CHAPTER 3

FUNCTIONS

3.1 INTRODUCTION

In this chapter, we shall study about one of the most important concepts in mathematics known as a function. Functions form one of the most important building blocks of Mathematics. The word "Function" is derived from a Latin word meaning operation and the words mapping and map are synonimus to it. Functions play a very important role in differential and integral calculus. In this chapter, we shall introduce the concept of a function as a correspondence between two sets. We shall also study function as a relation from one set to the other set.

3.2 FUNCTION AS A SPECIAL KIND OF RELATION

DEFINITION *Let A and B be two non-empty sets. A relation f from A to B, i.e., a sub-set of $A \times B$, is called a function (or a mapping or a map) from A to B, if*

(i) *for each $a \in A$ there exists $b \in B$ such that $(a, b) \in f$*

(ii) *$(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$.*

Thus, a non-void subset f of $A \times B$ is a function from A to B if each element of A appears in some ordered pair in f and no two ordered pairs in f have the same first element.

If $(a, b) \in f$, then b is called the image of a under f.

ILLUSTRATION 1 *Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and f_1, f_2 and f_3 be three subsets of $A \times B$ as given below:*

$$f_1 = \{(1, 2), (2, 3), (3, 4)\},\ f_2 = \{(1, 2), (1, 3), (2, 3), (3, 4)\},\ f_3 = \{(1, 3), (2, 4)\}.$$

Then, f_1 is a function from A to B but f_2 and f_3 are not functions from A to B. f_2 is not a function from A to B, because $1 \in A$ has two images 2 and 3 in B and f_3 is not a function from A to B because $3 \in A$ has no image in B.

If a function f is expressed as the set of ordered pairs, the domain of f is the set of all first components of members of f and the range of f is the set of second components of members of f i.e. Domain of $f = \{a : (a, b) \in f\}$, and Range of $f = \{b : (a, b) \in f\}$

ILLUSTRATION 2 If $x, y \in \{1, 2, 3, 4\}$, then which of the following are functions in the given set?

(a) $f_1 = \{(x, y) : y = x + 1\}$ (b) $f_2 = \{(x, y) : x + y > 4\}$

(c) $f_3 = \{(x, y) : y < x\}$ (d) $f_4 = \{(x, y) : x + y = 5\}$

Also, in case of a function give its range.

SOLUTION If we express f_1, f_2, f_3 and f_4 as sets of ordered pairs, then we have

$$f_1 = \{(1, 2), (2, 3), (3, 4)\},$$

$$f_2 = \{(1, 4), (4, 1), (2, 3), (3, 2), (2, 4), (4, 2), (3, 4), (4, 3)\},$$

$$f_3 = \{(2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\} \text{ and } f_4 = \{(1, 4), (2, 3), (3, 2), (4, 1)\}.$$

(a) We have, $f_1 = \{(1, 2), (2, 3), (3, 4)\}$. We observe that an element 4 of the given set has not appeared in first place of any ordered pair of f_1. So, f_1 is not a function from the given set to itself.

(b) We have, $f_2 = \{(1, 4), (4, 1), (2, 3), (3, 2), (2, 4), (4, 2), (3, 4), (4, 3)\}$. We observe that 2, 3, 4 have appeared more than once as first components of the ordered pairs in f_2. So, f_2 is not a function.

(c) We have, $f_3 = \{(2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\}$. We observe that 3 and 4 have appeared more than once as first components of the ordered pairs in f_3. So, f_3 is not a function.

(d) We have, $f_4 = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$. We observe that each element of the given set has appeared as first components in one and only one ordered pair of f_4. So, f_4 is a function in the given set. In this case, Range of $f = \{1, 2, 3, 4\}$.

ILLUSTRATION 3 *Let f be a relation on the set N of natural numbers defined by $f = \{(n, 3n) : n \in N\}$. Is f a function from N to N. If so, find the range of f.*

SOLUTION We find that for each $n \in N$, there exists a unique $3n \in N$ such that $(n, 3n) \in f$. Therefore, f is a function from N to N. Clearly, Range of $f = \{f(n) : n \in N\} = \{3n : n \in N\}$.

ILLUSTRATION 4 *Let f be a subset of $Z \times Z$ defined by $f = \{(ab, (a + b)) : a, b \in Z\}$. Is f a function from Z into Z. Justify your answers.* **[NCERT]**

SOLUTION We observe that: $1 \times 6 = 6$ and $2 \times 3 = 6$

$$(1 \times 6, 1 + 6) \in f \text{ and } (2 \times 3, 2 + 3) \in f \Rightarrow (6, 7) \in f \text{ and } (6, 5) \in f$$

Thus, $(6, 7) \in f$ and $(6, 5) \in f$ but $5 \neq 7$. Hence, f is not a function from Z to Z.

3.3 FUNCTION AS A CORRESPONDENCE

DEFINITION *Let A and B be two non-empty sets. Then a function 'f' from set A to set B is a rule or method or correspondence which associates elements of set A to elements of set B such that:*

(i) *all elements of set A are associated to elements in set B.*

(ii) *an element of set A is associated to a unique element in set B.*

In other words, a function 'f' from a set A to a set B associates each element of set A to a unique element of set B.

Terms such as "map" (or "mapping"), "correspondence" are used as synonyms for "function". If f is a function from a set A to a set B, then we write $f : A \to B$ or $A \to B$, which is read as *f is a function from A to B* or *f maps A to B*.

If an element $a \in A$ is associated to an element $b \in B$, then b is called *'the f-image of a'* or *'image of a under f* or *'the value of the function f at a'*. Also, a is called the pre-image of b under the function f. We write it as: $b = f(a)$

ILLUSTRATION *Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d, e\}$ be two sets and let f_1, f_2, f_3 and f_4 be rules associating elements of set A to elements of set B as shown in figures 3.1 – 3.4.*

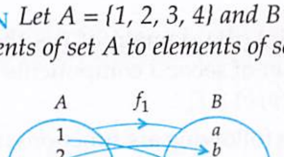

Fig. 3.1

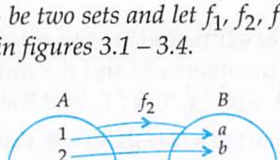

Fig. 3.2

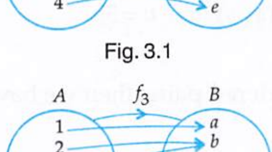

Fig. 3.3

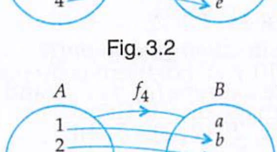

Fig. 3.4

We observe that f_1 is not a function from set A to set B, because there is an element $3 \in A$ which is not associated to any element of B.

Also, f_2 is not a function from A to B because an element $4 \in A$ is associated to two elements c and e in B. But, f_3 and f_4 are functions from A to B, because under f_3 and f_4 each element in A is associated to a unique element in B.

3.3.1 DESCRIPTION OF A FUNCTION

Let $f : A \to B$ be a function such that the set A consists of a finite number of elements. Then, $f(x)$ can be described by listing the values which it attains at different points of its domain. For example, if $A = \{-1, 1, 2, 3\}$ and B is the set of real numbers, then a function $f : A \to B$ can be described as $f(-1) = 3, f(1) = 0, f(2) = 3/2$ and $f(3) = 0$. In case, A is an infinite set, then f cannot be described by listing the images at points in its domain. In such cases functions are generally described by a formula. For example, $f : Z \to Z$ given by $f(x) = x^2 + 1$ or $f : R \to R$ given by $f(x) = e^x$ etc.

3.3.2 DOMAIN, CO-DOMAIN AND RANGE OF A FUNCTION

Let $f : A \to B$. Then, the set A is known as the domain of f and the set B is known as the co-domain of f. The set of all f-images of elements of A is known as the range of f or image set of A under f and is denoted by $f(A)$.

Thus, $f(A) = \{f(x) : x \in A\}$ = Range of f. Clearly, $f(A) \subseteq B$.

ILLUSTRATION 1 *Let $A = \{-2, -1, 0, 1, 2\}$ and $B = \{0, 1, 2, 3, 4, 5, 6\}$. Consider a rule $f(x) = x^2$. Under this rule, we obtain $f(-2) = (-2)^2 = 4$, $f(-1) = (-1)^2 = 1$, $f(0) = 0^2 = 0$, $f(1) = 1^2 = 1$ and $f(2) = 2^2 = 4$. We observe that each element of A is associated to a unique element of B. So, $f : A \to B$ given by $f(x) = x^2$ is a function. Clearly, domain $(f) = A = \{-2, -1, 0, 1, 2\}$ and range $(f) = \{0, 1, 4\}$.*

ILLUSTRATION 2 *Consider a rule $f(x) = 2x - 3$ associating elements of N (set of natural numbers) to elements of N. This rule does not define a function from N to itself, because $f(1) = 2 \times 1 - 3 = -1 \notin N$ i.e. $1 \in N$ (domain) is not associated to any element of N (co-domain). Thus, every rule relating elements of one set to elements of another set need not be a function.*

ILLUSTRATION 3 *Let $A = \{-2, -1, 0, 1, 2\}$ and $f : A \to Z$ be given by $f(x) = x^2 - 2x - 3$. Find:*
(i) the range of f (ii) pre-images of 6, −3 and 5.

SOLUTION (i) We have, $f(x) = x^2 - 2x - 3$.

$\therefore$ $f(-2) = (-2)^2 - 2(-2) - 3 = 5$, $f(-1) = (-1)^2 - 2(-1) - 3 = 0$, $f(0) = -3$,

$f(1) = 1^2 - 2 \times 1 - 3 = -4$ and $f(2) = 2^2 - 2 \times 2 - 3 = -3$.

So, range $(f) = \{f(-2), f(-1), f(0), f(1), f(2)\} = \{0, 5, -3, -4\}$

(ii) Let x be a pre-image of 6. Then,

$$f(x) = 6 \Rightarrow x^2 - 2x - 3 = 6 \Rightarrow x^2 - 2x - 9 = 0 \Rightarrow x = 1 \pm \sqrt{10}$$

Since $x = 1 \pm \sqrt{10} \notin A$. So, there is no pre-image of 6.

Let x be a pre-image of − 3. Then,

$$f(x) = -3 \Rightarrow x^2 - 2x - 3 = -3 \Rightarrow x^2 - 2x = 0 \Rightarrow x = 0, 2.$$

Clearly, $0, 2 \in A$. So, 0 and 2 are pre-images of − 3.

Let x be a pre-image of 5. Then,

$$f(x) = 5 \Rightarrow x^2 - 2x - 3 = 5 \Rightarrow x^2 - 2x - 8 = 0 \Rightarrow (x - 4)(x + 2) = 0 \Rightarrow x = 4, -2.$$

Since, $-2 \in A$ but $4 \notin A$. So, −2 is a pre-image of 5.

3.4 EQUAL FUNCTIONS

DEFINITION *Two functions f and g are said to be equal iff*

(i) *domain of f = domain of g* (ii) *co-domain of f = co-domain of g,*

and (iii) *$f(x) = g(x)$ for every x belonging to their common domain.*

If two functions f and g are equal, then we write $f = g$.

ILLUSTRATION 1 *Let $A = \{1, 2\}$, $B = \{3, 6\}$ and $f : A \to B$ given by $f(x) = x^2 + 2$ and $g : A \to B$ given by $g(x) = 3x$. Then, we observe that f and g have the same domain and co-domain. Also we have,* $f(1) = 3 = g(1)$ *and* $f(2) = 6 = g(2)$. Hence, $f = g$.

ILLUSTRATION 2 *Let $f : R - \{2\} \to R$ be defined by $f(x) = \dfrac{x^2 - 4}{x - 2}$ and $g : R \to R$ be defined by $g(x) = x + 2$. Find whether $f = g$ or not.*

SOLUTION We have, $f(x) = \dfrac{x^2 - 4}{x - 2}, x \neq 2$.

$$\Rightarrow \quad f(x) = \frac{(x-2)(x+2)}{x-2} = x + 2 \text{ for all } x \neq 2. \Rightarrow f(x) = g(x) \text{ for all } x \neq 2.$$

Thus, $f(x) = g(x)$ for all $x \in R - \{2\}$. But, $f(x)$ and $g(x)$ have different domains. Infact, domain of $f = R - \{2\}$ and domain of $g = R$. Therefore, $f \neq g$.

ILLUSTRATION 3 *Let $f : Z \to Z$ and $g : Z \to Z$ be functions defined by $f = \{(n, n^2) : n \in Z\}$ and, $g = \{(n, |n|^2) : n \in Z\}$. Show that: $f = g$.*

SOLUTION Clearly, Domain of f = Domain of $g = Z$ and, Co-domain of f = Co-domain of $g = Z$.

We have, $f(n) = n^2$ and $g(n) = |n|^2 = n^2$ $\quad [\because |n|^2 = n^2]$

$\therefore \quad f(n) = g(n)$ for all $n \in Z$.

Hence, $f = g$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Express the following functions as sets of ordered pairs and determine their ranges*

(i) $f : A \to R, f(x) = x^2 + 1$, *where* $A = \{-1, 0, 2, 4\}$.

(ii) $g : A \to N, g(x) = 2x$, *where* $A = \{x : x \in N, x \leq 10\}$.

SOLUTION (i) We have, $f(x) = x^2 + 1$.

$$\therefore \quad f(-1) = (-1)^2 + 1 = 2, f(0) = 0^2 + 1 = 1, f(2) = 2^2 + 1 = 5 \text{ and } f(4) = 4^2 + 1 = 17$$

So, $\quad f = \{(x, f(x)) : x \in A\} = \{(-1, 2), (0, 1), (2, 5), (4, 17)\}$. Hence, Range of $(f) = \{2, 1, 5, 17\}$

(ii) We have, $g(x) = 2x$ and $A = \{1, 2, 3, ... 10\}$. Therefore,

$g(1) = 2 \times 1 = 2, \quad g(2) = 2 \times 2 = 4, \; g(3) = 2 \times 3 = 6, \; g(4) = 2 \times 4 = 8, g(5) = 2 \times 5 = 10,$

$g(6) = 2 \times 6 = 12, \quad g(7) = 2 \times 7 = 14, g(8) = 2 \times 8 = 16, \; g(9) = 2 \times 9 = 18$ and, $g(10) = 2 \times 10 = 20.$

$\therefore g = \{(x, g(x) : x \in A\} = \{(1, 2), (2, 4), (3, 6), ..., (10, 20)\}$.

Hence, Range of $g = g(A) = \{g(x) : x \in A\} = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$.

EXAMPLE 2 *Find the domain for which the functions $f(x) = 2x^2 - 1$ and $g(x) = 1 - 3x$ are equal.*

[NCERT EXEMPLAR]

SOLUTION The values of x for which $f(x)$ and $g(x)$ are equal are given by

$f(x) = g(x) \Rightarrow 2x^2 - 1 = 1 - 3x \Rightarrow 2x^2 + 3x - 2 = 0 \Rightarrow (x+2)(2x-1) = 0 \Rightarrow x = -2, 1/2.$

Thus, $f(x)$ and $g(x)$ are equal on the set $\{-2, 1/2\}$.

EXAMPLE 3 *Is $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ a function? If this is described by the formula, $g(x) = \alpha x + \beta$, then what values should be assigned to α and β?* **[NCERT EXEMPLAR]**

SOLUTION Since no two ordered pairs in g have the same first component. So, g is a function such that $g(1) = 1$, $g(2) = 3$, $g(3) = 5$ and $g(4) = 7$.

It is given that $g(x) = \alpha x + \beta$.

$\therefore \quad g(1) = 1$ and $g(2) = 3 \Rightarrow \alpha + \beta = 1$ and $2\alpha + \beta = 3 \Rightarrow \alpha = 2, \beta = -1$.

EXAMPLE 4 *Given $A = \{-1, 0, 2, 5, 6, 11\}$, $B = \{-2, -1, 0, 18, 28, 108\}$ and $f(x) = x^2 - x - 2$. Is $f(A) = B$? Find $f(A)$.*

SOLUTION We have, $f(x) = x^2 - x - 2$.

$\therefore \quad f(-1) = (-1)^2 - (-1) - 2 = 0, \quad f(0) = 0^2 - 0 - 2 = -2, \quad f(2) = 2^2 - 2 - 2 = 0,$

$f(5) = 5^2 - 5 - 2 = 18, \quad f(6) = 6^2 - 6 - 2 = 28$ and $f(11) = 11^2 - 11 - 2 = 108$.

Hence, $f(A) = \{f(x) : x \in A\} = \{f(-1), f(0), f(2), f(5), f(6), f(11)\} = \{0, -2, 18, 28, 108\}$

We observe that $-1 \in B$, but $-1 \notin f(A)$. So, $f(A) \neq B$.

EXAMPLE 5 *Let $f : R \to R$ be given by $f(x) = x^2 + 3$. Find : (i) $\{x : f(x) = 28\}$ (ii) the pre-images of 39 and 2 under f.*

SOLUTION (i) We have, $f(x) = x^2 + 3$

$\therefore \quad f(x) = 28 \Rightarrow x^2 + 3 = 28 \Rightarrow x^2 = 25 \Rightarrow x = \pm 5$

Hence, $\{x : f(x) = 28\} = \{-5, 5\}$.

(ii) Let x be a pre-image of 39. Then, the image of x under f is 39.

i.e. $\quad f(x) = 39 \Rightarrow x^2 + 3 = 39 \Rightarrow x^2 = 36 \Rightarrow x = \pm 6$

So, pre-images of 39 are -6 and 6.

Let x be a pre-image of 2. Then, the image of x under f is 2.

i.e. $\quad f(x) = 2 \Rightarrow x^2 + 3 = 2 \Rightarrow x^2 = -1$

We find that no real value of x satisfies the equation $x^2 = -1$. Therefore, 2 does not have any pre-image under f.

EXAMPLE 6 *Let $f : R \to R$ be a function given by $f(x) = x^2 + 1$. Find:*

(i) $f^{-1}\{-5\}$ (ii) $f^{-1}\{26\}$ (iii) $f^{-1}\{10, 37\}$

SOLUTION Recall that if $f : A \to B$ is a function and $y \in B$. Then, $f^{-1}(y) = \{x \in A : f(x) = y\}$. In other words, $f^{-1}(y)$ is the set of pre-images of y.

(i) Let $f^{-1}(-5) = x$. Then, $f(x) = -5 \Rightarrow x^2 + 1 = -5 \Rightarrow x^2 = -6$. Clearly, this equation is not solvable in R. Therefore, there is no pre-image of -5. So, $f^{-1}\{-5\} = \phi$.

