{ and, } \lim_{h \to 0} n(1-h) + m = \lim_{h \to 0} n(1+h)^3 + m$$

$$\Leftrightarrow \quad n = m, \text{ and } n + m = n + m \Leftrightarrow m = n$$

Hence, $\lim_{x \to 0} f(x)$ and $\lim_{x \to 1} f(x)$ both sides for $n = m$.

EXAMPLE 8 *If* $f(x) = \begin{cases} |x| + 1, & x < 0 \\ 0, & x = 0 \\ |x| - 1, & x > 0 \end{cases}$. *For what value (s) of 'a' does* $\lim_{x \to a} f(x)$ *exist?*

SOLUTION We have, [NCERT]

$$f(x) = \begin{cases} |x| + 1, & x < 0 \\ 0, & x = 0 \\ |x| - 1, & x > 0 \end{cases} \Rightarrow f(x) = \begin{cases} -x + 1, & x < 0 \\ 0, & x = 0 \\ x - 1, & x > 0 \end{cases} \qquad \left[\because |x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}\right]$$

Clearly, $\lim_{x \to a} f(x)$ exists for all $a \neq 0$. So, let us see whether $\lim_{x \to 0} f(x)$ exist or not.

We observe that:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} f(0-h) = \lim_{h \to 0} -(-h) + 1 = 1$$

and, $$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} h - 1 = -1$$

$$\therefore \quad \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

So, $\lim_{x \to 0} f(x)$ does not exist. Hence, $\lim_{x \to a} f(x)$ exists for all $a \neq 0$.

EXAMPLE 9 *Suppose* $f(x) = \begin{cases} a + bx, & x < 1 \\ 4, & x = 1 \\ b - ax, & x > 1 \end{cases}$

and, if $\lim_{x \to 1} f(x) = f(1)$. *What are possible values of a and b?* [NCERT]

SOLUTION We find that:

$$\lim_{x \to 1} f(x) = f(1)$$

$$\Leftrightarrow \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$$\Leftrightarrow \quad \lim_{x \to 1^-} f(x) = f(1) \text{ and, } \lim_{x \to 1^+} f(x) = f(1)$$

$$\Leftrightarrow \quad \lim_{h \to 0} f(1-h) = 4 \text{ and, } \lim_{h \to 0} f(1+h) = 4$$

$$\Leftrightarrow \quad \lim_{h \to 0} \left\{a + b(1-h)\right\} = 4 \text{ and, } \lim_{h \to 0} \left\{b - a(1+h)\right\} = 4$$

$$\Leftrightarrow \quad a + b = 4 \text{ and, } b - a = 4 \Leftrightarrow a = 0, b = 4$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 10 *Find the left hand and right hand limits of the greatest integer function $f(x) = [x]$ = greatest integer less than or equal to x, at $x = k$, where k is an integer. Also, show that* $\lim_{x \to k} f(x)$ *does not exist.*

SOLUTION We find that:

$$(\text{LHL at } x=k) = \lim_{x \to k^-} f(x) = \lim_{h \to 0} f(k-h) = \lim_{h \to 0} [k-h]$$

$$= \lim_{h \to 0} k-1 = k-1 \qquad [\because\ k-1 < k-h < k \ \therefore [k-h] = k-1]$$

$$(\text{RHL at } x=k) = \lim_{x \to k^+} f(x) = \lim_{h \to 0} f(k+h) = \lim_{h \to 0} [k+h]$$

$$= \lim_{h \to 0} k = k \qquad [\because\ k < k+h < k+1 \ \therefore [k+h] = k]$$

Clearly, $\lim\limits_{x \to k^-} f(x) \neq \lim\limits_{x \to k^+} f(x)$. Hence, $\lim\limits_{x \to k} f(x)$ does not exist.

EXAMPLE 11 *Prove that* $\lim\limits_{x \to a^+} [x] = [a]$ *for all* $a \in R$, *where* [.] *denotes the greatest integer function.*

SOLUTION For every real number a there exists an integer k such that $k \le a < k+1$.

$$\text{Now, } \lim_{x \to a^+} [x] = \lim_{h \to 0} [a+h] = k \qquad [\because\ k \le a < k+1 \ \therefore k \le a+h < k+1 \Rightarrow [a+h] = k]$$

$$= [a] \qquad [\because\ k \le a < k+1 \Rightarrow [a] = k]$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 12 *Show that* $\lim\limits_{x \to 0} \dfrac{e^{1/x}-1}{e^{1/x}+1}$ *does not exist.*

SOLUTION Let $f(x) = \dfrac{e^{1/x}-1}{e^{1/x}+1}$. Then,

(LHL of $f(x)$ at $x=0$)

$$= \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} \frac{e^{-1/h}-1}{e^{-1/h}+1}$$

$$= \lim_{h \to 0} \left(\frac{\dfrac{1}{e^{1/h}} - 1}{\dfrac{1}{e^{1/h}} + 1} \right) = \frac{0-1}{0+1} = -1 \qquad \left[\because\ h \to 0 \Rightarrow \frac{1}{h} \to \infty \Rightarrow e^{1/h} \to \infty \Rightarrow \frac{1}{e^{1/h}} \to 0\right]$$

and, (RHL of $f(x)$ at $x=0$)

$$= \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \frac{e^{1/h}-1}{e^{1/h}+1} = \lim_{h \to 0} \left(\frac{1 - \dfrac{1}{e^{1/h}}}{1 + \dfrac{1}{e^{1/h}}} \right) = \frac{1-0}{1+0} = 1$$

Clearly, $\lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$. Hence, $\lim\limits_{x \to 0} f(x)$ does not exist.

EXAMPLE 13 *If f is an odd function and if* $\lim\limits_{x \to 0} f(x)$ *exists. Prove that this limit must be zero.*

SOLUTION It is given that

$$\lim_{x \to 0} f(x) \text{ exists}$$

$$\Rightarrow \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$$

$$\Rightarrow \quad \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(0+h)$$

$$\Rightarrow \quad \lim_{h \to 0} f(-h) = \lim_{h \to 0} f(h)$$

$$\Rightarrow \quad -\lim_{h \to 0} f(h) = \lim_{h \to 0} f(h) \qquad [\because f(x) \text{ is odd } \therefore f(-h) = -f(h)]$$

$$\Rightarrow \quad 2\lim_{h \to 0} f(h) = 0 \Rightarrow \lim_{h \to 0} f(h) = 0 \Rightarrow \lim_{x \to 0} f(x) = 0$$

EXAMPLE 14 *If f is an even function, then prove that* $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$.

SOLUTION We find that

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} f(h) \qquad [\because f \text{ is even } \therefore f(-h) = f(h)]$$

$$= \lim_{h \to 0} f(0+h) = \lim_{x \to 0^+} f(x).$$

EXERCISE 28.1

BASIC

1. Show that $\lim_{x \to 0} \frac{x}{|x|}$ does not exist. [NCERT]

2. Find k so that $\lim_{x \to 2} f(x)$ may exist, where $f(x) = \begin{cases} 2x+3, & x \le 2 \\ x+k, & x > 2 \end{cases}$.

3. Show that $\lim_{x \to 0} \frac{1}{x}$ does not exist.

4. Let $f(x)$ be a function defined by $f(x) = \begin{cases} \dfrac{3x}{|x|+2x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$.

   Show that $\lim_{x \to 0} f(x)$ does not exist.

5. Let $f(x) = \begin{cases} x+1, & \text{if } x > 0 \\ x-1, & \text{if } x < 0 \end{cases}$. Prove that $\lim_{x \to 0} f(x)$ does not exist.

6. Let $f(x) = \begin{cases} x+5, & \text{if } x > 0 \\ x-4, & \text{if } x < 0 \end{cases}$. Prove that $\lim_{x \to 0} f(x)$ does not exist.

7. Find $\lim_{x \to 3} f(x)$, where $f(x) = \begin{cases} 4, & \text{if } x > 3 \\ x+1, & \text{if } x < 3 \end{cases}$

8. If $f(x) = \begin{cases} 2x+3, & x \le 0 \\ 3(x+1), & x > 0 \end{cases}$. Find $\lim_{x \to 0} f(x)$ and $\lim_{x \to 1} f(x)$. [NCERT]

9. Find $\lim_{x \to 1} f(x)$, if $f(x) = \begin{cases} x^2-1, & x \le 1 \\ -x^2-1, & x > 1 \end{cases}$. [NCERT]

10. Evaluate $\lim_{x \to 0} f(x)$, where $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$ [NCERT]

BASED ON LOTS

11. Let $a_1, a_2, \ldots, a_n$ be fixed real numbers such that $f(x) = (x-a_1)(x-a_2)\ldots(x-a_n)$

    What is $\lim_{x \to a_1} f(x)$? For $a \ne a_1, a_2, \ldots, a_n$ compute $\lim_{x \to a} f(x)$. [NCERT]

12. Find $\lim\limits_{x \to 1^+} \dfrac{1}{x-1}$.

13. Evaluate the following one sided limits:

(i) $\lim\limits_{x \to 2^+} \dfrac{x-3}{x^2-4}$ (ii) $\lim\limits_{x \to 2^-} \dfrac{x-3}{x^2-4}$ (iii) $\lim\limits_{x \to 0^+} \dfrac{1}{3x}$

(iv) $\lim\limits_{x \to -8^+} \dfrac{2x}{x+8}$ (v) $\lim\limits_{x \to 0^+} \dfrac{2}{x^{1/5}}$ (vi) $\lim\limits_{x \to \frac{\pi}{2}^-} \tan x$

(vii) $\lim\limits_{x \to -\pi/2^+} \sec x$ (viii) $\lim\limits_{x \to 0^-} \dfrac{x^2-3x+2}{x^3-2x^2}$ (ix) $\lim\limits_{x \to -2^+} \dfrac{x^2-1}{2x+4}$

(x) $\lim\limits_{x \to 0^-} (2 - \cot x)$ (xi) $\lim\limits_{x \to 0^-} 1 + \operatorname{cosec} x$

BASED ON HOTS

14. Show that $\lim\limits_{x \to 0} e^{-1/x}$ does not exist.

15. Find: (i) $\lim\limits_{x \to 2} [x]$ (ii) $\lim\limits_{x \to \frac{5}{2}} [x]$ (iii) $\lim\limits_{x \to 1} [x]$

16. Prove that $\lim\limits_{x \to a^+} [x] = [a]$ for all $a \in R$. Also, prove that $\lim\limits_{x \to 1^-} [x] = 0$.

17. Show that $\lim\limits_{x \to 2^-} \dfrac{x}{[x]} \neq \lim\limits_{x \to 2^+} \dfrac{x}{[x]}$.

18. Find $\lim\limits_{x \to 3^+} \dfrac{x}{[x]}$. Is it equal to $\lim\limits_{x \to 3^-} \dfrac{x}{[x]}$.

19. Find $\lim\limits_{x \to -5/2} [x]$.

20. Evaluate $\lim\limits_{x \to 2} f(x)$ (if it exists), where $f(x) = \begin{cases} x - [x] &, \quad x < 2 \\ 4 &, \quad x = 2 \\ 3x - 5 &, \quad x > 2 \end{cases}$.

21. Show that $\lim\limits_{x \to 0} \sin \dfrac{1}{x}$ does not exist.

22. Let $f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{where } x \neq \dfrac{\pi}{2} \\ 3, & \text{where } x = \dfrac{\pi}{2} \end{cases}$ and if $\lim\limits_{x \to \pi/2} f(x) = f\left(\dfrac{\pi}{2}\right)$, find the value of k.

ANSWERS

2. $k = 5$ 7. 4 8. 3, 6 9. Does not exist 10. Does not exist

11. $0, (a - a_1)(a - a_2) \ldots (a - a_n)$ 12. ∞

13. (i) $-\infty$ (ii) ∞ (iii) ∞ (iv) $-\infty$ (v) ∞ (vi) ∞
(vii) ∞ (viii) $-\infty$ (ix) ∞ (x) ∞ (xi) $-\infty$

15. (i) Does not exist (ii) 2 (iii) Does not exist 18. 1, No

19. -3 20. 1 22. $k = 6$

HINTS TO SELECTED PROBLEMS

1. $\lim\limits_{x \to 0^-} \dfrac{x}{|x|} = \lim\limits_{x \to 0^-} \dfrac{x}{-x} = -1$ and, $\lim\limits_{x \to 0^+} = \dfrac{x}{|x|} = \lim\limits_{x \to 0^+} \dfrac{x}{x} = 1$

$\therefore \quad \lim\limits_{x \to 0^-} \dfrac{x}{|x|} \neq \lim\limits_{x \to 0^+} \dfrac{x}{|x|}$. Hence, $\lim\limits_{x \to 0} \dfrac{x}{|x|}$ does not exist.

3. $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{h \to 0} f(0-h) = \lim\limits_{h \to 0} \dfrac{1}{-h} = -\infty$ and, $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{h \to 0} f(0+h) = \lim\limits_{h \to 0} \dfrac{1}{h} = \infty$

Clearly, $\lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$. Hence, $\lim\limits_{x \to 0} f(x)$ does not exist.

8. We have, $f(x) = \begin{cases} 2x+3, & x \le 0 \\ 3(x+1) & x \ge 0 \end{cases}$

$\therefore \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (2x+3) = 2 \times 0 + 3 = 3$ and, $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} 3(x+1) = 3(0+1) = 3$

So, $\lim\limits_{x \to 0} f(x)$ exists and is equal to 3.

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} 2x+3 = 2 \times 1 + 3 = 5$, $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} 3(x+1) = 3(1+1) = 6.$

$\therefore \quad \lim\limits_{x \to 1^-} f(x) \neq \lim\limits_{x \to 1^+} f(x)$. Hence, $\lim\limits_{x \to 1} f(x)$ does not exists.

9. We have, $f(x) = \begin{cases} x^2-1, & x \le 1 \\ -x^2-1, & x \ge 1 \end{cases}$

$\because \quad \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x^2 - 1 = 1^2 - 1 = 0$ and, $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} -x^2 - 1 = -1 - 1 = -2$

$\therefore \quad \lim\limits_{x \to 1^-} f(x) \neq \lim\limits_{x \to 1^+} f(x)$. Hence, $\lim\limits_{x \to 1} f(x)$ does not exist.

10. We have, $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$\because \quad \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \dfrac{|x|}{x} = \lim\limits_{x \to 0^-} \dfrac{-x}{x} = -1$ and, $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \dfrac{|x|}{x} = \lim\limits_{x \to 0} \dfrac{x}{x} = 1.$

$\therefore \quad \lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$. Hence, $\lim\limits_{x \to 0} f(x)$ does not exist.

11. We have, $f(x) = (x-a_1)(x-a_2)\ldots(x-a_n)$

$\therefore \lim\limits_{x \to a_1^-} f(x) = \lim\limits_{h \to 0} f(a_1 - h) = \lim\limits_{h \to 0} -h(a_1 - h - a_2)(a_1 - h - a_3)\ldots(a_1 - h - a_n)$

$= 0(a_1 - a_2)(a_1 - a_3)\ldots.(a_1 - a_n) = 0$

$\lim\limits_{x \to a_1^+} f(x) = \lim\limits_{h \to 0} (a_1 + h - a_1)(a_1 + h - a_2)(a_1 + h - a_3)\ldots(a_1 + h - a_n)$

$$= \lim_{h \to 0} h(a_1 + h - a_2)(a_1 + h - a_3) \ldots (a_1 + h - a_n) = 0\,(a_1 - a_2)(a_1 - a_3) \ldots (a_1 - a_n) = 0$$

$\therefore \lim_{x \to a_1^-} f(x) = \lim_{x \to a_1^+} f(x) = 0$. Hence, $\lim_{x \to a_1} f(x) = 0$.

For any $a \neq a_1, a_2, \ldots a_n$, we find that

$$\lim_{x \to a} f(x) = \lim_{x \to a} (x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n) = (a - a_1)(a - a_2)(a - a_3) \ldots (a - a_n)$$

12. $\lim_{x \to 1^+} \frac{1}{x-1} = \lim_{h \to 0} \frac{1}{1+h-1} = \lim_{h \to 0} \frac{1}{h} = \infty.$

14. We find that: $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} e^{1/h} = \infty$

and, $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} e^{-1/h} = \lim_{h \to 0} \frac{1}{e^{1/h}} = 0$

21. We we find that:

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h - 0} \sin\left(\frac{1}{-h}\right) = -\lim_{h \to 0} \sin \frac{1}{h}$$

$\Rightarrow \lim_{x \to 0^-} f(x) = -$(An oscillating number which oscillates between -1 and 1).

So, $\lim_{x \to 0^-} f(x)$ does not exist. Similarly, $\lim_{x \to 0^+} f(x)$ does not exist.

28.3 DIFFERENCE BETWEEN THE VALUE OF A FUNCTION AT A POINT AND THE LIMIT AT THAT POINT

Let $f(x)$ be a function and let a be a point. Then, we have the following possibilities:

(i) $\lim_{x \to a} f(x)$ *exists but* $f(a)$ *(the value of* $f(x)$ *at* $x = a$*) does not exist*:

Consider the function $f(x)$ defined by $f(x) = \frac{x^2 - 9}{x - 3}$. Clearly, this function is not defined at $x = 3$ i.e. $f(3)$ does not exist, because it attains the form $\frac{0}{0}$. But, it can be easily seen that $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 6$. So, $\lim_{x \to 3} f(x)$ exists. Thus, the $\lim_{x \to 3} f(x)$ exists but $f(3)$ does not exist.

(ii) *The value* $f(a)$ *exists but* $\lim_{x \to a} f(x)$ *does not exist:* In Example 10 on page 28.7, we have seen that $\lim_{x \to k} f(x)$ does not exist but $f(k) = k$ exists.

(iii) $\lim_{x \to a} f(x)$ *and* $f(a)$ *both exist but are unequal*: Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x \neq 2 \\ 3, & x = 2 \end{cases}$$

It can be easily seen that $\lim_{x \to 2^-} f(x) = 4 = \lim_{x \to 2^+} f(x)$. So, $\lim_{x \to 2} f(x)$ exists and is equal to 4. Also, the value $f(2)$ exists and is equal to 3.

Thus, $\lim\limits_{x \to 2} f(x)$ and $f(2)$ both exist but are unequal.

(iv) $\lim\limits_{x \to a} f(x)$ *and* $f(a)$ *both exist and are equal*: Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} &, x \neq 2 \\ 4 &, x = 2 \end{cases}$$

For this function, it can be easily seen that $\lim\limits_{x \to 2} f(x)$ and $f(2)$ both exist and are equal to 4.

28.4 THE ALGEBRA OF LIMITS

Let f and g be two real functions with common domain D. In the chapter on functions, we have defined four new function $f \pm g, fg, f/g$ on domain D by setting

(i) $(f \pm g)(x) = f(x) \pm g(x)$, (ii) $(fg)(x) = f(x)\, g(x)$

(iii) $(f/g)(x) = f(x)/g(x)$, if $g(x) \neq 0$ for any $x \in D$.

Following are some results concerning the limits of these functions.

Let $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$. If l and m exist, then

(i) $\lim\limits_{x \to a} (f \pm g)(x) = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x) = l \pm m$

(ii) $\lim\limits_{x \to a} (fg)(x) = \lim\limits_{x \to a} f(x) \lim\limits_{x \to a} g(x) = lm$

(iii) $\lim\limits_{x \to a} \left(\dfrac{f}{g}\right)(x) = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \dfrac{l}{m}$, provided $m \neq 0$.

(iv) $\lim\limits_{x \to a} k\, f(x) = k \lim\limits_{x \to a} f(x)$, where k is constant

(v) $\lim\limits_{x \to a} |f(x)| = |\lim\limits_{x \to a} f(x)| = |l|$ (vi) $\lim\limits_{x \to a} \left\{f(x)\right\}^{g(x)} = l^m$

(vii) If $f(x) \leq g(x)$ for every x in the deleted neighbourhood of a, then $\lim\limits_{x \to a} f(x) \leq \lim\limits_{x \to a} g(x)$.

(viii) If $f(x) \leq g(x) \leq h(x)$ for every x in the deleted neighbourhood of a and

$\lim\limits_{x \to a} f(x) = l = \lim\limits_{x \to a} h(x)$, then $\lim\limits_{x \to a} g(x) = l$.

This result is often stated as *Sandwich Theorem*.

(ix) If $\lim\limits_{x \to a} f(x) = +\infty$ or $-\infty$, then $\lim\limits_{x \to a} \dfrac{1}{f(x)} = 0$.

28.5 INDETERMINATE FORMS AND EVALUATION OF LIMITS

Uptill now we have been discussing left hand and right hand limits and the existence of limits. In what follows, we will be assuming that the limit of a function at a given point exists. In the previous section, we have stated that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}, \text{ provided that } \lim_{x \to a} g(x) \neq 0.$$

An interesting situation now arises. If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)}$ takes the form $\frac{0}{0}$, which is undefined or meaningless. But, this does not imply that $\lim_{x \to a} \frac{f(x)}{g(x)}$ is meaningless or it does not exist. In fact, in many cases this limit exists and has a finite value. The determination of limit in such a case is traditionally referred to as the evaluation of the indeterminate form $\frac{0}{0}$, though literally speaking nothing is indeterminate involved here. Sometimes $\frac{0}{0}$ is referred to as undetermined form or illusory form. In addition to $\frac{0}{0}$ there are six other indeterminate forms, namely, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0, ∞^0 and 1^∞. Among all these seven indeterminate forms $\frac{0}{0}$ is the fundamental one because all the remaining six forms can easily be reduced to this form. In this chapter, we shall study how to evaluate a limit which belongs to one of following in determinate forms:

$\frac{0}{0}$, $0 \times \infty$ and $\infty - \infty$.

To facilitate the job of evaluation of limits we categorize problems on limits in the following categories:

(i) Algebraic Limits. (ii) Non-algebraic Limits.

If a problem on limits does not involve trigonometric, inverse trigonometric, exponential and logarithmic function, then it is a problem on algebraic limits, otherwise, it is a problem on non-algebraic limits.

For example,

(i) $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$ (ii) $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$ (iii) $\lim_{x \to 2} \frac{x^5 - 32}{x - 2}$

(iv) $\lim_{x \to \infty} \frac{x^2 + x + 1}{2x^2 + 5}$ etc. are problems on algebraic limits.

Following are some examples of non-algebraic limits:

(i) $\lim_{x \to 0} \frac{e^{\sin x} - 1}{x}$ (ii) $\lim_{x \to 0} \frac{3 \sin^{-1} 2x}{\sin x}$ (iii) $\lim_{x \to \pi/4} \frac{\sin x - \cos x}{4x - \pi}$ (iv) $\lim_{x \to 0} \frac{2^x - 3^x}{x}$

28.6 EVALUATION OF ALGEBRAIC LIMITS

In order to evaluate algebraic limits we have the following methods.

(i) Direct substitution method. (ii) Factorisation method.

(iii) Rationalisation method. (iv) By using some standard limits.

(v) Method of evaluation of algebraic limits at infinity.

Let us now discuss these methods with suitable illustrations in the following sub- sections.

28.6.1 DIRECT SUBSTITUTION METHOD

Consider the limits: $\lim_{x \to a} f(x)$ and $\lim_{x \to a} \frac{\Phi(x)}{\psi(x)}$. If $f(a)$ and $\frac{\varphi(a)}{\psi(a)}$ exist and are fixed real numbers, then we say that

$$\lim_{x \to a} f(x) = f(a) \text{ and } \lim_{x \to a} \frac{\varphi(x)}{\psi(a)} = \frac{\phi(a)}{\psi(a)}$$

In other words, if the direct substitution of the point, to which the variable tends to, we obtain a fixed real number, then the number obtained is the limit of the function. In fact, if the point to which the variable tends to is a point in the domain of the function, then the value of the function at that point is its limit.

Following examples will illustrate the above method.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate :* $\lim_{x \to 1} 3x^2 + 4x + 5.$

SOLUTION $\lim_{x \to 1} 3x^2 + 4x + 5 = 3(1)^2 + 4(1) + 5 = 12.$

EXAMPLE 2 *Evaluate:* $\lim_{x \to 2} \frac{x^2 - 4}{x + 3}$

SOLUTION Using direct substitution method, we find that:

$$\lim_{x \to 2} \frac{x^2 - 4}{x + 3} = \frac{4 - 4}{2 + 3} = \frac{0}{5} = 0.$$

EXAMPLE 3 *Evaluate:* $\lim_{x \to 0} \frac{\sqrt{1 + x} + \sqrt{1 - x}}{1 + x}$

SOLUTION Using direct substitution method, we obtain

$$\lim_{x \to 0} \frac{\sqrt{1 + x} + \sqrt{1 - x}}{1 + x} = \frac{\sqrt{1 + 0} + \sqrt{1 - 0}}{1 + 0} = \frac{1 + 1}{1} = 2.$$

EXERCISE 28.2

BASIC

Evaluate the following limits:

1. $\lim_{x \to 1} \frac{x^2 + 1}{x + 1}$
2. $\lim_{x \to 0} \frac{2x^2 + 3x + 4}{x^2 + 3x + 2}$
3. $\lim_{x \to 3} \frac{\sqrt{2x + 3}}{x + 3}$
4. $\lim_{x \to 1} \frac{\sqrt{x + 8}}{\sqrt{x}}$
5. $\lim_{x \to a} \frac{\sqrt{x} + \sqrt{a}}{x + a}$
6. $\lim_{x \to 1} \frac{1 + (x - 1)^2}{1 + x^2}$
7. $\lim_{x \to 0} \frac{x^{2/3} - 9}{x - 27}$
8. $\lim_{x \to 0} 9$
9. $\lim_{x \to 2} (3 - x)$
10. $\lim_{x \to -1} (4x^2 + 2)$
11. $\lim_{x \to -1} \frac{x^3 - 3x + 1}{x - 1}$
12. $\lim_{x \to 0} \frac{3x + 1}{x + 3}$
13. $\lim_{x \to 3} \frac{x^2 - 9}{x + 2}$
14. $\lim_{x \to 0} \frac{ax + b}{cx + d}, d \neq 0$

ANSWERS

1. 1 2. 2 3. $\frac{1}{2}$ 4. 3 5. $\frac{1}{\sqrt{a}}$ 6. $\frac{1}{2}$ 7. $\frac{1}{3}$ 8. 9
9. 1 10. 6 11. $\frac{-3}{2}$ 12. $\frac{1}{3}$ 13. 0 14. $\frac{b}{d}$

28.6.2 FACTORIZATION METHOD

Consider the limit: $\lim_{x \to a} \frac{f(x)}{g(x)}$. If by substituting $x = a$, $\frac{f(x)}{g(x)}$, reduces to the form $\frac{0}{0}$, then $(x - a)$ is a factor of $f(x)$ and $g(x)$ both. So, we first factorize $f(x)$ and $g(x)$ and then cancel out the common factor to evaluate the limit.

Following algorithm may be used to evaluate the limit by factorization method.

ALGORITHM

Step I *Obtain the problem, say,* $\lim_{x \to a} \frac{f(x)}{g(x)}$, *where* $\lim_{x \to a} f(x) = 0$ *and* $\lim_{x \to a} g(x) = 0$.

Step II *Factorize $f(x)$ and $g(x)$.*

Step III *Cancel out the common factor(s) of $f(x)$ and $g(x)$.*

Step IV *Use direct substitution method to obtain the limit.*

Some useful results to remember:

(i) $a^2 - b^2 = (a-b)(a+b)$ (ii) $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

(iii) $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ (iv) $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2) = (a+b)(a-b)(a^2 + b^2)$

(v) If $f(\alpha) = 0$, then $x - \alpha$ is a factor of $f(x)$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:* $\lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - 4}$.

SOLUTION When $x = 2$ the expression $\frac{x^2 - 5x + 6}{x^2 - 4}$ assumes the indeterminate form $\frac{0}{0}$. Therefore, $(x - 2)$ is a common factor in numerator and denominator. Factorising the numerator and denominator, we obtain

$$\lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - 4} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 2} \frac{(x-2)(x-3)}{(x+2)(x-2)} = \lim_{x \to 2} \frac{x-3}{x+2} = \frac{2-3}{2+2} = -\frac{1}{4}.$$

EXAMPLE 2 *Evaluate:* $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$.

SOLUTION When $x = 1$ the expression $\frac{x^3 - 1}{x - 1}$ assumes the indeterminate form $\frac{0}{0}$. Therefore, $(x - 1)$ is a common factor in numerator and denominator. Factorising the numerator and denominator, we obtain

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 1} \frac{(x-1)(x^2+x+1)}{(x-1)} = \lim_{x \to 1} x^2 + x + 1 = 1^2 + 1 + 1 = 3.$$

EXAMPLE 3 *Evaluate:* $\lim_{x \to 2} \dfrac{x^3 - 3x^2 + 4}{x^4 - 8x^2 + 16}$.

SOLUTION When $x = 2$, the expression $\dfrac{x^3 - 3x^2 + 4}{x^4 - 8x^2 + 16}$ assumes the indeterminate form $\dfrac{0}{0}$. Therefore, $(x - 2)$ is a factor common to numerator and denominator. Factorising the numerator and denominator, we obtain

$$\lim_{x \to 2} \frac{x^3 - 3x^2 + 4}{x^4 - 8x^2 + 16} = \lim_{x \to 2} \frac{(x-2)(x^2 - x - 2)}{(x^2 - 4)^2}$$

$$= \lim_{x \to 2} \frac{(x-2)(x-2)(x+1)}{(x-2)^2 (x+2)^2} = \lim_{x \to 2} \frac{x+1}{(x+2)^2} = \frac{2+1}{(2+2)^2} = \frac{3}{16}$$

EXAMPLE 4 *Evaluate:* $\lim_{x \to 2} \dfrac{x^3 - 6x^2 + 11x - 6}{x^2 - 6x + 8}$.

