CHAPTER 4

MEASUREMENT OF ANGLES

4.1 INTRODUCTION

The word *'Trigonometry'* is derived from two Greek words : (i) trigonon and, (ii) metron. The word trigonon means a triangle and the word metron means a measure. Hence, trigonometry means the science of measuring triangles. In broader sense it is that branch of Mathematics which deals with the measurement of the sides and the angles of a triangle and the problems *allied* with angles.

4.2 ANGLES

ANGLE *Consider a ray $\overrightarrow{OA}$. If this ray rotates about its initial point O and takes the position OB, then we say that the angle $\angle AOB$ has been generated.*

Thus, an angle is considered as the figure obtained by rotating a given ray about its initial point.

The revolving ray is called the generating line of the angle. The initial position *OA* is called the *initial side* and the final position *OB* is called *terminal side* of the angle. The initial point *O* on the initial side about which the ray rotates is called the vertex of the angle.

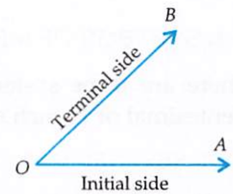


Fig. 4.1

MEASURE OF AN ANGLE *The measure of an angle is the amount of rotation performed to get the terminal side.*

SENSE OF AN ANGLE *The sense of an angle is determined by the direction of rotation of the initial side into the terminal side. The sense of an angle is said to be positive or negative according as the initial side rotates in anticlockwise or clockwise direction to get to the terminal side.*

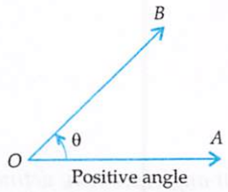


Fig. 4.2 Positive angle

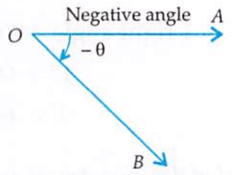


Fig. 4.3 Negative angle

RIGHT ANGLE *If the revolving ray starting from its initial position to final position describes one quarter of a circle, then we say that the measure of the angle formed is a right angle.*

4.3 SOME USEFUL TERMS

QUADRANTS *Let X'OX and YOY' be two lines at right angles in the plane of the paper. These lines divide the plane of the paper into four equal parts which are known as quadrants. The lines X'OX and YOY' are known as x-axis and y-axis respectively. These two lines taken together are known as the coordinate axes. The regions XOY, YOX', X'OY' and Y'OX are known as the first, the second, the third and the fourth quadrant respectively.*

ANGLE IN STANDARD POSITION *An angle is said to be in standard position if its vertex coincides with the origin O and the initial side coincides with OX i.e. the positive direction of x-axis.*

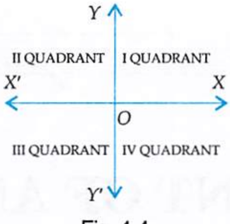


Fig. 4.4

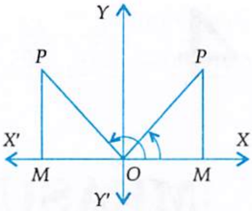


Fig. 4.5

ANGLE IN A QUADRANT *An angle in standard position is said to be in a particular quadrant, if the terminal side of the angle in standard position lies in that quadrant.*

QUADRANT ANGLE *An angle in standard position is said to be a quadrant angle, if the terminal side coincides with one of the axes.*

TRIANGLE OF REFERENCE *If from any point P on the terminal side of an angle in standard position a perpendicular PM is drawn on x-axis, then the right angled triangle OMP, thus formed, is called the triangle of reference of the $\angle XOP$. (See Fig. 4.5)*

CO-TERMINAL ANGLES *Two angles with different measures but having the same initial sides and the same terminal sides are known as co-terminal angles.*

Angles of measure 30°, 390° and – 330° are co-terminal angles.

4.4 SYSTEMS OF MEASUREMENT OF ANGLES

There are three systems for measuring angles, viz. (i) Sexagesimal or English system, (ii) Centesimal or French system, (iii) Circular system.