(ii) Let $f^{-1}(26) = x$. Then, $f(x) = 26 \Rightarrow x^2 + 1 = 26 \Rightarrow x = \pm 5$. So, pre-images of 26 are -5 and 5. i.e. $f^{-1}\{26\} = \{-5, 5\}$.

(iii) Let $f^{-1}(10) = x$. Then, $f(x) = 10 \Rightarrow x^2 + 1 = 10 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$. So, pre-images of 10 are -3 and 3. i.e. $f^{-1}(10) = \{-3, 3\}$.

Let $f^{-1}(37) = x$. Then, $f(x) = 37 \Rightarrow x^2 + 1 = 37 \Rightarrow x^2 = 36 \Rightarrow x = \pm 6$. So, pre-images of 37 are -6 and 6. Hence, $f^{-1}\{10, 37\} = \{3, -3, 6, -6\}$.

EXAMPLE 7 *Let $f = \{(1, 1), (2, 3), (0, -1), (-1, -3)\}$ be a function described by the formula $f(x) = ax + b$ for some integers a, b. Determine a, b.* [NCERT]

SOLUTION Clearly, $f(1) = 1, f(2) = 3, f(0) = -1$ and $f(-1) = -3$. It is given that $f(x) = ax + b$.

$\therefore$ $f(1) = 1$ and $f(2) = 3 \Rightarrow a + b = 1$ and $2a + b = 3 \Rightarrow a = 2, b = -1$.

Substituting the values of a and b in $f(x) = ax + b$, we get: $f(x) = 2x - 1$.

$\therefore$ Clearly, $f(0) = -1$ and $f(-1) = -3$ are true.

Hence, $a = 2$ and $b = -1$.

EXAMPLE 8 *If $f : R \to R$ be defined as* $f(x) = \begin{cases} 1, & \text{if } x \in Q \\ -1, & \text{if } x \notin Q. \end{cases}$

Find (i) $f(1/2), f(\pi), f(\sqrt{2})$ (ii) Range of f (iii) pre-images of 1 *and* –1.

SOLUTION (i) It is evident from the definition of f that at every rational point the function attains value 1 and at every irrational point attains value –1.

$$\therefore \quad \frac{1}{2} \in Q \Rightarrow f\left(\frac{1}{2}\right) = 1, \pi \notin Q \Rightarrow f(\pi) = -1 \text{ and } \sqrt{2} \notin Q \Rightarrow f(\sqrt{2}) = -1.$$

(ii) We find that $f(x)$ attains values 1 or –1 according as x is rational or irrational and a real number is either rational or irrational. Thus, all rational numbers have image 1 and all irrational numbers have image -1. Hence, Range of $f = \{1, -1\}$.

(iii) Since $f(x) = 1$ for all $x \in Q$. Therefore, pre-images of 1 are rational numbers i.e $f^{-1}(1) = Q$.

Also, –1 is the image of every real number which is not rational. Therefore,

$f^{-1}(-1) = R - Q =$ Set of irrational numbers.

EXAMPLE 9 *Let $f : R \to R$ be such that $f(x) = 2^x$. Determine:*

(i) *Range of f* (ii) $\{x : f(x) = 1\}$ (iii) *whether $f(x + y) = f(x) f(y)$ holds.*

SOLUTION (i) Since 2^x is positive for every $x \in R$. So, $f(x) = 2^x$ is a positive real number for every $x \in R$. Moreover, for every positive real number x, there exist $\log_2 x \in R$ such that

$$f(\log_2 x) = 2^{\log_2 x} = x \qquad [\because a^{\log_a x} = x]$$

Hence, the range of f is the set of all positive real numbers.

(ii) $f(x) = 1 \Rightarrow 2^x = 1 \Rightarrow 2^x = 2^0 \Rightarrow x = 0$. Therefore, $\{x : f(x) = 1\} = \{0\}$.

(iii) We have, $f(x) = 2^x$.

$$\therefore \quad f(x + y) = 2^{x+y} = 2^x \times 2^y = f(x) f(y)$$

Hence, $f(x + y) = f(x) f(y)$ holds for all $x, y \in R$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 10 *Let A be the set of two positive integers. Let $f : A \to Z^+$ (set of positive integers) be defined by $f(n) = p$, where p is the highest prime factor of n. If range of $f = \{3\}$. Find set A. Is A uniquely determined?*

SOLUTION It is given that the set A consists of two positive integers. So, let $A = \{n, m\}$. Since range of $f = \{3\}$.

$\therefore$ $f(n) = 3$ and $f(m) = 3$

$\Rightarrow$ Highest prime factors of n and m both are equal to 3.

$\Rightarrow$ $(n = 3$ and $m = 6)$ or $(n = 3$ and $m = 9)$ or $(n = 3$ and $m = 12)$ or $(n = 6$ and $m = 12)$ etc.

$\Rightarrow$ Either $A = \{3, 6\}$, or $A = \{3, 9\}$, or $A = \{3, 12\}$, or $A = \{6, 12\}$ etc.

Clearly, A is not uniquely determined.

EXAMPLE 11 *Let $A \subseteq N$ and $f : A \to A$ be defined by : $f(n) =$ the highest prime factor of n. If range of f is A, determine A. Is A uniquely determined?*

SOLUTION For any $n \in A$, we have

$f(n) =$ Highest prime factor of n

$\Rightarrow$ $f(n)$ takes prime values only $\Rightarrow$ Range of f consists of prime numbers only

But, it is given that range of f is A. Therefore, set A consists of prime numbers only.

Hence, $A =$ Set of some prime numbers. Clearly, A is not uniquely determined.

EXERCISE 3.1

BASIC

1. Define a function as a set of ordered pairs.
2. Define a function as a correspondence between two sets.
3. What is the fundamental difference between a relation and a function ? Is every relation a function ?
4. Let $A = \{-2, -1, 0, 1, 2\}$ and $f : A \to Z$ be a function defined by $f(x) = x^2 - 2x - 3$. Find:
   (i) range of f i.e. $f(A)$ (ii) pre-images of 6, –3 and 5.
5. If a function $f : R \to R$ be defined by $f(x) = \begin{cases} 3x - 2, & x < 0 \\ 1, & x = 0 \\ 4x + 1, & x > 0 \end{cases}$. Find: $f(1), f(-1), f(0), f(2)$.
6. A function $f : R \to R$ is defined by $f(x) = x^2$. Determine
   (i) range of f (ii) $\{x : f(x) = 4\}$ (iii) $\{y : f(y) = -1\}$.
7. Let $f : R^+ \to R$, where R^+ is the set of all positive real numbers be such that $f(x) = \log_e x$. Determine: (i) the image set of the domain of f (ii) $\{x : f(x) = -2\}$ (iii) whether $f(xy) = f(x) + f(y)$ holds.
8. Write the following relations as sets of ordered pairs and find which of them are functions:
   (i) $\{(x, y) : y = 3x, x \in \{1, 2, 3\}, y \in \{3, 6, 9, 12\}\}$.
   (ii) $\{(x, y) : y > x + 1, x = 1, 2$ and $y = 2, 4, 6\}$ (iii) $\{(x, y) : x + y = 3, x, y \in \{0, 1, 2, 3\}\}$
9. Let $f : R \to R$ and $g : C \to C$ be two functions defined as $f(x) = x^2$ and $g(x) = x^2$. Are they equal functions?
10. If f, g, h are three functions defined from R to R as follows:
    (i) $f(x) = x^2$ (ii) $g(x) = \sin x$ (iii) $h(x) = x^2 + 1$
    Find the range of each function.
11. Let $X = \{1, 2, 3, 4\}$ and $Y = \{1, 5, 9, 11, 15, 16\}$. Determine which of the following sets are functions from X to Y
    (i) $f_1 = \{(1, 1), (2, 11), (3, 1), (4, 15)\}$ (ii) $f_2 = \{(1, 1), (2, 7), (3, 5)\}$
    (iii) $f_3 = \{(1, 5), (2, 9) (3, 1), (4, 5), (2, 11)\}$. **[NCERT]**
12. Let $A = \{12, 13, 14, 15, 16, 17\}$ and $f : A \to Z$ be a function given by $f(x) =$ highest prime factor of x. Find range of f.
13. If $f : R \to R$ be defined by $f(x) = x^2 + 1$, then find $f^{-1}\{17\}$ and $f^{-1}\{-3\}$.
14. Let $A = \{p, q, r, s\}$ and $B = \{1, 2, 3\}$. Which of the following relations from A to B is not a function?

(i) $R_1 = \{(p, 1), (q, 2), (r, 1), (s, 2)\}$ (ii) $R_2 = \{(p, 1), (q, 1), (r, 1), (s, 1)\}$
(iii) $R_3 = \{(p, 1), (q, 2), (p, 2), (s, 3)\}$ (iv) $R_4 = \{(p, 2), (q, 3), (r, 2), (s, 2)\}$.

15. Let $A = \{9, 10, 11, 12, 13\}$ and let $f : A \to N$ be defined by $f(n)$ = the highest prime factor of n. Find the range of f. **[NCERT]**

16. The function f is defined by $f(x) = \begin{cases} x^2, & 0 \le x \le 3 \\ 3x, & 3 \le x \le 10 \end{cases}$. The relation g is defined by

$g(x) = \begin{cases} x^2, & 0 \le x \le 2 \\ 3x, & 2 \le x \le 10 \end{cases}$ Show that f is a function and g is not a function. **[NCERT]**

17. If $f(x) = x^2$, find $\dfrac{f(1.1) - f(1)}{(1.1) - 1}$. **[NCERT]**

18. Express the function $f : X \to R$ given by $f(x) = x^3 + 1$ as set of ordered pairs, where $X = \{-1, 0, 3, 9, 7\}$. **[NCERT EXEMPLAR]**

ANSWERS

4. (i) $f(A) = \{-4, -3, 0, 5\}$ (ii) ϕ, $\{0, 2\}$, -2. 5. $f(1) = 5$, $f(-1) = -5$, $f(0) = 1$, $f(2) = 9$
6. (i) R^+ (set of all real numbers greater than or equal to zero) (ii) $\{-2, 2\}$ (iii) ϕ
7. (i) R (ii) $\{e^{-2}\}$ (iii) Yes
8. (i) $\{(1, 3), (2, 6), (3, 9)\}$; Function (ii) $\{(1, 4), (1, 6), (2, 4), (2, 6)\}$; Not a function
(iii) $\{(0, 3), (1, 2), (2, 1), (3, 0)\}$; Function.
9. No, Since domain of $f \ne$ domain of g.
10. (i) $R^+ = \{x \in R \mid x \ge 0\}$ (ii) $\{x \in R : -1 \le x \le 1\}$ (iii) $\{x \in R : x \ge 1\}$.
11. (i) 12. $\{3, 13, 7, 5, 2, 17\}$ 13. $f^{-1}(17) = \{-4, 4\}$, $f^{-1}(-3) = \phi$
14. (iii) 15. Range $(f) = \{3, 5, 11, 13\}$. 17. 2.1
18. $f = \{(-1, 0), (0, 1), (3, 28), (9, 730), (7, 344)\}$

HINTS TO SELECTED PROBLEMS

11. (iii) $f_3 = \{(1,5), (2, 9), (3, 1), (4, 5), (2, 11)\}$ is not a function because $(2, 9)$ and $(2, 11) \in f_3$, which means that 2 is related to two elements in y.

12. Clearly, $f(12)$ = highest prime factor of 12 = 3. Similarly, $f(13) = 13$, $f(14) = 7$, $f(15) = 5$, $f(16) = 2$ and $f(17) = 17$. Hence, range $(f) = \{3, 13, 7, 5, 2, 17\}$.

15. $A = \{9, 10, 11, 12, 13\}$ and $f : A \to N$ is defined by $f(n)$ = the highest prime factor of n.
$\therefore$ $f(9) = 3$, $f(10) = 5$, $f(11) = 11$, $f(12) = 3$ and $f(13) = 13$
Hence, range $(f) = \{f(9), f(10), f(11), f(12), f(13)\} = \{3, 5, 11, 13\}$

16. We observe that $f(x) = \begin{cases} x^2, & 0 \le x \le 3 \\ 3x, & 3 \le x \le 10 \end{cases}$ associates all numbers in $[0, 10]$ to numbers in R and no number in $[0, 10]$ is associated to two or more numbers. Hence, f is a function. But, g is not a function because 2 is associated to two distinct elements viz. 4 and 6.

17. We have, $f(x) = x^2$

$$\therefore \quad \frac{f(1.1) - f(1)}{(1.1) - 1} = \frac{(1.1)^2 - 1^2}{(1.1) - 1} = \frac{(1.1 + 1)(1.1 - 1)}{(1.1 - 1)} = 2.1$$

3.5 REAL FUNCTIONS

In this section, we will discuss functions having domain and co-domain both as subsets of the set R of all real numbers. Such functions are called real functions or real valued functions of the real variable as defined below.

REAL VALUED FUNCTION *A function $f : A \to B$ is called a real valued function, if B is a subset of R (set of all real numbers).*

If A and B both are subsets of R, then f is called a real function.

In section 3.3.1, we have discussed the description of a function. Generally, domain and co-domain both are infinite subsets of R in case of real functions of real variable. Therefore, a real function is generally described by some general formula. In other words, images of various elements in the domain of a real function are provided by some general formula. For example, $f : R \to R$ given by $f(x) = x^2 + x + 1$ or, $f : A \to B$ given by $f(x) = \dfrac{x-1}{x^2-4}$ etc. In practice, real functions are described by giving the general expressions or formulae describing them without mentioning their domains and co-domains. Following are some examples of real functions.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *If $f(x) = 3x^4 - 5x^2 + 9$, find $f(x-1)$.*

SOLUTION We have, $f(x) = 3x^4 - 5x^2 + 9$. Replacing x by $(x-1)$, we obtain

$$f(x-1) = 3(x-1)^4 - 5(x-1)^2 + 9 = 3x^4 - 12x^3 + 13x^2 - 2x + 7$$

EXAMPLE 2 *If $f(x) = x + \dfrac{1}{x}$, prove that $\{f(x)\}^3 = f(x^3) + 3 f\left(\dfrac{1}{x}\right)$.*

SOLUTION We have,

$$f(x) = x + \frac{1}{x} \Rightarrow f(x^3) = x^3 + \frac{1}{x^3},\ \{f(x)\}^3 = \left(x + \frac{1}{x}\right)^3 \text{ and } f\left(\frac{1}{x}\right) = \frac{1}{x} + x.$$

$$\therefore \quad \left(x + \frac{1}{x}\right)^3 = \left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right) \Rightarrow \{f(x)\}^3 = f(x^3) + 3 f(x)$$

EXAMPLE 3 *If $f(x) = \dfrac{1}{2x+1}$, $x \neq -\dfrac{1}{2}$, then show that $f\left(f(x)\right) = \dfrac{2x+1}{2x+3}$, provided that $x \neq -\dfrac{3}{2}$.*

SOLUTION We have, $f(x) = \dfrac{1}{2x+1}$

$$\Rightarrow \quad f(f(x)) = f\left(\frac{1}{2x+1}\right) = \frac{1}{2\left(\dfrac{1}{2x+1}\right)+1} = \frac{1}{\dfrac{2}{2x+1}+1} = \frac{2x+1}{2+2x+1} = \frac{2x+1}{2x+3}$$

Clearly, $f(f(x)) = \dfrac{2x+1}{2x+3}$ is real for $2x + 3 \neq 0$ i.e. $f(f(x))$ is defined for $2x + 3 \neq 0$ i.e. $x \neq -\dfrac{3}{2}$.

Hence, $f(f(x)) = \dfrac{2x+1}{2x+3}$, provided that $x \neq -\dfrac{3}{2}$.

EXAMPLE 4 *If $f(x) = \dfrac{x-1}{x+1}$, $x \neq -1$, then show that $f(f(x)) = -\dfrac{1}{x}$, provided that $x \neq 0$.*

SOLUTION We have, $f(x) = \dfrac{x-1}{x+1}, x \neq -1.$

$$\therefore \quad f(f(x)) = f\left(\frac{x-1}{x+1}\right) = \frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1} \qquad \left[\text{Replacing } x \text{ by } \frac{x-1}{x+1} \text{ in the formula for } f(x)\right]$$

$$= \frac{x-1-x-1}{x-1+x+1} = \frac{-2}{2x} = -\frac{1}{x}.$$

We find that $-\dfrac{1}{x}$ is meaningful for $x \neq 0$. Hence, $f(f(x)) = -\dfrac{1}{x}$, provided that $x \neq 0$.

EXAMPLE 5 *Let f be defined by* $f(x) = x-4$ *and g be defined by* $g(x) = \begin{cases} \dfrac{x^2-16}{x+4}, & x \neq -4 \\ \lambda, & x = -4 \end{cases}$

Find λ *such that* $f(x) = g(x)$ *for all x.*

SOLUTION We have,

$$f(x) = g(x) \quad \text{for all } x \in R$$

$$\Rightarrow \quad f(-4) = g(-4) \Rightarrow -4-4 = \lambda \Rightarrow \lambda = -8. \qquad [\because f(x) = x-4 \therefore f(-4)-4 = -8]$$

EXAMPLE 6 *If f is a real function defined by* $f(x) = \dfrac{x-1}{x+1}$, *then prove that:* $f(2x) = \dfrac{3f(x)+1}{f(x)+3}$.

SOLUTION We have, $f(x) = \dfrac{x-1}{x+1}$

$$\Rightarrow \quad \frac{f(x)}{1} = \frac{x-1}{x+1} \Rightarrow \frac{f(x)+1}{f(x)-1} = \frac{x-1+x+1}{x-1-x-1} \qquad \text{[Applying componendo and dividendo]}$$

$$\Rightarrow \quad \frac{f(x)+1}{f(x)-1} = -x \Rightarrow x = \frac{f(x)+1}{1-f(x)} \qquad \ldots\text{(i)}$$

Again, $f(x) = \dfrac{x-1}{x+1}$

$$\Rightarrow \quad f(2x) = \frac{2x-1}{2x+1} = \frac{2\left\{\frac{f(x)+1}{1-f(x)}\right\}-1}{2\left\{\frac{f(x)+1}{1-f(x)}\right\}+1} \qquad \text{[Using (i)]}$$

$$= \frac{2f(x)+2-1+f(x)}{2f(x)+2+1-f(x)} = \frac{3f(x)+1}{f(x)+3}$$

EXERCISE 3.2

BASIC

1. If $f(x) = x^2 - 3x + 4$, then find the values of x satisfying the equation $f(x) = f(2x+1)$.
2. If $f(x) = (x-a)^2 (x-b)^2$, find $f(a+b)$.
3. If $y = f(x) = \dfrac{ax-b}{bx-a}$, show that $x = f(y)$. [NCERT EXEMPLAR]
4. If $f(x) = \dfrac{1}{1-x}$, show that $f[f\{f(x)\}] = x$.