SOLUTION When $x = 2$, the expression $\dfrac{x^3 - 6x^2 + 11x - 6}{x^2 - 6x + 8}$ assumes the form $\dfrac{0}{0}$. Therefore, $(x - 2)$ is a factor common to numerator and denominator. Factorising the numerator and denominator, we get

$$\lim_{x \to 2} \frac{x^3 - 6x^2 + 11x - 6}{x^2 - 6x + 8}$$

$$= \lim_{x \to 2} \frac{(x-1)(x-2)(x-3)}{(x-2)(x-4)} = \lim_{x \to 2} \frac{(x-1)(x-3)}{(x-4)} = \frac{(2-1)(2-3)}{(2-4)} = \frac{1}{2}.$$

EXAMPLE 5 *Evaluate:* $\lim_{x \to 1/2} \dfrac{8x^3 - 1}{16x^4 - 1}$.

SOLUTION When $x = 1/2$, the expression $\dfrac{8x^3 - 1}{16x^4 - 1}$ assumes the form $\dfrac{0}{0}$. Therefore, $\left(x - \dfrac{1}{2}\right)$ or, $2x - 1$ is a factor common to numerator and denominator. Factorising the numerator and denominator, we obtain

$$\lim_{x \to 1/2} \frac{8x^3 - 1}{16x^4 - 1} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 1/2} \frac{(2x)^3 - 1^3}{(4x^2)^2 - 1^2} = \lim_{x \to 1/2} \frac{(2x-1)(4x^2 + 2x + 1)}{(4x^2+1)(4x^2-1)} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 1/2} \frac{(2x-1)(4x^2 + 2x + 1)}{(4x^2+1)(2x-1)(2x+1)} = \lim_{x \to 1/2} \frac{4x^2 + 2x + 1}{(4x^2+1)(2x+1)} = \frac{3}{4}.$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 6 *Evaluate:* $\lim_{x \to 1} \left(\frac{2}{1-x^2} + \frac{1}{x-1}\right)$.

SOLUTION We have,

$$\lim_{x \to 1} \left(\frac{2}{1-x^2} + \frac{1}{x-1}\right) = \lim_{x \to 1} \left(\frac{2}{1-x^2} - \frac{1}{1-x}\right)$$

When $x = 1$, the expression $\frac{2}{1-x^2} - \frac{1}{1-x}$ assumes the form $\infty - \infty$. So, we need some simplification to express it in the form $\frac{0}{0}$. Taking LCM, we get

$$\lim_{x \to 1}\left(\frac{2}{1-x^2} - \frac{1}{1-x}\right) \qquad (\infty - \infty \text{ form})$$

$$= \lim_{x \to 1} \frac{2-(1+x)}{1-x^2} = \lim_{x \to 1} \frac{1-x}{1-x^2} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 1} \frac{1}{1+x} = \frac{1}{2}.$$

EXAMPLE 7 *Evaluate:* $\lim_{x \to 1} \left(\frac{1}{x^2+x-2} - \frac{x}{x^3-1}\right)$.

SOLUTION When $x = 1$, the expression $\frac{1}{x^2+x-2} - \frac{x}{x^3-1}$ assumes the indeterminate form $\infty - \infty$. So, we need simplification to reduce the expression in the indeterminate form $\frac{0}{0}$.

$$\lim_{x \to 1} \left(\frac{1}{x^2+x-2} - \frac{x}{x^3-1}\right) \qquad (\infty - \infty \text{ form})$$

$$= \lim_{x \to 1} \left\{\frac{1}{(x+2)(x-1)} - \frac{x}{(x-1)(x^2+x+1)}\right\} \qquad (\infty - \infty \text{ form})$$

$$= \lim_{x \to 1} \left\{\frac{(x^2+x+1) - x(x+2)}{(x+2)(x-1)(x^2+x+1)}\right\} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 1} \frac{-(x-1)}{(x+2)(x-1)(x^2+x+1)} = \lim_{x \to 1} \frac{-1}{(x+2)(x^2+x+1)} = -\frac{1}{9}.$$

EXAMPLE 8 *Evaluate:* $\lim_{x \to \sqrt{2}} \frac{x^4-4}{x^2+3x\sqrt{2}-8}$. **[NCERT EXEMPLAR]**

SOLUTION When $x = \sqrt{2}$, the expression $\frac{x^4-4}{x^2+3x\sqrt{2}-8}$ assumes the indeterminate form $\frac{0}{0}$. So, $(x-\sqrt{2})$ is a factor of numerator and denominator. Factorising the numerator and denominator, we get

$$\lim_{x \to \sqrt{2}} \frac{x^4-4}{x^2+3x\sqrt{2}-8} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to \sqrt{2}} \frac{(x^2-2)(x^2+2)}{(x+4\sqrt{2})(x-\sqrt{2})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to \sqrt{2}} \frac{(x-\sqrt{2})(x+\sqrt{2})(x^2+2)}{(x+4\sqrt{2})(x-\sqrt{2})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to \sqrt{2}} \frac{(x+\sqrt{2})(x^2+2)}{(x+4\sqrt{2})} = \frac{(2\sqrt{2})(2+2)}{5\sqrt{2}} = \frac{8}{5}.$$

EXAMPLE 9 *Evaluate:* $\lim_{x \to 4} \frac{(x^2-x-12)^{18}}{(x^3-8x^2+16x)^9}$.

SOLUTION When $x = 4$, the expression $\frac{(x^2-x-12)^{18}}{(x^3-8x^2+16x)^9}$ assumes the form $\frac{0}{0}$. So, $(x-4)$ is a common factor in numerator and denominator. Factorising the numerator and denominator, we get

$$\lim_{x \to 4} \frac{(x^2-x-12)^{18}}{(x^3-8x^2+16x)^9}$$

$$= \lim_{x \to 4} \frac{[(x-4)(x+3)]^{18}}{[x(x^2-8x+16)]^9} = \lim_{x \to 4} \frac{[(x-4)(x+3)]^{18}}{x^9(x-4)^{18}}$$

$$= \lim_{x \to 4} \frac{(x-4)^{18}(x+3)^{18}}{x^9(x-4)^{18}} = \lim_{x \to 4} \frac{(x+3)^{18}}{x^9} = \frac{7^{18}}{4^9}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 10 *Evaluate:* $\lim_{x \to 3} \frac{x^3-7x^2+15x-9}{x^4-5x^3+27x-27}$.

SOLUTION When $x = 3$, the expression $\frac{x^3-7x^2+15x-9}{x^4-5x^3+27x-27}$ assumes the form $\frac{0}{0}$. So, $(x-3)$ is a factor of numerator and denominator. Factorising the numerator and denominator, we get

$$\lim_{x \to 3} \frac{x^3-7x^2+15x-9}{x^4-5x^3+27x-27} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 3} \frac{(x-3)(x^2-4x+3)}{(x-3)(x^3-2x^2-6x+9)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 3} \frac{x^2-4x+3}{x^3-2x^2-6x+9} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 3} \frac{(x-3)(x-1)}{(x-3)(x^2+x-3)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 3} \frac{x-1}{x^2+x-3} = \frac{3-1}{9+3-3} = \frac{2}{9}.$$

EXAMPLE 11 *Evaluate:* $\lim_{x \to \sqrt{2}} \frac{x^9-3x^8+x^6-9x^4-4x^2-16x+84}{x^5-3x^4-4x+12}$

SOLUTION When $x = \sqrt{2}$, the expression $\dfrac{x^9 - 3x^8 + x^6 - 9x^4 - 4x^2 - 16x + 84}{x^5 - 3x^4 - 4x + 12}$ assume the form $\dfrac{0}{0}$. Therefore, $(x - \sqrt{2})$ is a factor of numerator and denominator. But, irrational roots occur in pairs. So, $(x + \sqrt{2})$ will also be a factor of both numerator and denominator, consequently, $(x^2 - 2)$ will be a common factor of numerator and denominator, Dividing numerator and denominator by $(x^2 - 2)$, we get

$$\lim_{x \to \sqrt{2}} \frac{x^9 - 3x^8 + x^6 - 9x^4 - 4x^2 - 16x + 84}{x^5 - 3x^4 - 4x + 12}$$

$$= \lim_{x \to \sqrt{2}} \frac{(x^2 - 2)\ (x^7 - 3x^6 + 2x^5 - 5x^4 + 4x^3 - 19x^2 + 8x - 42)}{(x^2 - 2)\,(x^3 - 3x^2 + 2x - 6)}$$

$$= \lim_{x \to \sqrt{2}} \frac{x^7 - 3x^6 + 2x^5 - 5x^4 + 4x^3 - 19x^2 + 8x - 42}{x^3 - 3x^2 + 2x - 6}$$

$$= \frac{8\sqrt{2} - 24 + 8\sqrt{2} - 20 + 8\sqrt{2} - 38 + 8\sqrt{2} - 42}{2\sqrt{2} - 6 + 2\sqrt{2} - 6} = \frac{8\sqrt{2} - 31}{\sqrt{2} - 3}.$$

EXERCISE 28.3

BASIC

Evaluate the following limits:

1. $\lim\limits_{x \to -5} \dfrac{2x^2 + 9x - 5}{x + 5}$
2. $\lim\limits_{x \to 3} \dfrac{x^2 - 4x + 3}{x^2 - 2x - 3}$
3. $\lim\limits_{x \to 3} \dfrac{x^4 - 81}{x^2 - 9}$
4. $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x^2 - 4}$
5. $\lim\limits_{x \to -1/2} \dfrac{8x^3 + 1}{2x + 1}$
6. $\lim\limits_{x \to 4} \dfrac{x^2 - 7x + 12}{x^2 - 3x - 4}$
7. $\lim\limits_{x \to 2} \dfrac{x^4 - 16}{x - 2}$
8. $\lim\limits_{x \to 5} \dfrac{x^2 - 9x + 20}{x^2 - 6x + 5}$
9. $\lim\limits_{x \to -1} \dfrac{x^3 + 1}{x + 1}$
10. $\lim\limits_{x \to 5} \dfrac{x^3 - 125}{x^2 - 7x + 10}$

BASED ON LOTS

11. $\lim\limits_{x \to \sqrt{2}} \dfrac{x^2 - 2}{x^2 + \sqrt{2}x - 4}$
12. $\lim\limits_{x \to \sqrt{3}} \dfrac{x^2 - 3}{x^2 + 3\sqrt{3}\,x - 12}$
13. $\lim\limits_{x \to \sqrt{3}} \dfrac{x^4 - 9}{x^2 + 4\sqrt{3}x - 15}$
14. $\lim\limits_{x \to 2} \left(\dfrac{x}{x - 2} - \dfrac{4}{x^2 - 2x} \right)$
15. $\lim\limits_{x \to 1} \left(\dfrac{1}{x^2 + x - 2} - \dfrac{x}{x^3 - 1} \right)$
16. $\lim\limits_{x \to 3} \left(\dfrac{1}{x - 3} - \dfrac{2}{x^2 - 4x + 3} \right)$
17. $\lim\limits_{x \to 2} \left(\dfrac{1}{x - 2} - \dfrac{2}{x^2 - 2x} \right)$
18. $\lim\limits_{x \to 1/4} \dfrac{4x - 1}{2\sqrt{x} - 1}$
19. $\lim\limits_{x \to 4} \dfrac{x^2 - 16}{\sqrt{x} - 2}$
20. $\lim\limits_{x \to 0} \dfrac{(a + x)^2 - a^2}{x}$
21. $\lim\limits_{x \to 2} \left(\dfrac{1}{x - 2} - \dfrac{4}{x^3 - 2x^2} \right)$

22. $\lim_{x \to 3} \left(\frac{1}{x-3} - \frac{3}{x^2-3x} \right)$ 23. $\lim_{x \to 1} \left(\frac{1}{x-1} - \frac{2}{x^2-1} \right)$ 24. $\lim_{x \to 3} (x^2-9)\left(\frac{1}{x+3} + \frac{1}{x-3} \right)$

BASED ON HOTS

25. $\lim_{x \to 1} \frac{x^4 - 3x^3 + 2}{x^3 - 5x^2 + 3x + 1}$ 26. $\lim_{x \to 2} \frac{x^3 + 3x^2 - 9x - 2}{x^3 - x - 6}$ 27. $\lim_{x \to 1} \frac{1 - x^{-1/3}}{1 - x^{-2/3}}$

28. $\lim_{x \to 3} \frac{x^2 - x - 6}{x^3 - 3x^2 + x - 3}$ 29. $\lim_{x \to -2} \frac{x^3 + x^2 + 4x + 12}{x^3 - 3x + 2}$ 30. $\lim_{x \to 1} \frac{x^3 + 3x^2 - 6x + 2}{x^3 + 3x^2 - 3x - 1}$

31. $\lim_{x \to 2} \left\{ \frac{1}{x-2} - \frac{2(2x-3)}{x^3 - 3x^2 + 2x} \right\}$ [NCERT EXEMPLAR] 32. $\lim_{x \to 1} \frac{\sqrt{x^2-1} + \sqrt{x-1}}{\sqrt{x^2-1}}, x > 1$

33. $\lim_{x \to 1} \left\{ \frac{x-2}{x^2-x} - \frac{1}{x^3 - 3x^2 + 2x} \right\}$ [NCERT] 34. $\lim_{x \to 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$ [NCERT EXEMPLAR]

ANSWERS

1. -11 2. $\frac{1}{2}$ 3. 18 4. 3 5. 3 6. $\frac{1}{5}$ 7. 32 8. $\frac{1}{4}$
9. 3 10. 25 11. $\frac{2}{3}$ 12. $\frac{2}{5}$ 13. 2 14. 2 15. $-\frac{1}{9}$ 16. $\frac{1}{2}$
17. $\frac{1}{2}$ 18. 2 19. 32 20. $2a$ 21. 1 22. $\frac{1}{3}$ 23. $\frac{1}{2}$ 24. 6
25. $\frac{5}{4}$ 26. $\frac{15}{11}$ 27. $\frac{1}{2}$ 28. $\frac{1}{2}$ 29. $\frac{4}{3}$ 30. $\frac{1}{2}$ 31. $-\frac{1}{2}$ 32. $\frac{\sqrt{2}+1}{\sqrt{2}}$
33. 2 34. 1

HINTS TO SELECTED PROBLEMS

31. $\lim_{x \to 2} \left\{ \frac{1}{x-2} - \frac{2(2x-3)}{x(x^2-3x+2)} \right\} = \lim_{x \to 2} \left\{ \frac{1}{x-2} - \frac{2(2x-3)}{x(x-1)(x-2)} \right\}$

$= \lim_{x \to 2} \left\{ \frac{x(x-1) - 2(2x-3)}{x(x-1)(x-2)} \right\} = \lim_{x \to 2} \left\{ \frac{x^2 - 5x + 6}{x(x-1)(x-2)} \right\} = \lim_{x \to 2} \frac{(x-3)(x-2)}{x(x-1)(x-2)}$

$= \lim_{x \to 2} \frac{x-3}{x(x-1)} = -\frac{1}{2}$

33. $\lim_{x \to 1} \left(\frac{x-2}{x^2-x} - \frac{1}{x^3 - 3x^2 + 2x} \right) = \lim_{x \to 1} \left\{ \frac{x-2}{x(x-1)} - \frac{1}{x(x-1)(x-2)} \right\}$

$= \lim_{x \to 1} \frac{(x-2)^2 - 1^2}{x(x-1)(x-2)} = \lim_{x \to 1} \frac{(x-3)(x-1)}{x(x-1)(x-2)} = \lim_{x \to 1} \frac{x-3}{x(x-2)} = \frac{-2}{-1} = 2$

34. $\lim_{x \to 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2} = \lim_{x \to 1} \frac{(x-1)(x^6 + x^5 - x^4 - x^3 - x^2 - x - 1)}{(x-1)(x^2 - 2x - 2)}$

$= \lim_{x \to 1} \frac{(x^6 + x^5 - x^4 - x^3 - x^2 - x - 1)}{x^2 - 2x - 2} = \frac{-3}{-3} = 1$

28.6.3 RATIONALISATION METHOD

This is particularly used when either the numerator or denominator or both involve expression consisting of square roots and substituting the value of x the rational expression takes the form $\frac{0}{0}$, $\frac{\infty}{\infty}$ etc. Following examples illustrate the procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:* $\lim\limits_{x \to 0} \frac{\sqrt{2+x} - \sqrt{2}}{x}$.

SOLUTION When $x = 0$, the expression $\frac{\sqrt{2+x} - \sqrt{2}}{x}$ takes the form $\frac{0}{0}$. Rationalising the numerator, we get

$$\lim_{x \to 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} = \lim_{x \to 0} \frac{(\sqrt{2+x} - \sqrt{2})(\sqrt{2+x} + \sqrt{2})}{x(\sqrt{2+x} + \sqrt{2})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2 + x - 2}{x(\sqrt{2+x} + \sqrt{2})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{2+x} + \sqrt{2}} = \frac{1}{2\sqrt{2}}.$$

EXAMPLE 2 *Evaluate:* $\lim\limits_{x \to 0} \frac{x}{\sqrt{a+x} - \sqrt{a-x}}$.

SOLUTION When $x = 0$, the expression $\frac{x}{\sqrt{a+x} - \sqrt{a-x}}$ takes the form $\frac{0}{0}$. Rationalising the denominator, we get

$$\lim_{x \to 0} \frac{x}{\sqrt{a+x} - \sqrt{a-x}} = \lim_{x \to 0} \frac{x}{(\sqrt{a+x} - \sqrt{a-x})} \times \frac{(\sqrt{a+x} + \sqrt{a-x})}{(\sqrt{a+x} + \sqrt{a-x})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{x(\sqrt{a+x} + \sqrt{a-x})}{(a + x - a + x)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{(\sqrt{a} + x + \sqrt{a-x})}{2} = \frac{2\sqrt{a}}{2} = \sqrt{a}$$

EXAMPLE 3 *Evaluate:* $\lim\limits_{x \to 0} \frac{\sqrt{a^2+x^2} - \sqrt{a^2-x^2}}{x^2}$.

SOLUTION When $x = 0$, the expression $\frac{\sqrt{a^2+x^2} - \sqrt{a^2-x^2}}{x^2}$ takes the form $\frac{0}{0}$. Rationalising the numerator, we get

$$\lim_{x \to 0} \frac{\sqrt{a^2+x^2} - \sqrt{a^2-x^2}}{x^2} = \lim_{x \to 0} \frac{(\sqrt{a^2+x^2} - \sqrt{a^2-x^2})}{x^2} \frac{(\sqrt{a^2+x^2} + \sqrt{a^2-x^2})}{(\sqrt{a^2+x^2} + \sqrt{a^2-x^2})} \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{a^2 + x^2 - a^2 + x^2}{x^2(\sqrt{a^2+x^2} + \sqrt{a^2-x^2})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2}{(\sqrt{a^2 + x^2} + \sqrt{a^2 - x^2})} = \frac{2}{\sqrt{a^2} + \sqrt{a^2}} = \frac{1}{a}$$

EXAMPLE 4 *Evaluate:* $\lim\limits_{x \to 4} \dfrac{x^2 - 16}{\sqrt{x^2 + 9} - 5}$.

SOLUTION When $x = 4$, the expression $\dfrac{x^2 - 16}{\sqrt{x^2 + 9} - 5}$ assumes the form $\dfrac{0}{0}$. Rationalising the denominator, we get

$$\lim_{x \to 4} \frac{x^2 - 16}{\sqrt{x^2 + 9} - 5} = \lim_{x \to 4} \frac{(x^2 - 16)}{(\sqrt{x^2 + 9} - 5)} \frac{(\sqrt{x^2 + 9} + 5)}{(\sqrt{x^2 + 9} + 5)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 4} \frac{(x^2 - 16)(\sqrt{x^2 + 9} + 5)}{(x^2 + 9 - 25)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 4} \left(\sqrt{x^2 + 9} + 5\right) = \left(\sqrt{16 + 9} + 5\right) = 5 + 5 = 10$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 5 *Evaluate:* $\lim\limits_{x \to a} \dfrac{\sqrt{a + 2x} - \sqrt{3x}}{\sqrt{3a + x} - 2\sqrt{x}}$. **[NCERT EXEMPLAR]**

SOLUTION When $x = a$, the expression $\dfrac{\sqrt{a + 2x} - \sqrt{3x}}{\sqrt{3a + x} - 2\sqrt{x}}$ assumes the form $\dfrac{0}{0}$.
Rationalising the numerator, we get

$$\lim_{x \to a} \frac{\sqrt{a + 2x} - \sqrt{3x}}{\sqrt{3a + x} - 2\sqrt{x}} = \lim_{x \to a} \frac{(\sqrt{a + 2x} - \sqrt{3x})(\sqrt{a + 2x} + \sqrt{3x})}{(\sqrt{3a + x} - 2\sqrt{x})(\sqrt{3a + x} + 2\sqrt{x})} \frac{(\sqrt{3a + x} + 2\sqrt{x})}{(\sqrt{a + 2x} + \sqrt{3x})} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to a} \frac{(a + 2x - 3x)}{(3a + x - 4x)} \frac{(\sqrt{3a + x} + 2\sqrt{x})}{(\sqrt{a + 2x} + \sqrt{3x})}$$

$$= \lim_{x \to a} \frac{\sqrt{3a + x} + 2\sqrt{x}}{3\left\{\sqrt{a + 2x} + \sqrt{3x}\right\}} = \frac{\sqrt{3a + a} + 2\sqrt{a}}{3(\sqrt{a + 2a} + \sqrt{3a})} = \frac{1}{3} \times \frac{4\sqrt{a}}{2\sqrt{3a}} = \frac{2}{3\sqrt{3}}$$

EXAMPLE 6 *Evaluate:* $\lim\limits_{x \to 4} \dfrac{3 - \sqrt{5 + x}}{1 - \sqrt{5 - x}}$.

SOLUTION When $x = 4$, the expression $\dfrac{3 - \sqrt{5 + x}}{1 - \sqrt{5 - x}}$ assumes the form $\dfrac{0}{0}$. Rationalising the numerator and denominator, we get

$$\lim_{x \to 4} \frac{3 - \sqrt{5 + x}}{1 - \sqrt{5 - x}} = \lim_{x \to 4} \frac{(3 - \sqrt{5 + x})(3 + \sqrt{5 + x})}{(1 - \sqrt{5 - x})(1 + \sqrt{5 - x})} \frac{(1 + \sqrt{5 - x})}{(3 + \sqrt{5 + x})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 4} \frac{(9 - 5 - x)}{(1 - 5 + x)} \left(\frac{1 + \sqrt{5 - x}}{3 + \sqrt{5 + x}}\right) \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 4} \frac{-(x - 4)(1 + \sqrt{5 - x})}{(x - 4)(3 + \sqrt{5 + x})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 4} \frac{-(1+\sqrt{5-x})}{(3+\sqrt{5+x})} = \frac{-(1+1)}{(3+3)} = -\frac{1}{3}$$

EXAMPLE 7 *Evaluate:* $\lim_{x \to 2} \dfrac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}$. **[NCERT EXEMPLAR]**

SOLUTION When $x = 2$, the expression $\dfrac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}$ assumes the form $\dfrac{0}{0}$. Rationalising the denominator, we get

$$\lim_{x \to 2} \frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}} = \lim_{x \to 2} \frac{(x-2)(x+2)(\sqrt{3x-2}+\sqrt{x+2})}{(\sqrt{3x-2}-\sqrt{x+2})(\sqrt{3x-2}+\sqrt{x+2})} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 2} \frac{(x-2)(x+2)\left(\sqrt{3x-2}+\sqrt{x+2}\right)}{(3x-2-x-2)} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 2} \frac{(x-2)(x+2)(\sqrt{3x-2}+\sqrt{x+2})}{2(x-2)} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 2} \frac{(x+2)\left(\sqrt{3x-2}+\sqrt{x+2}\right)}{2} = \frac{(2+2)(2+2)}{2} = 8.$$

EXAMPLE 8 *Evaluate:* $\lim_{x \to 1} \dfrac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$.

SOLUTION When $x = 1$, the expression $\dfrac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$ takes the form $\dfrac{0}{0}$. Rationalising $(\sqrt{x}-1)$ in the numerator, we get

$$\lim_{x \to 1} \frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3} = \lim_{x \to 1} \frac{(2x-3)(\sqrt{x}-1)(\sqrt{x}+1)}{(\sqrt{x}+1)(2x+3)(x-1)} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 1} \frac{(2x-3)(x-1)}{(\sqrt{x}+1)(2x+3)(x-1)} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 1} \frac{2x-3}{(\sqrt{x}+1)(2x+3)} = -\frac{1}{10}.$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 9 *Evaluate:* $\lim_{x \to \sqrt{10}} \dfrac{\sqrt{7-2x}-(\sqrt{5}-\sqrt{2})}{x^2-10}$

SOLUTION We have,

$$\lim_{x \to \sqrt{10}} \frac{\sqrt{7-2x}-(\sqrt{5}-\sqrt{2})}{x^2-10}$$

$$= \lim_{x \to \sqrt{10}} \frac{\sqrt{7-2x}-\sqrt{(\sqrt{5}-\sqrt{2})^2}}{x^2-10} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to \sqrt{10}} \frac{\sqrt{7-2x}-\sqrt{7-2\sqrt{10}}}{x^2-10} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to \sqrt{10}} \frac{\sqrt{7-2x} - \sqrt{7-2\sqrt{10}}}{x^2 - 10} \times \frac{\sqrt{7-2x} + \sqrt{7-2\sqrt{10}}}{\sqrt{7-2x} + \sqrt{7-2\sqrt{10}}}$$

$$= \lim_{x \to \sqrt{10}} \frac{(7-2x) - (7-2\sqrt{10})}{(x-\sqrt{10})(x+\sqrt{10})\left\{\sqrt{7-2x} + \sqrt{7-2\sqrt{10}}\right\}}$$

$$= \lim_{x \to \sqrt{10}} \frac{-2x + 2\sqrt{10}}{(x-\sqrt{10})(x+\sqrt{10})\left\{\sqrt{7-2x} + \sqrt{7-2\sqrt{10}}\right\}}$$

$$= \lim_{x \to \sqrt{10}} \frac{-2(x-\sqrt{10})}{(x-\sqrt{10})(x+\sqrt{10})\left\{\sqrt{7-2x} + \sqrt{7-2\sqrt{10}}\right\}}$$

$$= \lim_{x \to \sqrt{10}} \frac{-2}{(x+\sqrt{10})\left\{\sqrt{7-2x} + \sqrt{7-2\sqrt{10}}\right\}} = \lim_{x \to \sqrt{10}} \frac{-2}{2\sqrt{10}\left\{\sqrt{7-2\sqrt{10}} + \sqrt{7-2\sqrt{10}}\right\}}$$

$$= \frac{-1}{\sqrt{10} \times 2 \times \sqrt{7-2\sqrt{10}}} = \frac{-1}{2\sqrt{10}(\sqrt{5}-\sqrt{2})} \qquad \left[\because (\sqrt{5}-\sqrt{2})^2 = 7-2\sqrt{10}\right]$$

$$= \frac{-1}{2\sqrt{10}} \times \frac{(\sqrt{5}+\sqrt{2})}{3} = -\frac{(\sqrt{5}+\sqrt{2})}{6\sqrt{10}}$$

EXERCISE 28.4

BASIC

Evaluate the following limits:

1. $\lim_{x \to 0} \frac{\sqrt{1+x+x^2} - 1}{x}$
2. $\lim_{x \to 0} \frac{2x}{\sqrt{a+x} - \sqrt{a-x}}$
3. $\lim_{x \to 0} \frac{\sqrt{a^2+x^2} - a}{x^2}$
4. $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{2x}$
5. $\lim_{x \to 2} \frac{\sqrt{3-x} - 1}{2-x}$
6. $\lim_{x \to 3} \frac{x-3}{\sqrt{x-2} - \sqrt{4-x}}$
7. $\lim_{x \to 0} \frac{x}{\sqrt{1+x} - \sqrt{1-x}}$
8. $\lim_{x \to 1} \frac{\sqrt{5x-4} - \sqrt{x}}{x-1}$
9. $\lim_{x \to 1} \frac{x-1}{\sqrt{x^2+3} - 2}$
10. $\lim_{x \to 3} \frac{\sqrt{x+3} - \sqrt{6}}{x^2 - 9}$
11. $\lim_{x \to 1} \frac{\sqrt{5x-4} - \sqrt{x}}{x^2 - 1}$
12. $\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$ [NCERT]
13. $\lim_{x \to 2} \frac{\sqrt{x^2+1} - \sqrt{5}}{x-2}$
14. $\lim_{x \to 2} \frac{x-2}{\sqrt{x} - \sqrt{2}}$

BASED ON LOTS

15. $\lim_{x \to 7} \frac{4 - \sqrt{9+x}}{1 - \sqrt{8-x}}$
16. $\lim_{x \to 0} \frac{\sqrt{a+x} - \sqrt{a}}{x\sqrt{a^2+ax}}$
17. $\lim_{x \to 5} \frac{x-5}{\sqrt{6x-5} - \sqrt{4x+5}}$
18. $\lim_{x \to 1} \frac{\sqrt{5x-4} - \sqrt{x}}{x^3 - 1}$
19. $\lim_{x \to 2} \frac{\sqrt{1+4x} - \sqrt{5+2x}}{x-2}$
20. $\lim_{x \to 1} \frac{\sqrt{3+x} - \sqrt{5-x}}{x^2 - 1}$