4.4.1 SEXAGESIMAL SYSTEM

In this system a right angle is divided into 90 equal parts, called degrees. The symbol $1°$ is used to denote one degree. Thus, one degree is one-ninetieth part of a right angle. Each degree is divided into 60 equal parts, called minutes. The symbol $1'$ is used to denote one minute. And each minute is divided into 60 equal parts, called seconds. The symbol $1''$ is used to denote one second.

Thus,

$$1 \text{ right angle} = 90 \text{ degrees } (90°)$$
$$1° = 60 \text{ minutes } (= 60')$$
$$1' = 60 \text{ seconds } (= 60'')$$

REMARK *Instead of defining degree as $\left(\frac{1}{90}\right)^{\text{th}}$ part of a right angle, we may define it in terms of one complete revolution as follows:*

ONE COMPLETE REVOLUTION *If the terminal side of an angle coincides with the initial side after rotation in anticlockwise direction, then we say that the terminal side has made one complete revolution*

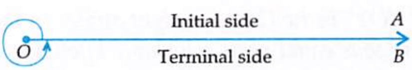
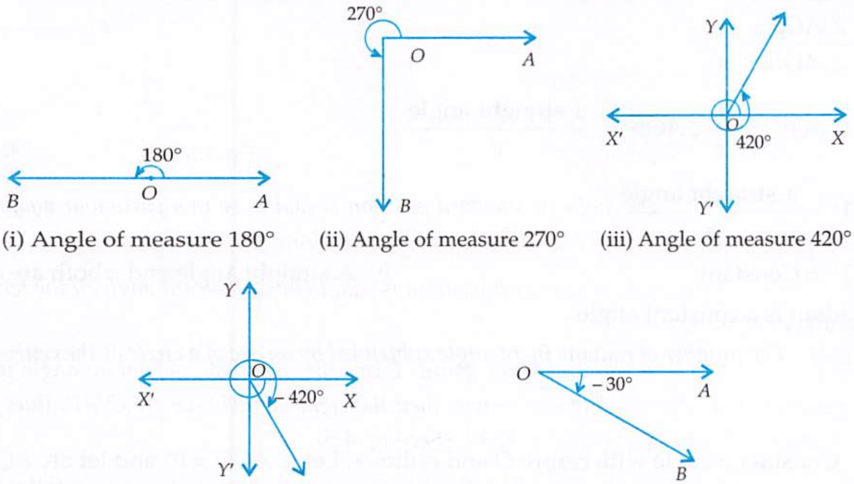


Fig. 4.6

DEGREE MEASURE *If a rotation from the initial side to terminal side is $\left(\frac{1}{360}\right)^{\text{th}}$ of a revolution, the angle is said to have a measure of one degree.*

It is evident from this definition of degree measure that 1 complete revolution = 360°

The angles of measures 180°, 270°, 420°, – 30°, – 420° are shown in the following figures.



(i) Angle of measure 180° (ii) Angle of measure 270° (iii) Angle of measure 420°

(iv) Angle of measure – 420° (v) Angle of measure – 30°

Fig. 4.7 Angle of measure

4.4.2 CENTESIMAL SYSTEM

In this system a right angle is divided into 100 equal parts, called grades; each grade is subdivided into 100 minutes, and each minute into 100 seconds.

The symbols 1^g , 1′ and 1′′ are used to denote a grade, a minute, and a second respectively.

Thus,
$$1 \text{ right angle} = 100 \text{ grades } (= 100^g)$$
$$1 \text{ grade} = 100 \text{ minutes } (= 100')$$
$$1 \text{ minute} = 100 \text{ seconds } (= 100'')$$

4.4.3 CIRCULAR SYSTEM

In this system the unit of measurement is radian as defined below.

RADIAN *One radian, written as* 1^c, *is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.*

Consider a circle of radius r having centre at O (see Fig. 4.8). Let A be a point on the circle. Now, cut off an arc AP whose length is equal to the radius r of the circle. Then by the definition the measure of $\angle AOP$ is 1 radian $(=1^c)$.