5. If $f(x)=\dfrac{x+1}{x-1}$, show that $f[f(x)]=x$.

6. If $f(x) = \begin{cases} x^2, \text{ when } x<0 \\ x, \text{ when } 0 \le x < 1 \\ \dfrac{1}{x}, \text{ when } x>1 \end{cases}$ **[NCERT]**

Find: (i) $f(1/2)$ (ii) $f(-2)$ (iii) $f(1)$ (iv) $f(\sqrt{3})$ and (v) $f(\sqrt{-3})$.

7. If $f(x)=x^3-\dfrac{1}{x^3}$, show that $f(x)+f\left(\dfrac{1}{x}\right)=0$.

8. If $f(x)=\dfrac{2x}{1+x^2}$, show that $f(\tan\theta)=\sin 2\theta$.

9. If $f(x)=\dfrac{x-1}{x+1}$ then show that: (i) $f\left(\dfrac{1}{x}\right)=-f(x)$ (ii) $f\left(-\dfrac{1}{x}\right)=-\dfrac{1}{f(x)}$

BASED ON HOTS

10. If $f(x)=(a-x^n)^{1/n}$, $a>0$ and $n \in N$, then prove that $f(f(x))=x$ for all x.

11. If for non-zero x, $a\,f(x)+b\,f\left(\dfrac{1}{x}\right)=\dfrac{1}{x}-5$, where $a \ne b$, then find $f(x)$.

ANSWERS

1. $x=-1, 2/3$ 2. $a^2 b^2$

6. (i) $\dfrac{1}{2}$ (ii) 4. (iii) 1 (iv) $\dfrac{1}{\sqrt{3}}$ (v) does not exist 11. $\dfrac{1}{a^2-b^2}\left\{\dfrac{a}{x}-bx\right\}-\dfrac{5}{a+b}$

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6. We have, $f(x)=\begin{cases} x^2, & \text{when } x<0 \\ x, & \text{when } 0 \le x < 1 \\ \dfrac{1}{x}, & \text{when } x \ge 1 \end{cases}$. Therefore,

(i) $f\left(\dfrac{1}{2}\right)=\dfrac{1}{2}$ (ii) $f(-2)=(-2)^2=4$ (iii) $f(1)=\dfrac{1}{1}=1$

(iv) $f(\sqrt{3})=\dfrac{1}{\sqrt{3}}$ (iv) $f(-\sqrt{3})=(-\sqrt{3})^2=3$

11. We have,

$$a\,f(x)+b\,f\left(\frac{1}{x}\right)=\frac{1}{x}-5 \quad \text{... (i)}$$

$$\Rightarrow a\,f\left(\frac{1}{x}\right)+b\,f(x)=x-5 \quad \left[\text{Replacing } x \text{ by } \frac{1}{x}\right] \text{ ... (ii)}$$

Adding (i) and (ii), we get

$$\left\{f(x)+f\left(\frac{1}{x}\right)\right\}(a+b)=x+\frac{1}{x}-10$$

$$\Rightarrow \quad f(x)+f\left(\frac{1}{x}\right) = \frac{1}{a+b}\left\{x+\frac{1}{x}-10\right\} \qquad \text{...(iii)}$$

Subtracting (ii) from (i), we get

$$f(x)-f\left(\frac{1}{x}\right)=\frac{1}{a-b}\left\{\frac{1}{x}-x\right\} \qquad \text{... (iv)}$$

Add (iii) and (iv) to obtain $f(x)$.

3.6 DOMAIN AND RANGE OF A REAL A FUNCTION

Mathematically to define a function one has to provide its domain, co-domain and the images of elements in its domain either by giving a general formula or by listing them one by one. As the domain and co-domain of real functions are subsets of R. Therefore, conventionally, real functions are described by providing the general formula for finding the images of elements in its domain. In such cases, the domain of the real function $f(x)$ is the set of all those real numbers for which the expression for $f(x)$ or the formula for $f(x)$ assumes real values only. In otherwords, the domain of $f(x)$ is the set of all those real numbers for which $f(x)$ is meaningful.

For example, a real function $f(x)$ described by the general formula $f(x)=\dfrac{3x-2}{x^2-1}$ assumes real values for all $x \in R$ except for $x=\pm 1$, because denominator of $\dfrac{3x-2}{x^2-1}$ becomes zero for $x=\pm 1$.

So, domain of $f(x)$ is the set of all real numbers other than -1 and 1 i.e. domain $(f)=R-\{-1, 1\}$.

Following examples will illustrate the procedure for finding the domain of a real function of a real variable.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the domain of each of the following real valued functions:*

(i) $f(x)=\dfrac{1}{x+2}$ (ii) $f(x)=\dfrac{x-1}{x-3}$

(iii) $f(x)=\dfrac{2x-3}{x^2-3x+2}$ (iv) $f(x)=\dfrac{x^2+3x+5}{x^2-5x+4}$ [NCERT]

SOLUTION (i) We have, $f(x)=\dfrac{1}{x+2}$. Clearly, $f(x)$ assumes real values for all real values of x except for the values of x satisfying $x+2=0$ i.e. $x=-2$. Hence, Domain $(f)=R-\{-2\}$.

(ii) We have, $f(x)=\dfrac{x-1}{x-3}$. We observe that $f(x)$ is a rational function of x as $\dfrac{x-1}{x-3}$ is a rational expression. Clearly, $f(x)$ assumes real values for all x except for the values of x for which $x-3=0$ i.e. $x=3$. Hence, Domain $(f)=R-\{3\}$.

(iii) We have, $f(x)=\dfrac{2x-3}{x^2-3x+2}$. Clearly, $f(x)$ is a rational function of x as $\dfrac{2x-3}{x^2-3x+2}$ is a rational expression. We observe that $f(x)$ assumes real values for all x except for all those values of x for which $x^2-3x+2=0$ i.e. $x=1, 2$. Hence, Domain $(f)=R-\{1, 2\}$.

(iv) We have, $f(x)=\dfrac{x^2+3x+5}{x^2-5x+4}$. Clearly, $f(x)$ is a rational function of x as $\dfrac{x^2+3x+5}{x^2-5x+4}$ is a rational expression in x. We observe that $f(x)$ assumes real values for all x except for all those values of x for which $x^2-5x+4=0$ i.e. $x=1, 4$. Hence, Domain $(f) = R-\{1, 4\}$.

EXAMPLE 2 *Find the domain of each of the following functions:*

(i) $f(x) = \sqrt{x-2}$ (ii) $f(x) = \dfrac{1}{\sqrt{1-x}}$ (iii) $f(x) = \sqrt{4-x^2}$

SOLUTION (i) We have, $f(x)=\sqrt{x-2}$. Clearly, $f(x)$ assumes real values for all x satisfying $x-2\geq 0 \Rightarrow x\geq 2 \Rightarrow x\in[2,\infty)$. Hence, Domain $(f) = [2, \infty)$.

(ii) We have, $f(x)=\dfrac{1}{\sqrt{1-x}}$. Clearly, $f(x)$ assumes real values for all x satisfying $1-x>0 \Rightarrow 1>x \Rightarrow x<1 \Rightarrow x\in(-\infty, 1)$. Hence, Domain $(f)=(-\infty, 1)$.

(iii) We have, $f(x)=\sqrt{4-x^2}$. Clearly, $f(x)$ assumes real values for all x satisfying

$$4-x^2\geq 0 \Rightarrow -(x^2-4)\geq 0 \Rightarrow x^2-4\leq 0 \Rightarrow (x-2)(x+2)\leq 0 \Rightarrow x\in[-2, 2].$$

Hence, Domain $(f) = [-2, 2]$.

EXAMPLE 3 *Find the domain of the function $f(x)$ defined by* $f(x)=\sqrt{4-x}+\dfrac{1}{\sqrt{x^2-1}}$.

SOLUTION Clearly, $f(x)$ is defined for all x satisfying

$4-x\geq 0$ and $x^2-1>0$

$\Rightarrow x-4\leq 0$ and $(x-1)(x+1)>0 \Rightarrow x\leq 4$ and $(x<-1$ or $x>1) \Rightarrow x\in(-\infty,-1)\cup(1, 4]$.

Hence, Domain $(f) = (-\infty, -1)\cup(1, 4]$.

3.6.1 RANGE OF REAL FUNCTIONS

The range of a real function of a real variable is the set of all real values taken by $f(x)$ at points in its domain. In order to find the range of a real function $f(x)$, we may use the following algorithm.

ALGORITHM

Step I *Put $y=f(x)$.*

Step II *Solve the equation $y=f(x)$ for x in terms of y. Let $x=\phi(y)$.*

Step III *Find the values of y for which the values of x, obtained from $x=\phi(y)$, are real and in the domain of f.*

Step IV *The set of values of y obtained in step III is the range of f.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the domain and range of the function $f(x)$ given by* $f(x) = \dfrac{x-2}{3-x}$.

SOLUTION We have, $f(x) = \dfrac{x-2}{3-x}$.

Domain of f: Clearly, $f(x)$ is defined for all x satisfying $3-x\neq 0$ i.e. $x\neq 3$. Hence, Domain $(f)=R-\{3\}$.

Range of f: Let $y = f(x)$. Then,

$$y = \frac{x-2}{3-x} \Rightarrow 3y - xy = x - 2 \Rightarrow x(y+1) = 3y + 2 \Rightarrow x = \frac{3y+2}{y+1}$$

Clearly, x assumes real values for all y except $y + 1 = 0$ i.e. $y = -1$. Hence, Range $(f) = R - \{-1\}$.

EXAMPLE 2 *Find the range of each of the following functions:*

(i) $f(x) = \frac{1}{\sqrt{x-5}}$ (ii) $f(x) = \sqrt{16 - x^2}$ (iii) $f(x) = \frac{x}{1+x^2}$ (iv) $f(x) = \frac{3}{2-x^2}$

SOLUTION (i) We have, $f(x) = \frac{1}{\sqrt{x-5}}$. Clearly, $f(x)$ takes real values for all x satisfying $x - 5 > 0 \Rightarrow x > 5 \Rightarrow x \in (5, \infty)$. Therefore, Domain $(f) = (5, \infty)$.

For any $x > 5$, we have

$$x - 5 > 0 \Rightarrow \sqrt{x-5} > 0 \Rightarrow \frac{1}{\sqrt{x-5}} > 0 \Rightarrow f(x) > 0.$$

Thus, $f(x)$ takes all real values greater than zero. Hence, Range $(f) = (0, \infty)$.

(ii) We have, $f(x) = \sqrt{16 - x^2}$. We observe that $f(x)$ is defined for all x satisfying

$$16 - x^2 \geq 0 \Rightarrow x^2 - 16 \leq 0 \Rightarrow (x-4)(x+4) \leq 0 \Rightarrow -4 \leq x \leq 4 \Rightarrow x \in [-4, 4]$$

$\therefore$ Domain $(f) = [-4, 4]$.

Let $y = f(x)$. Then,

$$y = \sqrt{16 - x^2} \Rightarrow y^2 = 16 - x^2 \Rightarrow x^2 = 16 - y^2 \Rightarrow x = \sqrt{16 - y^2}$$

Clearly, x will take real values, if

$$16 - y^2 \geq 0 \Rightarrow y^2 - 16 \leq 0 \Rightarrow (y-4)(y+4) \leq 0 \Rightarrow -4 \leq y \leq 4 \Rightarrow y \in [-4, 4]$$

Also, $y = \sqrt{16 - x^2} \geq 0$ for all $x \in [-4, 4]$. Therefore, $y \in [0, 4]$ for all $x \in [-4, 4]$.

Hence, Range $f) = [0, 4]$.

(iii) We have, $f(x) = \frac{x}{1+x^2}$. We observe that $f(x)$ takes real values for all $x \in R$. Hence, domain $(f) = R$. Let $y = f(x)$. Then,

$$y = f(x) \Rightarrow y = \frac{x}{1+x^2} \Rightarrow x^2 y - x + y = 0 \Rightarrow x = \frac{1 \pm \sqrt{1 - 4y^2}}{2y}$$

Clearly, x will assume real values, if

$$1 - 4y^2 \geq 0 \text{ and } y \neq 0 \Rightarrow 4y^2 - 1 \leq 0 \text{ and } y \neq 0 \Rightarrow y^2 - \frac{1}{4} \leq 0 \text{ and } y \neq 0$$

$$\Rightarrow \left(y - \frac{1}{2}\right)\left(y + \frac{1}{2}\right) \leq 0 \text{ and } y \neq 0 \Rightarrow -\frac{1}{2} \leq y \leq \frac{1}{2} \text{ and } y \neq 0 \Rightarrow y \in \left[-\frac{1}{2}, \frac{1}{2}\right] - \{0\}$$

Also, $y = 0$ for $x = 0$. Hence, Range $(f) = \left[-\frac{1}{2}, \frac{1}{2}\right]$.

(iv) We have, $f(x) = \frac{3}{2-x^2}$. For $f(x)$ to be real, we must have $2 - x^2 \neq 0 \Rightarrow x \neq \pm\sqrt{2}$.

$\therefore$ Domain $(f) = R - \{-\sqrt{2}, \sqrt{2}\}$. Let $f(x) = y$. Then,

$$y = f(x) \Rightarrow y = \frac{3}{2-x^2} \Rightarrow 2y - x^2 y = 3 \Rightarrow x^2 y = 2y - 3 \Rightarrow x = \pm\sqrt{\frac{2y-3}{y}}$$

We observe that x will take real values other than $-\sqrt{2}$ and $\sqrt{2}$, if

$$\frac{2y-3}{y} > 0 \Rightarrow y \in (-\infty, 0) \cup [3/2, \infty) \qquad \text{[See Fig. 3.5]}$$

Fig. 3.5 Signs of $\frac{2y-3}{y}$

Hence, $\text{range}(f) = (-\infty, 0) \cup [3/2, \infty)$.

EXAMPLE 3 *Find the domain and range of the function* $f(x) = \frac{x^2-9}{x-3}$.

SOLUTION We have, $f(x) = \frac{x^2-9}{x-3}$.

Domain of f: Clearly, $f(x)$ is not defined for $x - 3 = 0$ i.e. $x = 3$. Therefore, Domain $(f) = R - \{3\}$.

Range of f: Let $f(x) = y$. Then, $f(x) = y \Rightarrow \frac{x^2-9}{x-3} = y \Rightarrow x + 3 = y \qquad [\because x \neq 3]$

It follows from the above relation that y takes all real values except 6 when x takes values in the set $R - \{3\}$. Therefore, $\text{Range}(f) = R - \{6\}$.

EXAMPLE 4 *Find the domain and range of the real valued function $f(x)$ given by* $f(x) = \frac{4-x}{x-4}$.

SOLUTION We have, $f(x) = \frac{4-x}{x-4}$.

Domain of f: We observe that $f(x)$ is defined for all x except at $x = 4$. At $x = 4$, $f(x)$ takes the indeterminate form $\frac{0}{0}$. Therefore, Domain $(f) = R - \{4\}$.

Range of f: For any $x \in$ Domain (f) i.e. for any $x \neq 4$, we have

$$f(x) = \frac{4-x}{x-4} = \frac{-(x-4)}{x-4} = -1. \text{ Therefore, Range}(f) = \{-1\}.$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 5 *Let* $f = \left\{\left(x, \frac{x^2}{1+x^2}\right) : x \in R\right\}$ *be a function from R into R. Determine the range of f.*

[NCERT]

SOLUTION We have, $f(x) = \frac{x^2}{x^2+1}$

Domain of f: Clearly, $f(x)$ is defined for all $x \in R$ as $x^2 + 1 \neq 0$ for any $x \in R$. So, Domain$(f) = R$.

Range of f: Let $f(x) = y$. Then,

$$f(x) = y$$

$$\Rightarrow \frac{x^2}{x^2+1} = y \Rightarrow x^2 = x^2 y + y \Rightarrow x^2(1-y) = y \Rightarrow x^2 = \frac{y}{1-y} \Rightarrow x = \pm\sqrt{\frac{y}{1-y}}$$

Clearly, x will take real values, if

$$\frac{y}{1-y} \geq 0$$

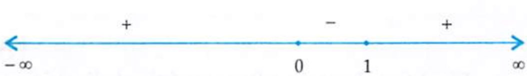

Fig. 3.6 Signs of $\frac{y}{y-1}$

$\Rightarrow \quad \dfrac{y-0}{y-1} \le 0 \Rightarrow 0 \le y < 1 \Rightarrow y \in [0, 1)$ [See Fig. 3.6]

Hence, range $(f) = [0, 1)$.

EXAMPLE 6 *Find the domain and range of the function* $f = \left\{ \left(x : \dfrac{1}{1-x^2} \right) : x \in R, x \ne \pm 1 \right\}$.

SOLUTION We have, $f(x) = \dfrac{1}{1-x^2}$

Domain of f: Clearly, $f(x)$ is defined for all $x \in R$ except for which $x^2 - 1 \ne 0$ i.e. $x = \pm 1$. Therefore, Domain of $f = R - \{-1, 1\}$.

Range of f: Let $f(x) = y$. Then,

$$f(x) = y \Rightarrow \frac{1}{1-x^2} = y \Rightarrow 1 - x^2 = \frac{1}{y} \Rightarrow x^2 = 1 - \frac{1}{y} = \frac{y-1}{y} \Rightarrow x = \pm\sqrt{\frac{y-1}{y-0}}$$

Clearly, x will take real values, if

$$\frac{y-1}{y-0} \ge 0$$

Fig. 3.7 Signs of $\frac{y-1}{y}$

$\Rightarrow \quad y < 0$ or $y \ge 1 \Rightarrow y \in (-\infty, 0) \cup [1, \infty)$ [See Fig. 3.7]

Hence, range $(f) = (-\infty, 0) \cup [1, \infty)$.

EXAMPLE 7 *Find the domain and range of the function* $f(x) = \dfrac{1}{2 - \sin 3x}$.

SOLUTION We have, $f(x) = \dfrac{1}{2 - \sin 3x}$

Domain of f: We know that

$-1 \le \sin 3x \le 1$ for all $x \in R$

$\Rightarrow \quad -1 \le -\sin 3x \le 1$ for all $x \in R$

$\Rightarrow \quad 1 \le 2 - \sin 3x \le 3$ for all $x \in R$ [Adding 2 throughout]

$\Rightarrow \quad 2 - \sin 3x \ne 0$ for any $x \in R \Rightarrow f(x) = \dfrac{1}{2 - \sin 3x}$ is defined for all $x \in R$

Hence, domain $(f) = R$.