21. $\lim_{x \to 0} \frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{x}$

22. $\lim_{x \to 0} \frac{\sqrt{1+x+x^2}-\sqrt{x+1}}{2x^2}$

23. $\lim_{x \to 4} \frac{2-\sqrt{x}}{4-x}$

24. $\lim_{x \to a} \frac{x-a}{\sqrt{x}-\sqrt{a}}$

25. $\lim_{x \to 0} \frac{\sqrt{1+3x}-\sqrt{1-3x}}{x}$

26. $\lim_{x \to 0} \frac{\sqrt{2-x}-\sqrt{2+x}}{x}$

27. $\lim_{x \to 1} \frac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1}$

28. $\lim_{x \to 1} \frac{(2x-3)(\sqrt{x}-1)}{3x^2+3x-6}$

29. $\lim_{x \to 0} \frac{\sqrt{1+x^2}-\sqrt{1+x}}{\sqrt{1+x^3}-\sqrt{1+x}}$

30. $\lim_{x \to 1} \frac{x^2-\sqrt{x}}{\sqrt{x}-1}$ [NCERT EXEMPLAR]

31. $\lim_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}, x \neq 0$ [NCERT EXEMPLAR]

BASED ON HOTS

32. $\lim_{x \to \sqrt{10}} \frac{\sqrt{7+2x}-(\sqrt{5}+\sqrt{2})}{x^2-10}$

33. $\lim_{x \to \sqrt{6}} \frac{\sqrt{5+2x}-(\sqrt{3}+\sqrt{2})}{x^2-6}$

34. $\lim_{x \to \sqrt{2}} \frac{\sqrt{3+2x}-(\sqrt{2}+1)}{x^2-2}$

ANSWERS

1. $\frac{1}{2}$ 2. $2\sqrt{a}$ 3. $\frac{1}{2a}$ 4. $\frac{1}{2}$ 5. $\frac{1}{2}$ 6. 1 7. 1 8. 2

9. 2 10. $\frac{1}{12\sqrt{6}}$ 11. 1 12. $\frac{1}{2}$ 13. $\frac{2}{\sqrt{5}}$ 14. $2\sqrt{2}$ 15. $-\frac{1}{4}$ 16. $\frac{1}{2a\sqrt{a}}$

17. 5 18. $\frac{2}{3}$ 19. $\frac{1}{3}$ 20. $\frac{1}{4}$ 21. 0 22. $\frac{1}{4}$ 23. $\frac{1}{4}$ 24. $2\sqrt{a}$

25. 3 26. $-\frac{1}{\sqrt{2}}$ 27. $\frac{1}{4}$ 28. $-\frac{1}{18}$ 29. 1 30. 3 31. $\frac{1}{2\sqrt{x}}$ 32. $\frac{(\sqrt{5}-\sqrt{2})}{6\sqrt{10}}$

33. $\frac{\sqrt{3}-\sqrt{2}}{2\sqrt{6}}$ 34. $\frac{\sqrt{2}-1}{2\sqrt{2}}$

HINTS TO SELECTED PROBLEMS

12. $$\lim_{x \to 0} = \frac{\sqrt{1+x}-1}{x} = \lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} \times \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} = \lim_{x \to 0} \frac{1+x-1}{x(\sqrt{1+x}+1)} = \lim_{x \to 0} \frac{1}{\sqrt{1+x}+1} = \frac{1}{2}$$

33. $$\lim_{x \to \sqrt{6}} = \frac{\sqrt{5+2x}-(\sqrt{3}+\sqrt{2})}{x^2-6} = \lim_{x \to \sqrt{6}} \frac{\sqrt{5+2x}-\sqrt{(\sqrt{3}+\sqrt{2})^2}}{x^2-6} = \lim_{x \to \sqrt{6}} \frac{\sqrt{5+2x}-\sqrt{5+2\sqrt{6}}}{x^2-6}$$

$$= \lim_{x \to \sqrt{6}} \frac{(\sqrt{5+2x})-(\sqrt{5+2\sqrt{6}})}{(x-\sqrt{6})(x+\sqrt{6})(\sqrt{5+2x}+\sqrt{5+2\sqrt{6}})} = \lim_{x \to \sqrt{6}} \frac{2(x-\sqrt{6})}{(x-\sqrt{6})(x+\sqrt{6})(\sqrt{5+2x}+\sqrt{5+2\sqrt{6}})}$$

$$= \lim_{x \to \sqrt{6}} \frac{2}{(x+\sqrt{6})(\sqrt{5+2x}+\sqrt{5+2\sqrt{6}})} = \frac{2}{(2\sqrt{6})(2\sqrt{5+2\sqrt{6}})} = \frac{1}{2\sqrt{6}(\sqrt{3}+\sqrt{2})}$$

28.6.4 EVALUATION OF ALGEBRAIC LIMITS BY USING SOME STANDARD LIMITS

Following theorem will be used to evaluate some algebraic limits.

THEOREM If $n \in Q$, then $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$.

PROOF We have,

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a^+} \frac{x^n - a^n}{x - a} \qquad \left[\because \lim_{x \to a} f(x) \text{ exists } \therefore \lim_{x \to a} f(x) = \lim_{x \to a^+} f(x)\right]$$

$$= \lim_{h \to 0} \frac{(a+h)^n - a^n}{a + h - a} = \lim_{h \to 0} \frac{a^n \left\{\left(1 + \frac{h}{a}\right)^n - 1\right\}}{h}$$

$$= a^n \lim_{h \to 0} \frac{\left\{1 + n\frac{h}{a} + \frac{n(n-1)}{2!}\frac{h^2}{a^2} + \ldots - 1\right\}}{h} \qquad \left[\because (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\right]$$

$$= a^n \lim_{h \to 0} \left\{\frac{n}{a} + \frac{n(n-1)}{2!}\frac{h}{a^2} + \ldots\right\} = a^n \times \frac{n}{a} = n\,a^{n-1}$$

Q.E.D.

Following examples will illustrate the use of the above result in evaluating algebraic limits.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:* $\lim\limits_{x \to 2} \dfrac{x^{10} - 1024}{x - 2}$

SOLUTION When $x = 2$, the expression $\dfrac{x^{10} - 1024}{x - 2}$ assumes the form $\dfrac{0}{0}$.

$$\therefore \quad \lim_{x \to 2} \frac{x^{10} - 1024}{x - 2} = \lim_{x \to 2} \frac{x^{10} - 2^{10}}{x - 2} = 10(2^{10-1}) = 5120$$

EXAMPLE 2 *Evaluate:* $\lim\limits_{x \to 2} \dfrac{x^{10} - 1024}{x^5 - 32}$.

SOLUTION When $x = 2$, the expression $\dfrac{x^{10} - 1024}{x^5 - 32}$ assumes the indeterminate form $\dfrac{0}{0}$.

Now, $$\lim_{x \to 2} \frac{x^{10} - 1024}{x^5 - 32} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 2} \frac{x^{10} - 2^{10}}{x^5 - 2^5} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 2} \frac{\dfrac{x^{10} - 2^{10}}{x - 2}}{\dfrac{x^5 - 2^5}{x - 2}} = \lim_{x \to 2} \frac{x^{10} - 2^{10}}{x - 2} \div \lim_{x \to 2} \frac{x^5 - 2^5}{x - 2} = 10 \cdot 2^{10-1} \div 5 \cdot 2^{5-1} = 64.$$

EXAMPLE 3 *Evaluate:* $\lim\limits_{x \to 9} \dfrac{x^{3/2} - 27}{x - 9}$.

SOLUTION When $x = 9$, the expression $\dfrac{x^{3/2} - 27}{x - 9}$ assumes the form $\dfrac{0}{0}$

$$\therefore \quad \lim_{x \to 9} \frac{x^{3/2} - 27}{x - 9} = \lim_{x \to 9} \frac{x^{3/2} - 9^{3/2}}{x - 9} = \frac{3}{2}(9)^{3/2-1} = \frac{3}{2}(3) = \frac{9}{2}$$

EXAMPLE 4 *Evaluate:* $\lim\limits_{x \to a} \dfrac{x\sqrt{x} - a\sqrt{a}}{x - a}$.

SOLUTION We have,

$$\lim_{x \to a} \frac{x\sqrt{x} - a\sqrt{a}}{x - a} = \lim_{x \to a} \frac{x^{3/2} - a^{3/2}}{x - a} = \frac{3}{2} a^{3/2 - 1} = \frac{3}{2}\sqrt{a}$$

EXAMPLE 5 *Evaluate:* $\lim\limits_{x \to a} \dfrac{x^m - a^m}{x^n - a^n}$.

SOLUTION We have,

$$\lim_{x \to a} \frac{x^m - a^m}{x^n - a^n} = \lim_{x \to a} \left\{ \frac{x^m - a^m}{x - a} \cdot \frac{x - a}{x^n - a^n} \right\} = \lim_{x \to a} \left\{ \frac{x^m - a^m}{x - a} \div \frac{x^n - a^n}{x - a} \right\}$$

$$= \lim_{x \to a} \frac{x^m - a^m}{x - a} \div \lim_{x \to a} \frac{x^n - a^n}{x - a} = ma^{m-1} \div na^{n-1} = \frac{m}{n} a^{m-n}.$$

EXAMPLE 6 *Evaluate:* $\lim\limits_{x \to 2} \dfrac{x - 2}{\sqrt[3]{x} - \sqrt[3]{2}}$.

SOLUTION We have,

$$\lim_{x \to 2} \frac{x - 2}{x^{1/3} - 2^{1/3}} = \frac{1}{\lim\limits_{x \to 2} \dfrac{x^{1/3} - 2^{1/3}}{x - 2}} = \frac{1}{\dfrac{1}{3}\left(2^{1/3 - 1}\right)} = \frac{1}{\dfrac{1}{3} \times (2^{-2/3})} = 3\,(2^{2/3})$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 7 *Evaluate:* $\lim\limits_{x \to 0} \dfrac{(1 - x)^n - 1}{x}$.

SOLUTION We have,

$$\lim_{x \to 0} \frac{(1 - x)^n - 1}{x} = -\lim_{x \to 0} \frac{(1 - x)^n - 1}{(1 - x) - 1} = -\lim_{y \to 1} \frac{y^n - 1^n}{y - 1}, \text{ where } y = 1 - x. \; [\because \; x \to 0 \Rightarrow y \to 1]$$

$$= -n\,(1)^{n-1} = -n.$$

EXAMPLE 8 *Evaluate:* $\lim\limits_{x \to a} \dfrac{(x + 2)^{5/3} - (a + 2)^{5/3}}{x - a}$.

SOLUTION We have,

$$\lim_{x \to a} \frac{(x + 2)^{5/3} - (a + 2)^{5/3}}{x - a}$$

$$= \lim_{x \to a} \frac{(x+2)^{5/3} - (a+2)^{5/3}}{(x+2)-(a+2)} = \lim_{y \to b} \frac{y^{5/3} - b^{5/3}}{y-b}, \text{ where } x+2 = y \text{ and } a+2 = b.$$

$$= \frac{5}{3} b^{5/3-1} = \frac{5}{3} b^{2/3} = \frac{5}{3}(a+2)^{2/3}$$

EXAMPLE 9 *Find the value of k, if* $\lim_{x \to 1} \frac{x^4-1}{x-1} = \lim_{x \to k} \frac{x^3-k^3}{x^2-k^2}$. **[NCERT EXEMPLAR]**

SOLUTION We have,

$$\lim_{x \to 1} \frac{x^4-1}{x-1} = \lim_{x \to 1} \frac{x^4-1^4}{x-1} = 4(1)^{4-1} = 4$$

and,
$$\lim_{x \to k} \frac{x^3-k^3}{x^2-k^2} = \lim_{x \to k} \frac{x^3-k^3}{x-k} \times \frac{x-k}{x^2-k^2} = \lim_{x \to k} \frac{x^3-k^3}{x-k} \div \frac{x^2-k^2}{x-k}$$

$$= \lim_{x \to k} \frac{x^3-k^3}{x-k} \div \lim_{x \to k} \frac{x^2-k^2}{x-k} = 3k^{3-1} \div 2k^{2-1} = \frac{3}{2}k$$

$$\therefore \quad \lim_{x \to 1} \frac{x^4-1}{x-1} = \lim_{x \to k} \frac{x^3-k^3}{x^2-k^2} \Rightarrow 4 = \frac{3k}{2} \Rightarrow k = \frac{8}{3}$$

EXAMPLE 10 *If* $\lim_{x \to 2} \frac{x^n - 2^n}{x-2} = 80$ *and* $n \in N$, *find n.* **[NCERT EXEMPLAR]**

SOLUTION It is given that:

$$\lim_{x \to 2} \frac{x^n-2^n}{x-2} = 80 \Rightarrow n \times 2^{n-1} = 80 \Rightarrow n \times 2^{n-1} = 5 \times 2^{5-1} \Rightarrow n = 5.$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 11 *If* $\lim_{x \to -a} \frac{x^9+a^9}{x+a} = 9$, *find the real values of a.*

SOLUTION We have,

$$\lim_{x \to -a} \frac{x^9+a^9}{x+a} = 9$$

$$\Rightarrow \quad \lim_{x \to -a} \frac{x^9-(-a)^9}{x-(-a)} = 9 \Rightarrow 9(-a)^{9-1} = 9 \Rightarrow 9a^8 = 9 \Rightarrow a^8 = 1 \Rightarrow a = \pm 1$$

EXAMPLE 12 *Evaluate:* $\lim_{x \to 1} \frac{(x+x^2+x^3+\ldots+x^n)-n}{x-1}$.

SOLUTION We find that

$$\lim_{x \to 1} \frac{(x+x^2+x^3+\ldots+x^n)-n}{x-1} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 1} \frac{(x-1)+(x^2-1)+(x^3-1)+\ldots+(x^n-1)}{x-1}$$

$$= \lim_{x\to 1}\frac{x-1}{x-1}+\lim_{x\to 1}\frac{x^2-1^2}{x-1}+\lim_{x\to 1}\frac{x^3-1}{x-1}+\ldots+\lim_{x\to 1}\frac{x^n-1^n}{x-1}$$

$$= 1+2(1)^{2-1}+3(1)^{3-1}+\ldots+n(1)^{n-1}=1+2+3+\ldots+n=\frac{n(n+1)}{2}.$$

EXERCISE 28.5

BASIC

Evaluate the following limits:

1. $\lim\limits_{x\to a}\dfrac{(x+2)^{5/2}-(a+2)^{5/2}}{x-a}$
2. $\lim\limits_{x\to a}\dfrac{(x+2)^{3/2}-(a+2)^{3/2}}{x-a}$
3. $\lim\limits_{x\to 0}\dfrac{(1+x)^6-1}{(1+x)^2-1}$
4. $\lim\limits_{x\to a}\dfrac{x^{2/7}-a^{2/7}}{x-a}$
5. $\lim\limits_{x\to a}\dfrac{x^{5/7}-a^{5/7}}{x^{2/7}-a^{2/7}}$
6. $\lim\limits_{x\to -1/2}\dfrac{8x^3+1}{2x+1}$
7. $\lim\limits_{x\to 27}\dfrac{(x^{1/3}+3)(x^{1/3}-3)}{x-27}$
8. $\lim\limits_{x\to 4}\dfrac{x^3-64}{x^2-16}$
9. $\lim\limits_{x\to 1}\dfrac{x^{15}-1}{x^{10}-1}$
10. $\lim\limits_{x\to -1}\dfrac{x^3+1}{x+1}$
11. $\lim\limits_{x\to a}\dfrac{x^{2/3}-a^{2/3}}{x^{3/4}-a^{3/4}}$

BASED ON LOTS

12. If $\lim\limits_{x\to 3}\dfrac{x^n-3^n}{x-3}=108$, find the value of n.
13. If $\lim\limits_{x\to a}\dfrac{x^9-a^9}{x-a}=9$, find all possible values of a.
14. If $\lim\limits_{x\to a}\dfrac{x^5-a^5}{x-a}=405$, find all possible values of a.
15. If $\lim\limits_{x\to a}\dfrac{x^9-a^9}{x-a}=\lim\limits_{x\to 5}(4+x)$, find all possible values of a.
16. If $\lim\limits_{x\to a}\dfrac{x^3-a^3}{x-a}=\lim\limits_{x\to 1}\dfrac{x^4-1}{x-1}$, find all possible values of a.

ANSWERS

1. $\frac{5}{2}(a+2)^{3/2}$ 2. $\frac{3}{2}(a+2)^{1/2}$ 3. 3 4. $\frac{2}{7}a^{-5/7}$ 5. $\frac{5}{2}a^{3/7}$
6. 3 7. $\frac{2}{9}$ 8. 6 9. $\frac{3}{2}$ 10. 3 11. $\frac{8}{9}a^{-1/12}$ 12. 4
13. 1, –1 14. $a=3,-3$ 15. 1, –1 16. $\pm\frac{2}{\sqrt{3}}$

28.6.5 METHOD OF EVALUATION OF ALGEBRAIC LIMITS AT INFINITY

Consider the functions $f(x)=\dfrac{1}{x}$ and $g(x)=\dfrac{1}{x^2}$. Graphs of these functions are shown in Figures 28.6 and 28.7.

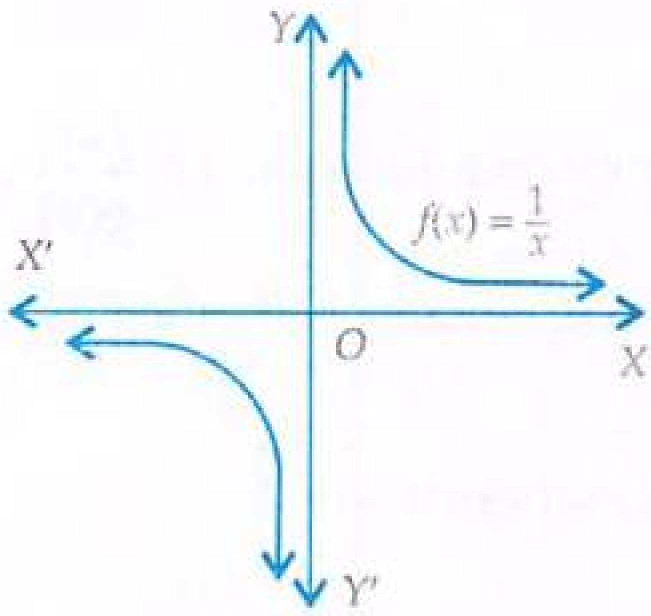

Fig. 28.6 Graph of $f(x) = \frac{1}{x}$

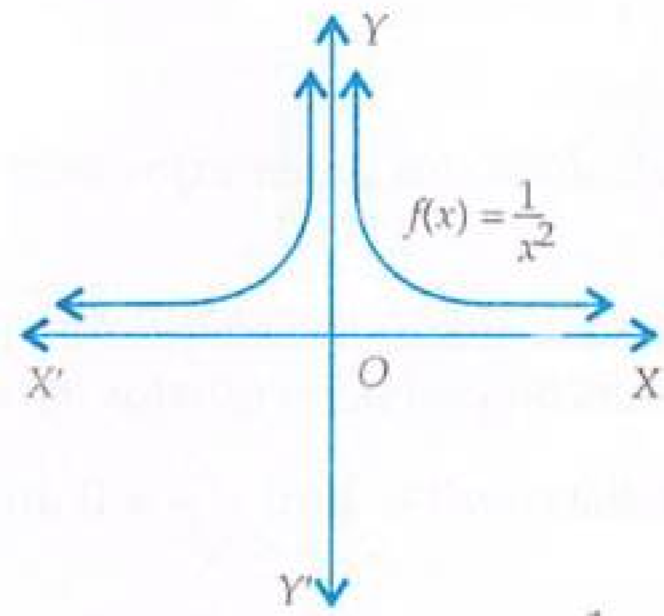

Fig. 28.7 Graph of $f(x) = \frac{1}{x^2}$

We observe from the graphs that as x increases, the values of $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$ decrease rapidly and when x is indefinitely large $\frac{1}{x}$ and $\frac{1}{x^2}$ are indefinitely small i.e. very close to zero. In such cases, we write

$$\lim_{x \to +\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x \to +\infty} \frac{1}{x^2} = 0.$$

We also observe from the graphs of these two functions that as x decreases and is very small negative real number, then also the values of $\frac{1}{x}$ and $\frac{1}{x^2}$ approach to zero. So, we write

$$\lim_{x \to -\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x^2} = 0.$$

It follows from the above discussion that:

(i) $\lim_{x \to \infty} c = c$ (ii) $\lim_{x \to -\infty} c = c$

(iii) $\lim_{x \to \infty} \frac{c}{x^n} = 0, n > 0$ (iv) $\lim_{x \to -\infty} \frac{c}{x^n} = 0, n \in N$

From the graphs of real functions, we obtain the following useful results:

(i) $\lim_{x \to +\infty} x \to +\infty$ (ii) $\lim_{x \to -\infty} x \to -\infty$

(iii) $\lim_{x \to +\infty} x^2 \to +\infty$ (iv) $\lim_{x \to -\infty} x^2 \to +\infty$ and so on.

(v) $\lim_{x \to \infty} e^x \to \infty$ or, $\lim_{x \to -\infty} e^{-x} \to \infty$

(vi) $\lim_{x \to \infty} e^{-x} \to 0$ or, $\lim_{x \to -\infty} e^x \to 0$

(vii) $\lim_{x \to \infty} a^x \to 0$, if $|a| < 1$ (viii) $\lim_{x \to \infty} a^x \to \infty$, if $a > 1$

(ix) $\lim_{x \to 0^+} \log_a x \to -\infty$ and $\lim_{x \to \infty} \log_a x \to \infty$, where $a > 1$

(x) $\lim_{x \to 0^+} \log_a x \to \infty$ and $\lim_{x \to \infty} \log_a x \to -\infty$, if $0 < a < 1$.

We use these results to evaluate limits at infinity. Following algorithm may be used to evaluate algebraic limits at infinity.

ALGORITHM

Step I *Write down the given expression in the form of a rational function. i.e.* $\frac{f(x)}{g(x)}$ *, if it is not so.*

Step II *If k is the highest power of x in numerator and denominator both, then divide each term in numerator and denominator by* x^k.

Step III *Use the results* $\lim_{x \to \infty} \frac{c}{x^n} = 0$ and $\lim_{x \to \infty} c = c$, where $n > 0$.

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:* $\lim_{x \to \infty} \frac{ax^2 + bx + c}{dx^2 + ex + f}$.

SOLUTION Here the expression assumes the form $\frac{\infty}{\infty}$. We notice that the highest power of x in both the numerator and denominator is 2. So we divide each term in both the numerator and denominator by x^2.

$$\therefore \quad \lim_{x \to \infty} \frac{ax^2 + bx + c}{dx^2 + ex + f} = \lim_{x \to \infty} \frac{a + \frac{b}{x} + \frac{c}{x^2}}{d + \frac{e}{x} + \frac{f}{x^2}} = \frac{a + 0 + 0}{d + 0 + 0} = \frac{a}{d}.$$

EXAMPLE 2 *Evaluate:* $\lim_{x \to \infty} \frac{5x - 6}{\sqrt{4x^2 + 9}}$.

SOLUTION Dividing each term in numerator and denominator, we find that

$$\lim_{x \to \infty} \frac{5x - 6}{\sqrt{4x^2 + 9}} = \lim_{x \to \infty} \frac{5 - 6/x}{\sqrt{4 + 9/x^2}} = \frac{5 - 0}{\sqrt{4 + 0}} = \frac{5}{2}$$

EXAMPLE 3 *Evaluate:* $\lim_{x \to \infty} \frac{\sqrt{3x^2 - 1} + \sqrt{2x^2 - 1}}{4x + 3}$

SOLUTION Dividing each term in the numerator and denominator by x, we get

$$\lim_{x \to \infty} \frac{\sqrt{3x^2 - 1} + \sqrt{2x^2 - 1}}{4x + 3} = \lim_{x \to \infty} \frac{\sqrt{3 - 1/x^2} + \sqrt{2 - 1/x^2}}{4 + 3/x} = \frac{\sqrt{3} + \sqrt{2}}{4}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *Evaluate:* $\lim_{x \to \infty} \sqrt{x}\,(\sqrt{x + c} - \sqrt{x})$.

SOLUTION The given expression is of the form $\infty - \infty$. So we first write it in the rational form $\frac{f(x)}{g(x)}$. So that it reduces to either $\frac{0}{0}$ form or $\frac{\infty}{\infty}$ form.

$$\therefore \lim_{x \to \infty} \sqrt{x}\left\{\sqrt{x+c} - \sqrt{x}\right\} = \lim_{x \to \infty} \frac{\sqrt{x}\left\{\sqrt{x+c} - \sqrt{x}\right\}\left\{\sqrt{x+c} + \sqrt{x}\right\}}{\left\{\sqrt{x+c} + \sqrt{x}\right\}}$$

$$= \lim_{x \to \infty} \frac{\sqrt{x}\,(x+c-x)}{\sqrt{x+c}+\sqrt{x}}$$

$$= \lim_{x \to \infty} \frac{c\sqrt{x}}{\sqrt{x+c}+\sqrt{x}} \qquad \left(\text{form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \infty} \frac{c}{\sqrt{1+\frac{c}{x}}+1} = \frac{c}{\sqrt{1+0}+1} = \frac{c}{2} \qquad [\text{Diving } N^r \text{ and } D^r \text{ by } \sqrt{x}]$$

EXAMPLE 5 *Evaluate:* $\lim_{x \to \infty} \left(\sqrt{x^2+x+1} - \sqrt{x^2+1}\right)$.

SOLUTION Here the expression assumes the form $\infty - \infty$ as $x \to \infty$. So, we first reduce it to the rational form $\frac{f(x)}{g(x)}$.

$$\lim_{x \to \infty} \sqrt{x^2+x+1} - \sqrt{x^2+1} = \lim_{x \to \infty} \frac{\left\{\sqrt{x^2+x+1} - \sqrt{x^2+1}\right\}}{\left\{\sqrt{x^2+x+1} + \sqrt{x^2+1}\right\}}\left\{\sqrt{x^2+x+1} + \sqrt{x^2+1}\right\}$$

$$= \lim_{x \to \infty} \frac{x^2+x+1-x^2-1}{\sqrt{x^2+x+1}+\sqrt{x^2+1}} = \lim_{x \to \infty} \frac{x}{\sqrt{x^2+x+1}+\sqrt{x^2+1}}$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1+\frac{1}{x^2}}} \qquad [\text{Dividing } N^r \text{ and } D^r \text{ by } x]$$

$$= \frac{1}{1+1} = \frac{1}{2}$$

EXAMPLE 6 *Evaluate:* $\lim_{n \to \infty} \frac{1+2+3+\ldots+n}{n^2}$.

SOLUTION Using $1+2+3+\ldots+n = \frac{n(n+1)}{2}$, we obtain

$$\lim_{n \to \infty} \frac{1+2+3+\ldots+n}{n^2} = \lim_{n \to \infty} \frac{1}{n^2} \times \frac{n(n+1)}{2} = \lim_{n \to \infty} \frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{2}$$

EXAMPLE 7 *Evaluate:* $\lim_{n \to \infty} \frac{n!}{(n+1)!-n!}$

SOLUTION We find that

$$\lim_{n \to \infty} \frac{n!}{(n+1)!-n!} = \lim_{n \to \infty} \frac{n!}{(n+1)\,n!-n!} = \lim_{n \to \infty} \frac{1}{n+1-1} = \lim_{n \to \infty} \frac{1}{n} = 0$$

EXAMPLE 8 *Let* $f(x) = \frac{ax+b}{x+1}$, $\lim_{x \to 0} f(x) = 2$ *and* $\lim_{x \to \infty} f(x) = 1$ *prove that* $f(-2) = 0$.

SOLUTION Given that

$$\lim_{x \to 0} f(x) = 2 \Rightarrow \lim_{x \to 0} \frac{ax+b}{x+1} = 2 \Rightarrow \frac{b}{1} = 2 \Rightarrow b = 2$$

It is also given that

$$\lim_{x \to \infty} f(x) = 1 \Rightarrow \lim_{x \to \infty} \frac{ax+b}{x+1} = 1 \Rightarrow \lim_{x \to \infty} \frac{a + \frac{b}{x}}{1 + \frac{1}{x}} = 1 \Rightarrow \frac{a+0}{1+0} = 1 \Rightarrow a = 1.$$

Substituting the values of a and b in $f(x) = \frac{ax+b}{x+1}$, we obtain

$$f(x) = \frac{x+2}{x+1} \Rightarrow f(-2) = \frac{-2+2}{-2+1} = 0.$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 9 *Evaluate:* $\lim_{x \to -\infty} \left(\sqrt{x^2 - x + 1} + x\right)$.