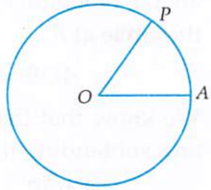


Fig. 4.8

Since a radian is chosen as the unit of measurement of an angle, therefore it should be a constant quantity. This we shall show in the following two theorems.

THEOREM 1 *Radian is a constant angle.*

PROOF Consider a circle with centre O and radius r. Take a point A on the circle and cut off an arc AP whose length is equal to the radius r. Join OA and OP. Then, by definition $\angle AOP = 1^c$.

Produce AO to meet the circle at B so that

$$\angle AOB = \text{a straight angle} = 2 \text{ right angles.}$$

Since the angles at the centre of a circle are proportional to the arcs subtending them.

$$\therefore \quad \frac{\angle AOP}{\angle AOB} = \frac{\text{arc } AP}{\text{arc } APB}$$

$$\Rightarrow \quad \frac{\angle AOP}{\angle AOB} = \frac{r}{\pi r} \qquad \left[\because \text{ arc } APB = \frac{1}{2}(2\pi r) = \pi r\right]$$

$$\Rightarrow \quad \frac{\angle AOP}{\angle AOB} = \frac{1}{\pi}$$

$$\Rightarrow \quad \angle AOP = \frac{1}{\pi}\angle AOB = \frac{\text{a straight angle}}{\pi}$$

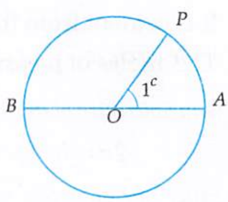


Fig. 4.9

$$\Rightarrow \quad 1^c = \frac{\text{a straight angle}}{\pi} \qquad [\because \angle AOP = 1^c]$$

$$\Rightarrow \quad 1^c = \text{Constant} \qquad [\because \text{ A straight angle and } \pi \text{ both are constants}]$$

Hence, radian is a constant angle **Q.E.D.**

THEOREM 2 *The number of radians in an angle subtended by an arc of a circle at the centre is equal to* $\dfrac{arc}{radius}$.

PROOF Consider a circle with centre O and radius r. Let $\angle AOQ = \theta^c$ and let arc $AQ = s$. Let P be a point on the arc AQ such that arc $AP = r$. Then, $\angle AOP = 1^c$. Since angles at the centre of a circle are proportional to the arcs subtending them.

$$\therefore \quad \frac{\angle AOQ}{\angle AOP} = \frac{\text{arc } AQ}{\text{arc } AP}$$

$$\Rightarrow \quad \angle AOQ = \left(\frac{\text{arc } AQ}{\text{arc } AP} \times 1\right)^c \qquad [\because \angle AOP = 1^c]$$

$$\Rightarrow \quad \theta = \frac{s}{r} \text{ radians}.$$

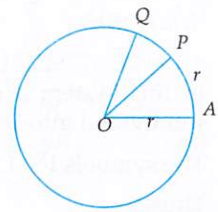


Fig. 4.10

Q.E.D

4.5 RELATION BETWEEN DEGREES AND RADIANS

Consider a circle with centre O and radius r. Let A be a point on the circle. Join OA and cut off an arc OP of length equal to the radius of the circle. Then, $\angle AOP = 1$ radian. Produce AO to meet the circle at B.

$$\therefore \quad \angle AOB = \text{a straight angle} = 2 \text{ right angles}$$

We know that the angles at the centre of a circle are proportional to the arcs subtending them.

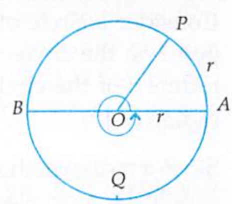


Fig. 4.11

$$\therefore \quad \frac{\angle AOP}{\angle AOB} = \frac{\text{arc } AP}{\text{arc } APB}$$

$$\Rightarrow \quad \frac{\angle AOP}{2 \text{ right angles}} = \frac{r}{\pi r} \qquad \left[\because \text{ arc } APB = \frac{1}{2}(\text{Circumference})\right]$$

$$\Rightarrow \quad \angle AOP = \frac{2 \text{ rightangles}}{\pi}$$

$$\Rightarrow \quad \angle AOP = \frac{180^\circ}{\pi}$$

Hence, One radian $= \dfrac{180^\circ}{\pi} \Rightarrow \pi$ radians $= 180^\circ$.