Range of f: As discussed above

$1 \le 2 - \sin 3x \le 3$ for all $x \in R$

$\Rightarrow \quad \dfrac{1}{3} \le \dfrac{1}{2 - \sin 3x} \le 1$ for all $x \in R \Rightarrow \dfrac{1}{3} \le f(x) \le 1$ for all $x \in R \Rightarrow f(x) \in \left[\dfrac{1}{3}, 1\right]$

Hence, range $(f) = \left[\dfrac{1}{3}, 1\right]$

EXERCISE 3.3

BASIC

1. Find the domain of each of the following real valued functions of real variable:

(i) $f(x) = \dfrac{1}{x}$ (ii) $f(x) = \dfrac{1}{x-7}$ (iii) $f(x) = \dfrac{3x-2}{x+1}$ (iv) $f(x) = \dfrac{2x+1}{x^2-9}$

(v) $f(x) = \dfrac{x^2 + 2x + 1}{x^2 - 8x + 12}$ [NCERT]

2. Find the domain of each of the following real valued functions of real variable:

(i) $f(x) = \sqrt{x-2}$ (ii) $f(x) = \dfrac{1}{\sqrt{x^2 - 1}}$ (iii) $f(x) = \sqrt{9 - x^2}$ (iv) $f(x) = \sqrt{\dfrac{x-2}{3-x}}$

BASED ON LOTS

3. Find the domain and range of each of the following real valued functions:

(i) $f(x) = \dfrac{ax + b}{bx - a}$ (ii) $f(x) = \dfrac{ax - b}{cx - d}$

(iii) $f(x) = \sqrt{x-1}$ [NCERT] (iv) $f(x) = \sqrt{x-3}$

(v) $f(x) = \dfrac{x-2}{2-x}$ (vi) $f(x) = |x - 1|$ [NCERT]

(vii) $f(x) = -|x|$ [NCERT] (viii) $f(x) = \sqrt{9 - x^2}$ [NCERT]

(ix) $f(x) = \dfrac{1}{\sqrt{16 - x^2}}$ (x) $f(x) = \sqrt{x^2 - 16}$

ANSWERS

1. Domain

(i) $R - \{0\}$

(ii) $R - \{7\}$

(iii) $R - \{-1\}$

(iv) $R - \{-3, 3\}$

(v) $R - \{2, 6\}$

2.

	Domain	Range
(i)	$[2, \infty)$	$[0, \infty)$
(ii)	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, -1] \cup (0, \infty)$
(iii)	$[-3, 3]$	$[0, 3]$
(iv)	$[2, 3)$	$[0, \infty)$

3.

	Domain	Range		Domain	Range
(i)	$R - \left\{\dfrac{a}{b}\right\}$	$R - \left\{\dfrac{a}{b}\right\}$	(ii)	$R - \left\{\dfrac{d}{c}\right\}$	$R - \left\{\dfrac{a}{c}\right\}$
(iii)	$[1, \infty)$	$[0, \infty)$	(iv)	$[3, \infty)$	$[0, \infty)$
(v)	$R - \{2\}$	$\{-1\}$	(vi)	R	$[0, \infty)$
(vii)	R	$(-\infty, 0]$	(viii)	$[-3, 3]$	$[0, 3]$
(ix)	$(-4, 4)$	$\left[\dfrac{1}{4}, \infty\right)$	(x)	$(-\infty, -4] \cup [4, \infty)$	$[0, \infty)$

HINTS TO SELECTED PROBLEMS

1. (v) $f(x) = \dfrac{x^2 + 2x + 1}{x^2 - 8x + 12} = \dfrac{(x+1)^2}{(x-6)(x-2)}$ is defined for all x satisfying

$(x-6)(x-2) \neq 0$ i..e. $x \neq 2, 6$. Therefore, Domain $(f) = R - \{2, 6\}$

3. (iii) $f(x) = \sqrt{x-1}$ is defined for all x satisfying $x - 1 \geq 0$ i.e. $x \geq 1$. So, domain $(f) = [1, \infty)$.

Let $y = \sqrt{x-1}$. Clearly, $y \geq 0$ for all $x \in [1, \infty)$. So, range $(f) = [0, \infty)$.

(vi) $f(x) = |x - 1|$. Clearly, $f(x)$ is defined for all $x \in R$. So, domain $(f) = R$.

Also, $f(x) = |x-1| \geq 0$ for all $x \in R$. So, range $(f) = [0, \infty)$.

(vii) $f(x) = -|x|$. We observe that $f(x)$ is defined for all $x \in R$. So, domain $(f) = R$.

Also, $|x| \geq 0$ for all $x \in R \Rightarrow -|x| \leq 0$ for all $x \in R \Rightarrow f(x) \leq 0$ for all $x \in R$.

So, range $(f) = (-\infty, 0]$.

(viii) We have, $f(x) = \sqrt{9-x^2}$. Clearly, $f(x)$ takes real values if

$$9 - x^2 \geq 0 \Rightarrow x^2 - 9 \leq 0 \Rightarrow (x-3)(x+3) \leq 0 \Rightarrow x \in [-3, 3]$$

$\therefore$ Domain $(f) = [-3, 3]$

Also, $f(x) = \sqrt{9-x^2} \geq 0$ for all $x \in [-3, 3]$.

Let $y = \sqrt{9-x^2}$. Then, $y^2 = 9 - x^2 \Rightarrow x^2 + y^2 = 9 \Rightarrow x = \sqrt{9-y^2}$

Clearly, $x \in R$, if $y \in [-3, 3]$. But, $y \geq 0$. Therefore, $y \in [0, 3]$. Hence, range $(f) = [0, 3]$

3.7 SOME STANDARD REAL FUNCTIONS AND THEIR GRAPHS

In this section, we shall discuss some standard real functions which frequently occur in the study of calculus.

CONSTANT FUNCTION *If k is a fixed real number, then a function $f(x)$ given by $f(x) = k$ for all $x \in R$ is called a constant function.*

Sometimes we also call it the constant function k.

We observe that the domain of the constant function $f(x) = k$ is the set R of all real numbers and range of f is the singleton set $\{k\}$.

The graph of a constant function $f(x) = k$ is a straight line parallel to x-axis (See Fig. 3.8) which is above or below x-axis according as k is positive or negative. If $k = 0$, then the straight line is coincident to x-axis.

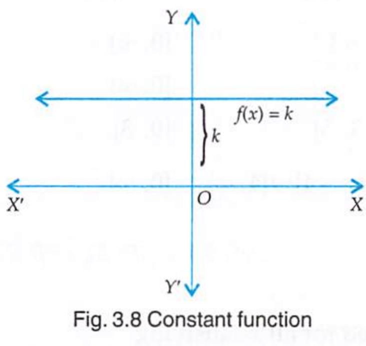

Fig. 3.8 Constant function

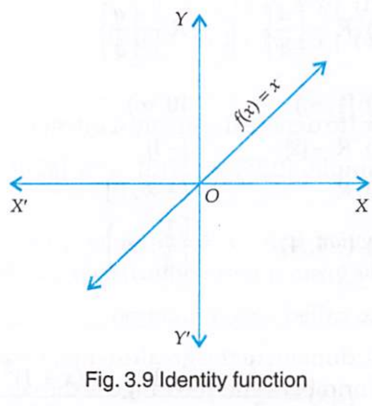

Fig. 3.9 Identity function

IDENTITY FUNCTION *The function that associates each real number to itself is called the identity function and is usually denoted by I.*

Thus, the function $I : R \to R$ defined by $I(x) = x$ for all $x \in R$ is called the identity function.

Clearly, the domain and range of the identity function are both equal to R.

The graph of the identity function is a straight line passing through the origin and inclined at an angle of 45° with X-axis.

MODULUS FUNCTION *The function $f(x)$ defined by* $f(x) = |x| = \begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0 \end{cases}$ *is called the modulus function.*

It is also called the absolute value function.

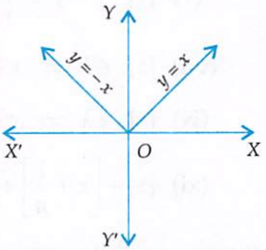

Fig. 3.10 Modulus function

We observe that the domain of the modulus function is the set R of all real numbers and the range is the set of all non-negative real numbers i.e. $R^+ = \{x \in R : x \geq 0\}$.

The graph of the modulus function is as shown in Fig. 3.10. for $x \geq 0$, the graph coincides with the graph of the identity function i.e. the line $y = x$ and for $x < 0$, it is coincident to the line $y = -x$.

PROPERTIES OF MODULUS FUNCTION The modulus function has the following properties:

(a) For any real number x, $\sqrt{x^2} = |x|$.

For example, $\sqrt{\cos^2 x} = |\cos x| = \begin{cases} \cos x, & 0 \leq x \leq \frac{\pi}{2} \\ -\cos x, & \frac{\pi}{2} < x \leq \pi \end{cases}$

(b) If a, b are positive real numbers, then

(i) $x^2 \leq a^2 \Leftrightarrow |x| \leq a \Leftrightarrow -a \leq x \leq a$ (ii) $x^2 \geq a^2 \Leftrightarrow |x| \geq a \Leftrightarrow x \leq -a$ or, $x \geq a$

(iii) $x^2 < a^2 \Leftrightarrow |x| < a \Leftrightarrow -a < x < a$ (iv) $x^2 > a^2 \Leftrightarrow |x| > a \Leftrightarrow x < -a$ or, $x > a$

(v) $a^2 \leq x^2 \leq b^2 \Leftrightarrow a \leq |x| \leq b \Leftrightarrow x \in [-b, -a] \cup [a, b]$

(vi) $a^2 < x^2 < b^2 \Leftrightarrow a < |x| < b \Leftrightarrow x \in (-b, -a) \cup (a, b)$

(c) For real numbers x and y, we have

(i) $|x + y| = |x| + |y| \Leftrightarrow (x \geq 0$ and $y \geq 0)$ or, $(x < 0$ and $y < 0)$

(ii) $|x - y| = |x| - |y| \Leftrightarrow (x \geq 0, y \geq 0$ and $|x| \geq |y|)$ or, $(x \leq 0, y \leq 0$ and $|x| \geq |y|)$

(iii) $|x \pm y| \leq |x| + |y|$

(iv) $|x \pm y| \geq \big||x| - |y|\big|$

GREATEST INTEGER FUNCTION (FLOOR FUNCTION) For any real number x, we use the symbol $[x]$ or, $\lfloor x \rfloor$ to denote the greatest integer less than or equal to x.

For example, $[2.75] = 2, [3] = 3, [0.74] = 0, [-7.45] = -8$ etc.

The function $f : R \to R$ defined by $f(x) = [x]$ for all $x \in R$ is called the greatest integer function or the floor function.

It is also called a step function.

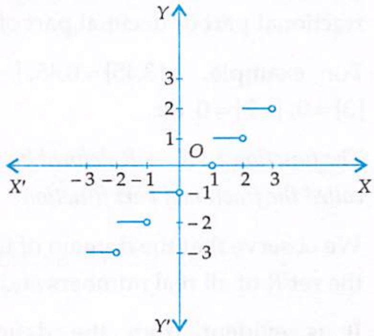

Fig. 3.11 Greatest integer function

Clearly, domain of the greatest integer function is the set R of all real numbers and the range is the set Z of all integers as it attains only integer values.

The graph of the greatest integer function is shown in Fig 3.11.

PROPERTIES OF GREATEST INTEGER FUNCTION *If n is an integer and x is a real number between n and $n+1$, then*

(i) $[-n] = -[n]$ (ii) $[x + k] = [x] + k$ for any integer k.

(iii) $[-x] = -[x]-1$ (iv) $[x]+[-x] = \begin{cases} -1, & \text{if } x \notin Z \\ 0, & \text{if } x \in Z \end{cases}$

(v) $[x]-[-x] = \begin{cases} 2[x]+1, & \text{if } x \notin Z \\ 2[x], & \text{if } x \in Z \end{cases}$ (vi) $[x] \geq k \Rightarrow x \geq k$, where $k \in Z$

(vii) $[x] \leq k \Rightarrow x < k+1$, where $k \in Z$ (viii) $[x] > k \Rightarrow x > k+1$, were $k \in Z$

(ix) $[x] < k \Rightarrow x < k$, where $k \in Z$ (x) $[x+y] = [x]+[y+x-[x]]$ for all $x, y \in R$

(xi) $[x]+\left[x+\dfrac{1}{n}\right]+\left[x+\dfrac{2}{n}\right]+\ldots+\left[x+\dfrac{n-1}{n}\right] = [nx], n \in N.$

SMALLEST INTEGER FUNCTION (CEILING FUNCTION) For any real number x, we use the symbol $\lceil x \rceil$ to denote the smallest integer greater than or equal to x.

For example, $\lceil 4.7 \rceil = 5, \lceil -7.2 \rceil = -7, \lceil 5 \rceil = 5, \lceil 0.75 \rceil = 1$ etc.

The function $f: R \to R$ defined by $f(x) = \lceil x \rceil$ for all $x \in R$ is called the smallest integer function or the ceiling function.

It is also a step function.

We observe that the domain of the smallest integer function is the set R of all real numbers and its range is the set Z of all integers. The graph of the smallest integer function is as shown in Fig. 3.12.

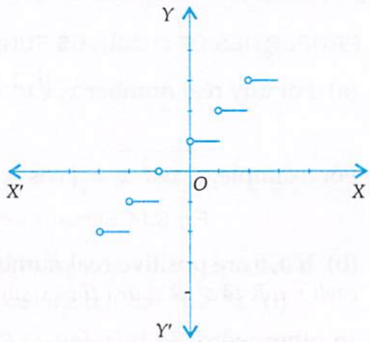

Fig. 3.12 Smallest integer function

PROPERTIES OF SMALLEST INTEGER FUNCTION Following are some properties of smallest integer function:

(i) $\lceil -n \rceil = -\lceil n \rceil$, where $n \in Z$ (ii) $\lceil -x \rceil = -\lceil x \rceil + 1$, where $x \in R - Z$

(iii) $\lceil x+n \rceil = \lceil x \rceil + n$, where $x \in R - Z$ and $n \in Z$

(iv) $\lceil x \rceil + \lceil -x \rceil = \begin{cases} 1, & \text{if } x \notin Z \\ 0, & \text{if } x \in Z \end{cases}$ (v) $\lceil x \rceil - \lceil -x \rceil = \begin{cases} 2\lceil x \rceil - 1, & \text{if } x \notin Z \\ 2\lceil x \rceil, & \text{if } x \in Z \end{cases}$

FRACTIONAL PART FUNCTION For any real number x, we use the symbol $\{x\}$ to denote the fractional part or decimal part of x.

For example, $\{3.45\} = 0.45, \{-2.75\} = 0.25, \{-0.55\} = 0.45$, $\{3\} = 0, \{-7\} = 0$ etc.

The function $f: R \to R$ defined by $f(x) = \{x\}$ for all $x \in R$ is called the fractional part function.

We observe that the domain of the fractional part function is the set R of all real numbers and the range is the set $[0, 1)$.

It is evident from the definition that $f(x) = \{x\} = x - [x]$ for all $x \in R$

The graph of the fractional part function is as shown in Fig. 3.13.

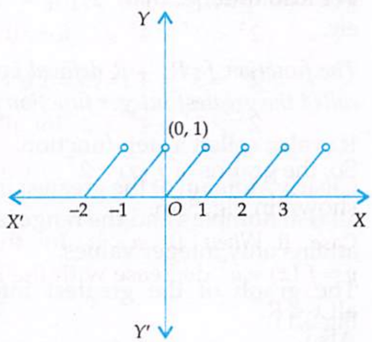

Fig. 3.13 Fractional part function

SIGNUM FUNCTION *The function f defined by* $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ or, $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$

is called the signum function.

The domain of the signum function is the set R of all real numbers and the range is the set $\{-1, 0, 1\}$. The graph of the signum function is as shown in Fig. 3.14.

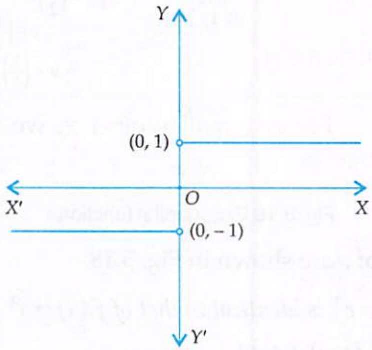

Fig. 3.14 Signum function

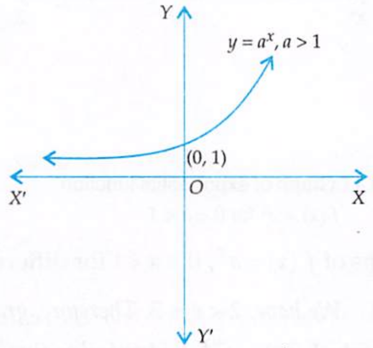

Fig. 3.15 Exponential function $f(x) = a^x$ for $a > 1$

EXPONENTIAL FUNCTION *If a is a positive real number other than unity, then a function that associates each* $x \in R$ *to* a^x *is called the exponential function.*

In other words, a function $f : R \to R$ defined by $f(x) = a^x$, where $a > 0$ and $a \neq 1$ is called the exponential function.

We observe that the domain of an exponential function is R the set of all real numbers and the range is the set $(0, \infty)$ as it attains only positive values.

As $a > 0$ and $a \neq 1$. So, we have the following cases:

Case I *When* $a > 1$: We observe that the values of $y = f(x) = a^x$ increase as the values of x increase.

Also, $f(x) = a^x \begin{cases} < 1 & \text{for } x < 0 \\ = 1 & \text{for } x = 0 \\ > 1 & \text{for } x > 0. \end{cases}$

Thus, the graph of $f(x) = a^x$ for $a > 1$ as shown in Fig. 3.15.

We also observe that:

$$2^x < 3^x < 4^x < \dots \text{ for all } x > 1$$

$$2^x = 3^x = 4^x = \dots = 1 \text{ for } x = 0$$

$$2^x > 3^x > 4^x > \dots \text{ for all } x < 1$$

So, the graphs of $f(x) = 2^x$, $f(x) = 3^x$, $f(x) = 4^x$ etc. are as shown in Fig. 3.16.

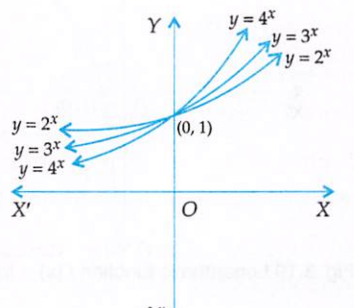

Fig. 3.16 Graphs of exponential functions

Case II *When* $0 < a < 1$: In this case, the values of $y = f(x) = a^x$ decrease with the increase in x and $y > 0$ for all $x \in R$.