SOLUTION We have,

$$\lim_{x \to -\infty} \left(\sqrt{x^2 - x + 1} + x\right) = \lim_{y \to \infty} \left(\sqrt{y^2 + y + 1} - y\right), \text{ where } y = -x$$

$$= \lim_{y \to \infty} \frac{\left\{\sqrt{y^2+y+1} - y\right\}\left\{\sqrt{y^2+y+1} + y\right\}}{\left\{\sqrt{y^2+y+1} + y\right\}}$$

$$= \lim_{y \to \infty} \frac{y^2 + y + 1 - y^2}{\sqrt{y^2+y+1} + y}$$

$$= \lim_{y \to \infty} \frac{y+1}{\sqrt{y^2+y+1} + y} = \lim_{y \to \infty} \frac{1 + \frac{1}{y}}{\sqrt{1 + \frac{1}{y} + \frac{1}{y^2}} + 1} = \frac{1}{2}$$

ALITER We have,

$$\lim_{x \to -\infty} \left(\sqrt{x^2 - x + 1} + x\right) = \lim_{x \to -\infty} \frac{\left\{\sqrt{x^2-x+1} + x\right\}\left\{\sqrt{x^2-x+1} - x\right\}}{\left\{\sqrt{x^2-x+1} - x\right\}}$$

$$= \lim_{x \to -\infty} \frac{x^2 - x + 1 - x^2}{\left\{\sqrt{x^2-x+1} - x\right\}} = \lim_{x \to -\infty} \frac{-x+1}{\sqrt{x^2-x+1} - x}$$

$$= \lim_{x \to -\infty} \frac{-\frac{x}{|x|} + \frac{1}{|x|}}{\frac{\sqrt{x^2-x+1}}{|x|} - \frac{x}{|x|}} \quad \left[\text{Dividing } N^r \text{ and } D^r \text{ by } |x|\right]$$

$$= \lim_{x \to -\infty} \frac{\frac{-x}{-x} - \frac{1}{x}}{\sqrt{\frac{x^2}{x^2} - \frac{x}{x^2} + \frac{1}{x^2}} + \frac{x}{x}} \quad \left[\because \sqrt{x^2} = |x| = -x \text{ for } x < 0\right]$$

$$= \lim_{x \to -\infty} \frac{1 - \frac{1}{x}}{\sqrt{1 - \frac{1}{x} + \frac{1}{x^2}} + 1} = \frac{1}{2}$$

EXAMPLE 10 *Evaluate:* $\lim_{x \to \infty} \frac{(x+1)^{10} + (x+2)^{10} + \dots + (x+100)^{10}}{x^{10} + 10^{10}}$

SOLUTION Dividing the numerator and denominator by x^{10}, we obtain

$$\lim_{x \to \infty} \frac{(x+1)^{10} + (x+2)^{10} + \dots + (x+100)^{10}}{x^{10} + 10^{10}}$$

$$= \lim_{x \to \infty} \frac{\left(1 + \frac{1}{x}\right)^{10} + \left(1 + \frac{2}{x}\right)^{10} + \dots + \left(1 + \frac{100}{x}\right)^{10}}{1 + \left(\frac{10}{x}\right)^{10}} = \frac{1 + 1 + \dots + 1 \text{ (100 - times)}}{1 + 0} = \frac{100}{1} = 100$$

EXERCISE 28.6

BASIC

Evaluate the following limits: (1-10)

1. $\lim_{x \to \infty} \frac{(3x-1)(4x-2)}{(x+8)(x-1)}$
2. $\lim_{x \to \infty} \frac{3x^3 - 4x^2 + 6x - 1}{2x^3 + x^2 - 5x + 7}$
3. $\lim_{x \to \infty} \frac{5x^3 - 6}{\sqrt{9 + 4x^6}}$
4. $\lim_{x \to \infty} \sqrt{x^2 + cx} - x$
5. $\lim_{x \to \infty} \sqrt{x+1} - \sqrt{x}$
6. $\lim_{x \to \infty} \sqrt{x^2 + 7x} - x$
7. $\lim_{x \to \infty} \frac{x}{\sqrt{4x^2 + 1} - 1}$
8. $\lim_{n \to \infty} \frac{n^2}{1 + 2 + 3 + \dots + n}$
9. $\lim_{x \to \infty} \frac{3x^{-1} + 4x^{-2}}{5x^{-1} + 6x^{-2}}$
10. $\lim_{x \to \infty} \frac{\sqrt{x^2 + a^2} + \sqrt{x^2 + b^2}}{\sqrt{x^2 + c^2} + \sqrt{x^2 + d^2}}$

BASED ON LOTS

Evaluate the following limits: (11-20)

11. $\lim_{n \to \infty} \frac{(n+2)! + (n+1)!}{(n+2)! - (n+1)!}$
12. $\lim_{x \to \infty} x \left\{\sqrt{x^2 + 1} - \sqrt{x^2 - 1}\right\}$
13. $\lim_{x \to \infty} \left\{\sqrt{x+1} - \sqrt{x}\right\} \sqrt{x+2}$
14. $\lim_{n \to \infty} \frac{1^2 + 2^2 + \dots + n^2}{n^3}$
15. $\lim_{n \to \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \dots + \frac{n-1}{n^2}\right)$
16. $\lim_{n \to \infty} \frac{1^3 + 2^3 + \dots + n^3}{n^4}$

17. $\lim_{n \to \infty} \frac{1^3 + 2^3 + \ldots + n^3}{(n-1)^4}$

18. $\lim_{x \to \infty} \sqrt{x}\left\{\sqrt{x+1} - \sqrt{x}\right\}$

19. $\lim_{n \to \infty} \left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^n}\right)$

20. $\lim_{x \to \infty} \frac{x^4 + 7x^3 + 46x + a}{x^4 + 6}$, where a is a non-zero real number.

21. $f(x) = \frac{ax^2 + b}{x^2 + 1}$, $\lim_{x \to 0} f(x) = 1$ and $\lim_{x \to \infty} f(x) = 1$, then prove that $f(-2) = f(2) = 1$.

22. Show that $\lim_{x \to \infty} (\sqrt{x^2 + x + 1} - x) \neq \lim_{x \to \infty} (\sqrt{x^2 + 1} - x)$

BASED ON HOTS

Evaluate the following limits:

23. $\lim_{x \to -\infty} \left(\sqrt{4x^2 - 7x} + 2x\right)$

24. $\lim_{x \to -\infty} \left(\sqrt{x^2 - 8x} + x\right)$

25. $\lim_{n \to \infty} \frac{1^4 + 2^4 + 3^4 + \ldots + n^4}{n^5} - \lim_{n \to \infty} \frac{1^3 + 2^3 + \ldots + n^3}{n^5}$

26. $\lim_{n \to \infty} \frac{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1)}{n^3}$

ANSWERS

1. 12 2. $\frac{3}{2}$ 3. $\frac{5}{2}$ 4. $\frac{c}{2}$ 5. 0 6. $\frac{7}{2}$ 7. $\frac{1}{2}$ 8. 2 9. $\frac{3}{5}$
10. 1 11. 1 12. 1 13. $\frac{1}{2}$ 14. $\frac{1}{3}$ 15. $\frac{1}{2}$ 16. $\frac{1}{4}$ 17. $\frac{1}{4}$ 18. $\frac{1}{2}$
19. $\frac{1}{2}$ 20. 1 23. $\frac{7}{4}$ 24. 4 25. $\frac{1}{5}$ 26. $\frac{1}{3}$

28.7 EVALUATION OF TRIGONOMETRIC LIMITS

In this section, we will be studying various methods of evaluating trigonometric limits. In order to evaluate trigonometric limits we will be using the following results which are stated and proved in the following theorem.

THEOREM *If angle θ is measured in radians, then*

(i) $\lim_{\theta \to 0} \sin\theta = 0$ (ii) $\lim_{\theta \to 0} \cos\theta = 1$ (iii) $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

(iv) $\lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1$ (v) $\lim_{\theta \to a} \frac{\sin(\theta - a)}{\theta - a} = 1$ (vi) $\lim_{\theta \to a} \frac{\tan(\theta - a)}{\theta - a} = 1$

PROOF (i) Let ABC be a right angled triangle such that $\angle C = \frac{\pi}{2}$ and $\angle ABC = \theta$. Then,

$$\sin\theta = \frac{CA}{BA} \text{ and, } \cos\theta = \frac{BC}{BA}.$$

Now, if we keep BC fixed and go on decreasing angle θ, then we find that A goes on coming nearer and nearer to C.

$\therefore \quad A \to C$ as $\theta \to 0$

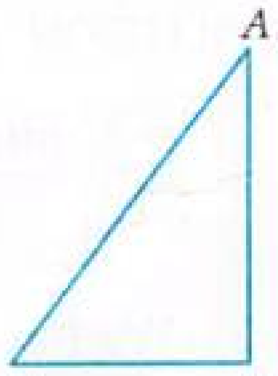

Fig. 28.8

This means that $CA \to 0$ and $BA \to BC$ as $\theta \to 0$.

$\Rightarrow \quad \dfrac{CA}{BA} \to 0$ and $\dfrac{BC}{BA} \to 1 \quad$ as $\theta \to 0$

$\Rightarrow \quad \sin\theta \to 0$ and $\cos\theta \to 1$ as $\theta \to 0$

(iii) Consider a circle of radius r. Let O be the centre of the circle such that $\angle AOB = \theta$ where θ is measured in radians and it is very small. Suppose the tangent at A meets OB produced at P. From the Fig. 28.9, we find that

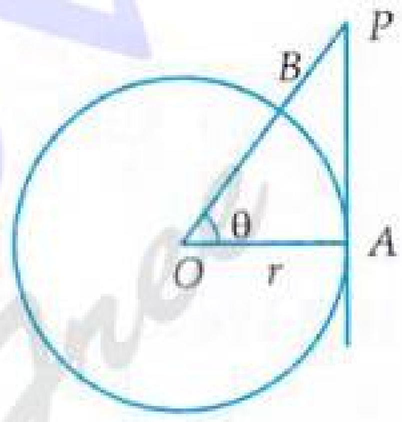

Fig. 28.9

Area of $\Delta OAB <$ Area of sector $OAB <$ Area of ΔOAP

$\Rightarrow \quad \dfrac{1}{2} OA \cdot OB \sin\theta < \dfrac{1}{2}(OA)^2 \theta < \dfrac{1}{2} OA \cdot AP$

$\Rightarrow \quad \dfrac{1}{2} r^2 \sin\theta < \dfrac{1}{2} r^2 \theta < \dfrac{1}{2} r^2 \tan\theta \quad [\text{In } \Delta OAP,\ AP = OA \tan\theta]$

$\Rightarrow \quad \sin\theta < \theta < \tan\theta$

$\Rightarrow \quad 1 < \dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta} \qquad [\because\ \theta \text{ is small} \therefore\ \sin\theta > 0]$

$\Rightarrow \quad 1 > \dfrac{\sin\theta}{\theta} > \cos\theta \Rightarrow 1 \geq \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} \geq \lim\limits_{\theta\to 0} \cos\theta$ or, $\lim\limits_{\theta\to 0}\cos\theta \leq \lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} \leq 1$

$\Rightarrow \quad 1 \leq \lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} \leq 1 \Rightarrow \lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$

(iv) $\quad \lim\limits_{\theta\to 0}\dfrac{\tan\theta}{\theta} = \lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} \times \dfrac{1}{\cos\theta} = \lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} \times \lim\limits_{\theta\to 0}\dfrac{1}{\cos\theta} = 1 \times 1 = 1$

(v) $\quad \lim\limits_{\theta\to a}\dfrac{\sin(\theta-a)}{\theta-a} = \lim\limits_{h\to 0}\dfrac{\sin(a+h-a)}{(a+h-a)} \quad \left[\text{Using: } \lim\limits_{\theta\to a}\dfrac{\sin(\theta-a)}{\theta-a} = \lim\limits_{\theta\to a^+}\dfrac{\sin(\theta-a)}{\theta-a}\right]$

$= \lim\limits_{h\to 0}\dfrac{\sin h}{h} = 1$

(vi) $\quad \lim\limits_{\theta\to a}\dfrac{\tan(\theta-a)}{\theta-a} = \lim\limits_{h\to 0}\dfrac{\tan(a+h-a)}{a+h-a} \quad \left[\text{Using: } \lim\limits_{\theta\to a}\dfrac{\tan(\theta-a)}{\theta-a} = \lim\limits_{\theta\to a^+}\dfrac{\tan(\theta-a)}{\theta-a}\right]$

$= \lim\limits_{h\to 0}\dfrac{\tan h}{h} = 1$

28.7.1 EVALUATION OF TRIGONOMETRIC LIMITS WHEN THE VARIABLE TENDS TO ZERO

For evaluating such type of limits, we use the formulae discussed above. Following examples will illustrate the same.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate the following limits:*

(i) $\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}$ (ii) $\lim\limits_{x\to 0}\dfrac{\sin 5x}{2x}$ (iii) $\lim\limits_{x\to 0}\dfrac{\sin ax}{\sin bx}$ **[NCERT]**

(iv) $\lim\limits_{x\to 0}\dfrac{\sin^2 ax}{\sin^2 bx}$ (v) $\lim\limits_{x\to 0}\dfrac{\sin^2 3x}{x^2}$

SOLUTION (i) We have,

$$\lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{x \to 0} \left(3 \times \frac{\sin 3x}{3x}\right) = 3 \lim_{x \to 0} \frac{\sin 3x}{3x} = 3(1) = 3 \qquad \left[\because \lim_{x \to 0} \frac{\sin 3x}{3x} = 1\right]$$

(ii) We have,

$$\lim_{x \to 0} \frac{\sin 5x}{2x} = \lim_{x \to 0} \left(\frac{5}{2} \times \frac{\sin 5x}{5x}\right) = \frac{5}{2} \lim_{x \to 0} \frac{\sin 5x}{5x} = \frac{5}{2}(1) = \frac{5}{2} \qquad \left[\because \lim_{x \to 0} \frac{\sin 5x}{5x} = 1\right]$$

(iii) We have,

$$\lim_{x \to 0} \frac{\sin ax}{\sin bx} = \lim_{x \to 0} \frac{\left(\frac{\sin ax}{ax}\right) ax}{\left(\frac{\sin bx}{bx}\right) bx} = \frac{a}{b} \frac{\lim_{x \to 0} \left(\frac{\sin ax}{ax}\right)}{\lim_{x \to 0} \left(\frac{\sin bx}{bx}\right)} = \frac{a}{b} \frac{(1)}{(1)} = \frac{a}{b}$$

(iv) We have,

$$\lim_{x \to 0} \frac{\sin^2 ax}{\sin^2 bx} = \lim_{x \to 0} \frac{\left(\frac{\sin ax}{ax}\right)(ax)\left(\frac{\sin ax}{ax}\right) ax}{\left(\frac{\sin bx}{bx}\right)(bx)\left(\frac{\sin bx}{bx}\right) bx} = \frac{a^2}{b^2} \lim_{x \to 0} \frac{\left(\frac{\sin ax}{ax}\right)\left(\frac{\sin ax}{ax}\right)}{\left(\frac{\sin bx}{bx}\right)\left(\frac{\sin bx}{bx}\right)} = \frac{a^2}{b^2} \times \frac{1}{1} = \frac{a^2}{b^2}$$

(v) We have,

$$\lim_{x \to 0} \frac{\sin^2 3x}{x^2} = \lim_{x \to 0} \frac{\sin 3x}{x} \times \frac{\sin 3x}{x} = \lim_{x \to 0} 3\left(\frac{\sin 3x}{3x}\right) \times 3\left(\frac{\sin 3x}{3x}\right)$$

$$= 3 \lim_{x \to 0} \frac{\sin 3x}{3x} \times 3 \lim_{x \to 0} \frac{\sin 3x}{3x} = (3)(3) = 9$$

EXAMPLE 2 *Evaluate:*

(i) $\lim_{x \to 0} \frac{\sin 2x + \sin 3x}{2x + \sin 3x}$ [NCERT EXEMPLAR] (ii) $\lim_{x \to 0} \frac{\sin 2x + \sin 6x}{\sin 5x - \sin 3x}$

SOLUTION (i) $\lim_{x \to 0} \frac{\sin 2x + \sin 3x}{2x + \sin 3x} = \lim_{x \to 0} \frac{\frac{\sin 2x}{x} + \frac{\sin 3x}{x}}{2 + \frac{\sin 3x}{x}}$ [Dividing numerator and denominator by x]

$$= \lim_{x \to 0} \frac{2\left(\frac{\sin 2x}{2x}\right) + 3\left(\frac{\sin 3x}{3x}\right)}{2 + 3\left(\frac{\sin 3x}{3x}\right)} = \frac{2 \times 1 + 3 \times 1}{2 + 3 \times 1} = \frac{5}{5} = 1$$

(ii) $\lim_{x \to 0} \frac{\sin 2x + \sin 6x}{\sin 5x - \sin 3x} = \lim_{x \to 0} \frac{\frac{\sin 2x}{x} + \frac{\sin 6x}{x}}{\frac{\sin 5x}{x} - \frac{\sin 3x}{x}}$ [Dividing numerator and denominator by x]

$$= \lim_{x \to 0} \frac{2\left(\frac{\sin 2x}{2x}\right) + 6\left(\frac{\sin 6x}{6x}\right)}{5\left(\frac{\sin 5x}{5x}\right) - 3\left(\frac{\sin 3x}{3x}\right)} = \frac{2 \times 1 + 6 \times 1}{5 \times 1 - 3 \times 1} = \frac{8}{2} = 4$$

ALITER See Example 6 (ii).

EXAMPLE 3 *Evaluate the following limits:*

(i) $\lim_{x \to 0} \frac{1-\cos 2x}{x^2}$ **[NCERT EXEMPLAR]** (ii) $\lim_{x \to 0} \frac{1-\cos 2x}{x}$

(iii) $\lim_{x \to 0} \frac{1-\cos x}{x^2}$ (iv) $\lim_{x \to 0} \frac{1-\cos 2mx}{1-\cos 2nx}$ **[NCERT EXEMPLAR]**

(v) $\lim_{x \to 0} \frac{1-\cos mx}{1-\cos nx}$ **[NCERT EXEMPLAR]**

SOLUTION (i) $$\lim_{x \to 0} \frac{1-\cos 2x}{x^2} = \lim_{x \to 0} \frac{2\sin^2 x}{x^2} = 2\lim_{x \to 0}\left(\frac{\sin x}{x} \times \frac{\sin x}{x}\right)$$

$$= 2\lim_{x \to 0}\frac{\sin x}{x} \times \lim_{x \to 0}\frac{\sin x}{x} = 2(1)(1) = 2$$

(ii) $$\lim_{x \to 0} \frac{1-\cos 2x}{x} = \lim_{x \to 0} \frac{2\sin^2 x}{x}$$

$$= 2\lim_{x \to 0}\left(\frac{\sin x}{x} \times \sin x\right) = 2\left(\lim_{x \to 0}\frac{\sin x}{x}\right)\left(\lim_{x \to 0}\sin x\right) = 2(1)(0) = 0$$

(iii) $$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{2\sin^2 x/2}{x^2} = 2\lim_{x \to 0}\left(\frac{\sin x/2}{x} \times \frac{\sin x/2}{x}\right)$$

$$= 2\lim_{x \to 0}\left(\frac{\sin x/2}{x/2} \times \frac{1}{2} \times \frac{1}{2}\,\frac{\sin x/2}{x/2}\right) = \frac{2}{4}\left(\lim_{x \to 0}\frac{\sin x/2}{x/2}\right) \times \left(\lim_{x \to 0}\frac{\sin x/2}{x/2}\right)$$

$$= \frac{2}{4}(1)(1) = \frac{1}{2}$$

(iv) $$\lim_{x \to 0} \frac{1-\cos 2mx}{1-\cos 2nx}$$

$$= \lim_{x \to 0} \frac{2\sin^2 mx}{2\sin^2 nx} = \lim_{x \to 0} \frac{\sin^2 mx}{\sin^2 nx}$$

$$= \lim_{x \to 0}\left\{\frac{\frac{\sin mx}{mx} \times mx}{\frac{\sin nx}{nx} \times nx} \times \frac{\frac{\sin mx}{mx} \times mx}{\frac{\sin nx}{nx} \times nx}\right\} = \frac{m^2}{n^2}\left\{\lim_{x \to 0}\frac{\frac{\sin mx}{mx}}{\frac{\sin nx}{nx}}\right\} \times \left\{\lim_{x \to 0}\frac{\frac{\sin mx}{mx}}{\frac{\sin nx}{nx}}\right\}$$

$$= \frac{m^2}{n^2}\left\{\frac{\lim_{x \to 0}\frac{\sin mx}{mx}}{\lim_{x \to 0}\frac{\sin nx}{nx}}\right\} \times \left\{\frac{\lim_{x \to 0}\frac{\sin mx}{mx}}{\lim_{x \to 0}\frac{\sin nx}{nx}}\right\} = \frac{m^2}{n^2}\left(\frac{1}{1}\right)\left(\frac{1}{1}\right) = \frac{m^2}{n^2}$$

(v) $$\lim_{x \to 0} \frac{1-\cos mx}{1-\cos nx} = \lim_{x \to 0} \frac{2\sin^2\left(\frac{mx}{2}\right)}{2\sin^2\left(\frac{nx}{2}\right)} = \lim_{x \to 0} \left(\frac{\sin \frac{mx}{2}}{\sin \frac{nx}{2}}\right)^2$$

$$= \lim_{x \to 0} \left\{\frac{\left(\frac{\sin \frac{mx}{2}}{\frac{mx}{2}}\right) \frac{mx}{2}}{\left(\frac{\sin \frac{nx}{2}}{\frac{nx}{2}}\right) \cdot \frac{nx}{2}}\right\}^2 = \left\{\frac{m}{n} \frac{\lim_{x \to 0} \left(\frac{\sin \frac{mx}{2}}{\frac{mx}{2}}\right)}{\lim_{x \to 0} \left(\frac{\sin \frac{nx}{2}}{\frac{nx}{2}}\right)}\right\}^2 = \left(\frac{m \times 1}{n \times 1}\right)^2 = \frac{m^2}{n^2}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *Evaluate the following limits:*

(i) $\lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x}$ [NCERT EXEMPLAR] (ii) $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$

SOLUTION (i) We have,

$$\lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x} = \lim_{x \to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{\sin^3 x} = \lim_{x \to 0} \frac{\sin x (1-\cos x)}{\cos x \sin^3 x}$$

$$= \lim_{x \to 0} \frac{1-\cos x}{\cos x \sin^2 x} = \lim_{x \to 0} \frac{1-\cos x}{\cos x (1-\cos^2 x)}$$

$$= \lim_{x \to 0} \frac{1-\cos x}{\cos x (1+\cos x)(1-\cos x)} = \lim_{x \to 0} \frac{1}{\cos x (1+\cos x)} = \frac{1}{2}$$

(ii) We have,

$$\lim_{x \to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x \to 0} \left(\frac{\sin x - \sin x \cos x}{x^3 \cos x}\right)$$

$$= \lim_{x \to 0} \left\{\frac{\sin x (1-\cos x)}{x^3 \cos x}\right\} = \lim_{x \to 0} \left\{\frac{\sin x}{x} \times \frac{1-\cos x}{x^2} \times \frac{1}{\cos x}\right\}$$

$$= \left\{\lim_{x \to 0} \frac{\sin x}{x}\right\} \times \left\{\lim_{x \to 0} \frac{2\sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2 \times 4}\right\} \times \lim_{x \to 0} \frac{1}{\cos x}$$

$$= \left\{\lim_{x \to 0} \frac{\sin x}{x}\right\} \times \frac{1}{2} \times \left\{\lim_{x \to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2\right\} \times \lim_{x \to 0} \frac{1}{\cos x} = 1 \times \frac{1}{2} (1)^2 \times \frac{1}{1} = \frac{1}{2}$$

ALITER $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x} \times \left(\frac{\sin x}{x}\right)^3 = \frac{1}{2} \times 1^3 = \frac{1}{2}$ [See Example 4 (i)]

EXAMPLE 5 *Evaluate:* $\lim_{x \to 0} \frac{\tan 2x - \sin 2x}{x^3}$

SOLUTION We have,

$$\lim_{x \to 0} \frac{\tan 2x - \sin 2x}{x^3} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{\frac{\sin 2x}{\cos 2x} - \sin 2x}{x^3} = \lim_{x \to 0} \frac{\sin 2x (1 - \cos 2x)}{x^3 \cos 2x} = \lim_{x \to 0} \frac{\sin 2x \times 2 \sin^2 x}{x^3 \cos 2x}$$

$$= 2 \lim_{x \to 0} \frac{\tan 2x \sin^2 x}{x^3} = 4 \lim_{x \to 0} \left(\frac{\tan 2x}{2x}\right)\left(\frac{\sin x}{x}\right)^2 = 4\,(1)\,(1)^2 = 4.$$

ALITER $\displaystyle \lim_{x \to 0} \frac{\tan 2x - \sin 2x}{x^3} = 8 \lim_{x \to 0} \frac{\tan 2x - \sin 2x}{(2x)^3} = 8 \times \frac{1}{2} = 4$ [See Example 4 (ii)]

EXAMPLE 6 *Evaluate the following limits:*

(i) $\displaystyle \lim_{x \to 0} \frac{\cos Ax - \cos Bx}{x^2}$ (ii) $\displaystyle \lim_{x \to 0} \frac{\sin 2x + \sin 6x}{\sin 5x - \sin 3x}$

SOLUTION (i) We have,

$$\lim_{x \to 0} \frac{\cos Ax - \cos Bx}{x^2} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2 \sin \left(\frac{A+B}{2}\right) x \, \sin \left(\frac{B-A}{2}\right) x}{x^2} \qquad \left[\because \cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}\right]$$

$$= 2 \lim_{x \to 0} \left\{ \frac{\sin \left(\frac{A+B}{2}\right) x}{\left(\frac{A+B}{2}\right) x} \times \left(\frac{A+B}{2}\right) \times \frac{\sin \left(\frac{B-A}{2}\right) x}{\left(\frac{B-A}{2}\right) x} \times \left(\frac{B-A}{2}\right) \right\} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= 2 \left(\frac{B+A}{2}\right) \left(\frac{B-A}{2}\right) \left\{ \lim_{x \to 0} \frac{\sin \left(\frac{A+B}{2}\right) x}{\left(\frac{A+B}{2}\right) x} \right\} \times \left\{ \lim_{x \to 0} \frac{\sin \left(\frac{B-A}{2}\right) x}{\left(\frac{B-A}{2}\right) x} \right\}$$

$$= \left(\frac{B^2 - A^2}{2}\right)(1)\,(1) = \frac{B^2 - A^2}{2}$$

(ii) We have,

$$\lim_{x \to 0} \frac{\sin 2x + \sin 6x}{\sin 5x - \sin 3x} = \lim_{x \to 0} \frac{2 \sin 4x \cos 2x}{2 \sin x \cos 4x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \left\{ \frac{\frac{\sin 4x}{4x} \times 4x \times \cos 2x}{\frac{\sin x}{x} \times x \times \cos 4x} \right\} = 4 \, \frac{\lim\limits_{x \to 0} \frac{\sin 4x}{4x} \times \lim\limits_{x \to 0} \cos 2x}{\lim\limits_{x \to 0} \frac{\sin x}{x} \times \lim\limits_{x \to 0} \cos 4x} = 4 \left(\frac{1 \times 1}{1 \times 1}\right) = 4$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 7 *Evaluate the following limits:*

(i) $\displaystyle \lim_{y \to 0} \frac{(x+y) \sec (x+y) - x \sec x}{y}$ **[NCERT EXEMPLAR]** (ii) $\displaystyle \lim_{x \to 0} \frac{\sec 4x - \sec 2x}{\sec 3x - \sec x}$

SOLUTION (i) We have,

$$\lim_{y \to 0} \frac{(x+y)\sec(x+y) - x\sec x}{y} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{y \to 0} \frac{x(\sec(x+y) - \sec x) + y\sec(x+y)}{y} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{y \to 0} x\left\{\frac{\sec(x+y) - \sec x}{y}\right\} + \lim_{y \to 0} \frac{y\sec(x+y)}{y}$$

$$= \lim_{y \to 0} x\left\{\frac{\cos x - \cos(x+y)}{y\cos x\cos(x+y)}\right\} + \lim_{y \to 0} \sec(x+y)$$

$$= \lim_{y \to 0} \left\{\frac{\cos x - \cos(x+y)}{y} \times \frac{x}{\cos x\cos(x+y)}\right\} + \lim_{y \to 0} \sec(x+y)$$

$$= \lim_{y \to 0} \left\{\frac{2\sin\left(x+\frac{y}{2}\right)\sin\left(\frac{y}{2}\right)}{2\left(\frac{y}{2}\right)} \times \frac{x}{\cos x\cos(x+y)}\right\} + \lim_{y \to 0} \sec(x+y)$$

$$= \lim_{y \to 0} \sin\left(x+\frac{y}{2}\right) \times \lim_{y \to 0} \frac{\sin\left(\frac{y}{2}\right)}{\left(\frac{y}{2}\right)} \times \lim_{y \to 0} \frac{x}{\cos x\cos(x+y)} + \lim_{y \to 0} \sec(x+y)$$

$$= \sin x \times 1 \times \frac{x}{\cos^2 x} + \sec x = x\tan x\sec x + \sec x$$

(ii) We have,

$$\lim_{x \to 0} \frac{\sec 4x - \sec 2x}{\sec 3x - \sec x} = \lim_{x \to 0} \left(\frac{\frac{\cos 2x - \cos 4x}{\cos 2x\cos 4x}}{\frac{\cos x - \cos 3x}{\cos x\cos 3x}}\right) \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \left(\frac{\cos 2x - \cos 4x}{\cos x - \cos 3x} \times \frac{\cos x\cos 3x}{\cos 2x\cos 4x}\right) = \lim_{x \to 0} \left(\frac{2\sin 3x\sin x}{2\sin 2x\sin x} \times \frac{\cos x\cos 3x}{\cos 2x\cos 4x}\right)$$

$$= \lim_{x \to 0} \left(\frac{\frac{\sin 3x}{3x} \times 3x}{\frac{\sin 2x}{2x} \times 2x} \times \frac{\cos x\cos 3x}{\cos 2x\cos 4x}\right) = \frac{3}{2} \frac{\lim\limits_{x \to 0} \frac{\sin 3x}{3x}}{\lim\limits_{x \to 0} \frac{\sin 2x}{2x}} \times \lim_{x \to 0} \frac{\cos x\cos 3x}{\cos 2x\cos 4x} = \frac{3}{2}\left(\frac{1}{1} \times \frac{1}{1}\right) = \frac{3}{2}$$