REMARK 1 *In Fig. 4.11, arc $(APBQA) = 2\pi r$. So, the measure of angle made by this arc at this centre O of the circle is $\frac{2\pi r}{r}$ radians i.e. 2π radians. Also, the measure of this angle is the measure of one revolution i.e. 360°.*

$\therefore \quad 2\pi \text{ radians} = 360° \Rightarrow \pi \text{ radians} = 180°$

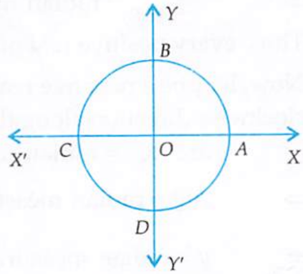


Fig. 4.12

Since one complete revolution subtends an angle of 2π radians at the centre of the unit circle shown in Fig. 4.12.

$\therefore \quad \angle AOB = \frac{1}{4} \times 2\pi = \frac{\pi}{2}$ *radians*, $\angle AOC = \frac{1}{2}(2\pi) = \pi$ *radians*

and, $\angle AOD = \frac{3}{4}(2\pi) = \frac{3\pi}{2}$ radians.

Thus, quadrant angles are integral multiples of $\frac{\pi}{2}$. All integral multiples of $\frac{\pi}{2}$ are called the quadrant angles.

REMARK 2 *When an angle is expressed in radians, the word radian is generally omitted.*

REMARK 3 *We know that $180° = \pi$ radians. Therefore, $1° = \frac{\pi}{180}$ radian.*

Hence, $30° = \frac{\pi}{180} \times 30 = \frac{\pi}{6}$ *radians*, $45° = \frac{\pi}{180} \times 45 = \frac{\pi}{4}$ *radians*, $60° = \frac{\pi}{180} \times 60 = \frac{\pi}{3}$ *radians*,

$90° = \frac{\pi}{180} \times 90 = \frac{\pi}{2}$ radians etc.

REMARK 4 *We have, π radians $= 180°$*

$\therefore \quad 1 \text{ radian} = \frac{180°}{\pi} = \left(\frac{180}{22} \times 7\right)^{\circ} = 57° 16' 22''$ *(approx).*

Again, $\quad 180° = \pi \text{ radians}$

$\therefore \quad 1° = \frac{\pi}{180} \text{ radian} = \left(\frac{22}{7 \times 180}\right) \text{radian} = 0.01746 \text{ radian.}$

4.6 RELATION BETWEEN RADIANS AND REAL NUMBERS

In this section, we shall show that radian measures and real numbers can be considered as one and the same i.e. every radian measure is a real number and every real number can be considered as radian measure of some angle. For this, let us consider the unit circle with centre *O* and let *A* be any point on it as shown in Fig. 4.13.

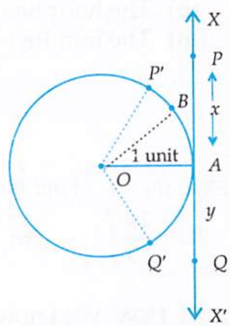


Fig. 4.13

Consider *OA* as initial side of an angle $\angle AOB$, where *B* is a point on the circle.

Now, $\quad \angle AOB = \frac{\text{arc } AB}{OA} \Rightarrow \angle AOB = \text{arc } AB \qquad [\because OA = 1 \text{ unit}]$

So, the length of arc *AB* is the radian measure of $\angle AOB$. Let XAX' be the tangent to the circle at *A*. Let the point *A* represent the real number zero, positive real number are represented by points on *AX* and negative real number by points on AX'. Let a positive real number *x* be represented by a point *P* on *AX*. Let us now rope the line *AP* in anticlockwise direction along the circle such that $AP = \text{arc } AP'$. Then, arc AP' = radian measure of $\angle AOP' \Rightarrow AP$ = radian measure of $\angle AOP'$

$\Rightarrow$ x = radian measure of $\angle AOP'$

Thus, every positive real number is the radian measure of some positive angle and vice-versa.