Also,

$$y = f(x) = a^x \begin{cases} > 1 & \text{for } x < 0 \\ = 1 & \text{for } x = 0 \\ < 1 & \text{for } x > 0 \end{cases}$$

Thus, the graph of $f(x) = a^x$ for $0 < a < 1$ is as shown in Fig. 3.17.

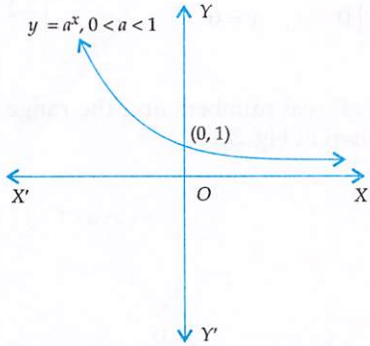

Fig. 3.17 Graph of exponential function $f(x) = a^x$ for $0 < a < 1$

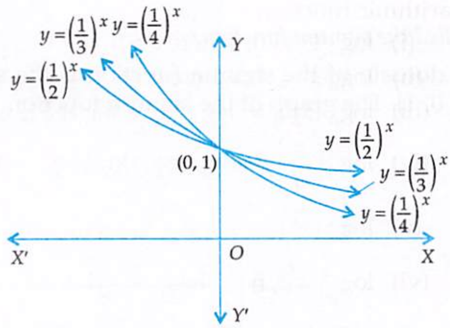

Fig. 3.18 Exponential functions

The graphs of $f(x) = a^x$, $0 < a < 1$ for different values of a are shown in Fig. 3.18.

REMARK 1 *We have,* $2 < e < 3$. *Therefore, graph of* $f(x) = e^x$ *is identical to that of* $f(x) = a^x$ *for* $a > 1$ *and the graph of* $f(x) = e^{-x}$ *is identical to that of* $f(x) = a^x$ *for* $0 < a < 1$.

LOGARITHMIC FUNCTION *If* $a > 0$ *and* $a \neq 1$, *then the function defined by* $f(x) = \log_a x$, $x > 0$ *is called the logarithmic function.*

We have learnt that the logarithmic function and the exponential function are inverse functions i.e. $\log_a x = y \Leftrightarrow x = a^y$.

We observe that the domain of the logarithmic function is the set of all non-negative real numbers i.e. $(0, \infty)$ and the range is the set R of all real numbers.

As $a > 0$ and $a \neq 1$. So, we have the following cases.

Case I *When* $a > 1$: In this case, we have $y = \log_a x \begin{cases} < 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ > 0 & \text{for } x > 1 \end{cases}$

Also, the values of y increase with the increase in x. So, the graph of $y = \log_a x$ is as shown in Fig.3.19.

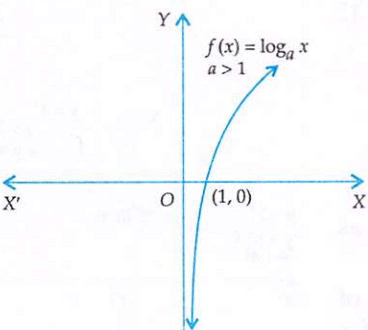

Fig. 3.19 Logarithmic function $f(x) = \log_a x$ for $a > 1$

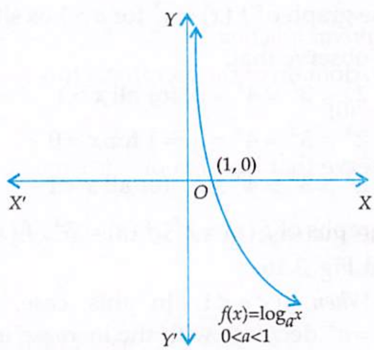

Fig. 3.20 Logarithmic function $f(x) = \log_a x$ for $0<a<1$

Case II *When* $0 < a < 1$: In this case, we have $y = \log_a x \begin{cases} > 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ < 0 & \text{for } x > 1 \end{cases}$

Also, the values of y decrease with the increase in x. So, the graph of $y = \log_a x$ is as shown in Fig. 3.20.

PROPERTIES OF LOGARITHMIC FUNCTION Following are some useful properties of logarithmic function:

(i) $\log_a 1 = 0$, where $a > 0, a \neq 1$

(ii) $\log_a a = 1$, where $a > 0, a \neq 1$

(iii) $\log_a (xy) = \log_a |x| + \log_a |y|$, where $a > 0, a \neq 1$ and $xy > 0$

(iv) $\log_a \left(\frac{x}{y}\right) = \log_a |x| - \log_a |y|$, where $a > 0, a \neq 1$ and $\frac{x}{y} > 0$

(v) $\log_a (x^n) = n \log_a |x|$, where $a > 0, a \neq 1$ and $x^n > 0$

(vi) $\log_{a^n} x^m = \frac{m}{n} \log_a x$, where $a > 0, a \neq 1$ and $x > 0$

(vii) $x^{\log_a y} = y^{\log_a x}$, where $x > 0, y > 0, a > 0, a \neq 1$

(viii) If $a > 1$, then the values of $f(x) = \log_a x$ increase with the increase in x.

i.e $x < y \Leftrightarrow \log_a x < \log_a y$. Also, $\log_a x \begin{cases} < 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ > 0 & \text{for } x > 1. \end{cases}$

(ix) If $0 < a < 1$, then the values of $f(x) = \log_a x$ decrease with the increase in x.

i.e $x < y \Leftrightarrow \log_a x > \log_a y$. Also, $\log_a x \begin{cases} > 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ < 0 & \text{for } x > 1 \end{cases}$

(x) $\log_a x = \frac{1}{\log_x a}$ for $a > 0, a \neq 1$ and $x > 0, x \neq 1$.

REMARK 2 *Functions $f(x) = \log_a x$ and $g(x) = a^x$ are inverse of each other. So, their graphs are mirror images of each other in the line mirror $y = x$.*

RECIPROCAL FUNCTION The function that associates a real number x to its reciprocal $1/x$ is called the reciprocal function. Since $1/x$ is not defined for $x = 0$. So, we define the reciprocal function as follows:

DEFINITION *The function $f : R - \{0\} \to R$ defined by $f(x) = \frac{1}{x}$ is called the reciprocal function.*

Clearly, domain of the reciprocal function is $R - \{0\}$ and its range is also $R - \{0\}$.

We observe that the sign of $\frac{1}{x}$ is same as that of x and $\frac{1}{x}$ decreases with the increase in x. So, the graph of $f(x) = \frac{1}{x}$ is as shown in Fig. 3.21.

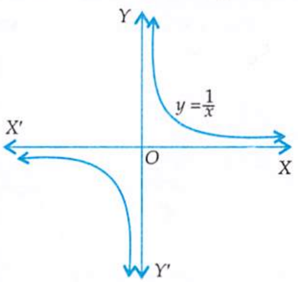

Fig. 3.21 Reciprocal function

SQUARE ROOT FUNCTION The function that associates a real number x *to* $+\sqrt{x}$ is called the square root function. Since $\sqrt{x}$ is real for $x > 0$. So, we defined the square root function as follows:

DEFINITION *The function $f : R^+ \to R$ defined by $f(x) = +\sqrt{x}$ is called the square root function.*

Clearly, domain of the square root function is R^+ *i.e.* $[0, \infty)$ and its range is also $[0, \infty)$.

We observe that the values of $f(x) = +\sqrt{x}$ increase with the increase in x. So, the graph of $f(x) = +\sqrt{x}$ is as shown in Fig. 3.22.

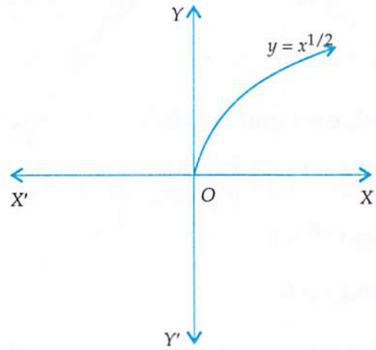

Fig. 3.22 Square root function

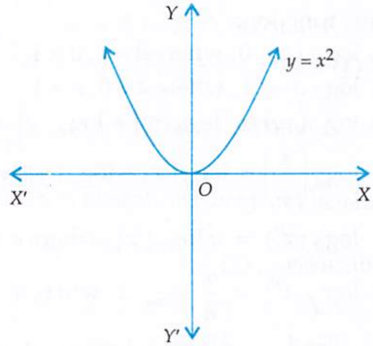

Fig. 3.23 Square function

SQUARE FUNCTION The function that associates a real number x to its square i.e. x^2 is called the square function. Since x^2 is defined for all $x \in R$. So, we define the square function as follows:

DEFINITION *The function $f : R \to R$ defined by $f(x) = x^2$ is called the square function.*

Clearly, domain of the square function is R and its range is the set of all non-negative real numbers i.e. $[0, \infty)$. The graph of $f(x) = x^2$ is parabola as shown in Fig. 3.23.

CUBE FUNCTION The function that associate a real number x to its cube is called the cube function. We observe that x^3 is meaningful for all $x \in R$. So, we define the cube function as follows:

DEFINITION *The function $f : R \to R$ defined by $f(x) = x^3$ is called the cube function.*

We observe that the sign of x^3 is same as that of x and the values of x^3 increase with the increase in x. So, the graph of $f(x) = x^3$ is as shown in Fig. 3.24. Clearly, the graph is symmetrical in opposite quadrants.

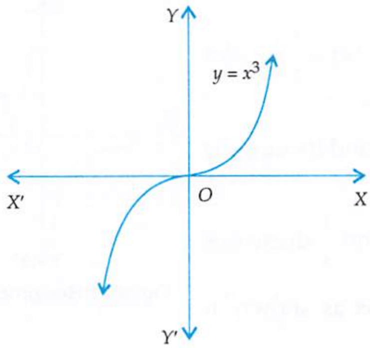

Fig. 3.24 Cube function

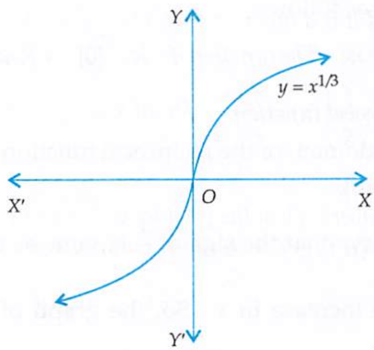

Fig. 3.25 Cube root function

CUBE ROOT FUNCTION The function that associates a real number x to its cube root $x^{1/3}$ is called the cube root function. Clearly, $x^{1/3}$ is defined for all $x \in R$. So, we define the cube root function as follows:

DEFINITION *The function $f : R \to R$ defined by $f(x) = x^{1/3}$ is called the cube root function.*

Clearly, domain and range of the cube root function are both equal to R.

Also, the sign of $x^{1/3}$ is same as that of x and $x^{1/3}$ increase with the increase in x. So, the graph of $f(x) = x^{1/3}$ is as shown in Fig. 3.25.

REMARK 3 *A function $f : R \to R$ is said to be a polynomial function if $f(x)$ is a polynomial in x. For example, $f(x) = x^2 - x + 4$, $g(x) = x^3 + 3x^2 + \sqrt{2}\,x - 1$ etc are polynomial functions.*

REMARK 4 *A function of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$, is called a rational function. The domain of a rational function $f(x) = \frac{p(x)}{q(x)}$ is the set of all real numbers, except points where $q(x) = 0$.*

3.8 OPERATIONS ON REAL FUNCTIONS

In this section, we shall introduce various operations, namely addition, subtraction, multiplication, division etc. on real functions.

ADDITION *Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions. Then, their sum $f + g$ is defined as that function from $D_1 \cap D_2$ to R which associates each $x \in D_1 \cap D_2$ to the number $f(x) + g(x)$.*

In other words, if $f : D_1 \to R$ and $g : D_2 \to R$ are two real functions, then their sum $f + g$ is a function from $D_1 \cap D_2$ to R such that

$$(f + g)(x) = f(x) + g(x) \quad \text{for all } x \in D_1 \cap D_2.$$

PRODUCT *Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions. Then, their product (or pointwise multiplication) $f\,g$ is a function from $D_1 \cap D_2$ to R and is defined as*

$$(f\,g)(x) = f(x)\,g(x) \quad \textit{for all } x \in D_1 \cap D_2$$

DIFFERENCE (SUBTRACTION) *Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions. Then the difference of g from f is denoted by $f - g$ and is defined as*

$$(f - g)(x) = f(x) - g(x) \quad \textit{for all } x \in D_1 \cap D_2$$

QUOTIENT *Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions. Then the quotient of f by g is denoted by $\frac{f}{g}$ and it is a function from $D_1 \cap D_2 - \{x : g(x) = 0\}$ to R defined by*

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \textit{for all } x \in D_1 \cap D_2 - \{x : g(x) = 0\}$$

MULTIPLICATION OF A FUNCTION BY A SCALAR *Let $f : D \to R$ ba a real function and α be a scalar (real number). Then the product $\alpha\,f$ is a function from D to R and is defined as*

$$(\alpha f)(x) = \alpha f(x) \quad \textit{for all } x \in D.$$

RECIPROCAL OF A FUNCTION *If $f : D \to R$ is a real function, then its reciprocal function $\frac{1}{f}$ is a function from $D - \{x : f(x) = 0\}$ to R and is defined as $\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)}$.*

REMARK 1 *The sum, difference product and quotient are defined for real functions only on their common domain. These operations do not make any sense for general functions even if their domains are same, because the sum, difference, product and quotient may or may not be meaningful for the elements in their common domain.*

REMARK 2 *For any real function $f : D \cap R$ and $n \in N$, we define*

$$\underbrace{(f f f \dots f)}_{n-\text{times}}(x) = \underbrace{f(x)\, f(x) \dots f(x)}_{n-\text{times}} = \{f(x)\}^n \quad \text{for all } x \in D$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the sum and difference of the identity function and the modulus function.* **[NCERT]**

SOLUTION We know that $f : R \to R$ defined by $f(x) = x$ is the identity function and $g : R \to R$ defined by $g(x) = |x|$ is the modulus function. Clearly, f and g have the same domain. Therefore, $f + g : R \to R$ and $f - g : R \to R$ such that

$$(f+g)(x) = f(x) + g(x) = x + |x| = \begin{cases} x + x, & \text{if } x \geq 0 \\ x - x, & \text{if } x < 0 \end{cases} = \begin{cases} 2x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

and, $$(f-g)(x) = f(x) - g(x) = x - |x| = \begin{cases} x - x, & \text{if } x \geq 0 \\ x - (-x), & \text{if } x < 0 \end{cases} = \begin{cases} 0, & \text{if } x \geq 0 \\ 2x, & \text{if } x < 0 \end{cases}$$

Thus, $f + g : R \to R$ and $f - g : R \to R$ are defined as

$$(f+g)(x) = \begin{cases} 2x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} \quad \text{and, } (f-g)(x) = \begin{cases} 0, & \text{if } x \geq 0 \\ 2x, & \text{if } x < 0 \end{cases}$$

EXAMPLE 2 *What are the sum and difference of the identity function and the reciprocal function?* **[NCERT]**

SOLUTION Let f and g denote respectively the identity function and the reciprocal function. Then, $f : R \to R$ and $g : R - \{0\} \to R$ such that $f(x) = x$ for all $x \in R$ and, $g(x) = \frac{1}{x}$ for all $x \in R - \{0\}$. The domains of f and g are R and $R - \{0\}$ respectively. Also, we have $R \cap R - \{0\} = R - \{0\}$. Therefore, $f + g : R - \{0\} \to R$ and $f - g : R - \{0\} \to R$ are given by

$$(f+g)(x) = f(x) + g(x) = x + \frac{1}{x} \text{ and, } (f-g)(x) = f(x) - g(x) = x - \frac{1}{x}$$

EXAMPLE 3 *Let $f : [2, \infty) \to R$ and $g : [-2, \infty) \to R$ be two real functions defined by $f(x) = \sqrt{x-2}$ and $g(x) = \sqrt{x+2}$. Find $f + g$ and $f - g$.*

SOLUTION Let $D_1 = [2, \infty)$ and $D_2 = [-2, \infty)$. Then, $D_1 \cap D_2 = [2, \infty)$. Thus, $f + g : [2, \infty) \to R$ and $f - g : [2, \infty) \to R$ are given by

$$(f+g)(x) = f(x) + g(x) = \sqrt{x-2} + \sqrt{x+2} \quad \text{for all } x \in [2, \infty)$$

and, $$(f-g)(x) = f(x) - g(x) = \sqrt{x-2} - \sqrt{x+2} \quad \text{for all } x \in [2, \infty).$$

EXAMPLE 4 *Find the product of the identity function and the modulus function.*

SOLUTION Let f and g denote respectively the identity function and modulus function. Then, $f : R \to R$ such that $f(x) = x$ for all x and, $g : R \to R$ such that $g(x) = |x|$ for all x. Clearly, f and g have the same domain. Therefore, the product $f\,g$ is a function from R to itself and is given by

$$(f\,g)(x) = f(x)\,g(x) = x|x| = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$$

EXAMPLE 5 *Find the quotient of the identity function by the modulus function.*

SOLUTION Let f and g denote respectively the identity function and the modulus function. Then, $f : R \to R$ is defined as $f(x) = x$ and, $g : R \to R$ is defined as $g(x) = |x|$. Clearly, f and g have the same domain.

And, $\quad g(x)=0 \Rightarrow |x|=0 \Rightarrow x=0.$

Therefore, the quotient of f by g i.e. $\frac{f}{g}$ is a function from $R-\{0\} \to R$ and is defined as

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{x}{|x|}=\begin{cases}\frac{x}{x}=1\ , & x>0\\ \frac{x}{-x}=-1\ , & x<0\end{cases}$$

EXAMPLE 6 *Find the product of the identity function and the reciprocal function.*

SOLUTION Let f and g denote respectively the identity function and the reciprocal function. Then, $f: R \to R$ is defined as $f(x)=x$ for all $x \in R$ and, $g: R-\{0\} \to R$ is defined as $g(x)=\frac{1}{x}$ for all $x \in R-\{0\}$. We find that Domain $(f) \cap$ Domain $(g) = R \cap R-\{0\} = R-\{0\}$. Therefore, the product fg is a function from $R-\{0\}$ to R and is defined as

$$(fg)(x)=f(x)\,g(x)=x\times\frac{1}{x}=1 \text{ for all } x\in R-\{0\}$$

Thus, $fg: R-\{0\} \to R$ is given by $(fg)(x)=1$ for all $x \in R-\{0\}$.