EXAMPLE 8 *Evaluate:* $\lim\limits_{x \to 0} \frac{\cot 2x - \operatorname{cosec} 2x}{x}$

SOLUTION We have,

$$\lim_{x \to 0} \frac{\cot 2x - \operatorname{cosec} 2x}{x} = \lim_{x \to 0} \frac{\frac{\cos 2x}{\sin 2x} - \frac{1}{\sin 2x}}{x} = \lim_{x \to 0} \frac{\cos 2x - 1}{x\sin 2x}$$

$$= \lim_{x \to 0} \frac{-(1 - \cos 2x)}{x\sin 2x} = -\lim_{x \to 0} \frac{2\sin^2 x}{x(2\sin x\cos x)} = -\lim_{x \to 0} \frac{\tan x}{x} = -1$$

EXAMPLE 9 *Evaluate:* $\lim\limits_{x \to 0} \frac{\sin x - 2\sin 3x + \sin 5x}{x}$ **[NCERT EXEMPLAR]**

SOLUTION We have,

$$\lim_{x \to 0} \frac{\sin x - 2 \sin 3x + \sin 5x}{x} = \lim_{x \to 0} \frac{(\sin x - \sin 3x) + (\sin 5x - \sin 3x)}{x}$$

$$= \lim_{x \to 0} \frac{-2 \sin x \cos 2x + 2 \sin x \cos 4x}{x} = \lim_{x \to 0} \frac{2 \sin x (\cos 4x - \cos 2x)}{x}$$

$$= 2 \lim_{x \to 0} \left(\frac{\sin x}{x}\right) \times \lim_{x \to 0} (\cos 4x - \cos 2x) = 2 \times 1 \times 0 = 0$$

ALITER $\lim_{x \to 0} \frac{\sin x - 2 \sin 3x + \sin 5x}{x}$

$$= \lim_{x \to 0} \left\{\left(\frac{\sin x}{x}\right) - 2\left(\frac{\sin 3x}{x}\right) + \left(\frac{\sin 5x}{x}\right)\right\} = \lim_{x \to 0} \frac{\sin x}{x} - 2 \lim_{x \to 0} \frac{\sin 3x}{x} + \lim_{x \to 0} \frac{\sin 5x}{x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} - 2 \times 3\left(\lim_{x \to 0} \frac{\sin 3x}{3x}\right) + 5\left(\lim_{x \to 0} \frac{\sin 5x}{5x}\right) = 1 - 2 \times 3 \times 1 + 5 \times 1 = 0$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 10 *Evaluate:* $\lim_{x \to 0} \frac{\tan x + 4 \tan 2x - 3 \tan 3x}{x^2 \tan x}$

SOLUTION We have,

$$\lim_{x \to 0} \frac{\tan x + 4 \tan 2x - 3 \tan 3x}{x^2 \tan x}$$

$$= \lim_{x \to 0} \frac{\tan x + 4\left(\frac{2 \tan x}{1 - \tan^2 x}\right) - 3\left(\frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}\right)}{x^2 \tan x}$$

$$= \lim_{x \to 0} \frac{1 + \frac{8}{1 - \tan^2 x} - 3\left(\frac{3 - \tan^2 x}{1 - 3 \tan^2 x}\right)}{x^2}$$

$$= \lim_{x \to 0} \frac{(1 - \tan^2 x)(1 - 3 \tan^2 x) + 8(1 - 3 \tan^2 x) - 3(3 - \tan^2 x)(1 - \tan^2 x)}{x^2 (1 - \tan^2 x)(1 - 3 \tan^2 x)}$$

$$= \lim_{x \to 0} \frac{1 - 4 \tan^2 x + 3 \tan^4 x + 8 - 24 \tan^2 x - 9 + 12 \tan^2 x - 3 \tan^4 x}{x^2 (1 - \tan^2 x)(1 - 3 \tan^2 x)}$$

$$= \lim_{x \to 0} \frac{-16 \tan^2 x}{x^2 (1 - \tan^2 x)(1 - 3 \tan^2 x)}$$

$$= -16 \lim_{x \to 0} \left(\frac{\tan x}{x}\right)^2 \times \frac{1}{1 - \tan^2 x} \times \frac{1}{1 - 3 \tan^2 x} = -16$$

EXAMPLE 11 *Evaluate:* $\lim_{x \to 0} \frac{1 - \cos x \sqrt{\cos 2x}}{x^2}$

SOLUTION We have,

$$\lim_{x \to 0} \frac{1 - \cos x \sqrt{\cos 2x}}{x^2} = \lim_{x \to 0} \frac{1 - \cos x \sqrt{\cos 2x}}{x^2} \times \frac{1 + \cos x \sqrt{\cos 2x}}{1 + \cos x \sqrt{\cos 2x}}$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x \cos 2x}{x^2 (1 + \cos x \sqrt{\cos 2x})} = \lim_{x \to 0} \frac{1 - \cos^2 x (2 \cos^2 x - 1)}{x^2 (1 + \cos x \sqrt{\cos 2x})}$$

$$= \lim_{x \to 0} \frac{1 - 2 \cos^4 x + \cos^2 x}{x^2 (1 + \cos x \sqrt{\cos 2x})} = \lim_{x \to 0} \frac{(1 - \cos^2 x)\ (1 + 2 \cos^2 x)}{x^2 (1 + \cos x \sqrt{\cos 2x})}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x^2} \times \lim_{x \to 0} \frac{1 + 2 \cos^2 x}{\left\{1 + \cos x \sqrt{\cos 2x}\right\}} = 1 \times \left(\frac{1+2}{1+1}\right) = \frac{3}{2}$$

EXAMPLE 12 *Evaluate:* $\lim_{x \to 0} \frac{8}{x^8} \left(1 - \cos \frac{x^2}{2} - \cos \frac{x^2}{4} + \cos \frac{x^2}{2} \cos \frac{x^2}{4}\right)$

SOLUTION We have,

$$\lim_{x \to 0} \frac{8}{x^8} \left(1 - \cos \frac{x^2}{2} - \cos \frac{x^2}{4} + \cos \frac{x^2}{2} \cos \frac{x^2}{4}\right)$$

$$= \lim_{x \to 0} \frac{8}{x^8} \left\{\left(1 - \cos \frac{x^2}{2}\right) - \cos \frac{x^2}{4} \left(1 - \cos \frac{x^2}{2}\right)\right\}$$

$$= \lim_{x \to 0} \frac{8}{x^8} \left(1 - \cos \frac{x^2}{2}\right)\left(1 - \cos \frac{x^2}{4}\right) = \lim_{x \to 0} 8 \left\{\frac{1 - \cos \frac{x^2}{2}}{x^4}\right\} \left\{\frac{1 - \cos \frac{x^2}{4}}{x^4}\right\}$$

$$= \lim_{x \to 0} 8 \times \frac{2 \sin^2 \frac{x^2}{4}}{x^4} \times \frac{2 \sin^2 \frac{x^2}{8}}{x^4} = \lim_{x \to 0} 32 \times \left\{\frac{\sin \frac{x^2}{4}}{x^2}\right\}^2 \times \left\{\frac{\sin \frac{x^2}{8}}{x^2}\right\}^2$$

$$= 32 \lim_{x \to 0} \left\{\frac{\sin \frac{x^2}{4}}{4\left(\frac{x^2}{4}\right)}\right\}^2 \times \left\{\frac{\sin \frac{x^2}{8}}{8\left(\frac{x^2}{8}\right)}\right\}^2 = 32 \times \left(\frac{1}{4}\right)^2 \times \left(\frac{1}{8}\right)^2 = 32 \times \frac{1}{16} \times \frac{1}{64} = \frac{1}{32}$$

EXAMPLE 13 *Evaluate:* $\lim_{x \to 0} \frac{1 - \cos x \cos 2x \cos 3x}{\sin^2 2x}$

SOLUTION We find that

$$\cos x \cos 2x \cos 3x = \frac{1}{2} \{2 \cos x \cos 2x \cos 3x\} = \frac{1}{2} \{(2 \cos x \cos 2x) \cos 3x\}$$

$$= \frac{1}{2} \left\{(\cos 3x + \cos x) \cos 3x\right\} = \frac{1}{2} \left\{\cos^2 3x + \cos 3x \cos x\right\}$$

$$= \frac{1}{4} \left\{2 \cos^2 3x + 2 \cos 3x \cos x\right\} = \frac{1}{4} \left\{1 + \cos 6x + \cos 4x + \cos 2x\right\}$$

$$\therefore \lim_{x \to 0} \frac{1 - \cos x \cos 2x \cos 3x}{\sin^2 2x} = \lim_{x \to 0} \frac{1 - \frac{1}{4} (1 + \cos 6x + \cos 4x + \cos 2x)}{\sin^2 2x}$$

$$= \lim_{x \to 0} \frac{4 - 1 - \cos 6x - \cos 4x - \cos 2x}{4 \sin^2 2x} = \lim_{x \to 0} \frac{(1 - \cos 6x) + (1 - \cos 4x) + (1 - \cos 2x)}{4 \sin^2 2x}$$

$$= \lim_{x \to 0} \frac{2 \sin^2 3x + 2 \sin^2 2x + 2 \sin^2 x}{4 \sin^2 2x} = \lim_{x \to 0} \frac{\frac{\sin^2 3x}{x^2} + \frac{\sin^2 2x}{x^2} + \frac{\sin^2 x}{x^2}}{2\left(\frac{\sin^2 2x}{x^2}\right)}$$

$$= \lim_{x \to 0} \frac{\left(\frac{\sin 3x}{x}\right)^2 + \left(\frac{\sin 2x}{x}\right)^2 + \left(\frac{\sin x}{x}\right)^2}{2\left(\frac{\sin 2x}{x}\right)^2} = \lim_{x \to 0} \frac{9 \times \left(\frac{\sin 3x}{3x}\right)^2 + 4 \times \left(\frac{\sin 2x}{2x}\right)^2 + \left(\frac{\sin x}{x}\right)^2}{2 \times 4\left(\frac{\sin 2x}{2x}\right)^2}$$

$$= \frac{9 \times 1 + 4 \times 1 + 1}{8} = \frac{14}{8} = \frac{7}{4}$$

EXERCISE 28.7

BASIC

Evaluate the following limits:

1. $\lim\limits_{x \to 0} \dfrac{\sin 3x}{5x}$
2. $\lim\limits_{x \to 0} \dfrac{\sin x^0}{x}$
3. $\lim\limits_{x \to 0} \dfrac{x^2}{\sin x^2}$
4. $\lim\limits_{x \to 0} \dfrac{\sin x \cos x}{3x}$
5. $\lim\limits_{x \to 0} \dfrac{3 \sin x - 4 \sin^3 x}{x}$
6. $\lim\limits_{x \to 0} \dfrac{\tan 8x}{\sin 2x}$
7. $\lim\limits_{x \to 0} \dfrac{\tan mx}{\tan nx}$
8. $\lim\limits_{x \to 0} \dfrac{\sin 5x}{\tan 3x}$
9. $\lim\limits_{x \to 0} \dfrac{\sin x^\circ}{x^\circ}$
10. $\lim\limits_{x \to 0} \dfrac{7x \cos x - 3 \sin x}{4x + \tan x}$
11. $\lim\limits_{x \to 0} \dfrac{\cos ax - \cos bx}{\cos cx - \cos dx}$
12. $\lim\limits_{x \to 0} \dfrac{\tan^2 3x}{x^2}$
13. $\lim\limits_{x \to 0} \dfrac{1 - \cos mx}{x^2}$
14. $\lim\limits_{x \to 0} \dfrac{3 \sin 2x + 2x}{3x + 2 \tan 3x}$
15. $\lim\limits_{x \to 0} \dfrac{\cos 3x - \cos 7x}{x^2}$
16. $\lim\limits_{\theta \to 0} \dfrac{\sin 3\theta}{\tan 2\theta}$
17. $\lim\limits_{x \to 0} \dfrac{\sin x^2 (1 - \cos x^2)}{x^6}$
18. $\lim\limits_{x \to 0} \dfrac{\sin^2 4x^2}{x^4}$
19. $\lim\limits_{x \to 0} \dfrac{x \cos x + 2 \sin x}{x^2 + \tan x}$
20. $\lim\limits_{x \to 0} \dfrac{2x - \sin x}{\tan x + x}$
21. $\lim\limits_{x \to 0} \dfrac{5x \cos x + 3 \sin x}{3x^2 + \tan x}$
22. $\lim\limits_{x \to 0} \dfrac{\sin 3x - \sin x}{\sin x}$
23. $\lim\limits_{x \to 0} \dfrac{\sin 5x - \sin 3x}{\sin x}$
24. $\lim\limits_{x \to 0} \dfrac{\cos 3x - \cos 5x}{x^2}$
25. $\lim\limits_{x \to 0} \dfrac{\tan 3x - 2x}{3x - \sin^2 x}$
26. $\lim\limits_{x \to 0} \dfrac{\sin (2 + x) - \sin (2 - x)}{x}$

BASED ON LOTS

27. $\lim\limits_{h \to 0} \dfrac{(a + h)^2 \sin (a + h) - a^2 \sin a}{h}$
28. $\lim\limits_{x \to 0} \dfrac{\tan x - \sin x}{\sin 3x - 3 \sin x}$
29. $\lim\limits_{x \to 0} \dfrac{\sec 5x - \sec 3x}{\sec 3x - \sec x}$
30. $\lim\limits_{x \to 0} \dfrac{1 - \cos 2x}{\cos 2x - \cos 8x}$

31. $\lim_{x \to 0} \frac{1 - \cos 2x + \tan^2 x}{x \sin x}$

32. $\lim_{x \to 0} \frac{\sin (a + x) + \sin (a - x) - 2 \sin a}{x \sin x}$

33. $\lim_{x \to 0} \frac{x^2 - \tan 2x}{\tan x}$

34. $\lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x}$ [NCERT EXEMPLAR]

35. $\lim_{x \to 0} \frac{x \tan x}{1 - \cos x}$

36. $\lim_{x \to 0} \frac{x^2 + 1 - \cos x}{x \sin x}$

37. $\lim_{x \to 0} \frac{\sin 2x (\cos 3x - \cos x)}{x^3}$

38. $\lim_{x \to 0} \frac{2 \sin x^\circ - \sin 2x^\circ}{x^3}$

39. $\lim_{x \to 0} \frac{x^3 \cot x}{1 - \cos x}$

40. $\lim_{x \to 0} \frac{x \tan x}{1 - \cos 2x}$

41. $\lim_{x \to 0} \frac{\sin (3 + x) - \sin (3 - x)}{x}$

42. $\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1}$

43. $\lim_{x \to 0} \frac{3 \sin^2 x - 2 \sin x^2}{3x^2}$

44. $\lim_{x \to 0} \frac{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}{x}$

45. $\lim_{x \to 0} \frac{1 - \cos 4x}{x^2}$

46. $\lim_{x \to 0} \frac{x \cos x + \sin x}{x^2 + \tan x}$

47. $\lim_{x \to 0} \frac{1 - \cos 2x}{3 \tan^2 x}$

48. $\lim_{\theta \to 0} \frac{1 - \cos 4\theta}{1 - \cos 6\theta}$

49. $\lim_{x \to 0} \frac{ax + x \cos x}{b \sin x}$ [NCERT]

50. $\lim_{\theta \to 0} \frac{\sin 4\theta}{\tan 3\theta}$

51. $\lim_{x \to 0} \frac{2 \sin x - \sin 2x}{x^3}$

52. $\lim_{x \to 0} \frac{1 - \cos 5x}{1 - \cos 6x}$

53. $\lim_{x \to 0} \frac{\operatorname{cosec} x - \cot x}{x}$

54. $\lim_{x \to 0} \frac{\sin 3x + 7x}{4x + \sin 2x}$

55. $\lim_{x \to 0} \frac{5x + 4 \sin 3x}{4 \sin 2x + 7x}$

56. $\lim_{x \to 0} \frac{3 \sin x - \sin 3x}{x^3}$

57. $\lim_{x \to 0} \frac{\tan 3x - \sin 3x}{x^3}$

58. $\lim_{x \to 0} \frac{\sin ax + bx}{ax + \sin bx}$ [NCERT]

BASED ON HOTS

59. $\lim_{x \to 0} (\operatorname{cosec} x - \cot x)$

60. $\lim_{x \to 0} \frac{x \{\sin (\alpha + \beta) x + \sin (\alpha - \beta) x + \sin 2\alpha x\}}{\cos^2 \beta x - \cos^2 \alpha x}$

61. $\lim_{x \to 0} \frac{\cos ax - \cos bx}{\cos cx - 1}$

62. $\lim_{h \to 0} \frac{(a + h)^2 \sin (a + h) - a^2 \sin a}{h}$

63. If $\lim_{x \to 0} kx \operatorname{cosec} x = \lim_{x \to 0} x \operatorname{cosec} kx$, find k.

ANSWERS

1. $\frac{3}{5}$ 2. $\frac{\pi}{180}$ 3. 1 4. $\frac{1}{3}$ 5. 3 6. 4 7. $\frac{m}{n}$ 8. $\frac{5}{3}$

9. 1 10. $\frac{4}{5}$ 11. $\frac{a^2 - b^2}{c^2 - d^2}$ 12. 9 13. $\frac{m^2}{2}$ 14. $\frac{8}{9}$ 15. 20

16. $\frac{3}{2}$ 17. $\frac{1}{2}$ 18. 16 19. 3 20. $\frac{1}{2}$ 21. 8 22. 2 23. 2

24. 8 25. $\frac{1}{3}$ 26. $2\cos 2$ 27. $2a\sin a + a^2\cos a$ 28. $\frac{-1}{8}$

29. 2 30. $\frac{1}{15}$ 31. 3 32. $-\sin a$ 33. -2 34. $\frac{1}{4\sqrt{2}}$ 35. 2

36. $\frac{3}{2}$ 37. -8 38. $\left(\frac{\pi}{180}\right)^3$ 39. 2 40. $\frac{1}{2}$ 41. $2\cos 3$ 42. 4 43. $\frac{1}{3}$

44. 1 45. 8 46. 2 47. $\frac{2}{3}$ 48. $\frac{4}{9}$ 49. $\frac{a+1}{b}$ 50. $\frac{4}{3}$ 51. 1

52. $\frac{25}{36}$ 53. $\frac{1}{2}$ 54. $\frac{5}{3}$ 55. $\frac{17}{15}$ 56. 4 57. $\frac{27}{2}$ 58. 1 59. 0

60. $\frac{4\alpha}{\alpha^2-\beta^2}$ 61. $\frac{a^2-b^2}{c^2}$ 62. $a^2\cos a + 2a\sin a$ 63. $k=\pm 1$

HINTS TO SELECTED PROBLEMS

42. $$\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}=\lim_{x\to 0}\frac{(2\cos^2 x-1)-1}{\cos x-1}=\lim_{x\to 0}\frac{2(\cos^2 x-1)}{\cos x-1}=\lim_{x\to 0}\frac{2(\cos x-1)(\cos x+1)}{\cos x-1}$$
$$=\lim_{x\to 0}2(\cos x+1)=4$$

49. $$\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}=\lim_{x\to 0}\frac{a+\cos x}{b\left(\frac{\sin x}{x}\right)}=\frac{a+1}{b\times 1}=\frac{a+1}{b}$$

58. Dividing numerator and denominator by x, we obtain
$$\lim_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}=\lim_{x\to 0}\frac{\frac{\sin ax}{x}+b}{a+\frac{\sin bx}{x}}=\lim_{x\to 0}\frac{a\left(\frac{\sin ax}{ax}\right)+b}{a+b\left(\frac{\sin bx}{bx}\right)}=\frac{a\times 1+b}{a+b\times 1}=\frac{a+b}{a+b}=1$$

28.7.2 EVALUATION OF TRIGONOMETRIC LIMITS WHEN VARIABLE TENDS TO A NON-ZERO QUANTITY

So far we have been discussing trigonometric limits when $x\to 0$. Now we will discuss evaluation of trigonometric limits when x tends to a non-zero real number. As we have already assumed that $\lim_{x\to a} f(x)$ always exists.

$$\therefore \quad \lim_{x\to a} f(x)=\lim_{x\to a^+} f(x) \qquad \left[\because \lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)\Leftrightarrow \lim_{x\to a} f(x)\right]$$
$$=\lim_{h\to 0} f(a+h)$$

Thus, we have the following algorithm to evaluate the said type of limits.

ALGORITHM

Step I *Obtain the problem. Suppose $x\to a$, where a is a non-zero real number.*

Step II *Replace $x\to a$ by $h\to 0$ and x by $(a+h)$.*

Step III *Solve the problem by using formulae discussed in the previous section.*

Following examples will illustrate the above procedure.

REMARK *In order to evaluate $\lim_{x\to a} f(x)$, where a is a non-zero real number, we may use the following results:*

(i) $\lim_{x\to a}\frac{\sin(x-a)}{x-a}=1$ (ii) $\lim_{x\to a}\frac{\tan(x-a)}{x-a}=1$

ILLUSTRATIVE EXAMPLES

BASIC

EXAMPLE 1 *Evaluate:*

(i) $\lim\limits_{x \to \pi} \dfrac{\sin x}{\pi - x}$ (ii) $\lim\limits_{x \to \pi} \dfrac{\sin 3x - 3\sin x}{(\pi - x)^3}$

(iii) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cos x}{\pi - 2x}$ (iv) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cos 3x + 3\cos x}{\left(\frac{\pi}{2} - x\right)^3}$

SOLUTION (i) $\lim\limits_{x \to \pi} \dfrac{\sin x}{(\pi - x)} = \lim\limits_{h \to 0} \dfrac{\sin(\pi + h)}{(\pi - (\pi + h))} = \lim\limits_{h \to 0} \dfrac{-\sin h}{-h} = \lim\limits_{h \to 0} \dfrac{\sin h}{h} = 1.$

ALITER $\lim\limits_{x \to \pi} \dfrac{\sin x}{\pi - x} = \lim\limits_{x \to \pi} \dfrac{\sin(\pi - x)}{(\pi - x)} = 1.$

(ii) $\lim\limits_{x \to \pi} \dfrac{\sin 3x - 3\sin x}{(\pi - x)^3} = \lim\limits_{x \to \pi} \dfrac{3\sin x - 4\sin^3 x - 3\sin x}{(\pi - x)^3} = -4 \lim\limits_{x \to \pi} \dfrac{\sin^3 x}{(\pi - x)^3}$

$$= -4 \lim_{h \to 0} \frac{\sin^3(\pi + h)}{\{\pi - (\pi + h)\}^3} = -4 \lim_{h \to 0} \frac{(-\sinh)^3}{(-h)^3} = -4 \lim_{h \to 0} \left(\frac{\sin h}{h}\right)^3$$

$$= -4 \times (1)^3 = -4$$

ALITER $\lim\limits_{x \to \pi} \dfrac{\sin 3x - 3\sin x}{(\pi - x)^3} = \lim\limits_{x \to \pi} \dfrac{3\sin x - 4\sin^3 x - 3\sin x}{(\pi - x)^3}$

$$= -4 \lim_{x \to \pi} \frac{\sin^3(\pi - x)}{(\pi - x)^3} = -4 \lim_{x \to \pi} \left\{\frac{\sin(\pi - x)}{\pi - x}\right\}^3 = -4 \times (1)^3 = -4$$

(iii) $\lim\limits_{x \to \pi/2} \dfrac{\cos x}{\pi - 2x} = \lim\limits_{h \to 0} \dfrac{\cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)} = \lim\limits_{h \to 0} \dfrac{-\sin h}{-2h} = \dfrac{1}{2} \lim\limits_{h \to 0} \dfrac{\sin h}{h} = \dfrac{1}{2} \times 1 = \dfrac{1}{2}$

ALITER $\lim\limits_{x \to \pi/2} \dfrac{\cos x}{\pi - 2x} = \lim\limits_{x \to \pi/2} \dfrac{\sin\left(\frac{\pi}{2} - x\right)}{2\left(\frac{\pi}{2} - x\right)} = \dfrac{1}{2} \lim\limits_{x \to \pi/2} \dfrac{\sin\left(\frac{\pi}{2} - x\right)}{\left(\frac{\pi}{2} - x\right)} = \dfrac{1}{2} \times 1 = \dfrac{1}{2}$

(iv) $\lim\limits_{x \to \pi/2} \dfrac{\cos 3x + 3\cos x}{\left(\frac{\pi}{2} - x\right)^3} = \lim\limits_{x \to \pi/2} \dfrac{4\cos^3 x - 3\cos x + 3\cos x}{\left(\frac{\pi}{2} - x\right)^3} = 4 \lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cos^3 x}{\left(\frac{\pi}{2} - x\right)^3}$

$$= 4 \lim_{h \to 0} \frac{\cos^3\left(\frac{\pi}{2} + h\right)}{\left\{\frac{\pi}{2} - \left(\frac{\pi}{2} + h\right)\right\}^3} = 4 \lim_{h \to 0} \frac{(-\sin h)^3}{(-h)^3} = 4 \lim_{h \to 0} \left(\frac{\sin h}{h}\right)^3 = 4 \times (1)^3 = 4$$

ALITER $\lim\limits_{x \to \pi/2} \dfrac{\cos 3x + 3\cos x}{\left(\frac{\pi}{2} - x\right)^3} = \lim\limits_{x \to \pi/2} \dfrac{4\cos^3 x - 3\cos x + 3\cos x}{\left(\frac{\pi}{2} - x\right)^3} = 4 \lim\limits_{x \to \pi/2} \dfrac{\cos^3 x}{\left(\frac{\pi}{2} - x\right)^3}$

$$= 4 \lim_{x \to \pi/2} \frac{\sin^3\left(\frac{\pi}{2} - x\right)}{\left(\frac{\pi}{2} - x\right)^3} = 4 \lim_{x \to \pi/2} \left\{ \frac{\sin\left(\frac{\pi}{2} - x\right)}{\left(\frac{\pi}{2} - x\right)} \right\}^3 = 4 \times (1)^3 = 4$$

EXAMPLE 2 *Evaluate:*

(i) $\lim_{x \to \frac{\pi}{2}} \frac{\cot x}{\frac{\pi}{2} - x}$ (ii) $\lim_{x \to \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}}$ (iii) $\lim_{x \to \frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \tan x$ [NCERT]

SOLUTION (i) $\lim_{x \to \frac{\pi}{2}} \frac{\cot x}{\left(\frac{\pi}{2} - x\right)} = \lim_{h \to 0} \frac{\cot\left(\frac{\pi}{2} + h\right)}{\frac{\pi}{2} - \left(\frac{\pi}{2} + h\right)} = \lim_{h \to 0} \frac{-\tan h}{-h} = \lim_{h \to 0} \frac{\tan h}{h} = 1$

ALITER $\lim_{x \to \frac{\pi}{2}} \frac{\cot x}{\left(\frac{\pi}{2} - x\right)} = \lim_{x \to \frac{\pi}{2}} \frac{\tan\left(\frac{\pi}{2} - x\right)}{\left(\frac{\pi}{2} - x\right)} = 1$

(ii) $\lim_{x \to \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}} = \lim_{h \to 0} \frac{\tan 2\left(\frac{\pi}{2} + h\right)}{\left(\frac{\pi}{2} + h\right) - \frac{\pi}{2}} = \lim_{h \to 0} \frac{\tan(\pi + 2h)}{h} = \lim_{h \to 0} \frac{\tan 2h}{h} = 2 \lim_{h \to 0} \frac{\tan 2h}{2h} = 2 \times 1 = 2$

ALITER $\lim_{x \to \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}} = \lim_{x \to \frac{\pi}{2}} -\frac{\tan(\pi - 2x)}{x - \frac{\pi}{2}} = -\lim_{x \to \frac{\pi}{2}} \frac{\tan 2\left(\frac{\pi}{2} - x\right)}{-\left(\frac{\pi}{2} - x\right)} = 2 \lim_{x \to \frac{\pi}{2}} \frac{\tan 2\left(\frac{\pi}{2} - x\right)}{2\left(\frac{\pi}{2} - x\right)}$

$= 2 \times 1 = 2$

(iii) $\lim_{x \to \frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \tan x = \lim_{h \to 0} \left\{\frac{\pi}{2} - \left(\frac{\pi}{2} + h\right)\right\} \tan\left(\frac{\pi}{2} + h\right) = \lim_{h \to 0} -h \times -\cot h = \lim_{h \to 0} \frac{h}{\tan h} = 1$

ALITER $\lim_{x \to \frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \tan x \ (0 \times \infty \text{ form}) = \lim_{x \to \frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)}{\cot x} \left(\frac{0}{0} \text{ form}\right) = \lim_{x \to \frac{\pi}{2}} \frac{\frac{\pi}{2} - x}{\tan\left(\frac{\pi}{2} - x\right)} = 1.$

EXAMPLE 3 *Evaluate the following limits:*

(i) $\lim_{n \to \infty} 2^n \sin \frac{a}{2^n}$ (ii) $\lim_{x \to \infty} x \tan \frac{1}{x}$ (iii) $\lim_{x \to \infty} 2^{x-1} \tan\left(\frac{a}{2^x}\right)$