Now, let y be a negative real number represented by a point Q on AX'. If we rope the line AQ in clockwise direction along the circle such that AQ = arc AQ', then

arc AQ' = radian measure of $\angle AOQ'$

$\Rightarrow$ AQ = radian measure of $\angle AOQ'$ $\left[\because \angle AOQ' = \frac{\text{Arc } AQ'}{1} = \text{Arc } AQ'\right]$

$\Rightarrow$ y = radian measure of $\angle AOQ'$

Thus, every negative real number is the radian measure of some negative angle and vice-versa.

Hence, radian measures of angles and real numbers can be considered as one and the same.

4.7 RELATION BETWEEN THREE SYSTEMS OF MEASUREMENT OF AN ANGLE

Let D be the number of degrees, R be the number of radians and G be the number of grades in an angle θ.

$\because$ $90° = 1$ right angle $\Rightarrow 1° = \frac{1}{90}$ right angle $\Rightarrow D° = \frac{D}{90}$ right angles

$\Rightarrow$ $\theta = \frac{D}{90}$ right angles ... (ii)

Also, π radians = 2 right angles $\Rightarrow$ 1 radian = $\frac{2}{\pi}$ right angles $\Rightarrow R$ radians = $\frac{2R}{\pi}$ right angles

$\Rightarrow$ $\theta = \frac{2R}{\pi}$ right angles ... (ii)

And, 100 grades = 1 right angle

$\Rightarrow$ 1 grade = $\frac{1}{100}$ right angle $\Rightarrow$ G grades = $\frac{G}{100}$ right angles

$\Rightarrow$ $\theta = \frac{G}{100}$ right angles ... (iii)

From (i), (ii) and (iii), we obtain: $\frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi}$

This is the required relation between the three systems of measurement of an angle.

SOME USEFUL POINTS

(i) The angle between two consecutive digits in a clock is $30° \left(= \frac{\pi}{6} \text{ radians}\right)$.

(ii) The hour hand rotates through an angle of 30° in one hour i.e. $(1/2)°$ in one minute.

(iii) The minute hand rotates through an angle of 6° in one minute.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the degree measure corresponding to the following radian measures:*

(i) $\left(\frac{2\pi}{15}\right)^c$ (ii) $\left(\frac{\pi}{8}\right)^c$ (iii) $\left(\frac{1}{4}\right)^c$ (iv) -2^c (v) 6^c (vi) $\left(\frac{11}{16}\right)^c$

SOLUTION We know that π radians $= 180°$ and so, $1^c = \left(\frac{180}{\pi}\right)^\circ$. Therefore,

(i) $\left(\frac{2\pi}{15}\right)^c = \left(\frac{2\pi}{15} \times \frac{180}{\pi}\right)^\circ = 24°$

(ii) $\left(\frac{\pi}{8}\right)^c = \left(\frac{\pi}{8} \times \frac{180}{\pi}\right)^\circ = \left(\frac{45}{2}\right)^\circ = \left(22\frac{1}{2}\right)^\circ = 22^\circ\left(\frac{1}{2} \times 60\right)' = 22^\circ\ 30'$

(iii) $\left(\frac{1}{4}\right)^c = \left(\frac{1}{4} \times \frac{180}{\pi}\right)^\circ = \left(\frac{1}{4} \times \frac{180}{22} \times 7\right)^\circ = \left(\frac{315}{22}\right)^\circ = \left(14\frac{7}{22}\right)^\circ$

$= 14^\circ\left(\frac{7}{22} \times 60\right)' = 14^\circ\left(19\frac{1}{11}\right)' = 14^\circ\ 19'\left(\frac{1}{11} \times 60\right)'' = 14^\circ 19'\ 5''$