EXAMPLE 7 *Find the quotient of the identity function by the reciprocal function.*

SOLUTION Let f and g denote respectively the identity function and the reciprocal function. Then, $f: R \to R$ is defined as $f(x)=x$ for all $x \in R$ and, $g: R-\{0\}-R$ is defined as $g(x)=\frac{1}{x}$ for all $x \in R-\{0\}$. We find that Domain $(f) \cap$ Domain $(g)=R \cap R-\{0\}=R-\{0\}$. And, $g(x) \neq 0$ for any $x \in R-\{0\}$.

$\therefore \quad \frac{f}{g}$ is a function from $R-\{0\} \to R$ and is given by $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{x}{1/x}=x^2$

Hence, $\frac{f}{g}: R-\{0\} \to R$ is given by $\left(\frac{f}{g}\right)(x)=x^2$ for all $x \in R-\{0\}$.

EXAMPLE 8 *Let c be a non-zero real number and $f: R \to R$ be a function defined by $f(x)=\frac{x}{c}$ for all $x \in R$. Find* (i) cf (ii) $c^2 f$ (iii) $\left(\frac{1}{c}\right)f$.

SOLUTION Clearly, cf, $c^2 f$ and $\left(\frac{1}{c}\right)f$ are functions from R to itself such that

(i) $\quad (cf)(x)=c\,f(x)=c\times\frac{x}{c}=x$ for all $x \in R$

(ii) $\quad (c^2 f)(x)=c^2 f(x)=c^2\times\frac{x}{c}=cx$ for all $x\in R$

(iii) $\quad \left(\left(\frac{1}{c}\right)f\right)(x)=\left(\frac{1}{c}\right)f(x)=\frac{1}{c}\times\frac{x}{c}=\frac{x}{c^2}$ for all $x \in R$.

EXAMPLE 9 *Let f and g be two real functions defined by $f(x)=\frac{1}{x+4}$ and $g(x)=(x+4)^3$.*

Find the following: (i) $f+g$ (ii) $f-g$ (iii) $f\,g$ (iv) $\frac{f}{g}$ (v) $2f$ (vi) $\frac{1}{f}$ (vii) $\frac{1}{g}$

SOLUTION We observe that $f(x)=\frac{1}{x+4}$ is defined for all $x \neq -4$. So, domain $(f)=R-\{-4\}$.

Clearly, $g(x)=(x+4)^3$ is defined for all $x \in R$. So, domain $(g)=R$. We find that

Domain $(f) \cap$ Domain $(g) = R - \{-4\}$. Therefore,

(i) $f+g : R-\{-4\} \to R$ is given by $(f+g)(x) = f(x) + g(x) = \frac{1}{x+4} + (x+4)^3 = \frac{(x+4)^4+1}{x+4}$

(ii) $f-g : R-\{-4\} \to R$ is defined as $(f-g)(x) = f(x) - g(x) = \frac{1}{x+4} - (x+4)^3 = \frac{1-(x+4)^4}{x+4}$

(iii) $fg : R-\{-4\} \to R$ is given by $(fg)(x) = f(x)\, g(x) = \frac{1}{x+4} \times (x+4)^3 = (x+4)^2$

(iv) $g(x) = 0 \Rightarrow (x+4)^3 = 0 \Rightarrow x = -4$.

$\therefore$ Domain $\left(\frac{f}{g}\right) =$ Domain $(f) \cap$ Domain $(g) - \{x : g(x) = 0\} = R - \{-4\}$. Therefore,

$\frac{f}{g} : R-\{-4\} \to R$ is given by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{1}{(x+4)^4}$

(v) $2f : R-\{-4\} \to R$ is given by $(2f)(x) = 2(f(x)) = \frac{2}{x+4}$ for all $x \in R - \{-4\}$.

(vi) We observe that $f(x) \neq 0$ for any $x \in R - \{-4\}$. Therefore, $\frac{1}{f} : R-\{-4\} \to R$ is given by

$$\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)} = \frac{1}{1/(x+4)} = (x+4)$$

(vii) We observe that $g(x) = (x+4)^3 = 0$ for $x = -4$. Therefore, $\frac{1}{g} : R-\{-4\} \to R$ is given by

$$\left(\frac{1}{g}\right)(x) = \frac{1}{g(x)} = \frac{1}{(x+4)^3}$$

EXAMPLE 10 *Let f and g be real functions defined by $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{4-x^2}$. Then, find each of the following functions:*

(i) $f+g$ (ii) $f-g$ (iii) fg (iv) $\frac{f}{g}$ (v) ff (vi) gg

SOLUTION We have, $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{4-x^2}$. Clearly, $f(x)$ is defined for all x satisfying

$$x+2 \geq 0 \Rightarrow x \geq -2 \Rightarrow x \in [-2, \infty). \text{ Therefore, Domain } (f) = [-2, \infty)$$

We observe that $g(x)$ is defined for all x satisfying

$4 - x^2 \geq 0 \Rightarrow x^2 - 4 \leq 0 \Rightarrow (x-2)(x+2) \leq 0 \Rightarrow x \in [-2, 2]$. Therefore, Domain $(g) = [-2, 2]$.

We find that: Domain $(f) \cap$ Domain $(g) = [-2, \infty) \cap [-2, 2] = [-2, 2]$. Therefore,

(i) $f+g : [-2, 2] \to R$ is given by $(f+g)(x) = f(x) + g(x) = \sqrt{x+2} + \sqrt{4-x^2}$

(ii) $f-g : [-2, 2] \to R$ is given by $(f-g)(x) = f(x) - g(x) = \sqrt{x+2} - \sqrt{4-x^2}$

(iii) $fg : [-2, 2] \to R$ is given by

$$(fg)(x) = f(x)\, g(x) = \sqrt{x+2} \times \sqrt{4-x^2} = \sqrt{(x+2)^2 (2-x)} = (x+2)\sqrt{2-x}$$

(iv) We have, $g(x) = \sqrt{4-x^2}$. Therefore, $g(x) = 0 \Rightarrow 4 - x^2 = 0 \Rightarrow x = \pm 2$. Therefore,

$$\text{Domain}\left(\frac{f}{g}\right) = [-2, 2] - \{-2, 2\} = (-2, 2)$$

Thus, $\frac{f}{g}:(-2, 2) \to R$ is given by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{(x+2)}}{\sqrt{4-x^2}} = \frac{1}{\sqrt{2-x}}$

(v) Since domain $(f) = [-2, \infty)$. Therefore,

$$(ff)(x) = f(x)\, f(x) = [f(x)]^2 = (\sqrt{x+2})^2 = x + 2 \text{ for all } x \in [-2, \infty)$$

(vi) Since domain $(g) = [-2, 2)$. Therefore,

$$(gg)(x) = g(x)\, g(x) = [g(x)]^2 = \left(\sqrt{4-x^2}\right)^2 = 4 - x^2 \text{ for all } x \in [-2, 2]$$

EXAMPLE 11 *Let f be the exponential function and g be the logarithmic function. Find:*

(i) $(f+g)(1)$ (ii) $(fg)(1)$ (iii) $(3f)(1)$ (iv) $(5g)(1)$

SOLUTION We have, $f: R \to R$ given by $f(x) = e^x$ and, $g: R^+ \to R$ given by $g(x) = \log_e x$.

(i) We find that: Domain $(f) \cap$ Domain $(g) = R \cap R^+ = R^+$. Therefore, $f + g: R^+ \to R$ is given by

$$(f+g)(x) = f(x) + g(x) = e^x + \log_e x \text{ for all } x \in R^+$$

$$\Rightarrow \quad (f+g)(1) = e^1 + \log_e 1 = e + 0 = e.$$

(ii) Domain $(f) \cap$ Domain $(g) = R \cap R^+ = R^+$. Therefore, $fg: R^+ \to R$ is given by

$$(fg)(x) = f(x)\, g(x) = e^x \,.\, \log_e x \Rightarrow (fg)(1) = e^1 \times \log_e 1 = e \times 0 = 0$$

(iii) Clearly, $(3f)(x) = 3(f(x)) = 3e^x$. Therefore, $(3f)(1) = 3e^1 = 3e$

(iv) Clearly, $(5g)(x) = 5(g(x)) = 5 \log_e x$. Therefore, $(5g)(1) = 5 \log_e 1 = 5 \times 0 = 0$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 12 *Find the domain of each of the following functions given by*

(i) $f(x) = \frac{1}{\sqrt{x - |x|}}$ [NCERT EXEMPLAR] (ii) $f(x) = \frac{1}{\sqrt{x + |x|}}$ [NCERT EXEMPLAR]

(iii) $f(x) = \frac{1}{\sqrt{x - [x]}}$ (iv) $f(x) = \frac{1}{\sqrt{x + [x]}}$

SOLUTION (i) We have, $f(x) = \frac{1}{\sqrt{x - |x|}}$. We know that $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

$$\therefore \quad x - |x| = \begin{cases} x - x = 0, & \text{if } x \geq 0 \\ x + x = 2x, & \text{if } x < 0 \end{cases}$$

$$\Rightarrow \quad x - |x| \leq 0 \quad \text{for all } x$$

$\Rightarrow \quad \frac{1}{\sqrt{x - |x|}}$ does not take real values for any $x \in R \Rightarrow f(x)$ is not defined for any $x \in R$.

Hence, Domain $(f) = \phi$.

(ii) We have, $f(x) = \frac{1}{\sqrt{x + |x|}}$. We know that $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

$$\therefore \quad x + |x| = \begin{cases} x + x, & \text{if } x \geq 0 \\ x - x, & \text{if } x < 0 \end{cases} \Rightarrow x + |x| = \begin{cases} 2x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} \quad ...(i)$$

Thus, $f(x) = \frac{1}{\sqrt{x + |x|}}$ assumes real values, if $x + |x| > 0 \Rightarrow x > 0 \Rightarrow x \in (0, \infty)$. [Using (i)]

Hence, Domain $(f) = (0, \infty)$.

(iii) We have, $f(x) = \frac{1}{\sqrt{x-[x]}}$. We know that $0 \le x-[x] < 1$ for all $x \in R$. Also, $x-[x]=0$ for $x \in Z$. Thus, $f(x) = \frac{1}{\sqrt{x-[x]}}$ is defined, if

$$x-[x] > 0 \Rightarrow x \in R - Z \qquad [\because x-[x]=0 \text{ for } x \in Z \text{ and } 0 < x-[x] < 1 \text{ for } x \in R-Z]$$

Hence, Domain $(f) = R - Z$.

(iv) We have, $f(x) = \frac{1}{\sqrt{x+[x]}}$. We know that

$$\left.\begin{aligned} x+[x] &> 0 \quad \text{for all } x>0 \\ x+[x] &= 0 \quad \text{for } x=0 \\ x+[x] &< 0 \quad \text{for all } x<0 \end{aligned}\right\} \qquad ...(i)$$

Clearly, $f(x) = \frac{1}{\sqrt{x+[x]}}$ is defined for all x satisfying $x+[x]>0$. Therefore, from (i), we find that, Domain $(f) = (0, \infty)$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 13 *Find the domain of definition of the function $f(x)$ given by*

$$f(x) = \log_4\left\{\log_5\left(\log_3(18x - x^2 - 77\right)\right)\right\}$$

SOLUTION We have, $(x) = \log_4\left\{\log_5\left(\log_3(18x-x^2-77)\right)\right\}$. Since $\log_a x$ is defined for all $x>0$. Therefore, $f(x)$ is defined if

$$\log_5\{\log_3(18x-x^2-77)\} > 0 \text{ and } 18x - x^2 - 77 > 0$$

$$\Rightarrow \log_3(18x-x^2-77) > 5^0 \text{ and } x^2 - 18x + 77 < 0$$

$$\Rightarrow \log_3(18x-x^2-77) > 1 \text{ and } (x-11)(x-7) < 0$$

$$\Rightarrow 18x - x^2 - 77 > 3^1 \text{ and } 7 < x < 11$$

$$\Rightarrow 18x - x^2 - 80 > 0 \text{ and } 7 < x < 11$$

$$\Rightarrow x^2 - 18x + 80 < 0 \text{ and } 7 < x < 11$$

$$\Rightarrow (x-10)(x-8) < 0 \text{ and } 7 < x < 11 \Rightarrow 8 < x < 10 \text{ and } 7 < x < 11 \Rightarrow 8 < x < 10 \Rightarrow x \in (8, 10).$$

Hence, the domain of $f(x)$ is $(8, 10)$.

EXAMPLE 14 *Find the domain of definition of the function $f(x)$ given by* $f(x) = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2}$.

SOLUTION We have, $f(x) = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2}$. Let $g(x) = \frac{1}{\log_{10}(1-x)}$ and $h(x) = \sqrt{x+2}$.

Then, $f(x) = g(x) + h(x)$. Therefore, Domain (f) = Domain $(g) \cap$ Domain (h).

Now, $g(x) = \frac{1}{\log_{10}(1-x)}$ is defined for all x for which $\log_{10}(1-x)$ is defined and

$$\log_{10}(1-x) \neq 0 \Rightarrow 1-x > 0 \text{ and } 1-x \neq 1 \Rightarrow x < 1 \text{ and } x \neq 0 \Rightarrow x \in (-\infty, 0) \cup (0, 1)$$

$\therefore$ Domain $(g) = (-\infty, 0) \cup (0, 1)$.

And, $h(x) = \sqrt{x+2}$ is defined for all x satisfying $x + 2 \geq 0 \Rightarrow x \geq -2 \Rightarrow x \in [-2, \infty)$.

$\therefore$ Domain $(h) = [-2, \infty)$.

Hence, Domain (f) = Domain $(g) \cap$ domain $(h) = (-\infty, 0) \cup (0, 1) \cap [-2, \infty) = [-2, 0) \cup (0, 1)$

EXAMPLE 15 *Find the range of each of the following functions:*

(i) $f(x) = |x-3|$ **[NCERT EXEMPLAR]** (ii) $f(x) = 1 - |x-2|$ **[NCERT EXEMPLAR]**

(iii) $f(x) = \dfrac{|x-4|}{x-4}$ **[NCERT EXEMPLAR]**

SOLUTION (i) We have, $(x) = |x-3|$. Clearly, $f(x)$ is defined for all $x \in R$. Therefore, Domain $(f) = R$. We find that

$|x-3| > 0$ for all $x \in R$

$\Rightarrow \quad 0 \leq |x-3| < \infty$ for all $x \in R \Rightarrow 0 \leq f(x) < \infty$ for all $x \in R \Rightarrow f(x) \in [0, \infty)$ for all $x \in R$

Hence, Range $(f) = [0, \infty)$.

(ii) We have, $f(x) = 1 - |x-2|$. We observe that $f(x)$ is defined for all $x \in R$. Therefore, Domain $(f) = R$.

Now, $0 \leq |x-2| < \infty$ for all $x \in R$

$\Rightarrow \quad -\infty < -|x-2| \leq 0$ for all $x \in R$

$\Rightarrow \quad -\infty < 1 - |x-2| \leq 1$ for all $x \in R \Rightarrow -\infty < f(x) \leq 1$ for all $x \in R \Rightarrow f(x) \in (-\infty, 1]$

Hence, Range $(f) = (-\infty, 1]$

(iii) We have, $f(x) = \dfrac{|x-4|}{x-4}$. Clearly, $f(x)$ is defined for all $x \in R$ except at $x = 4$. Therefore,

Domain $(f) = R - \{4\}$. We find that

$$f(x) = \frac{|x-4|}{x-4} = \begin{cases} \dfrac{x-4}{x-4} = 1 \quad , & \text{if } x > 4 \\ \dfrac{-(x-4)}{x-4} = -1 \ , & \text{if } x < 4 \end{cases}$$

Thus, $f(x)$ takes only two values –1 and 1. Hence, Range $(f) = \{-1, 1\}$.

EXAMPLE 16 *Find the domain and range of each of the following functions given by*

(i) $f(x) = \dfrac{1}{\sqrt{x-[x]}}$ (ii) $f(x) = 1 - |x-3|$

SOLUTION (i) We have, $f(x) = \dfrac{1}{\sqrt{x-[x]}}$.

Domain of f: We know that $0 \leq x - [x] < 1$ for all $x \in R$ and, $x - [x] = 0$ for $x \in Z$.

$\therefore \quad 0 < x - [x] < 1$ for all $x \in R - Z \Rightarrow f(x) = \dfrac{1}{\sqrt{x-[x]}}$ exists for all $x \in R - Z$.

Hence, Domain $(f) = R - Z$.

Range of f: As discussed above

$0 < x - [x] < 1$ for all $x \in R - Z$

$\Rightarrow \ 0 < \sqrt{x-[x]} < 1$ for all $x \in R - Z$

$\Rightarrow \ 1 < \dfrac{1}{\sqrt{x-[x]}} < \infty$ for all $x \in R - Z \Rightarrow 1 < f(x) < \infty$ for all $x \in R - Z \Rightarrow$ Range $(f) = (1, \infty)$.

(ii) We have, $f(x) = 1 - |x - 3|$. Clearly, $f(x)$ is defined for all $x \in R$. Therefore, Domain $(f) = R$.

Range of f: For any $x \in R$, we find that

$$|x-3| \geq 0 \Rightarrow -|x-3| \leq 0 \Rightarrow 1-|x-3| \leq 1 \Rightarrow f(x) \leq 1 \Rightarrow f(x) \in (-\infty, 1]$$

Hence, Range $(f) = (-\infty, 1]$.

EXAMPLE 17 *Find the domain of the real function $f(x)$ defined by* $f(x) = \sqrt{\dfrac{1-|x|}{2-|x|}}$.

SOLUTION We have, $f(x) = \sqrt{\dfrac{1-|x|}{2-|x|}}$. We observe that $f(x)$ is defined for all x satisfying

$\dfrac{1-|x|}{2-|x|} \geq 0.$

Fig. 3.26 Signs of $\dfrac{1-|x|}{2-|x|}$ for different values of $|x|$

Now, $\dfrac{1-|x|}{2-|x|} \geq 0 \Rightarrow \dfrac{|x|-1}{|x|-2} \geq 0 \Rightarrow |x| \leq 1$ or, $|x| > 2$ [$\because$ See Fig. 3.26]

$\Rightarrow \quad x \in [-1, 1]$ or, $x \in (-\infty, -2) \cup (2, \infty) \Rightarrow x \in (-\infty, -2) \cup (2, \infty) \cup [-1, 1]$

Hence, domain $(f) = (-\infty, -2) \cup (2, \infty) \cup [-1, 1]$.