SOLUTION (i) $\lim_{n \to \infty} 2^n \sin \frac{a}{2n}$ $(\infty \times 0 \text{ form})$

$$= \lim_{n \to \infty} \frac{\sin\left(\frac{a}{2^n}\right)}{\left(\frac{1}{2^n}\right)} = \lim_{n \to \infty} \left\{ \frac{\sin\left(\frac{a}{2^n}\right)}{\left(\frac{a}{2^n}\right)} \right\} \times a = 1 \times a = a \qquad \left(\frac{0}{0} \text{ form}\right)$$

(ii) $\lim\limits_{x \to \infty} x \tan \dfrac{1}{x} = \lim\limits_{x \to \infty} \left\{ \dfrac{\tan\left(\dfrac{1}{x}\right)}{\left(\dfrac{1}{x}\right)} \right\} = 1$

(iii) $\lim\limits_{x \to \infty} 2^{x-1} \tan\left(\dfrac{a}{2^x}\right) = \dfrac{a}{2} \lim\limits_{x \to \infty} \dfrac{\tan\left(\dfrac{a}{2x}\right)}{\left(\dfrac{a}{2x}\right)} = \dfrac{a}{2} \times 1 = \dfrac{a}{2}$

EXAMPLE 4 *Evaluate the following limits:*

(i) $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{\sin x - \cos x}{x - \dfrac{\pi}{4}}$ [NCERT EXEMPLAR] (ii) $\lim\limits_{x \to \pi/6} \dfrac{\sqrt{3} \sin x - \cos x}{x - \dfrac{\pi}{6}}$ [NCERT EXEMPLAR]

(iii) $\lim\limits_{x \to a} \dfrac{\sin x - \sin a}{x - a}$ (iv) $\lim\limits_{x \to a} \dfrac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$ [NCERT EXEMPLAR]

SOLUTION (i) We find that:

$$\lim_{\pi \to \pi/4} \frac{\sin x - \cos x}{x - \dfrac{\pi}{4}}$$

$$= \lim_{h \to 0} \frac{\sin\left(\dfrac{\pi}{4} + h\right) - \cos\left(\dfrac{\pi}{4} + h\right)}{\dfrac{\pi}{4} + h - \dfrac{\pi}{4}} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{\sin \dfrac{\pi}{4} \cos h + \cos \dfrac{\pi}{4} \sin h - \cos \dfrac{\pi}{4} \cos h + \sin \dfrac{\pi}{4} \sin h}{h} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{2\left(\dfrac{1}{\sqrt{2}} \sin h\right)}{h} = \frac{2}{\sqrt{2}} \lim_{h \to 0} \frac{\sin h}{h} = \sqrt{2}\,(1) = \sqrt{2}$$

ALITER $\lim\limits_{\pi \to \pi/4} \dfrac{\sin x - \cos x}{x - \dfrac{\pi}{4}} = \lim\limits_{x \to \pi/4} \dfrac{\sqrt{2} \left\{\dfrac{1}{\sqrt{2}} \sin x - \dfrac{1}{\sqrt{2}} \cos x\right\}}{x - \dfrac{\pi}{4}}$

$$= \sqrt{2} \lim_{x \to \pi/4} \frac{\sin x \cos \dfrac{\pi}{4} - \cos x \sin \dfrac{\pi}{4}}{x - \dfrac{\pi}{4}} = \sqrt{2} \lim_{x \to \pi/4} \frac{\sin\left(x - \dfrac{\pi}{4}\right)}{x - \dfrac{\pi}{4}}$$

$$= \sqrt{2} \times 1 = \sqrt{2}$$

(ii) We have,

$$\lim_{x \to \frac{\pi}{6}} \frac{\sqrt{3} \sin x - \cos x}{x - \dfrac{\pi}{6}} = 2 \lim_{x \to \pi/6} \frac{\left(\sin x \cos \dfrac{\pi}{6} - \cos x \sin \dfrac{\pi}{6}\right)}{x - \dfrac{\pi}{6}} = 2 \lim_{x \to \pi/6} \frac{\sin\left(x - \dfrac{\pi}{6}\right)}{\left(x - \dfrac{\pi}{6}\right)} = 2.$$

(iii) We have,

$$\lim_{x \to a} \frac{\sin x - \sin a}{x - a} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{\sin(a+h) - \sin a}{a+h-a} = \lim_{h \to 0} \frac{\sin a \cos h + \cos a \sin h - \sin a}{h} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \cos a \frac{\sin h}{h} - \sin a \left(\frac{1 - \cos h}{h}\right) = \lim_{h \to 0} \cos a \frac{\sin h}{h} - \sin a \lim_{h \to 0} \left(\frac{1 - \cos h}{h}\right)$$

$$= \cos a \left(\lim_{h \to 0} \frac{\sin h}{h}\right) - \sin a \left(\lim_{h \to 0} \frac{2 \sin^2 \frac{h}{2}}{h}\right) = \cos a \lim_{h \to 0} \frac{\sin h}{h} - 2 \sin a \lim_{h \to 0} \left(\frac{\frac{\sin h/2}{h}}{h/2}\right)^2 \times \frac{h}{4}$$

$$= \cos a \times 1 - 2 \sin a (1)^2 \times 0 = \cos a$$

ALITER $$\lim_{x \to a} \frac{\sin x - \sin a}{x - a} = \lim_{x \to a} \frac{2 \sin\left(\frac{x-a}{2}\right) \cos\left(\frac{x+a}{2}\right)}{2\left(\frac{x-a}{2}\right)} = \lim_{x \to a} \left\{\frac{\sin\left(\frac{x-a}{2}\right)}{\frac{x-a}{2}}\right\} \times \cos\left(\frac{x+a}{2}\right)$$

$$= \cos a$$

(iv) We have,

$$\lim_{x \to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}} = \lim_{x \to a} \frac{\sin x - \sin a}{x - a} (\sqrt{x} + \sqrt{a})$$

$$= \lim_{x \to a} \frac{2 \sin\left(\frac{x-a}{2}\right) \cos\left(\frac{x+a}{2}\right)}{x - a} (\sqrt{x} + \sqrt{a})$$

$$= \lim_{x \to a} \frac{2 \sin\left(\frac{x-a}{2}\right)}{2\left(\frac{x-a}{2}\right)} \times \cos\left(\frac{x+a}{2}\right) \times (\sqrt{x} + \sqrt{a}) = 1 \times \cos a \times (\sqrt{a} + \sqrt{a}) = 2\sqrt{a} \cos a$$

EXAMPLE 5 *Evaluate the following limits:*

(i) $\lim_{x \to \frac{\pi}{2}} \frac{1 + \cos 2x}{(\pi - 2x)^2}$ (ii) $\lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{\left(\frac{\pi}{2} - x\right)^2}$

SOLUTION (i) We have,

$$\lim_{x \to \pi/2} \frac{1 + \cos 2x}{(\pi - 2x)^2} = \lim_{h \to 0} \frac{1 + \cos 2\left(\frac{\pi}{2} + h\right)}{\left\{\pi - 2\left(\frac{\pi}{2} + h\right)\right\}^2} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{1 + \cos(\pi + 2h)}{4h^2} = \lim_{h \to 0} \frac{1 - \cos 2h}{4h^2} = \lim_{h \to 0} \frac{2 \sin^2 h}{4h^2}$$

$$= \frac{2}{4} \lim_{h \to 0} \left(\frac{\sin h}{h}\right)\left(\frac{\sin h}{h}\right) = \frac{1}{2}\left(\lim_{h \to 0} \frac{\sin h}{h}\right)\left(\lim_{h \to 0} \frac{\sin h}{h}\right) = \frac{1}{2}$$

ALITER $\lim\limits_{x \to \pi/2} \dfrac{1+\cos 2x}{(\pi-2x)^2} = \lim\limits_{x \to \pi/2} \dfrac{1-\cos(\pi-2x)}{(\pi-2x)^2} = \lim\limits_{x \to \pi/2} \dfrac{2\sin^2\left(\frac{\pi}{2}-x\right)}{4\left(\frac{\pi}{2}-x\right)^2}$

$$= \frac{1}{2} \lim_{x \to \pi/2} \left\{ \frac{\sin\left(\frac{\pi}{2}-x\right)}{\left(\frac{\pi}{2}-x\right)} \right\}^2 = \frac{1}{2} \times (1)^2 = \frac{1}{2}$$

(ii) We have,

$$\lim_{x \to \frac{\pi}{2}} \frac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2} = \lim_{h \to 0} \frac{1-\sin\left(\frac{\pi}{2}+h\right)}{\left\{\frac{\pi}{2}-\left(\frac{\pi}{2}+h\right)\right\}^2} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{1-\cos h}{h^2} = \lim_{h \to 0} 2\frac{\sin^2 \frac{h}{2}}{h^2} = \frac{2}{4} \lim_{h \to 0} \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}}\right)\left(\frac{\sin \frac{h}{2}}{\frac{h}{2}}\right) = \frac{1}{2}(1)(1) = \frac{1}{2}$$

ALITER $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2} = \lim\limits_{x \to \frac{\pi}{2}} \dfrac{1-\cos\left(\frac{\pi}{2}-x\right)}{\left(\frac{\pi}{2}-x\right)^2} = \lim\limits_{x \to \frac{\pi}{2}} \dfrac{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{4\left(\frac{\pi}{4}-\frac{x}{2}\right)^2}$

$$= \frac{2}{4} \lim_{x \to \frac{\pi}{2}} \left\{ \frac{\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)}{\left(\frac{\pi}{4}-\frac{x}{2}\right)} \right\}^2 = \frac{2}{4} \times (1)^2 = \frac{1}{2}$$

EXAMPLE 6 *Evaluate the following limits:*

(i) $\lim\limits_{x \to a} \dfrac{\cos x - \cos a}{\cot x - \cot a}$ (ii) $\lim\limits_{x \to \pi/2} (\sec x - \tan x)$ **[NCERT EXEMPLAR]**

(iii) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cot x - \cos x}{(\pi-2x)^3}$

SOLUTION (i) We have,

$$\lim_{x \to a} \frac{\cos x - \cos a}{\cot x - \cot a} = \lim_{x \to a} \frac{\cos x - \cos a}{\cos x \sin a - \sin x \cos a} \cdot \sin x \sin a \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to a} \frac{-2\sin\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)}{-2\sin\left(\frac{x-a}{2}\right)\cos\left(\frac{x-a}{2}\right)} \sin x \sin a$$

$$= \lim_{x \to a} \frac{\sin\left(\frac{x+a}{2}\right)}{\cos\left(\frac{x-a}{2}\right)} \sin x \sin a = \frac{\sin a}{1} \times \sin a \sin a = \sin^3 a$$

(ii) We have,

$$\lim_{x \to \pi/2} (\sec x - \tan x) \qquad (\text{form } \infty - \infty)$$

$$= \lim_{x \to \pi/2} \left(\frac{1 - \sin x}{\cos x}\right) \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \left\{\frac{1 - \sin(\pi/2 + h)}{\cos\left(\frac{\pi}{2} + h\right)}\right\} = \lim_{h \to 0} \frac{1 - \cos h}{-\sin h} = \lim_{h \to 0} \frac{2\sin^2 \frac{h}{2}}{-2\sin h/2 \cos h/2}$$

$$= -\lim_{h \to 0} \tan h/2 = -\tan 0 = 0$$

ALITER $\displaystyle \lim_{x \to \pi/2} (\sec x - \tan x) = \lim_{x \to \pi/2} \frac{1 - \sin x}{\cos x} = \lim_{x \to \pi/2} \frac{1 - \cos\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right)}$

$$= \lim_{x \to \pi/2} \frac{2\sin^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)\cos\left(\frac{\pi}{4} - \frac{\pi}{2}\right)} = \lim_{x \to \pi/2} \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) = 0$$

(iii) We have,

$$\lim_{x \to \pi/2} \frac{\cot x - \cos x}{(\pi - 2x)^3} = \lim_{h \to 0} \frac{\cot\left(\frac{\pi}{2} + h\right) - \cos\left(\frac{\pi}{2} + h\right)}{\left\{\pi - 2\left(\frac{\pi}{2} + h\right)\right\}^3} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{-\tan h + \sin h}{-8h^3} = \lim_{h \to 0} \frac{-\sin h(1 - \cos h)}{\cos h(-8h^3)} = \frac{1}{8} \lim_{h \to 0} \frac{\tan h}{h} \times \frac{1 - \cos h}{h^2}$$

$$= \frac{1}{8} \lim_{h \to 0} \frac{\tan h}{h} \times \lim_{h \to 0} \frac{2\sin^2 \frac{h}{2}}{h^2} = \frac{1}{8} \times \frac{2}{4} \lim_{h \to 0} \frac{\sin^2 \frac{h}{2}}{\left(\frac{h}{2}\right)^2} = \frac{1}{16}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 7 *Evaluate:* $\displaystyle \lim_{x \to \frac{\pi}{6}} \frac{2 - \sqrt{3}\cos x - \sin x}{(6x - \pi)^2}$

SOLUTION We have,

$$\lim_{x \to \frac{\pi}{6}} \frac{2 - \sqrt{3}\cos x - \sin x}{(6x - \pi)^2} = \lim_{h \to 0} \frac{2 - \sqrt{3}\cos\left(\frac{\pi}{6} + h\right) - \sin\left(\frac{\pi}{6} + h\right)}{\left\{6\left(\frac{\pi}{6} + h\right) - \pi\right\}^2} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{2 - \sqrt{3}\left(\cos\frac{\pi}{6}\cos h - \sin\frac{\pi}{6}\sin h\right) - \left(\sin\frac{\pi}{6}\cos h + \cos\frac{\pi}{6}\sin h\right)}{36h^2}$$

$$= \lim_{h \to 0} \frac{2 - \frac{3}{2}\cos h + \frac{\sqrt{3}}{2}\sin h - \frac{1}{2}\cos h - \frac{\sqrt{3}}{2}\sin h}{36 h^2} = \lim_{h \to 0} \frac{2(1 - \cos h)}{36 h^2}$$

$$= \frac{1}{18} \lim_{h \to 0} \frac{2 \sin^2\left(\frac{h}{2}\right)}{h^2} = \frac{1}{9} \lim_{h \to 0} \left(\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right)^2 \times \frac{1}{4} = \frac{1}{9} \times (1)^2 \times \frac{1}{4} = \frac{1}{36}$$

ALITER $\lim_{x \to \pi/6} \frac{2 - \sqrt{3}\cos x - \sin x}{(6x - \pi)^2} = \lim_{x \to \pi/6} \frac{2 - 2\left\{\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x\right\}}{(6x - \pi)^2}$

$$= \lim_{x \to \pi/6} \frac{2 - 2\left\{\cos x \cos\frac{\pi}{6} + \sin x \sin\frac{\pi}{6}\right\}}{(6x - \pi)^2} = \lim_{x \to \pi/6} \frac{2 - 2\cos\left(x - \frac{\pi}{6}\right)}{(6x - \pi)^2}$$

$$= \lim_{x \to \pi/6} \frac{2 \times 2\sin^2\left(\frac{x}{2} - \frac{\pi}{12}\right)}{144\left(\frac{x}{2} - \frac{\pi}{12}\right)^2} = \frac{4}{144} \lim_{x \to \pi/6} \left\{\frac{\sin\left(\frac{x}{2} - \frac{\pi}{12}\right)}{\left(\frac{x}{2} - \frac{\pi}{12}\right)}\right\}^2 = \frac{1}{36} \times (1)^2 = \frac{1}{36}$$

EXAMPLE 8 *Prove that:* $\lim_{x \to \pi/4} \frac{\tan^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)} = -4$ [NCERT EXEMPLAR]

SOLUTION We have,

$$\lim_{x \to \pi/4} \frac{\tan^3 x - \tan x}{\cos(x + \pi/4)} = \lim_{x \to \pi/4} \frac{\tan x(\tan x - 1)(\tan x + 1)}{\cos(x + \pi/4)}$$

$$= \lim_{x \to \pi/4} \frac{\tan x(\sin x - \cos x)(\tan x + 1)}{\cos x \cos(x + \pi/4)}$$

$$= -\lim_{x \to \pi/4} \frac{\tan x(\cos x - \sin x)(\tan x + 1)}{\cos x \cos(x + \pi/4)}$$

$$= -\sqrt{2} \lim_{x \to \pi/4} \frac{\tan x\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)(\tan x + 1)}{\cos x \cos(x + \pi/4)}$$

$$= -\sqrt{2} \lim_{x \to \pi/4} \frac{\tan x \cos(x + \pi/4)(\tan x + 1)}{\cos x \cos(x + \pi/4)}$$

$$= -\sqrt{2} \times \lim_{x \to \pi/4} \frac{\tan x(\tan x + 1)}{\cos x} = -\sqrt{2} \times 2\sqrt{2} = -4.$$

EXAMPLE 9 *Evaluate :* $\lim_{n \to \infty} \left(\cos\frac{x}{2} \cos\frac{x}{2^2} \cos\frac{x}{2^3} \ldots \cos\frac{x}{2^n}\right)$

SOLUTION We have,

$$\lim_{n \to \infty} \left(\cos\frac{x}{2} \cos\frac{x}{2^2} \cos\frac{x}{2^3} \ldots \cos\frac{x}{2^n}\right)$$

$$= \lim_{n \to \infty} \frac{\sin\left(2^n \times \frac{x}{2^n}\right)}{2^n \sin\left(\frac{x}{2^n}\right)} \qquad \left[\because \cos A \cos 2A \ldots \cos 2^{n-1} A = \frac{\sin 2^n A}{2^n \sin A}\right]$$

$$= \lim_{n \to \infty} \frac{\sin x}{x \left\{\frac{\sin\left(\frac{x}{2^n}\right)}{\left(\frac{x}{2^n}\right)}\right\}} = \frac{\sin x}{x} \qquad \left[\because \lim_{n \to \infty} \frac{\sin\left(\frac{x}{2^n}\right)}{\left(\frac{x}{2^n}\right)} = 1\right]$$

EXAMPLE 10 *Evaluate :* $\lim_{x \to \pi/4} \frac{4\sqrt{2} - (\cos x + \sin x)^5}{1 - \sin 2x}$

SOLUTION We have,

$$\lim_{x \to \pi/4} \frac{4\sqrt{2} - (\cos x + \sin x)^5}{1 - \sin 2x} = \lim_{x \to \pi/4} \frac{2^{5/2} - \left\{(\cos x + \sin x)^2\right\}^{5/2}}{2 - (1 + \sin 2x)}$$

$$= \lim_{x \to \pi/4} \frac{2^{5/2} - (1 + \sin 2x)^{5/2}}{2 - (1 + \sin 2x)}$$

$$= \lim_{x \to \pi/4} \frac{(1 + \sin 2x)^{5/2} - 2^{5/2}}{(1 + \sin 2x) - 2}$$

$$= \lim_{u \to 2} \frac{u^{5/2} - 2^{5/2}}{u - 2}, \text{ where } u = 1 + \sin 2x$$

$$= \frac{5}{2} \times (2)^{5/2 - 1} = \frac{5}{2} \times 2^{3/2} = 5\sqrt{2}$$

EXAMPLE 11 *If* α, β *are the roots of* $ax^2 + bx + c$*, then evaluate* $\lim_{x \to \beta} \frac{1 - \cos(ax^2 + bx + c)}{(x - \beta)^2}$.

SOLUTION It is given that α and β are the roots of the given equation $ax^2 + bx + c = 0$. Therefore,

$$ax^2 + bx + c = a(x - \alpha)(x - \beta)$$

Now,

$$\lim_{x \to \beta} \frac{1 - \cos(ax^2 + bx + c)}{(x - \beta)^2} = \lim_{x \to \beta} \frac{1 - \cos\{a(x - \alpha)(x - \beta)\}}{(x - \beta)^2} = \lim_{x \to \beta} \frac{2 \sin^2 \left\{\frac{a(x - \alpha)(x - \beta)}{2}\right\}}{(x - \beta)^2}$$

$$= 2 \lim_{x \to \beta} \left[\frac{\sin\left\{\frac{a(x - \alpha)(x - \beta)}{2}\right\}}{\left\{\frac{a(x - \alpha)(x - \beta)}{2}\right\}}\right]^2 \times \frac{a^2 (x - \alpha)^2}{4}$$

$$= 2 \times \frac{a^2 (\beta - \alpha)^2}{4} = \frac{a^2 (\beta - \alpha)^2}{2}$$

EXERCISE 28.8

BASED ON LOTS

Evaluate the following limits:

1. $\lim\limits_{x \to \pi/2} \left(\frac{\pi}{2} - x\right) \tan x$

2. $\lim\limits_{x \to \frac{\pi}{2}} \frac{\sin 2x}{\cos x}$

3. $\lim\limits_{x \to \frac{\pi}{2}} \frac{\cos^2 x}{1 - \sin x}$

4. $\lim\limits_{x \to \frac{\pi}{3}} \frac{\sqrt{1 - \cos 6x}}{\sqrt{2}\,(\pi/3 - x)}$

5. $\lim\limits_{x \to a} \frac{\cos x - \cos a}{x - a}$

6. $\lim\limits_{x \to \frac{\pi}{4}} \frac{1 - \tan x}{x - \frac{\pi}{4}}$

7. $\lim\limits_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{\left(\frac{\pi}{2} - x\right)^2}$

8. $\lim\limits_{x \to \frac{\pi}{3}} \frac{\sqrt{3} - \tan x}{\pi - 3x}$

9. $\lim\limits_{n \to \infty} n \sin\left(\frac{\pi}{4n}\right) \cos\left(\frac{\pi}{4n}\right)$

10. $\lim\limits_{n \to \infty} 2^{n-1} \sin\left(\frac{a}{2^n}\right)$

11. $\lim\limits_{n \to \infty} \frac{\sin\left(\frac{a}{2^n}\right)}{\sin\left(\frac{b}{2^n}\right)}$

12. $\lim\limits_{x \to -1} \frac{x^2 - x - 2}{(x^2 + x) + \sin(x + 1)}$

13. $\lim\limits_{x \to 2} \frac{x^2 - x - 2}{x^2 - 2x + \sin(x - 2)}$

BASED ON HOTS

14. $\lim\limits_{x \to a} \frac{a \sin x - x \sin a}{ax^2 - xa^2}$

15. $\lim\limits_{x \to \pi/2} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x}$

16. $\lim\limits_{x \to \frac{\pi}{2}} \frac{\sqrt{2 - \sin x} - 1}{\left(\frac{\pi}{2} - x\right)^2}$

17. $\lim\limits_{x \to \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{\left(\frac{\pi}{4} - x\right)^2}$

18. $\lim\limits_{x \to \frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3}$

19. $\lim\limits_{x \to a} \frac{\cos x - \cos a}{\sqrt{x} - \sqrt{a}}$

20. $\lim\limits_{x \to \pi} \frac{\sqrt{5 + \cos x} - 2}{(\pi - x)^2}$

21. $\lim\limits_{x \to a} \frac{\cos\sqrt{x} - \cos\sqrt{a}}{x - a}$

22. $\lim\limits_{x \to a} \frac{\sin\sqrt{x} - \sin\sqrt{a}}{x - a}$

23. $\lim\limits_{x \to 1} \frac{1 - x^2}{\sin 2\pi x}$

24. $\lim\limits_{x \to \frac{\pi}{4}} \frac{f(x) - f\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}}$, where $f(x) = \sin 2x$

25. $\lim\limits_{x \to 1} \frac{1 + \cos \pi x}{(1 - x)^2}$

26. $\lim\limits_{x \to 1} \frac{1 - x^2}{\sin \pi x}$

27. $\lim\limits_{x \to \frac{\pi}{4}} \frac{1 - \sin 2x}{1 + \cos 4x}$

28. $\lim\limits_{x \to \pi} \frac{1 + \cos x}{\tan^2 x}$

29. $\lim\limits_{x \to 1} (1 - x) \tan\left(\frac{\pi x}{2}\right)$

30. $\lim\limits_{x \to \frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2} \sin x}$

31. $\lim\limits_{x \to \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2}$

32. $\lim\limits_{x \to \pi/4} \frac{\sqrt{\cos x} - \sqrt{\sin x}}{x - \pi/4}$

33. $\lim\limits_{x \to 1} \dfrac{1-\dfrac{1}{x}}{\sin \pi(x-1)}$

34. $\lim\limits_{x \to \frac{\pi}{6}} \dfrac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$ [NCERT EXEMPLAR]

35. $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{\sqrt{2} - \cos x - \sin x}{(4x-\pi)^2}$

36. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\left(\dfrac{\pi}{2} - x\right) \sin x - 2\cos x}{\left(\dfrac{\pi}{2} - x\right) + \cot x}$

37. $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{\cos x - \sin x}{\left(\dfrac{\pi}{4} - x\right)(\cos x + \sin x)}$

38. $\lim\limits_{x \to \pi} \dfrac{1 - \sin \dfrac{x}{2}}{\cos \dfrac{x}{2}\left(\cos \dfrac{x}{4} - \sin \dfrac{x}{4}\right)}$ [NCERT EXEMPLAR]

ANSWERS

1. 1 2. 2 3. 2 4. Does not exist 5. $-\sin a$ 6. -2 7. $\frac{1}{2}$

8. $\frac{4}{3}$ 9. $\frac{\pi}{4}$ 10. $\frac{a}{2}$ 11. $\frac{a}{b}$ 12. ∞ 13. 1 14. $\frac{a\cos a - \sin a}{a^2}$

15. $\frac{1}{4\sqrt{2}}$ 16. $\frac{1}{4}$ 17. $\frac{1}{\sqrt{2}}$ 18. $\frac{1}{16}$ 19. $-2\sqrt{a}\sin a$ 20. $\frac{1}{8}$

21. $-\frac{1}{2\sqrt{a}}\sin\sqrt{a}$ 22. $\frac{1}{2\sqrt{a}}\cos\sqrt{a}$ 23. $\frac{-1}{\pi}$ 24. 0 25. $\frac{\pi^2}{2}$ 26. $\frac{2}{\pi}$

27. $\frac{1}{4}$ 28. $\frac{1}{2}$ 29. $\frac{2}{\pi}$ 30. 2 31. $\frac{1}{4}$ 32. $-\frac{1}{2^{1/4}}$ 33. $\frac{1}{\pi}$ 34. 4

35. $\frac{1}{16\sqrt{2}}$ 36. $-\frac{1}{2}$ 37. 1 38. $\sqrt{2}$

28.7.3 EVALUATION OF TRIGONOMETRIC LIMITS BY FACTORISATION

Sometimes trigonometric limits can also be evaluated by factorisation method as illustrated in the following examples.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate the following limits:*

(i) $\lim\limits_{x \to \pi/4} \dfrac{\sec^2 x - 2}{\tan x - 1}$ (ii) $\lim\limits_{x \to \pi} \dfrac{1 + \sec^3 x}{\tan^2 x}$ (iii) $\lim\limits_{x \to \pi/2} \dfrac{1 - \sin^3 x}{\cos^2 x}$

(iv) $\lim\limits_{x \to \pi} \dfrac{1 + \cos^3 x}{\sin^2 x}$ (v) $\lim\limits_{x \to \pi/2} \dfrac{\sqrt{2} - \sqrt{1 + \sin x}}{\sqrt{2}\cos^2 x}$ (vi) $\lim\limits_{x \to 0} \dfrac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x}$

SOLUTION (i) $\lim\limits_{x \to \pi/4} \dfrac{\sec^2 x - 2}{\tan x - 1} = \lim\limits_{x \to \pi/4} \dfrac{1 + \tan^2 x - 2}{\tan x - 1} = \lim\limits_{x \to \pi/4} (\tan x + 1) = 2.$

(ii) $\lim\limits_{x \to \pi} \dfrac{1 + \sec^3 x}{\tan^2 x} = \lim\limits_{x \to \pi} \dfrac{(1 + \sec^3 x)}{(\sec^2 x - 1)} = \lim\limits_{x \to \pi} \dfrac{(\sec x + 1)(\sec^2 x - \sec x + 1)}{(\sec x + 1)(\sec x - 1)}$ $\left(\text{form } \dfrac{0}{0}\right)$

$$= \lim_{x \to \pi} \frac{\sec^2 x - \sec x + 1}{\sec x - 1} = \frac{1 + 1 + 1}{-2} = -\frac{3}{2}$$

(iii) $\lim\limits_{x \to \pi/2} \dfrac{1-\sin^3 x}{\cos^2 x} = \lim\limits_{x \to \pi/2} \dfrac{(1-\sin x)(1+\sin x+\sin^2 x)}{(1-\sin x)(1+\sin x)} = \lim\limits_{x \to \pi/2} \dfrac{1+\sin x+\sin^2 x}{1+\sin x}$

$$= \frac{1+1+1}{1+1} = \frac{3}{2}$$

(iv) $\lim\limits_{x \to \pi} \dfrac{1+\cos^3 x}{\sin^2 x} = \lim\limits_{x \to \pi} \dfrac{(1+\cos x)(1-\cos x+\cos^2 x)}{(1-\cos x)(1+\cos x)} = \lim\limits_{x \to \pi} \dfrac{1-\cos x+\cos^2 x}{1-\cos x}$

$$= \frac{1+1+1}{1+1} = \frac{3}{2}$$

(v) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\sqrt{2}-\sqrt{1+\sin x}}{\sqrt{2}\cos^2 x} = \lim\limits_{x \to \frac{\pi}{2}} \dfrac{\sqrt{2}-\sqrt{1+\sin x}}{\sqrt{2}\cos^2 x} \times \dfrac{\sqrt{2}+\sqrt{1+\sin x}}{\sqrt{2}+\sqrt{1+\sin x}}$

$$= \lim_{x \to \frac{\pi}{2}} \frac{2-(1+\sin x)}{\sqrt{2}(1-\sin^2 x)} \times \frac{1}{\sqrt{2}+\sqrt{1+\sin x}} = \lim_{x \to \frac{\pi}{2}} \frac{(1-\sin x)}{\sqrt{2}(1-\sin x)(1+\sin x)} \times \frac{1}{\sqrt{2}+\sqrt{1+\sin x}}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{1}{\sqrt{2}(1+\sin x)} \times \frac{1}{\sqrt{2}+\sqrt{1+\sin x}} = \frac{1}{2\sqrt{2}} \times \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{8}$$

(vi) $\lim\limits_{x \to 0} \dfrac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x} = \lim\limits_{x \to 0} \dfrac{\sqrt{2}-\sqrt{2}\cos\frac{x}{2}}{(1-\cos x)(1+\cos x)}$

$$= \lim_{x \to 0} \frac{\sqrt{2}\left(1-\cos\frac{x}{2}\right)}{2\sin^2\frac{x}{2}(1+\cos x)} = \frac{1}{\sqrt{2}} \lim_{x \to 0} \frac{1-\cos\frac{x}{2}}{\left(1-\cos\frac{x}{2}\right)\left(1+\cos\frac{x}{2}\right)(1+\cos x)}$$

$$= \frac{1}{\sqrt{2}} \lim_{x \to 0} \frac{1}{\left(1+\cos\frac{x}{2}\right)(1+\cos x)} = \frac{1}{\sqrt{2}} \frac{1}{(1+1)(1+1)} = \frac{1}{4\sqrt{2}}$$

EXAMPLE 2 *Evaluate:* $\lim\limits_{x \to \pi/6} \dfrac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1}$

SOLUTION We find that:

$$\lim_{x \to \pi/6} \frac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1} = \lim_{x \to \pi/6} \frac{(2\sin x-1)(\sin x+1)}{(2\sin x-1)(\sin x-1)} = \lim_{x \to \pi/6} \frac{\sin x+1}{\sin x-1} = \frac{\frac{1}{2}+1}{\frac{1}{2}-1} = -3$$

EXERCISE 28.9

BASIC

Evaluate the following limits:

1. $\lim\limits_{x \to \pi} \dfrac{1+\cos x}{\tan^2 x}$
2. $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{\operatorname{cosec}^2 x-2}{\cot x-1}$
3. $\lim\limits_{x \to \frac{\pi}{6}} \dfrac{\cot^2 x-3}{\operatorname{cosec} x-2}$
4. $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{2-\operatorname{cosec}^2 x}{1-\cot x}$
5. $\lim\limits_{x \to \pi} \dfrac{\sqrt{2+\cos x}-1}{(\pi-x)^2}$
6. $\lim\limits_{x \to \frac{3\pi}{2}} \dfrac{1+\operatorname{cosec}^3 x}{\cot^2 x}$

ANSWERS

1. $\frac{1}{2}$ 2. 2 3. 4 4. 2 5. $\frac{1}{4}$ 6. $-\frac{3}{2}$

28.8 EVALUATION OF EXPONENTIAL AND LOGARITHMIC LIMITS

Sometimes, following expansions are useful in evaluating limits. Students are advised to learn these expansions.

1. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$

2. $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \dots$

3. $\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \dots$

4. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots$

5. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \dots$

6. $a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \dots$

7. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots$

8. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots$

9. $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots$

THEOREM Prove that:

(i) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$ (ii) $\lim_{x \to 0} \frac{\log_e(1+x)}{x} = 1$ (iii) $\lim_{x \to 0} \frac{\log_a(1+x)}{x} = \log_a e$

PROOF (i) Using expansion of a^x, we obtain

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{x \to 0} \frac{1 + x\log_e a + \frac{x^2}{2!}(\log_e a)^2 + \dots - 1}{x} = \lim_{x \to 0} \left\{\log_e a + \frac{x}{2!}(\log_e a)^2 + \dots\right\}$$

$$= \log_e a.$$

(ii) Using expansion of $\log_e(1+x)$, we obtain

$$\lim_{x \to 0} \frac{\log_e(1+x)}{x} = \lim_{x \to 0} \frac{x - \frac{x^2}{2} + \frac{x^3}{3} - \dots}{x} = \lim_{x \to 0} 1 - \frac{x}{2} + \frac{x^2}{3} \dots = 1.$$

(iii) $\lim\limits_{x\to 0} \dfrac{\log_a(1+x)}{x} = \lim\limits_{x\to 0} \dfrac{\log_e(1+x)}{x\log_e a} = \dfrac{1}{\log_e a} \lim\limits_{x\to 0} \dfrac{\log_e(1+x)}{x} = \dfrac{1}{\log_e a} = \log_a e$

COROLLARY Putting $a = e$ in (i) we get

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \log_e e = 1.$$

REMARK Throughout this text $\log x$ stands for $\log_e x$ or $\ln x$ unless stated otherwise.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate the following limits:*

(i) $\lim\limits_{x\to 0} \dfrac{a^x - b^x}{x}$ (ii) $\lim\limits_{x\to 0} \dfrac{a^x - b^x}{\sin x}$ (iii) $\lim\limits_{x\to 0} \dfrac{2^x - 1}{\sqrt{1+x}-1}$

(iv) $\lim\limits_{x\to 0} \dfrac{3^x - 2^x}{\tan x}$ (v) $\lim\limits_{x\to 0} \dfrac{a^{\sin x} - 1}{\sin x}$ (vi) $\lim\limits_{x\to 1} \dfrac{a^{x-1} - 1}{\sin \pi x}$

SOLUTION (i) We have,

$$\lim_{x\to 0} \frac{a^x - b^x}{x} = \lim_{x\to 0} \frac{(a^x-1)-(b^x-1)}{x} = \lim_{x\to 0} \frac{a^x-1}{x} - \lim_{x\to 0} \frac{b^x-1}{x} = \log_e a - \log_e b = \log_e\left(\frac{a}{b}\right).$$

ALITER $\lim\limits_{x\to 0} \dfrac{a^x - b^x}{x} = \lim\limits_{x\to 0} \dfrac{b^x\left\{\left(\frac{a}{b}\right)^x - 1\right\}}{x} = b^0 \times \log_e\left(\dfrac{a}{b}\right) = \log_e\left(\dfrac{a}{b}\right)$

(ii) We have,

$$\lim_{x\to 0} \frac{a^x - b^x}{\sin x} = \lim_{x\to 0} \left\{\left(\frac{a^x-1}{\sin x}\right) - \left(\frac{b^x-1}{\sin x}\right)\right\} = \lim_{x\to 0}\left(\frac{a^x-1}{x} \times \frac{x}{\sin x}\right) - \lim_{x\to 0}\left(\frac{b^x-1}{x}\times\frac{x}{\sin x}\right)$$

$$= \lim_{x\to 0} \frac{a^x-1}{x} \times \lim_{x\to 0} \frac{x}{\sin x} - \lim_{x\to 0} \frac{b^x-1}{x} \times \lim_{x\to 0} \frac{x}{\sin x}$$

$$= (\log_e a) \times 1 - (\log_e b) \times 1 = \log_e\left(\frac{a}{b}\right).$$

(iii) We have,

$$\lim_{x\to 0} \frac{2^x - 1}{\sqrt{1+x}-1} = \lim_{x- 0} \frac{2^x-1}{\sqrt{1+x}-1} \times \frac{(\sqrt{1+x}+1)}{(\sqrt{1+x}+1)} = \lim_{x- 0} \frac{2^x-1}{x} \times \left\{\sqrt{1+x}+1\right\}$$

$$= \lim_{x\to 0} \frac{2^x-1}{x} \times \lim_{x\to 0} (\sqrt{1+x}+1) = (\log_e 2)\, 2 = 2\log_e 2.-\log_e 4$$

(iv) $\lim\limits_{x\to 0} \dfrac{3^x - 2^x}{\tan x} = \lim\limits_{x\to 0} \dfrac{(3^x-1)-(2^x-1)}{x} \times \dfrac{x}{\tan x}$

$$= \left\{\lim_{x\to 0} \frac{3^x-1}{x} - \lim_{x\to 0} \frac{2^x-1}{x}\right\} \times \lim_{x\to 0} \frac{x}{\tan x} = (\log_e 3 - \log_e 2) \times 1 = \log_e\left(\frac{3}{2}\right)$$

(v) Let $y = \sin x$. Then, $y \to 0$ as $x \to 0$.

$\therefore$ $\lim\limits_{x\to 0} \dfrac{a^{\sin x} - 1}{\sin x} = \lim\limits_{y\to 0} \dfrac{a^y - 1}{y} = \log_e a$

(vi) We have,

$$\lim_{x\to 1}\frac{a^{x-1}-1}{\sin \pi x}=\lim_{h\to 0}\frac{a^{1+h-1}-1}{\sin \pi(1+h)}=\lim_{h\to 0}\frac{a^h-1}{-\sin \pi h}$$

$$=\frac{-1}{\pi}\lim_{h\to 0}\left(\frac{a^h-1}{h}\right)\left(\frac{\pi h}{\sin \pi h}\right)=-\frac{1}{\pi}\left(\lim_{h\to 0}\frac{a^h-1}{h}\right)\left(\lim_{h\to 0}\frac{\pi h}{\sin \pi h}\right)=-\frac{1}{\pi}\log_e a$$

ALITER $\displaystyle\lim_{x\to 1}\frac{a^{x-1}-1}{\sin \pi x}=\lim_{x\to 1}\frac{a^{x-1}-1}{x-1}\times\frac{x-1}{\sin(\pi-\pi x)}$

$$=\lim_{x\to 1}\frac{a^{x-1}-1}{x-1}\times\frac{\pi(x-1)}{-\pi\sin \pi(x-1)}=\log_e a\times\frac{1}{-\pi}=-\frac{1}{\pi}\log_e a$$

BASED ON LOTS

EXAMPLE 2 *Evaluate the following limits:*

(i) $\displaystyle\lim_{x\to 0}\frac{10^x-2^x-5^x+1}{x\tan x}$ (ii) $\displaystyle\lim_{x\to 0}\frac{3^x+3^{-x}-2}{x^2}$ (iii) $\displaystyle\lim_{x\to 0}\frac{2^{3x}-3^x}{\sin 3x}$

(iv) $\displaystyle\lim_{x\to 0}\frac{3^{2x}-2^{3x}}{x}$ (v) $\displaystyle\lim_{x\to 0}\frac{3^{2x}-1}{2^{3x}-1}$ (vi) $\displaystyle\lim_{x\to 0}\frac{\sin 3x}{3^x-1}$

SOLUTION (i) We have,

$$\lim_{x\to 0}\frac{10^x-2^x-5^x+1}{x\tan x}=\lim_{x\to 0}\frac{5^x\cdot 2^x-2^x-5^x+1}{x\tan x}$$

$$=\lim_{x\to 0}\frac{5^x(2^x-1)-(2^x-1)}{x\tan x}=\lim_{x\to 0}\frac{(5^x-1)(2^x-1)}{x\tan x}=\lim_{x\to 0}\frac{5^x-1}{x}\times\frac{2^x-1}{x}\times\frac{x}{\tan x}$$

$$=\lim_{x\to 0}\frac{5^x-1}{x}\times\lim_{x\to 0}\frac{2^x-1}{x}\times\lim_{x\to 0}\frac{x}{\tan x}=(\log_e 5)(\log_e 2)(1)=(\log_e 5)(\log_e 2)$$

(ii) We have,

$$\lim_{x\to 0}\frac{3^x+3^{-x}-2}{x^2}$$

$$=\lim_{x\to 0}\frac{3^{2x}-2\times 3^x+1}{3^x\times x^2}=\lim_{x\to 0}\left(\frac{3^x-1}{x}\right)^2\times\frac{1}{3^x}=(\log_e 3)^2\times\left(\frac{1}{3}\right)^0=(\log_e 3)^2$$

(iii) We have,

$$\lim_{x\to 0}\frac{2^{3x}-3^x}{\sin 3x}=\lim_{x\to 0}\left\{\frac{(2^{3x}-1)}{\sin 3x}-\frac{(3^x-1)}{\sin 3x}\right\}$$

$$=\lim_{x\to 0}\left(\frac{2^{3x}-1}{3x}\times\frac{3x}{\sin 3x}\right)-\left(\lim_{x\to 0}\frac{3^x-1}{3x}\times\frac{3x}{\sin 3x}\right)$$

$$=(\log_e 2)\times 1-\frac{1}{3}(\log_e 3)=\log_e 2-\frac{1}{3}\log_e 3.$$

(iv) We have,

$$\lim_{x\to 0}\frac{3^{2x}-2^{3x}}{x}=\lim_{x\to 0}\left\{\left(\frac{3^{2x}-1}{x}\right)-\left(\frac{2^{3x}-1}{x}\right)\right\}$$

$$= \lim_{x\to 0}\left(\frac{3^{2x}-1}{2x}\times 2\right)-\lim_{x\to 0}\left(\frac{2^{3x}-1}{3x}\times 3\right)=2\log_e 3-3\log_e 2=\log_e 9-\log_e 8=\log_e\left(\frac{9}{8}\right)$$

(v) We have,

$$\lim_{x\to 0}\frac{3^{2x}-1}{2^{3x}-1}=\lim_{x\to 0}\frac{\left(\frac{3^{2x}-1}{2x}\right)\times 2x}{\left(\frac{2^{3x}-1}{3x}\right)\times 3x}=\frac{2}{3}\times\frac{\lim_{x\to 0}\left(\frac{3^{2x}-1}{2x}\right)}{\lim_{x\to 0}\left(\frac{2^{3x}-1}{3x}\right)}=\frac{2}{3}\left(\frac{\log_e 3}{\log_e 2}\right)$$

$$=\frac{\log_e 3^2}{\log_e 2^3}=\frac{\log_e 9}{\log_e 8}.$$

(vi) $$\lim_{x\to 0}\frac{\sin 3x}{3^x-1}=\lim_{x\to 0}\frac{\sin 3x}{3x}\times\frac{3x}{3^x-1}=3\lim_{x\to 0}\frac{\sin 3x}{3x}\times\lim_{x\to 0}\frac{x}{3^x-1}=\frac{3}{\log 3}$$

EXAMPLE 3 *Evaluate the following limits:*

(i) $\lim_{x\to 0}\frac{e^{-x}-1}{x}$ (ii) $\lim_{x\to 0}\frac{e^x-e^{-x}}{x}$ (iii) $\lim_{x\to 1}\frac{x-1}{\log_e x}$ (iv) $\lim_{x\to 0}\frac{e^x+e^{-x}-2}{x^2}$

SOLUTION (i) Putting $-x=y$, we obtain

$$\lim_{x\to 0}\frac{e^{-x}-1}{x}=\lim_{y\to 0}\frac{e^y-1}{-y}=-\lim_{y\to 0}\frac{e^y-1}{y}=-1$$

(ii) We have,

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{x}=\lim_{x\to 0}\left\{\left(\frac{e^x-1}{x}\right)-\left(\frac{e^{-x}-1}{x}\right)\right\}=\left(\lim_{x\to 0}\frac{e^x-1}{x}\right)+\left(\lim_{x\to 0}\frac{e^{-x}-1}{-x}\right)=1+1=2$$

(iii) We have,

$$\lim_{x\to 1}\frac{x-1}{\log_e x}=\lim_{h\to 0}\frac{1+h-1}{\log_e(1+h)}=\lim_{h\to 0}\frac{h}{\log_e(1+h)}=\frac{1}{\lim_{h\to 0}\frac{\log(1+h)}{h}}=\frac{1}{1}=1$$

ALITER $$\lim_{x\to 1}\frac{x-1}{\log_e x}=\lim_{x\to 1}\frac{x-1}{\log_e\{1+(x-1)\}}=\lim_{x\to 1}\frac{1}{\frac{\log_e\{1+(x-1)\}}{x-1}}=\frac{1}{1}=1$$

(iv) We have,

$$\lim_{x\to 0}\frac{e^x+e^{-x}-2}{x^2}$$

$$=\lim_{x\to 0}\frac{e^{2x}+1-2e^x}{x^2e^x}=\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^2\times e^x=\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^2\times\lim_{x\to 0}e^x=(1^2)\times e^0=1.$$

EXAMPLE 4 *Evaluate the following limits:*

(i) $\lim_{x\to 0}\frac{\log_e(5+x)-\log_e(5-x)}{x}$ (ii) $\lim_{x\to 5}\frac{\log_e x-\log_e 5}{x-5}$

SOLUTION (i) We find that:

$$\lim_{x\to 0}\frac{\log_e(5+x)-\log_e(5-x)}{x}=\lim_{x\to 0}\frac{\log_e\left\{5\left(1+\frac{x}{5}\right)\right\}-\log_e\left\{5\left(1-\frac{x}{5}\right)\right\}}{x}$$

$$= \lim_{x \to 0} \frac{\left\{\log_e 5 + \log_e\left(1+\frac{x}{5}\right)\right\} - \left\{\log_e 5 + \log_e\left(1+\frac{x}{5}\right)\right\}}{x} = \lim_{x \to 0} \frac{\log_e\left(1+\frac{x}{5}\right) - \log_e\left(1-\frac{x}{5}\right)}{x}$$

$$= \lim_{x \to 0} \frac{1}{5} \times \frac{\log_e\left(1+\frac{x}{5}\right)}{x/5} - \lim_{x \to 0} \frac{\log_e\left(1-\frac{x}{5}\right)}{-x/5} \times \frac{1}{(-5)} = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

(ii) $$\lim_{x \to 5} \frac{\log_e x - \log_e 5}{x-5} = \lim_{h \to 0} \frac{\log_e (5+h) - \log_e 5}{h} = \lim_{h \to 0} \frac{\log_e\left(\frac{5+h}{5}\right)}{h} = \lim_{h \to 0} \frac{\log_e\left(1+\frac{h}{5}\right)}{\frac{h}{5}} \times \frac{1}{5} = \frac{1}{5}$$

ALITER $$\lim_{x \to 5} \frac{\log_e x - \log_e 5}{x-5} = \lim_{x \to 5} \frac{\log_e\left(\frac{x}{5}\right)}{x-5} = \lim_{x \to 5} \frac{\log_e\left(1+\frac{x}{5}-1\right)}{x-5} = \lim_{x \to 5} \frac{\log_e\left(1+\frac{x-5}{5}\right)}{\left(\frac{x-5}{5}\right)} \times \frac{1}{5}$$

$$= 1 \times \frac{1}{5} = \frac{1}{5}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 5 *Prove that:* $\lim_{x \to 2} \frac{3^x + 3^{3-x} - 12}{3^{3-x} - 3^{x/2}} = -\frac{4}{3}.$

SOLUTION $$\lim_{x \to 2} \frac{3^x + 3^{3-x} - 12}{3^{3-x} - 3^{x/2}} = \lim_{x \to 2} \frac{(3^x)^2 - 12(3^x) + 27}{3^3 - (3^{x/2})^3} = \lim_{x \to 2} \frac{(3^x - 3)\ (3^x - 9)}{(3 - 3^{x/2})(9 + 3 \times 3^{x/2} + 3^x)}$$

$$= -\lim_{x \to 2} \frac{(3^x - 3)(3^{x/2} - 3)(3^{x/2} + 3)}{(3^{x/2} - 3)(3^x + 3 \times 3^{x/2} + 9)} = -\lim_{x \to 2} \frac{(3^x - 3)(3^{x/2} + 3)}{(3^x + 3 \times 3^{x/2} + 9)} = -\frac{(9-3)(3+3)}{(9+9+9)} = -\frac{4}{3}$$

EXAMPLE 6 *Evaluate:* $\lim_{x \to e} \frac{\log_e x - 1}{x - e}$

SOLUTION We have,

$$\lim_{x \to e} \frac{\log_e x - 1}{x - e} = \lim_{h \to 0} \frac{\log_e (e+h) - 1}{e + h - e} = \lim_{h \to 0} \frac{\log_e (e+h) - \log_e e}{h} \qquad [\because \log e = 1]$$

$$= \lim_{h \to 0} \frac{\log_e\left(1+\frac{h}{e}\right)}{h} = \lim_{h \to 0} \frac{\log_e\left(1+\frac{h}{e}\right)}{\frac{h}{e}} \times \frac{1}{e} = 1 \times \frac{1}{e} = \frac{1}{e} \qquad \left[\because \lim_{x \to 0} \frac{\log_e(1+x)}{x} = 1\right]$$

ALITER $$\lim_{x \to e} \frac{\log_e x - 1}{x - e} = \lim_{x \to e} \frac{\log_e x - \log_e e}{x - e} = \lim_{x \to e} \frac{\log_e\left(\frac{x}{e}\right)}{x - e}$$

$$= \lim_{x \to e} \frac{\log_e\left(1+\frac{x}{e}-1\right)}{x-e} = \lim_{x \to e} \frac{\log_e\left(1+\frac{x-e}{e}\right)}{\left(\frac{x-e}{e}\right)e} = \frac{1}{e} \times 1 = \frac{1}{e}$$

EXERCISE 28.10

BASIC

Evaluate the following limits:

1. $\lim\limits_{x \to 0} \dfrac{5^x - 1}{\sqrt{4 + x} - 2}$

2. $\lim\limits_{x \to 0} \dfrac{\log(1 + x)}{3^x - 1}$

3. $\lim\limits_{x \to 0} \dfrac{a^{mx} - 1}{b^{nx} - 1}$, $n \neq 0$

4. $\lim\limits_{x \to 0} \dfrac{a^x + b^x - 2}{x}$

5. $\lim\limits_{x \to 0} \dfrac{a^{mx} - b^{nx}}{x}$

6. $\lim\limits_{x \to 0} \dfrac{a^x + b^x + c^x - 3}{x}$

7. $\lim\limits_{x \to 0} \dfrac{5^x + 3^x + 2^x - 3}{x}$

8. $\lim\limits_{x \to 0} \dfrac{a^{mx} - b^{nx}}{\sin kx}$

9. $\lim\limits_{x \to 0} \dfrac{e^x - 1 + \sin x}{x}$

10. $\lim\limits_{x \to 0} \dfrac{\sin 2x}{e^x - 1}$

11. $\lim\limits_{x \to 0} \dfrac{e^{\sin x} - 1}{x}$

12. $\lim\limits_{x \to 0} \dfrac{\log(a + x) - \log a}{x}$

13. $\lim\limits_{x \to 0} \dfrac{8^x - 2^x}{x}$

14. $\lim\limits_{x \to 0} \dfrac{\sqrt{1 + x} - 1}{\log(1 + x)}$

15. $\lim\limits_{x \to 0} \dfrac{a^x + a^{-x} - 2}{x^2}$

16. $\lim\limits_{x \to 0} \dfrac{9^x - 2.6^x + 4^x}{x^2}$

17. $\lim\limits_{x \to 0} \dfrac{8^x - 4^x - 2^x + 1}{x^2}$

18. $\lim\limits_{x \to 2} \dfrac{x - 2}{\log_a (x - 1)}$

19. $\lim\limits_{x \to \infty} (a^{1/x} - 1)\, x$

20. $\lim\limits_{x \to 0} \dfrac{a^x + b^x - c^x - d^x}{x}$

21. $\lim\limits_{x \to 0} \dfrac{e^{2x} - e^x}{\sin 2x}$

22. $\lim\limits_{x \to a} \dfrac{\log x - \log a}{x - a}$

23. $\lim\limits_{x \to 0} \dfrac{\log(a + x) - \log(a - x)}{x}$

24. $\lim\limits_{x \to 0} \dfrac{\log(2 + x) + \log 0.5}{x}$

25. $\lim\limits_{x \to 0} \dfrac{\log(3 + x) - \log(3 - x)}{x}$

BASED ON LOTS

26. $\lim\limits_{x \to 0} \dfrac{x(2^x - 1)}{1 - \cos x}$

27. $\lim\limits_{x \to 0} \dfrac{\log|1 + x^3|}{\sin^3 x}$

28. $\lim\limits_{x \to \pi/2} \dfrac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x}$

29. $\lim\limits_{x \to 0} \dfrac{e^x - 1}{\sqrt{1 - \cos x}}$

30. $\lim\limits_{x \to 5} \dfrac{e^x - e^5}{x - 5}$

31. $\lim\limits_{x \to 0} \dfrac{e^{x+2} - e^2}{x}$

32. $\lim\limits_{x \to \pi/2} \dfrac{e^{\cos x} - 1}{\cos x}$

33. $\lim\limits_{x \to 0} \dfrac{e^{3+x} - \sin x - e^3}{x}$

34. $\lim\limits_{x \to 0} \dfrac{e^x - x - 1}{2}$

35. $\lim\limits_{x \to 0} \dfrac{e^{3x} - e^{2x}}{x}$

36. $\lim\limits_{x \to 0} \dfrac{e^{\tan x} - 1}{\tan x}$

37. $\lim\limits_{x \to 0} \dfrac{e^{bx} - e^{ax}}{x}$ where $0 < a < b$

38. $\lim\limits_{x \to 0} \dfrac{e^{\tan x} - 1}{x}$

39. $\lim\limits_{x \to 0} \dfrac{e^x - e^{\sin x}}{x - \sin x}$

40. $\lim\limits_{x \to 0} \dfrac{3^{2+x} - 9}{x}$

41. $\lim\limits_{x \to 0} \dfrac{a^x - a^{-x}}{x}$

42. $\lim\limits_{x \to 0} \dfrac{x(e^x - 1)}{1 - \cos x}$

43. $\lim\limits_{x \to \pi/2} \dfrac{2^{-\cos x} - 1}{x\left(x - \dfrac{\pi}{2}\right)}$

ANSWERS

1. $4 \log 5$

2. $\dfrac{1}{\log_e 3}$

3. $\dfrac{m}{n} \dfrac{\log_e a}{\log_e b}$

4. $\log_e (ab)$

5. $\log_e\left(\frac{a^m}{b^n}\right)$	6. $\log_e (abc)$	7. $\log_e 30$	8. $\frac{1}{k}\log_e\left(\frac{a^m}{b^n}\right)$
9. 2	10. 2	11. 1	12. $\frac{1}{a}$
13. $\log_e 4$	14. $\frac{1}{2}$	15. $(\log_e a)^2$	16. $\left(\log_e \frac{3}{2}\right)^2$
17. $(\log_e 4)(\log_e 2)$	18. $\log_e a$	19. $\log_e a$	20. $\log_e\left(\frac{ab}{cd}\right)$
21. $\frac{1}{2}$	22. $\frac{1}{a}$	23. $\frac{2}{a}$	24. $\frac{1}{2}$
25. $\frac{2}{3}$	26. $\log_e 4$	27. 1	28. $\log_e a$
29. Does not exist	30. e^5	31. e^2	32. 1
33. $e^3 - 1$	34. 0	35. 1	36. 1
37. $(b-a)$	38. 1	39. 1	40. $9 . \log_e 3$
41. $2\log_e 2$	42. 2	43. $\frac{2}{\pi}\log_e 2$	

HINTS TO SELECTED PROBLEMS

10. $$\lim_{x\to 2}\frac{x-2}{\log_a (x-1)} = \lim_{x\to 2}\frac{(x-2)}{\log_e (x-1)\times \log_a e} = \lim_{x\to 2}\frac{1}{\frac{\log_e \{1+(x-2)\}}{x-2}}\times \log_e a$$

$$= \frac{1}{1}\times \log_e a = \log_e a$$

29. $$\lim_{x\to 0}\frac{e^x-1}{\sqrt{1-\cos x}} = \lim_{x\to 0}\frac{e^x-1}{\sqrt{2}\left|\sin\frac{x}{2}\right|}$$

Now, $$\lim_{x\to 0^-}\frac{e^x-1}{\sqrt{2}\left|\sin\frac{x}{2}\right|} = -\frac{1}{\sqrt{2}}\lim_{x\to 0}\frac{e^x-1}{\sin\frac{x}{2}} = -\frac{2}{\sqrt{2}}\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)\times\left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right) = -\sqrt{2}$$

and, $$\lim_{x\to 0^+}\frac{e^x-1}{\sqrt{2}\left|\sin\frac{x}{2}\right|} = \frac{2}{\sqrt{2}}\lim_{x\to 0^+}\left(\frac{e^x-1}{x}\right)\left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right) = \sqrt{2}.$$

$\therefore$ $$\lim_{x\to 0^-}\frac{e^x-1}{\sqrt{2}\left|\sin\frac{x}{2}\right|} \neq \lim_{x\to 0^+}\frac{e^x-1}{\sqrt{2}\left(\sin\frac{x}{2}\right)}$$

33. $$\lim_{x\to 0}\left\{\frac{e^{3+x}-e^3}{x}-\frac{\sin x}{x}\right\} = \lim_{x\to 0}\left\{e^3\left(\frac{e^x-1}{x}\right)-\frac{\sin x}{x}\right\} = e^3\times 1 - 1 = e^3-1$$

39. $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x} = \lim_{x \to 0} e^{\sin x} \left(\frac{e^{x - \sin x} - 1}{x - \sin x} \right) = e^0 \times 1 = 1$

42. $\lim_{x \to 0} x \left(\frac{e^x - 1}{1 - \cos x} \right) = \lim_{x \to 0} x^2 \left(\frac{\frac{e^x - 1}{x}}{2 \sin^2 \frac{x}{2}} \right) = 2 \lim_{x \to 0} \left(\frac{e^x - 1}{x} \right) \left(\frac{\frac{x}{2}}{\sin \frac{x}{2}} \right)^2 = 2 \times 1 \times 1^2 = 2.$

43. $\lim_{x \to \pi/2} \frac{2^{-\cos x} - 1}{x\left(x - \frac{\pi}{2}\right)} = \lim_{x \to \pi/2} \frac{2^{\sin(x - \pi/2)} - 1}{x - \frac{\pi}{2}} \times \frac{1}{x} = \lim_{x \to \pi/2} \frac{2^{\sin\left(x - \frac{\pi}{2}\right)} - 1}{\sin\left(x - \frac{\pi}{2}\right)} \times \frac{\sin\left(x - \frac{\pi}{2}\right)}{x - \frac{\pi}{2}} \times \frac{1}{x}$

$$= \log_e 2 \times 1 \times \frac{2}{\pi} = \frac{2}{\pi} \log_e 2$$

28.9 EVALUATION OF LIMITS OF THE FORM 1^∞

To evaluate exponential limits of the form 1^∞, we use the following result which is stated and proved as a theorem.