(iv) $(-2)^c = \left(\frac{180}{\pi} \times -2\right)^\circ = \left(\frac{180}{22} \times 7 \times (-2)\right)^\circ = \left(-114\frac{6}{11}\right)^\circ = \left\{-114^\circ\left(\frac{6}{11} \times 60\right)'\right\}$

$= -\left[114^\circ\left(32\frac{8}{11}\right)'\right] = -\left[114^\circ\ 32'\left(\frac{8}{11} \times 60\right)''\right] = -(114^\circ\ 32'\ 44'')$

(v) $6^c = \left(\frac{180}{\pi} \times 6\right)^\circ = \left(\frac{180}{22} \times 7 \times 6\right)^\circ = \left(\frac{90 \times 7 \times 6}{11}\right)^\circ = \left(\frac{3780}{11}\right)^\circ = \left(343\frac{7}{11}\right)^\circ$

$= 343^\circ\left(\frac{7}{11} \times 60\right)' = 343^\circ\left(\frac{420}{11}\right)' = 343'\ 38'\left(\frac{2}{11} \times 60\right)'' = 343^\circ\ 38'\ 11''$

(vi) $\left(\frac{11}{16}\right)^c = \left(\frac{180}{\pi} \times \frac{11}{16}\right)^\circ = \left(\frac{180}{22} \times 7 \times \frac{11}{16}\right)^\circ = \left(\frac{315}{8}\right)^\circ = \left(39\frac{3}{8}\right)^\circ$

$= 39^\circ\left(\frac{3}{8} \times 60\right)' = 39^\circ\ 22'\left(\frac{1}{2} \times 60\right)'' = 39^\circ\ 22'\ 30''$

EXAMPLE 2 *Find the radian measures corresponding to the following degree measures :*

(i) 340° (ii) 75° (iii) $-37^\circ\ 30'$ (iv) $5^\circ\ 37'\ 30''$ (v) $40^\circ\ 20'$ (vi) 520°

SOLUTION We know that $180^\circ = \pi^c$ and, $1^\circ = \left(\frac{\pi}{180}\right)^c$. Therefore,

(i) $340^\circ = \left(340 \times \frac{\pi}{180}\right)^c = \left(\frac{17\pi}{9}\right)^c$

(ii) $75^\circ = \left(75 \times \frac{\pi}{180}\right)^c = \left(\frac{5\pi}{12}\right)^c$

(iii) Clearly, $30' = \left(\frac{30}{60}\right)^\circ = \frac{1}{2}^\circ$.

$\therefore \quad -37^\circ\ 30' = -\left(37\frac{1}{2}\right)^\circ = -\left(\frac{75}{2}\right)^\circ = -\left(\frac{75}{2} \times \frac{\pi}{180}\right)^c = -\left(\frac{5\pi}{24}\right)^c$

(iv) Clearly, $30'' = \left(\frac{30}{60}\right)' = \left(\frac{1}{2}\right)'$. Therefore, $37'\ 30'' = \left(37\frac{1}{2}\right)' = \left(\frac{75}{2}\right)' = \left(\frac{75}{2} \times \frac{1}{60}\right)^\circ = \left(\frac{5}{8}\right)^\circ$

So, $5^\circ\ 37'\ 30'' = \left(5\frac{5}{8}\right)^\circ = \left(\frac{45}{8}\right)^\circ = \left(\frac{45}{8} \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{32}\right)^c$

(v) Clearly, $20' = \left(\frac{20}{60}\right)^\circ = \frac{1}{3}^\circ$. Therefore, $40^\circ\ 20' = \left(40\frac{1}{3}\right)^\circ = \left(\frac{121}{3}\right)^\circ = \left(\frac{121}{3} \times \frac{\pi}{180}\right)^c = \left(\frac{121\pi}{540}\right)^c$

(vi) Clearly, $520^\circ = \left(520 \times \frac{\pi}{180}\right)^c = \left(\frac{26\pi}{9}\right)^c$

EXAMPLE 3 *Find the length of an arc of a circle of radius 5 cm subtending a central angle measuring* 15°.