EXAMPLE 18 *Find the domain of the function f given by* $f(x) = \dfrac{1}{\sqrt{[x]^2 - [x] - 6}}$ **[NCERT EXEMPALR]**

SOLUTION We have, $f(x) = \dfrac{1}{\sqrt{[x]^2 - [x] - 6}}$

Clearly, $f(x)$ is defined for all x satisfying

$[x]^2 - [x] - 6 > 0$

$\Rightarrow \quad ([x]-3)([x]+2) > 0$

$\Rightarrow \quad [x] < -2$ or $[x] > 3$

$\Rightarrow \quad x \in (-\infty, -2)$ or $x \in [4, \infty) \Rightarrow x \in (-\infty, -2) \cup [4, \infty)$

Hence, domain $(f) = (-\infty, -2) \cup [4, \infty)$.

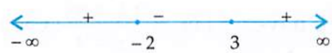

Fig. 3.27 Signs of $[x]^2 - [x] - 6$ for different values of x

EXERCISE 3.4

BASIC

1. Find $f+g$, $f-g$, cf $(c \in R, c \neq 0)$, fg, $\dfrac{1}{f}$ and $\dfrac{f}{g}$ in each of the following:

   (i) $f(x) = x^3 + 1$ and $g(x) = x + 1$ (ii) $f(x) = \sqrt{x-1}$ and $g(x) = \sqrt{x+1}$.

2. Let $f(x) = 2x + 5$ and $g(x) = x^2 + x$. Describe (i) $f+g$ (ii) $f-g$ (iii) fg (iv) f/g. Find the domain in each case.

3. If $f(x)$ be defined on $[-2, 2]$ and is given by $f(x) = \begin{cases} -1 & , \quad -2 \leq x \leq 0 \\ x-1 & , \quad 0 < x \leq 2 \end{cases}$

   and $g(x) = f(|x|) + |f(x)|$. Find $g(x)$.

4. Let f, g be two real functions defined by $f(x) = \sqrt{x+1}$ and $g(x) = \sqrt{9-x^2}$. Then, describe each of the following functions:

   (i) $f+g$ (ii) $g-f$ (iii) fg (iv) f/g

   (v) $\dfrac{g}{f}$ (vi) $2f - \sqrt{5}\, g$ (vii) $f^2 + 7f$ (viii) $\dfrac{5}{g}$

5. If $f(x) = \log_e(1-x)$ and $g(x) = [x]$, then determine each of the following functions:
(i) $f+g$ (ii) fg (iii) $\frac{f}{g}$ (iv) $\frac{g}{f}$

Also, find $(f+g)(-1)$, $(fg)(0)$, $\left(\frac{f}{g}\right)\left(\frac{1}{2}\right)$, $\left(\frac{g}{f}\right)\left(\frac{1}{2}\right)$.

BASED ON LOTS

6. If f, g, h are real functions defined by $f(x) = \sqrt{x+1}$, $g(x) = \frac{1}{x}$ and $h(x) = 2x^2 - 3$, then find the values of $(2f + g - h)(1)$ and $(2f + g - h)(0)$.

7. The function f is defined by $f(x) = \begin{cases} 1-x, & x<0 \\ 1, & x=0 \\ x+1, & x>0 \end{cases}$. Draw the graph of $f(x)$. [NCERT]

8. Let $f, g: R \to R$ be defined, respectively by $f(x) = x+1$, $g(x) = 2x-3$. Find $f+g$, $f-g$ and $\frac{f}{g}$. [NCERT]

9. Let $f:[0,\infty) \to R$ and $g: R \to R$ be defined by $f(x) = \sqrt{x}$ and $g(x) = x$. Find $f+g$, $f-g$, fg and $\frac{f}{g}$. [NCERT]

10. Let $f(x) = x^2$ and $g(x) = 2x+1$ be two real functions. Find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$. [NCERT]

ANSWERS

1. (i) $f+g: R \to R$ given by $(f+g)(x) = x^3 + x + 2$

$f-g: R \to R$ given by $(f-g)(x) = x^3 - x$

$cf: R \to R$ given by $(cf)(x) = c(x^3+1)$

$fg: R \to R$ given by $(fg)(x) = (x+1)^2(x^2-x+1)$

$\frac{1}{f}: R - \{-1\} \to R$ given by $\left(\frac{1}{f}\right)(x) = \frac{1}{x^3+1}$

$\frac{f}{g}: R - \{-1\} \to R$ given by $\left(\frac{f}{g}\right)(x) = x^2 + x + 1$

(ii) $f \pm g: [1,\infty) \to R$ defined by $(f+g)(x) = \sqrt{x-1} \pm \sqrt{x+1}$

$cf: [1,\infty) \to R$ defined by $(cf)(x) = c\sqrt{x-1}$

$fg: [1,\infty) \to R$ defined by $(fg)(x) = \sqrt{x^2-1}$

$\frac{1}{f}: (1,\infty) \to R$ defined by $\left(\frac{1}{f}\right)(x) = \frac{1}{\sqrt{x-1}}$

$\frac{f}{g}: [1,\infty) \to R$ defined by $\left(\frac{f}{g}\right)(x) = \sqrt{\frac{x-1}{x+1}}$

2. (i) $(f+g)(x) = x^2 + 3x + 5$; dom $(f+g) = R$ (ii) $(f-g)(x) = 5 + x - x^2$; dom $(f-g) = R$

(iii) $(fg)(x) = 2x^3 + 7x^2 + 5x$; dom $(fg) = R$ (iv) $\left(\frac{f}{g}\right)(x) = \frac{2x+5}{x^2+x}$, dom $\left(\frac{f}{g}\right) = R - \{0, 1\}$

3. $g(x) = \begin{cases} -x, & -2 \le x \le 0 \\ 0, & 0 < x < 1 \\ 2(x-1), & 1 \le x \le 2 \end{cases}$

4. (i) $f + g : [-1, 3] \to R$ defined by $(f+g)(x) = \sqrt{x+1} + \sqrt{9-x^2}$

(ii) $g - f : [-1, 3] \to R$ defined by $(g-f)(x) = \sqrt{9-x^2} - \sqrt{x+1}$

(iii) $fg : [-1, 3] \to R$ defined by $(fg)(x) = \sqrt{9 + 9x - x^2 - x^3}$

(iv) $\frac{f}{g} : [-1, 3] \to R$ defined by $\left(\frac{f}{g}\right)(x) = \sqrt{\frac{x+1}{9-x^2}}$

(v) $\frac{g}{f} : (-1, 3] \to R$ defined by $\left(\frac{g}{f}\right)(x) = \sqrt{\frac{9-x^2}{x+1}}$

(vi) $2f - \sqrt{5}\,g : [-1, 3] \to R$ defined by $(2f - \sqrt{5}\,g)(x) = 2\sqrt{x+1} - \sqrt{45 - 5x^2}$

(vii) $f^2 + 7f : [-1, \infty) \to R$ defined by $(f^2 + 7f)(x) = x + 1 + 7\sqrt{x+1}$

(viii) $\frac{5}{g} : (-3, 3) \to R$ defined by $\left(\frac{5}{g}\right)(x) = \frac{5}{\sqrt{9-x^2}}$

5. (i) $f + g : (-\infty, 1) \to R$ defined by $(f+g)(x) = \log_e(1-x) + [x]$

(ii) $fg : (-\infty, 1) \to R$ defined by $(fg)(x) = [x]\log_e(1-x)$

(iii) $\frac{f}{g} : (-\infty, 0) \to R$ defined by $\left(\frac{f}{g}\right)(x) = \frac{\log_e(1-x)}{[x]}$

(iv) $\frac{g}{f} : (-\infty, 0) \cup (0, 1) \to R$ defined by $\left(\frac{g}{f}\right)(x) = \frac{[x]}{\log_e(1-x)}$

$(f+g)(-1) = \log_e 2 - 1$ and, $(fg)(0) = 0$, $\left(\frac{f}{g}\right)\left(\frac{1}{2}\right)$ does not exist $\left(\frac{g}{f}\right)\left(\frac{1}{2}\right) = 0$

6. $2(\sqrt{2}+1)$, 0, does not exist.

8. $f + g : R \to R$ defined by $(f+g)(x) = 3x - 2$; $f - g : R \to R$ defined by $(f-g)(x) = -x + 4$

$\frac{f}{g} : R - \left\{\frac{3}{2}\right\} \to R$ defined by $\left(\frac{f}{g}\right)(x) = \frac{x+1}{2x-3}$

9. $f + g : [0, \infty) \to R$ defined by $(f+g)(x) = \sqrt{x} + x$; $f - g : [0, \infty) \to$ defined by $(f-g)(x) = \sqrt{x} - x$

$fg : [0, \infty) \to R$ defined by $(fg)(x) = x^{3/2}$; $\frac{f}{g} : (0, \infty) \to R$ defined by $\left(\frac{f}{g}\right)(x) = \frac{1}{\sqrt{x}}$

10. $(f+g)(x) = (x+1)^2$, $(f-g)(x) = x^2 - 2x - 1$, $(fg)(x) = 2x^3 + x^2$, $\left(\frac{f}{g}\right)(x) = \frac{x^2}{2x+1}$.

HINTS TO SELECTED PROBLEMS

7. We have, $f(x) = \begin{cases} 1-x, & x<0 \\ 1, & x=0 \\ x+1, & x>0 \end{cases}$.

Let $f(x) = y$. Then, $y = \begin{cases} 1-x, & x<0 \\ 1, & x=0 \\ x+1, & x>0 \end{cases}$

The graph of $f(x)$ is as shown in Fig. 3.28.

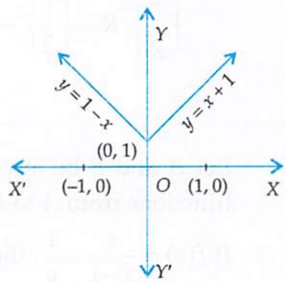

Fig. 3.28

8. $f : R \to R$ and $g : R \to R$ are given by $f(x) = x+1$ and $g(x) = 2x-3$. Clearly, $D(f) = R$ and $D(g) = R$. Therefore,

(i) $D(f+g) = D(f) \cap D(g) = R$ and, $(f+g)(x) = f(x) + g(x) = x+1+2x-3 = 3x-2$

(ii) $D(f-g) = D(f) \cap D(g) = R$ and, $(f-g)(x) = f(x) - g(x) = x+1-2x+3 = -x+4$

(iii) $D(fg) = D(f) \cap D(g) = R$ and, $(fg)(x) = f(x)\,g(x) = (x+1)(2x-3) = 2x^2 - x - 3$

(iv) $D\left(\frac{f}{g}\right) = D(f) \cap D(g) - \{x : g(x) = 0\} = R - \left\{\frac{3}{2}\right\}$ and, $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x+1}{2x-3}$

9. It is given that $f : [0, \infty) \to R$ and $g : R \to R$ are such that $f(x) = \sqrt{x}$ and $g(x) = x$. We find that

$D(f+g) = [0, \infty) \cap R = [0, \infty)$. Therefore,

$f+g : [0, \infty) \to R$ is given by $(f+g)(x) = f(x) + g(x) = \sqrt{x} + x$

$D(f-g) = D(f) \cap D(g) = [0, \infty) \cap R = [0, \infty)$. Therefore,

$f-g : [0, \infty) \to R$ is given by $(f-g)(x) = f(x) - g(x) = \sqrt{x} - x$

$D(fg) = D(f) \cap D(g) = [0, \infty) \cap R = [0, \infty)$. Therefore,

$fg : [0, \infty) \to R$ is given by $(fg)(x) = f(x)\,g(x) = \sqrt{x}\; x = x^{3/2}$

$D\left(\frac{f}{g}\right) = D(f) \cap D(g) - \{x : g(x) = 0\} = (0, \infty)$. Therefore,

$\frac{f}{g} : (0, \infty) \to R$ is given by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$

10. We have, $f(x) = x^2$ and $g(x) = 2x+1$. Clearly, $D(f) = R$ and $D(g) = R$.

$\therefore \quad D(f \pm g) = D(f) \cap D(g) = R \cap R = R$

$D(fg) = D(f) \cap D(g) = R \cap R = R$

$D\left(\frac{f}{g}\right) = D(f) \cap D(g) - \{x : g(x) = 0\} = R \cap R - \left\{-\frac{1}{2}\right\} = R - \left\{-\frac{1}{2}\right\}$

Thus, $f+g : R \to R$ is given by $(f+g)(x) = f(x) + g(x) = x^2 + 2x + 1$

$f-g : R \to R$ is given by $(f-g)(x) = f(x) - g(x) = x^2 - 2x - 1$

$(fg): R \to R$ is given by $(fg)(x) = f(x)\, g(x) = x(2x+1)$

$\left(\dfrac{f}{g}\right): R - \left\{\dfrac{1}{2}\right\} \to R$ is given by $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{x}{2x+1}$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. Let A and B be any two sets such that $n(A) = p$ and $n(B) = q$, then the total number of functions from A to B is equal to ....
2. If $f(x) = \dfrac{x}{x-1} = \dfrac{1}{y}$, then $f(y) =$ ....
3. If $y = f(x) = \dfrac{ax+b}{cx-d}$, then $f(y) =$ ....
4. The domain of the function $f(x) = \dfrac{1}{\sqrt{|x| - x}}$ is ....
5. The range of the function $f(x) = [x] - x$ is ....
6. The range of the function $f(x) = \dfrac{x+2}{|x+2|}$ is ....
7. The range of the function $f(x) = \log_a x, a > 0$ is ....
8. Let f and g be two functions given by $f = \{(2, 4), (5, 6), (8, -1), (10, -3)\}$ and $g = \{(2, 5), (7, 1)\ (8, 4), (10, 13), (11, -5)\}$. Then, domain of $f + g$ is ....
9. Let f and g be two real functions given by $f = \{(10, 1), (2, 0), (3, -4), (4, 2), (5, 1)\}$ and $g = \{(1, 0), (2, 2), (3, -1), (4, 4), (5, 3)\}$. Then the domain fg is given by ....
10. The domain for which the functions $f(x) = 3x^2 - 1$ and $g(x) = 3 + x$ are equal is ....
11. The domain of the function $f(x) = \dfrac{x^2+1}{x^2 - 3x + 2}$ is ....
12. If $f(x) = \dfrac{x-1}{x+1}$, then $f\left(\dfrac{1}{x}\right) + f(x)$ is equal to ....
13. If $f(x) = \dfrac{x-1}{x+1}$, then $f(x)\ f\left(-\dfrac{1}{x}\right)$ is equal to ....
14. If $f(x) = [x]^2 - 5[x] + 6$, then the set of values of x satisfying $f(x) = 0$ is ....
15. The domain of the function $f(x) = \sqrt{9-x} + \dfrac{1}{\sqrt{x^2 - 16}}$ is equal to ....
16. The domain and range of the function $f(x) = \dfrac{2-x}{x-2}$ are ....and................. respectively.
17. The domain of the function $f(x) = \dfrac{1}{\sqrt{[x]^2 - 3[x] + 2}}$ is ....
18. The range of the function $f(x) = \dfrac{|x-4|}{x-4}$ is ....
19. The domain of the function $f(x) = x + [x]$ is ....
20. The range of the function $f(x) = \sqrt{1 - x^2}$ is ....

21. The domain of the function $f(x)=\sum_{n=1}^{10}\frac{1}{|2x-n|}$ is ....

22. The domain of the function $f(x)=\frac{|x|-2}{|x|-3}$ is ....

23. If $f(2x+3)=4x^2+12x+15$, then the value of $f(3x+2)$ is ....

24. The number of elements of an identity function defined on a set containing four elements is ....

ANSWERS

1. q^p
2. $1-x$
3. x
4. $(-\infty, 0)$
5. $(-1, 0]$
6. $\{-1, 1\}$
7. R
8. $\{2, 8, 10\}$
9. $\{2, 3, 4, 5\}$
10. $\left\{-1, \frac{4}{3}\right\}$
11. $R-\{1, 2\}$
12. 0
13. -1
14. $[2, 4)$
15. $(-\infty, -4)\cup(4, 9]$
16. $R-\{2\}, \{-1\}$
17. $(-\infty, 1)\cup[3, \infty)$
18. $\{-1, 1\}$
19. R
20. $[0, 1]$
21. $R-\left\{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, 5\right\}$
22. $R-\{-3, 3\}$
23. $9x^2-12x+24$
24. 4

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the range of the real function $f(x) = |x|$.
2. If f is a real function satisfying $f\left(x+\frac{1}{x}\right) = x^2+\frac{1}{x^2}$ for all $x\in R-\{0\}$, then write the expression for $f(x)$.
3. Write the range of the function $f(x)=\sin[x]$, where $\frac{-\pi}{4}\le x\le\frac{\pi}{4}$.
4. If $f(x) = \cos[\pi^2]x+\cos[-\pi^2]x$, where $[x]$ denotes the greatest integer less than or equal to x, then write the value of $f(\pi)$.
5. Write the range of the function $f(x) = \cos[x]$, where $-\frac{\pi}{2}<x<\frac{\pi}{2}$.
6. Write the range of the function $f(x) = e^{x-[x]}, x\in R$.
7. Let $f(x)=\frac{\alpha x}{x+1}$, $x\ne -1$. Then write the value of α satisfying $f(f(x)) = x$ for all $x\ne -1$.
8. If $f(x) = 1-\frac{1}{x}$, then write the value of $f\left(f\left(\frac{1}{x}\right)\right)$.
9. Write the domain and range of the function $f(x)=\frac{x-2}{2-x}$.
10. If $f(x)=4x-x^2, x\in R$, then write the value of $f(a+1)-f(a-1)$.
11. If f, g, h are real functions given by $f(x) = x^2$, $g(x) = \tan x$ and, $h(x) = \log_e x$, then write the value of (hogof) $\left(\sqrt{\frac{\pi}{4}}\right)$.

12. Write the domain and range of function $f(x)$ given by $f(x) = \dfrac{1}{\sqrt{x - |x|}}$.
13. Write the domain and range of $f(x) = \sqrt{x - [x]}$.
14. Write the domain and range of function $f(x)$ given by $f(x) = \sqrt{[x] - x}$.
15. Let A and B be two sets such that $n(A) = p$ and $n(B) = q$, write the number of functions from A to B.
16. Let f and g be two functions given by

$$f = \{(2, 4), (5, 6), (8, -1), (10, -3)\} \text{ and } g = \{(2, 5), (7, 1), (8, 4), (10, 13), (11, -5)\}.$$

Find the domain of $f + g$.
17. Find the set of values of x for which the functions $f(x) = 3x^2 - 1$ and $g(x) = 3 + x$ are equal.
18. Let f and g be two real functions given by

$$f = \{(0, 1), (2, 0), (3, -4), (4, 2), (5, 1)\} \text{ and } g = \{(1, 0), (2, 2), (3, -1), (4, 4), (5, 3)\}.$$

Find the domain of fg.