THEOREM *If* $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ *such that* $\lim_{x \to a} \frac{f(x)}{g(x)}$ *exists, then*

$$\lim_{x \to a} \left\{1 + f(x)\right\}^{\frac{1}{g(x)}} = e^{\lim_{x \to a} \frac{f(x)}{g(x)}}$$

PROOF Let $A = \lim_{x \to a} \left\{1 + f(x)\right\}^{\frac{1}{g(x)}}$. Then,

$$\log_e A = \lim_{x \to a} \frac{\log\{1 + f(x)\}}{g(x)} = \lim_{x \to a} \frac{\log\{1 + f(x)\}}{f(x)} \times \frac{f(x)}{g(x)}$$

$$\Rightarrow \quad \log_e A = \lim_{x \to a} \frac{f(x)}{g(x)} \qquad \left[\because \lim_{x \to a} \frac{\log\{1 + f(x)\}}{f(x)} = 1\right]$$

$$\Rightarrow \quad A = e^{\lim_{x \to a} \frac{f(x)}{g(x)}}$$

Q.E.D

REMARK *The above result can also be restated in the following form:*

If $\lim_{x \to a} f(x) = 1$ *and* $\lim_{x \to a} g(x) = \infty$ *such that* $\lim_{x \to a} \{f(x) - 1\} g(x)$ *exists, then*

$$\lim_{x \to a} \{f(x)\}^{g(x)} = e^{\lim_{x \to a} \{f(x) - 1\} g(x)}$$

PARTICULAR CASES

(i) $\lim_{x \to 0} (1 + x)^{1/x} = e$ (ii) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$ (iii) $\lim_{x \to 0} (1 + \lambda x)^{1/x} = e^\lambda$ (iv) $\lim_{x \to \infty} \left(1 + \frac{\lambda}{x}\right)^x = e^\lambda$

Following examples will illustrate the applications of the above mentioned results.

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 1 *Evaluate the following limits:*

(i) $\lim\limits_{x \to \infty} \left(1 + \frac{2}{x}\right)^x$ (ii) $\lim\limits_{x \to 0} (1 + \sin x)^{2 \cot x}$

(iii) $\lim\limits_{x \to 1} (\log_3 3x)^{\log_x 3}$ (iv) $\lim\limits_{x \to 0} (\cos x)^{\cot x}$

SOLUTION (i) $\lim\limits_{x \to \infty} \left(1 + \frac{2}{x}\right)^x = e^{\lim\limits_{x \to \infty} \frac{2}{x} \times x} = e^2$

(ii) $\lim\limits_{x \to 0} (1 + \sin x)^{2 \cot x} = e^{\lim\limits_{x \to 0} \sin x \times 2 \cot x} = e^{\lim\limits_{x \to 0} 2 \cos x} = e^2$

(iii) $\lim\limits_{x \to 1} (\log_3 3x)^{\log_x 3} = \lim\limits_{x \to 1} (\log_3 3 + \log_3 x)^{\log_x 3} = \lim\limits_{x \to 1} (1 + \log_3 x)^{\frac{1}{\log_3 x}}$

$$= e^{\lim\limits_{x \to 1} \log_3 x \times \frac{1}{\log_3 x}} = e^1 = e$$

(iv) We have,

$$\lim_{x \to 0} (\cos x)^{\cot x} = \lim_{x \to 0} \{1 + \cos x - 1\}^{\cot x} = \lim_{x \to 0} \{1 - (1 - \cos x)\}^{\cot x}$$

$$= \lim_{x \to 0} \left\{1 - 2 \sin^2 \left(\frac{x}{2}\right)\right\}^{\cot x} = e^{\lim\limits_{x \to 0} -2 \sin^2(x/2) \times \cot x}$$

$$= e^{\lim\limits_{x \to 0} \frac{-2 \sin^2(x/2) \cos x}{2 \sin(x/2) \cos x/2}} = e^{\lim\limits_{x \to 0} -\tan(x/2) \times \cos x} = e^0 = 1$$

EXAMPLE 2 *Evaluate the following limits:*

(i) $\lim\limits_{x \to 0} \left(\frac{a^x + b^x + c^x}{3}\right)^{1/x}$ (ii) $\lim\limits_{x \to a} \left(2 - \frac{a}{x}\right)^{\tan \pi x/2a}$

SOLUTION (i) We have,

$$\lim_{x \to 0} \left(\frac{a^x + b^x + c^x}{3}\right)^{1/x}$$

$$= \lim_{x \to 0} \left\{1 + \frac{a^x + b^x + c^x - 3}{3}\right\}^{1/x} = \lim_{x \to 0} \left\{1 + \frac{(a^x - 1) + (b^x - 1) + (c^x - 1)}{3}\right\}^{1/x}$$

$$= e^{\lim\limits_{x \to 0} \frac{a^x - 1}{3x} + \frac{b^x - 1}{3x} + \frac{c^x - 1}{3x}} = e^{\frac{1}{3}\left\{\lim\limits_{x \to 0} \frac{a^x - 1}{x} + \lim\limits_{x \to 0} \frac{b^x - 1}{x} + \lim\limits_{x \to 0} \frac{c^x - 1}{x}\right\}}$$

$$= e^{\frac{1}{3}\{\log a + \log b + \log c\}} = e^{\log (abc)^{1/3}} = (abc)^{1/3}$$

(ii) We have,

$$\lim_{x \to a} \left(2 - \frac{a}{x}\right)^{\tan \pi x/2a} = \lim_{x \to a} \left\{1 + \left(1 - \frac{a}{x}\right)\right\}^{\tan \pi x/2a}$$

$$= e^{\lim_{x \to a} \left(1 - \frac{a}{x}\right) \tan \frac{\pi x}{2a}} = e^{\lim_{x \to a} \left(\frac{x-a}{x}\right) \tan \frac{\pi x}{2a}} = e^{l}, \text{ where } l = \lim_{x \to a} \left(\frac{x-a}{x}\right) \tan \frac{\pi x}{2a}$$

Now, $l = \lim_{x \to a} \left(\frac{x-a}{x}\right) \tan \frac{\pi x}{2a} = \lim_{x \to a} \left(\frac{x-a}{x}\right) \cot \left(\frac{\pi}{2} - \frac{\pi x}{2a}\right) = \lim_{x \to a} \left(\frac{x-a}{x}\right) \cot \frac{\pi}{2a}(a-x)$

$$\Rightarrow \quad l = \lim_{x \to a} \left(\frac{x-a}{x}\right) \times \frac{1}{\tan \frac{\pi}{2a}(a-x)} = -\lim_{x \to a} \frac{(a-x)}{\tan \frac{\pi}{2a}(a-x)} \times \frac{1}{x}$$

$$\Rightarrow \quad l = -\lim_{x \to a} \frac{\frac{\pi}{2a}(a-x)}{\tan \frac{\pi}{2a}(a-x)} \times \frac{2a}{\pi x} = -\frac{2}{\pi}$$

Hence, $\lim_{x \to a} \left(2 - \frac{a}{x}\right)^{\tan \frac{\pi x}{2a}} = e^{-2/\pi}$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 3 *Evaluate the following limits*:

(i) $\lim_{x \to \infty} \left(\frac{x+5}{x-1}\right)^x$ (ii) $\lim_{x \to \infty} \left(\frac{x^2+4x+3}{x^2+2x-5}\right)^x$

SOLUTION (i) We have,

$$\lim_{x \to \infty} \left(\frac{x+5}{x-1}\right)^x = \lim_{x \to \infty} \left(1 + \frac{6}{x-1}\right)^x = e^{\lim_{x \to \infty} \frac{6x}{x-1}} = e^6$$

(ii) We have

$$\lim_{x \to \infty} \left(\frac{x^2+4x-3}{x^2+2x-5}\right)^x = \lim_{x \to \infty} \left\{1 + \frac{2x+8}{x^2+2x-5}\right\}^x = e^{\lim_{x \to \infty} \frac{2x^2+8x}{x^2+2x-5}} = e^2$$

EXAMPLE 4 *Evaluate the following limits:*

(i) $\lim_{x \to 0} \left\{\tan \left(\frac{\pi}{4} + x\right)\right\}^{1/x}$ (ii) $\lim_{x \to 0} (\cos 2x)^{1/x^2}$

SOLUTION (i) We have,

$$\lim_{x \to 0} \left\{\tan \left(\frac{\pi}{4} + x\right)\right\}^{1/x}$$

$$= \lim_{x \to 0} \left\{\frac{1 + \tan x}{1 - \tan x}\right\}^{1/x}$$

$$= \lim_{x \to 0} \left\{1 + \frac{2 \tan x}{1 - \tan x}\right\}^{1/x} = e^{\lim_{x \to 0} \frac{2 \tan x}{1 - \tan x} \times \frac{1}{x}} = e^{\lim_{x \to 0} \frac{2}{1 - \tan x} \times \frac{\tan x}{x}} = e^2$$

(ii) We have,

$$\lim_{x \to 0} (\cos 2x)^{1/x^2} = \lim_{x \to 0} \left\{1 + (\cos 2x - 1)\right\}^{1/x^2} = e^{\lim_{x \to 0} \frac{\cos 2x - 1}{x^2}} = e^{-\lim_{x \to 0} \frac{2 \sin^2 x}{x^2}} = e^{-2}$$

EXERCISE 28.11

BASIC

Evaluate the following limits:

1. $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$

2. $\lim_{x \to 0^+} \left\{1 + \tan^2 \sqrt{x}\right\}^{1/2x}$

3. $\lim_{x \to 0} (\cos x)^{1/\sin x}$

4. $\lim_{x \to 0} (\cos x + \sin x)^{1/x}$

5. $\lim_{x \to 0} (\cos x + a \sin bx)^{1/x}$

6. $\lim_{x \to \infty} \left\{\frac{x^2 + 2x + 3}{2x^2 + x + 5}\right\}^{\frac{3x - 2}{3x + 2}}$

7. $\lim_{x \to 1} \left\{\frac{x^3 + 2x^2 + x + 1}{x^2 + 2x + 3}\right\}^{\frac{1 - \cos(x - 1)}{(x - 1)^2}}$

8. $\lim_{x \to 0} \left\{\frac{e^x + e^{-x} - 2}{x^2}\right\}^{1/x^2}$

9. $\lim_{x \to a} \left\{\frac{\sin x}{\sin a}\right\}^{\frac{1}{x - a}}$

10. $\lim_{x \to \infty} \left\{\frac{3x^2 + 1}{4x^2 - 1}\right\}^{\frac{x^3}{1 + x}}$

ANSWERS

1. e^x 2. $\sqrt{e}$ 3. 1 4. e 5. e^{ab} 6. $\frac{1}{2}$ 7. $\sqrt{\frac{5}{6}}$ 8. $e^{1/12}$
9. $e^{\cot a}$ 10. 0

FILL IN THE BLANKS TYPES QUESTIONS (FBQs)

1. If $f(x) = \frac{\tan x}{x - \pi}$, then $\lim_{x \to \pi} f(x) =$ ....

2. $\lim_{x \to 3^-} \frac{x}{[x]} =$ ....

3. If $\lim_{x \to 0} \left(\sin mx \cot \frac{x}{\sqrt{3}}\right) = 2$, then $m =$ ....

4. $\lim_{x \to 0} \frac{|\sin x|}{x}$ ....

5. $\lim_{x \to \frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1}$ is equal to ....

6. $\lim_{x \to 1} \frac{x^m - 1}{x^n - 1}$ is equal to ....

7. $\lim\limits_{n \to \infty} \dfrac{1+2+3+\ldots+n}{n^2+100}$ ....

8. $\lim\limits_{x \to 0} \dfrac{e^{2\tan x}-1}{x}$ is equal to ....

9. $\lim\limits_{x \to 1} \dfrac{\log_e x}{x-1}$ is equal to ....

10. Let $f(x) = \begin{cases} x^2-1, & 0 < x < 2 \\ 2x+3, & 2 \le x < 3 \end{cases}$, the quadratic equation whose roots are $\lim\limits_{x \to 2^-} f(x)$ and $\lim\limits_{x \to 2^+} f(x)$ is ....

11. $\lim\limits_{x \to 1} \dfrac{\tan(x^2-1)}{x-1}$ is equal to ....

12. $\lim\limits_{x \to 0} \dfrac{\tan x^\circ}{x}$ is equal to ....

13. $\lim\limits_{x \to 0} \dfrac{2x+3\tan x}{3x-2\sin x}$ is equal to ....

14. $\lim\limits_{x \to 0} \dfrac{5x\cos x - 2\sin x}{3x+\tan x} =$ ....

15. If $f(x) = \begin{cases} x, & x > 1 \\ x^2, & x < 1 \end{cases}$, then $\lim\limits_{x \to 1} f(x) =$ ....

16. $\lim\limits_{x \to \infty} \left(\sqrt{x^2+1} - x\right)$ is equal to ....

17. The value of $\lim\limits_{x \to 2} \dfrac{3^{x/2}-3}{3^x-9}$ is ....

ANSWERS

1. 1 2. 1 3. $\dfrac{2}{\sqrt{3}}$ 4. Does not exist 5. 2 6. $\dfrac{m}{n}$ 7. $\dfrac{1}{2}$
8. 2 9. 1 10. $x^2-12x+35=0$ 11. 2 12. $\dfrac{\pi}{180}$ 13. 5
14. $\dfrac{3}{4}$ 15. 1 16. 0 17. $\dfrac{1}{6}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the value of $\lim\limits_{x \to 0} \dfrac{\sqrt{1-\cos 2x}}{x}$.
2. Write the value of $\lim\limits_{x \to 0^-} [x]$.
3. Write the value of $\lim\limits_{x \to 0^+} [x]$.
4. Write the value of $\lim\limits_{x \to 1^-} x - [x]$.
5. Write the value of $\lim\limits_{x \to 0^-} \dfrac{\sin [x]}{[x]}$.
6. Write the value of $\lim\limits_{x \to \pi} \dfrac{\sin x}{x-\pi}$.

7. Write the value of $\lim_{x \to \infty} \frac{\sin x}{x}$.

8. Write the value of $\lim_{x \to 2} \frac{|x-2|}{x-2}$.

9. Write the value of $\lim_{x \to 0} \frac{\sin x^\circ}{x}$.

10. Write the value of $\lim_{x \to 0^-} \frac{\sin x}{\sqrt{x}}$.

11. Write the value of $\lim_{x \to 0} \frac{\sin x}{\sqrt{1+x}-1}$.

12. Write the value of $\lim_{x \to -\infty} (3x + \sqrt{9x^2 - x})$.

13. Write the value of $\lim_{n \to \infty} \frac{n! + (n+1)!}{(n+1)! + (n+2)!}$.

14. Write the value of $\lim_{x \to \pi/2} \frac{2x - \pi}{\cos x}$.

15. Write the value of $\lim_{n \to \infty} \frac{1+2+3+\ldots+n}{n^2}$.

ANSWERS

1. Does not exit 2. -1 3. 1 4. 1 5. $\sin 1$ 6. -1 7. 0
8. Does not exist 9. $\frac{\pi}{180}$ 10. Does not exist 11. 2 12. $\frac{1}{6}$ 13. 0
14. -2 15. $\frac{1}{2}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. $\lim_{n \to \infty} \frac{1^2 + 2^2 + 3^2 + \ldots + n^2}{n^3}$ is equal to
   (a) 1 (b) 1/2 (c) 1/3 (d) 0

2. $\lim_{x \to 0} \frac{\sin 2x}{x}$ is equal to
   (a) 0 (b) 1 (c) 1/2 (d) 2

3. If $f(x) = x \sin(1/x)$, $x \neq 0$, then $\lim_{x \to 0} f(x) =$
   (a) 1 (b) 0 (c) -1 (d) does not exist

4. $\lim_{x \to 0} \frac{1 - \cos 2x}{x}$ is
   (a) 0 (b) 1 (c) 2 (d) 4

5. $\lim_{x \to 0} \frac{(1 - \cos 2x) \sin 5x}{x^2 \sin 3x} =$
   (a) 10/3 (b) 3/10 (c) 6/5 (d) 5/6

6. $\lim_{x \to 0} \frac{x}{\tan x}$ is
   (a) 0 (b) 1 (c) 4 (d) not defined

7. $\lim_{n \to \infty} \left\{ \frac{1}{1-n^2} + \frac{2}{1-n^2} + \ldots + \frac{n}{1-n^2} \right\}$ is equal to
   (a) 0 (b) $-1/2$ (c) 1/2 (d) none of these

8. $\lim_{x \to \infty} \frac{\sin x}{x}$ equals
   (a) 0 (b) ∞ (c) 1 (d) does not exist

9. $\lim_{x \to 0} \frac{\sin x^0}{x}$ is equal to

(a) 1 (b) π (c) x (d) $\pi / 180$

10. $\lim_{x \to 3} \frac{x-3}{|x-3|}$, is equal to

(a) 1 (b) -1 (c) 0 (d) does not exist

11. $\lim_{x \to a} \frac{x^n - a^n}{x - a}$ is equal to

(a) na^n (b) na^{n-1} (c) na (d) 1

12. $\lim_{x \to \pi/4} \frac{\sqrt{2} \cos x - 1}{\cot x - 1}$ is equal to

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{2\sqrt{2}}$ (d) 1

13. $\lim_{x \to \infty} \frac{\sqrt{x^2 - 1}}{2x + 1}$ is equal to

(a) 1 (b) 0 (c) -1 (d) $1/2$

14. $\lim_{h \to 0} 2 \left\{ \frac{\sqrt{3} \sin (\pi/6 + h) - \cos (\pi/6 + h)}{\sqrt{3}\, h (\sqrt{3} \cos h - \sin h)} \right\}$ is equal to

(a) $2/3$ (b) $4/3$ (c) $-2\sqrt{3}$ (d) $-4/3$

15. $\lim_{h \to 0} \left\{ \frac{1}{h \sqrt[3]{8+h}} - \frac{1}{2h} \right\} =$

(a) $-1/12$ (b) $-4/3$ (c) $-16/3$ (d) $-1/48$

16. $\lim_{n \to \infty} \left\{ \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2n+1)(2n+3)} \right\}$ is equal to

(a) 0 (b) $1/2$ (c) $1/9$ (d) 2

17. $\lim_{x \to 1} \frac{\sin \pi x}{x - 1}$ is equal to

(a) $-\pi$ (b) π (c) $-\frac{1}{\pi}$ (d) $\frac{1}{\pi}$

18. If $\lim_{x \to 1} \frac{x + x^2 + x^3 + \ldots + x^n - n}{x - 1} = 5050$, then n equals

(a) 10 (b) 100 (c) 150 (d) none of these

19. The value of $\lim_{x \to \infty} \frac{\sqrt{1 + x^4} + (1 + x^2)}{x^2}$ is

(a) -1 (b) 1 (c) 2 (d) none of these

20. $\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$ is equal to

(a) $\frac{1}{2}$ (b) 2 (c) 0 (d) 1

21. $\lim_{x \to \pi/3} \frac{\sin\left(\frac{\pi}{3} - x\right)}{2\cos x - 1}$ is equal to

(a) $\sqrt{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{3}$

22. $\lim_{x \to 3} \frac{\sum_{r=1}^{n} x^r - \sum_{r=1}^{n} 3^r}{x - 3}$ is equal to

(a) $\frac{(2n-1) \times 3^n}{4}$ (b) $\frac{(2n-1) \times 3^n + 1}{4}$ (c) $(2n-1)\ 3^n + 1$ (d) $\frac{(2n-1) \times 3^n - 1}{4}$

23. $\lim_{n \to \infty} \frac{1 - 2 + 3 - 4 + 5 - 6 + ... + (2n-1) - 2n}{\sqrt{n^2+1} + \sqrt{n^2-1}}$ is equal to

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 1 (d) -1

24. If $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $\lim_{x \to 0} f(x)$ equals

(a) 1 (b) 0 (c) -1 (d) none of these.

[NCERT EXEMPLAR]

25. $\lim_{n \to \infty} \frac{n!}{(n+1)! + n!}$ is equal to

(a) $\frac{1}{2}$ (b) 0 (c) 2 (d) 1

26. $\lim_{x \to \pi/4} \frac{4\sqrt{2} - (\cos x + \sin x)^5}{1 - \sin 2x}$ is equal to

(a) $5\sqrt{2}$ (b) $3\sqrt{2}$ (c) $\sqrt{2}$ (d) none of these

27. $\lim_{x \to 2} \frac{\sqrt{1 + \sqrt{2+x}} - \sqrt{3}}{x - 2}$ is equal to

(a) $\frac{1}{8\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $8\sqrt{3}$ (d) $\sqrt{3}$

28. $\lim_{x \to \infty} a^x \sin\left(\frac{b}{a^x}\right)$, $a, b > 1$ is equal to

(a) b (b) a (c) $a \log_e b$ (d) $b \log_e a$

29. $\lim_{x \to 0} \frac{8}{x^8} \left\{1 - \cos\frac{x^2}{2} - \cos\frac{x^2}{4} + \cos\frac{x^2}{2} \cos\frac{x^2}{4}\right\}$ is equal to

(a) $1/16$ (b) $-1/16$ (c) $1/32$ (d) $-1/32$

30. If α is a repeated root of $ax^2 + bx + c = 0$, then $\lim_{x \to \alpha} \frac{\tan(ax^2 + bx + c)}{(x - \alpha)^2}$ is

(a) a (b) b (c) c (d) 0

31. The value of $\lim\limits_{x \to 0} \dfrac{\sqrt{a^2 - ax + x^2} - \sqrt{a^2 + ax + x^2}}{\sqrt{a + x} - \sqrt{a - x}}$ is

(a) a (b) $\sqrt{a}$ (c) $-a$ (d) $-\sqrt{a}$

32. The value of $\lim\limits_{x \to 0} \dfrac{1 - \cos x + 2 \sin x - \sin^3 x - x^2 + 3x^4}{\tan^3 x - 6 \sin^2 x + x - 5x^3}$ is

(a) 1 (b) 2 (c) -1 (d) -2

33. $\lim\limits_{\theta \to \pi/2} \dfrac{1 - \sin \theta}{(\pi/2 - \theta) \cos \theta}$ is equal to

(a) 1 (b) -1 (c) $1/2$ (d) $-1/2$

34. The value of $\lim\limits_{x \to \pi/2} (\sec x - \tan x)$ is

(a) 2 (b) -1 (c) 1 (d) 0

35. The value of $\lim\limits_{x \to \infty} \dfrac{n!}{(n+1)! - n!}$ is

(a) 0 (b) 1 (c) -1 (d) none of these

36. The value of $\lim\limits_{n \to \infty} \dfrac{(n+2)! + (n+1)!}{(n+2)! - (n+1)!}$ is

(a) 0 (b) -1 (c) 1 (d) none of these

37. The value of $\lim\limits_{x \to \infty} \dfrac{(x+1)^{10} + (x+2)^{10} + \ldots + (x+100)^{10}}{x^{10} + 10^{10}}$ is

(a) 10 (b) 100 (c) 10^{10} (d) none of these

38. The value of $\lim\limits_{n \to \infty} \left\{ \dfrac{1 + 2 + 3 + \ldots + n}{n + 2} - \dfrac{n}{2} \right\}$ is

(a) $1/2$ (b) 1 (c) -1 (d) $-1/2$

39. $\lim\limits_{x \to 1} [x - 1]$, where $[\cdot]$ is the greatest integer function, is equal to

(a) 1 (b) 2 (c) 0 (d) does not exist

40. $\lim\limits_{x \to 0} \dfrac{|x|}{x}$ is equal to [NCERT EXEMPLAR]

(a) 1 (b) -1 (c) 0 (d) does not exist

41. $\lim\limits_{x \to 0} \dfrac{|\sin x|}{x}$ is [NCERT EXEMPLAR]

(a) 1 (b) -1 (c) 0 (d) does not exist

42. If $f(x) = \begin{cases} \dfrac{\sin [x]}{[x]}, & [x] \neq 0 \\ 0, & [x] = 0 \end{cases}$, where $[\cdot]$ denotes the greatest integer function, then $\lim\limits_{x \to 0} f(x)$ is equal to

(a) 1 (b) 0 (c) -1 (d) none of these

[NCERT EXEMPLAR]

43. $\lim\limits_{x \to \pi} \dfrac{\sin x}{x - \pi}$ is

(a) 1 (b) 2 (c) -1 (d) -2

[NCERT EXEMPLAR]

44. $\lim\limits_{x \to 0} \dfrac{x^2 \cos x}{1-\cos x}$ is

(a) 2 (b) $\dfrac{3}{2}$ (c) $-\dfrac{3}{2}$ (d) 1 [NCERT EXEMPLAR]

45. $\lim\limits_{x \to 0} \dfrac{(1+x)^n - 1}{x}$ is

(a) n (b) 1 (c) $-n$ (d) 0 [NCERT EXEMPLAR]

46. $\lim\limits_{x \to 1} \dfrac{x^m - 1}{x^n - 1}$ is

(a) 1 (b) $\dfrac{m}{n}$ (c) $-\dfrac{m}{n}$ (d) $\dfrac{m^2}{n^2}$ [NCERT EXEMPLAR]

47. $\lim\limits_{\theta \to 0} \dfrac{1-\cos 4\theta}{1-\cos 6\theta}$ is

(a) $\dfrac{4}{9}$ (b) $\dfrac{1}{2}$ (c) $-\dfrac{1}{2}$ (d) -1 [NCERT EXEMPLAR]

48. $\lim\limits_{x \to 0} \dfrac{\operatorname{cosec} x - \cot x}{x}$ is

(a) $-\dfrac{1}{2}$ (b) 1 (c) $\dfrac{1}{2}$ (d) 1

49. $\lim\limits_{x \to \pi/4} \dfrac{\sec^2 x - 2}{\tan x - 1}$ is

(a) 3 (b) 1 (c) 2 (d) $\sqrt{2}$ [NCERT EXEMPLAR]

50. $\lim\limits_{x \to 0} \dfrac{\sin x}{\sqrt{1+x} - \sqrt{1-x}}$ is

(a) 2 (b) 0 (c) 1 (d) -1 [NCERT EXEMPLAR]

51. $\lim\limits_{x \to 1} \dfrac{(\sqrt{x}-1)(2x-3)}{2x^2+x-3}$ is

(a) $\dfrac{1}{10}$ (b) $-\dfrac{1}{10}$ (c) 1 (d) none of these

[NCERT EXEMPLAR]

ANSWERS

1. (c)	2. (d)	3. (b)	4. (a)	5. (a)	6. (b)	7. (b)	8. (a)
9. (d)	10. (d)	11. (b)	12. (b)	13. (d)	14. (b)	15. (d)	16. (b)
17. (a)	18. (b)	19. (b)	20. (a)	21. (c)	22. (b)	23. (b)	24. (b)
25. (a)	26. (a)	27. (a)	28. (a)	29. (c)	30. (a)	31. (d)	32. (b)
33. (c)	34. (d)	35. (a)	36. (c)	37. (b)	38. (d)	39. (d)	40. (d)
41. (d)	42. (d)	43. (c)	44. (a)	45. (a)	46. (b)	47. (a)	48. (c)
49. (c)	50. (c)	51. (b)					

SUMMARY

1. $\lim_{x \to a} f(x)$ exists $\Leftrightarrow \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

2. For a function $f(x)$ and a real number a, $\lim_{x \to a} f(x)$ and $f(a)$ may not be same. In fact:
   (i) $\lim_{x \to a} f(x)$ exists but $f(a)$ (the value of $f(x)$ at $x = a$) may not exists
   (ii) The value $f(a)$ exists but $\lim_{x \to a} f(x)$ does not exist
   (iii) $\lim_{x \to a} f(x)$ and $f(a)$ both exist but are unequal
   (iv) $\lim_{x \to a} f(x)$ and $f(a)$ both exist and are equal.

3. Let $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$. If l and m both exist, then
   (i) $\lim_{x \to a} k\,f(x) = k \lim_{x \to a} f(x)$
   (ii) $\lim_{x \to a} (f \pm g)(x) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = l + m$
   (iii) $\lim_{x \to a} (fg)(x) = \lim_{x \to a} f(x) \lim_{x \to a} g(x) = lm$
   (iv) $\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{l}{m}$
   (v) $\lim_{x \to a} \{f(a)\}^{g(x)} = l^m$

4. Following are some standard limits:
   (i) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$
   (ii) $\lim_{x \to 0} \frac{\sin x}{x} = 1$
   (iii) $\lim_{x \to 0} \frac{\tan x}{x} = 1$
   (iv) $\lim_{x \to a} \frac{\sin(x-a)}{x-a} = 1$
   (v) $\lim_{x \to a} \frac{\tan(x-a)}{x-a} = 1$
   (vi) $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$
   (vii) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a,\ a \neq 0,\ a > 1$
   (viii) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$