ANSWERS

1. $[0, \infty)$
2. $f(x) = x^2 - 2$, where $|x| > 2$
3. $\{-\sin 1, 0\}$
4. 0
5. $\{1, \cos 1, \cos 2\}$
6. $[1, e)$
7. $\alpha = -1$
8. $\dfrac{x}{x-1}$
9. $D(f) = R - \{2\}$, $R(f) = \{-1\}$
10. $4(2 - a)$
11. 0
12. $D(f) = \phi = R(f)$
13. $D(f) = R$, $R(f) = [0, 1)$
14. $D(f) = z$, $R(f) = \{0\}$
15. q^p
16. $\{2, 8, 10\}$
17. $\{-1, 4/3\}$
18. $\{2, 3, 4, 5\}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, then which of the following is a function from A to B?
   (a) $\{(1, 2), (1, 3), (2, 3), (3, 3)\}$
   (b) $\{(1, 3), (2, 4)\}$
   (c) $\{(1, 3), (2, 2), (3, 3)\}$
   (d) $\{(1, 2), (2, 3), (3, 2), (3, 4)\}$.
2. If $f : Q \to Q$ is defined as $f(x) = x^2$, then $f^{-1}(9)$ is equal to
   (a) 3
   (b) -3
   (c) $\{-3, 3\}$
   (d) ϕ
3. Which one of the following is not a function?
   (a) $\{(x, y) : x, y \in R, x^2 = y\}$
   (b) $\{(x, y) : x, y \in R, y^2 = x\}$
   (c) $\{(x, y) : x, y \in R, x = y^3\}$
   (d) $\{(x, y) : x, y \in R, y = x^3\}$
4. If $f(x) = \cos(\log x)$, then $f(x^2)\, f(y^2) - \dfrac{1}{2}\left\{ f\left(\dfrac{x^2}{y^2}\right) + f(x^2 y^2) \right\}$ has the value
   (a) -2
   (b) -1
   (c) $1/2$
   (d) none of these
5. If $f(x) = \cos(\log x)$, then $f(x)\, f(y) - \dfrac{1}{2}\left\{ f\left(\dfrac{x}{y}\right) + f(xy) \right\}$ has the value
   (a) -1
   (b) $1/2$
   (c) -2
   (d) none of these
6. Let $f(x) = |x - 1|$. Then,

(a) $f(x^2) = [f(x)]^2$ (b) $f(x+y) = f(x) f(y)$
(c) $f(|x|) = |f(x)|$ (d) none of these

7. The range of $f(x) = \cos[x]$, for $-\pi/2 < x < \pi/2$ is
(a) $\{-1, 1, 0\}$ (b) $\{\cos 1, \cos 2, 1\}$ (c) $\{\cos 1, -\cos 1, 1\}$ (d) $[-1, 1]$

8. Which of the following are functions?
(a) $\{(x, y) : y^2 = x, x, y \in R\}$ (b) $\{(x, y) : y = |x|, x, y \in R\}$
(c) $\{(x, y) : x^2 + y^2 = 1, x, y \in R\}$ (d) $\{(x, y) : x^2 - y^2 = 1, x, y \in R\}$

9. If $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$ and $g(x) = \dfrac{3x + x^3}{1 + 3x^2}$, then $f(g(x))$ is equal to
(a) $f(3x)$ (b) $\{f(x)\}^3$ (c) $3f(x)$ (d) $-f(x)$

10. If $A = \{1, 2, 3\}$, $B = \{x, y\}$, then the number of functions that can be defined from A into B is
(a) 12 (b) 8 (c) 6 (d) 3

11. If $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$, then $f\left(\dfrac{2x}{1+x^2}\right)$ is equal to
(a) $\{f(x)\}^2$ (b) $\{f(x)\}^3$ (c) $2f(x)$ (d) $3f(x)$

12. If $f(x) = \cos(\log x)$, then value of $f(x) f(4) - \dfrac{1}{2}\left\{f\left(\dfrac{x}{4}\right) + f(4x)\right\}$ is
(a) 1 (b) -1 (c) 0 (d) ± 1

13. If $f(x) = \dfrac{2^x + 2^{-x}}{2}$, then $f(x+y) f(x-y)$ is equals to
(a) $\dfrac{1}{2}\{f(2x) + f(2y)\}$ (b) $\dfrac{1}{2}\{f(2x) - f(2y)\}$
(c) $\dfrac{1}{4}\{f(2x) + f(2y)\}$ (d) $\dfrac{1}{4}\{f(2x) - f(2y)\}$

14. If $2f(x) - 3f\left(\dfrac{1}{x}\right) = x^2$ $(x \neq 0)$, then $f(2)$ is equal to
(a) $-\dfrac{7}{4}$ (b) $\dfrac{5}{2}$ (c) -1 (d) none of these

15. Let $f : R \to R$ be defined by $f(x) = 2x + |x|$. Then $f(2x) + f(-x) - f(x) =$
(a) $2x$ (b) $2|x|$ (c) $-2x$ (d) $-2|x|$

16. The range of the function $f(x) = \dfrac{x^2 - x}{x^2 + 2x}$ is
(a) R (b) $R - \{1\}$ (c) $R - \{-1/2, 1\}$ (d) none of these

17. If $x \neq 1$ and $f(x) = \dfrac{x+1}{x-1}$ is a real function, then $f(f(f(2))$ is
(a) 1 (b) 2 (c) 3 (d) 4

18. If $f(x) = \cos(\log_e x)$, then $f\left(\dfrac{1}{x}\right) f\left(\dfrac{1}{y}\right) - \dfrac{1}{2}\left\{f(xy) + f\left(\dfrac{x}{y}\right)\right\}$ is equal to
(a) $\cos(x - y)$ (b) $\log(\cos(x - y))$ (c) 1 (d) $\cos(x + y)$

19. Let $f(x) = x$, $g(x) = \dfrac{1}{x}$ and $h(x) = f(x) g(x)$. Then, $h(x) = 1$ for
(a) $x \in R$ (b) $x \in Q$ (c) $x \in R - Q$ (d) $x \in R, x \neq 0$

20. If $f(x) = \dfrac{\sin^4 x + \cos^2 x}{\sin^2 x + \cos^4 x}$ for $x \in R$, then $f(2002) =$

(a) 1 (b) 2 (c) 3 (d) 4

21. The function $f : R \to R$ is defined by $f(x) = \cos^2 x + \sin^4 x$. Then, $f(R) =$

(a) $[3/4, 1)$ (b) $(3/4, 1]$ (c) $[3/4, 1]$ (d) $(3/4, 1)$

22. Let $A = \{x \in R : x \neq 0, -4 \leq x \leq 4\}$ and $f : A \to R$ be defined by $f(x) = \dfrac{|x|}{x}$ for $x \in A$. Then A is

(a) $\{1, -1\}$ (b) $\{x : 0 \leq x \leq 4\}$ (c) $\{1\}$ (d) $\{x : -4 \leq x \leq 0\}$

23. If $f : R \to R$ and $g : R \to R$ are defined by $f(x) = 2x + 3$ and $g(x) = x^2 + 7$, then the values of x such that $g(f(x)) = 8$ are

(a) 1, 2 (b) −1, 2 (c) −1, −2 (d) 1, −2

24. If $f : [-2, 2] \to R$ is defined by $f(x) = \begin{cases} -1, & \text{for } -2 \leq x \leq 0 \\ x - 1, & \text{for } 0 \leq x \leq 2 \end{cases}$, then

$\{x \in [-2, 2] : x \leq 0 \text{ and } f(|x|) = x\} =$

(a) $\{-1\}$ (b) $\{0\}$ (c) $\{-1/2\}$ (d) ϕ

25. If $e^{f(x)} = \dfrac{10 + x}{10 - x}$, $x \in (-10, 10)$ and $f(x) = k f\left(\dfrac{200x}{100 + x^2}\right)$, then $k =$

(a) 0.5 (b) 0.6 (c) 0.7 (d) 0.8

26. If f is a real valued function given by $f(x) = 27x^3 + \dfrac{1}{x^3}$ and α, β are roots of $3x + \dfrac{1}{x} = 12$. Then,

(a) $f(\alpha) \neq f(\beta)$ (b) $f(\alpha) = 10$ (c) $f(\beta) = -10$ (d) none of these

27. If $f(x) = 64x^3 + \dfrac{1}{x^3}$ and α, β are the roots of $4x + \dfrac{1}{x} = 3$. Then,

(a) $f(\alpha) = f(\beta) = -9$ (b) $f(\alpha) = f(\beta) = 63$

(c) $f(\alpha) \neq f(\beta)$ (d) none of these

28. If $3f(x) + 5f\left(\dfrac{1}{x}\right) = \dfrac{1}{x} - 3$ for all non-zero x, then $f(x) =$

(a) $\dfrac{1}{14}\left(\dfrac{3}{x} + 5x - 6\right)$ (b) $\dfrac{1}{14}\left(-\dfrac{3}{x} + 5x - 6\right)$

(c) $\dfrac{1}{14}\left(-\dfrac{3}{x} + 5x + 6\right)$ (d) none of these

29. If $f : R \to R$ be given by $f(x) = \dfrac{4^x}{4^x + 2}$ for all $x \in R$. Then,

(a) $f(x) = f(1 - x)$ (b) $f(x) + f(1 - x) = 0$

(c) $f(x) + f(1 - x) = 1$ (d) $f(x) + f(x - 1) = 1$

30. If $f(x) = \sin[\pi^2]x + \sin[-\pi^2]x$, where $[x]$ denotes the greatest integer less than or equal to x, then

(a) $f(\pi/2) = 1$ (b) $f(\pi) = 2$ (c) $f(\pi/4) = -1$ (d) none of these

31. The domain of the function $f(x) = \sqrt{2 - 2x - x^2}$ is

(a) $[-\sqrt{3}, \sqrt{3}]$ (b) $[-1 - \sqrt{3}, -1 + \sqrt{3}]$

(c) $[-2, 2]$ (d) $[-2 - \sqrt{3}, -2 + \sqrt{3}]$

32. The domain of definition of $f(x)=\sqrt{\frac{x+3}{(2-x)(x-5)}}$ is

(a) $(-\infty,-3]\cup(2,5)$ (b) $(-\infty,-3)\cup(2,5)$
(c) $(-\infty,-3]\cup[2,5]$ (d) none of these

33. The domain of the function $f(x)=\sqrt{\frac{(x+1)(x-3)}{x-2}}$ is

(a) $[-1,2)\cup[3,\infty)$ (b) $(-1,2)\cup[3,\infty)$
(c) $[-1,2]\cup[3,\infty)$ (d) none of these

34. The domain of definition of the function $f(x)=\sqrt{x-1}+\sqrt{3-x}$ is
(a) $[1,\infty)$ (b) $(-\infty,3)$ (c) $(1,3)$ (d) $[1,3]$

35. The domain of definition of the function $f(x)=\sqrt{\frac{x-2}{x+2}}+\sqrt{\frac{1-x}{1+x}}$ is

(a) $(-\infty,-2]\cup[2,\infty)$ (b) $[-1,1]$
(c) ϕ (d) none of these

36. The domain of definition of the function $f(x)=\log|x|$ is
(a) R (b) $(-\infty,0)$ (c) $(0,\infty)$ (d) $R-\{0\}$

37. The domain of definition of $f(x)=\sqrt{4x-x^2}$ is
(a) $R-[0,4]$ (b) $R-(0,4)$ (c) $(0,4)$ (d) $[0,4]$

38. The domain of definition of $f(x)=\sqrt{x-3-2\sqrt{x-4}}-\sqrt{x-3+2\sqrt{x-4}}$ is
(a) $[4,\infty)$ (b) $(-\infty,4]$ (c) $(4,\infty)$ (d) $(-\infty,4)$

39. The domain of the function $f(x)=\sqrt{5|x|-x^2-6}$ is

(a) $(-3,-2)\cup(2,3)$ (b) $[-3,-2)\cup[2,3)$
(c) $[-3,-2]\cup[2,3]$ (d) none of these

40. The range of the function $f(x)=\frac{x}{|x|}$ is

(a) $R-\{0\}$ (b) $R-\{-1,1\}$ (c) $\{-1,1\}$ (d) none of these

41. The range of the function $f(x)=\frac{x+2}{|x+2|}$, $x\neq-2$ is

(a) $\{-1,1\}$ (b) $\{-1,0,1\}$ (c) $\{1\}$ (d) $(0,\infty)$

42. The range of the function $f(x)=|x-1|$ is
(a) $(-\infty,0)$ (b) $[0,\infty)$ (c) $(0,\infty)$ (d) R

43. Let $f(x)=\sqrt{x^2+1}$. Then, which of the following is correct?
(a) $f(xy)=f(x)f(y)$ (b) $f(xy)\geq f(x)f(y)$ (c) $f(xy)\leq f(x)f(y)$ (d) none of these
[NCERT EXEMPLAR]

44. If $[x]^2-5[x]+6=0$, where $[\cdot]$ denotes the greatest integer function, then
(a) $x\in[3,4]$ (b) $x\in(2,3]$ (c) $x\in[2,3]$ (d) $x\in[2,4)$
[NCERT EXEMPLAR]

45. The range of $f(x)=\frac{1}{1-2\cos x}$ is

(a) $[1/3,1]$ (b) $[-1,1/3]$
(c) $(-\infty,-1)\cup[1/3,\infty)$ (d) $[-1/3,1]$ [NCERT EXEMPLAR]

46. The domain of the function $f(x)=\sqrt{4-x}+\frac{1}{\sqrt{x^2-1}}$ is equal to

(a) $(-\infty, -1) \cup (1, 4)$ (b) $(-\infty, -1] \cup (1, 4]$
(c) $(-\infty, -1) \cup [1, 4]$ (d) $(-\infty, -1) \cup [1, 4)$ [NCERT EXEMPLAR]

47. Domain of $f(x) = \sqrt{a^2 - x^2}$, $a > 0$ is

(a) $(-a, a)$ (b) $[-a, a]$ (c) $[0, a]$ (d) $(-a, 0]$

[NCERT EXEMPLAR]

48. If $f(x) = ax + b$, where a and b are integers, $f(-1) = -5$ and $f(x) = 3$, then a and b are equal

(a) $a = -3, b = -1$ (b) $a = 2, b = -3$ (c) $a = 0, b = 2$ (d) $a = 2, b = 3$

[NCERT EXEMPLAR]

49. The domain and range of the real function of defined by $f(x) = \dfrac{4-x}{x-4}$ is given by

(a) Domain $= R$, Range $= \{-1, 1\}$ (b) Domain $= R - \{1\}$, Range $= R$
(c) Domain $= R - \{4\}$, Range $= \{-1\}$ (d) Domain $= R - \{-4\}$, Range $= \{-1, 1\}$

[NCERT EXEMPLAR]

50. The domain and range of real function f defined by $f(x) = \sqrt{x-1}$ is given by

(a) Domain $= (1, \infty)$, Range $= (0, \infty)$ (b) Domain $= [1, \infty)$, Range $= (0, \infty)$
(c) Domain $= [1, \infty)$, Range $= [0, \infty)$ (d) Domain $= [1, \infty)$, Range $= [0, \infty)$

[NCERT EXEMPLAR]

51. The domain of the function f given by $f(x) = \dfrac{x^2 + 2x + 1}{x^2 - x - 6}$

(a) $R - \{-2, 3\}$ (b) $R - \{-3, 2\}$ (c) $R - [-2, 3]$ (d) $R - (-2, 3)$

[NCERT EXEMPLAR]

52. The domain and range of the function f given by $f(x) = 2 - |x - 5|$, is

(a) Domain $= R^+$, Range $= (-\infty, 1]$ (b) Domain $= R$, Range $= (-\infty, 2]$
(c) Domain $= R$, Range $= (-\infty, 2)$ (d) Domain $= R^+$, Range $= (-\infty, 2]$

[NCERT EXEMPLAR]

53. If $f(x) = x^3 - \dfrac{1}{x^3}$, then $f(x) + f\left(\dfrac{1}{x}\right)$ is equal to

(a) $2x^3$ (b) $\dfrac{2}{x^3}$ (c) 0 (d) 1 [NCERT EXEMPLAR]

54. The domain of the function f defined by $f(x) = \dfrac{1}{\sqrt{x - |x|}}$ is

(a) R_0 (b) R^+ (c) R^- (d) none of these

[NCERT EXEMPLAR]

ANSWERS

1. (c)	2. (c)	3. (b)	4. (d)	5. (d)	6. (d)	7. (b)	8. (b)
9. (c)	10. (b)	11. (c)	12. (c)	13. (a)	14. (a)	15. (b)	16. (c)
17. (c)	18. (d)	19. (d)	20. (a)	21. (c)	22. (a)	23. (c)	24. (c)
25. (a)	26. (d)	27. (a)	28. (b)	29. (c)	30. (a)	31. (b)	32. (a)
33. (a)	34. (d)	35. (c)	36. (d)	37. (d)	38. (a)	39. (c)	40. (c)

41. (a) 42. (b) 43. (c) 44. (d) 45. (c) 46. (a) 47. (b) 48. (b)
49. (c) 50. (c) 51. (a) 52. (b) 53. (c) 54. (d)

SUMMARY

1. Let A and B be two non-empty sets. Then a relation f from A to B is a function, if
   (i) for each $a \in A$ there exists $b \in B$ such that $(a, b) \in f$
   (ii) $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$.

   In other words, f is a function from A to B if each element of A appears in some ordered pair in f and no two ordered pairs in f have the same first element.

   If $(a, b) \in f$, then b is called the image of a under f.
2. A function f from a set A to a set B is a rule associating elements of set A to elements of set B such that every element in set A is associated to a unique elements in set B.

   The set A is called the domain of f and the set B is called its co-domain.
3. The range of a function f is the set of images of elements in the domain.
4. A real function has the domain and co-domain both as subsets of set R.
5. If $f : D_1 \to R$ and $g : D_2 \to R$ are two real functions and $c \in R$, then
   (i) $f \pm g : D_1 \cap D_2 \to R$ is defined as $(f \pm g)(x) = f(x) \pm g(x)$
   (ii) $fg : D_1 \cap D_2 \to R$ is defined as $(fg)(x) = f(x)\, g(x)$
   (iii) $\frac{f}{g} : D_1 \cap D_2 - \{x : g(x) = 0\} \to R$ is defined as $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$
   (iv) $cf : D_1 \cap D_2 \to R$ is defined as $(cf)(x) = c\, f(x)$.