SOLUTION Let s be the length of the arc subtending an angle θ^c at the centre of a circle of radius r.

Then, $\theta = \frac{s}{r}$. Here, $r = 5$ cm and $\theta = 15° = \left(15 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{12}\right)^c$

$$\therefore \quad \theta = \frac{s}{r} \Rightarrow \frac{\pi}{12} = \frac{s}{5} \Rightarrow s = \frac{5\pi}{12} \text{ cm.}$$

EXAMPLE 4 *Find in degrees the angle subtended at the centre of a circle of diameter 50 cm by an arc of length 11 cm.*

SOLUTION Here, $r = 25$ cm and $s = 11$ cm

$$\therefore \quad \theta = \left(\frac{s}{r}\right)^c$$

$$\Rightarrow \quad \theta = \left(\frac{11}{25}\right)^c = \left(\frac{11}{25} \times \frac{180}{\pi}\right)^\circ = \left(\frac{11}{25} \times \frac{180}{22} \times 7\right)^\circ = \left(\frac{126}{5}\right)^\circ = \left(25\frac{1}{5}\right)^\circ = 25°\left(\frac{1}{5} \times 60\right)' = 25°\,12'$$

EXAMPLE 5 *Find in degrees the angle through which a pendulum swings if its length is 50 cm and the tip describes an arc of length 10 cm.*

SOLUTION Here, $r = 50$ cm and $s = 10$ cm.

$$\therefore \ \theta = \left(\frac{s}{r}\right)^c$$

$$\Rightarrow \theta = \left(\frac{10}{50}\right)^c = \left(\frac{1}{5}\right)^c = \left(\frac{1}{5} \times \frac{180}{\pi}\right)^\circ = \left(\frac{36}{22} \times 7\right)^\circ = \left(\frac{126}{11}\right)^\circ = \left(11\frac{5}{11}\right)^\circ = 11°\left(\frac{5}{11} \times 60\right)' = 11°\,27'\,16''$$

EXAMPLE 6 *A horse is tied to a post by a rope. If the horse moves along a circular path always keeping the rope tight and describes 88 metres when it has traced out* 72° *at the centre, find the length of the rope.*

SOLUTION Let the post be at point P and let PA be the length of the rope in tight position. Suppose the horse moves along the arc AB so that $\angle APB = 72°$ and arc $AB = 88$ m. Let r be the length of the rope i.e. $PA = r$ metres.

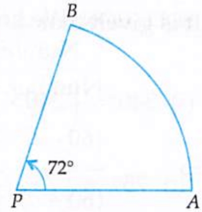


Fig. 4.14

Here, $\theta = 72° = \left(72 \times \frac{\pi}{180}\right)^c = \left(\frac{2\pi}{5}\right)^c$ and $s = 88$ m

$$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}} \Rightarrow \frac{2\pi}{5} = \frac{88}{r} \Rightarrow r = 88 \times \frac{5}{2\pi} = 70 \text{ metres.}$$

BASED ON LOTS

EXAMPLE 7 *In a circle of diameter 40 cm the length of a chord is 20 cm. Find the length of minor arc corresponding to the chord.*

SOLUTION Let arc $AB = s$. It is given that $OA = 20$ cm and chord $AB = 20$ cm. Therefore, ΔOAB is an equilateral triangle.

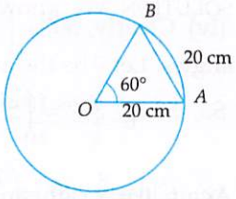


Fig. 4.15

Hence, $\angle AOB = 60° = \left(60 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c$

$$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}} \Rightarrow \frac{\pi}{3} = \frac{s}{20} \Rightarrow s = \frac{20\pi}{3} \text{ cm.}$$

EXAMPLE 8 *The angles of a triangle are in A.P. The number of degrees in the least is to the number of radians in the greatest as 60 : π. Find the angles in degrees.*

SOLUTION Let the measures of angles of the triangle be $(a-d)^\circ$, a° and $(a+d)^\circ$ Then,

$$(a-d)+a+(a+d) = 180^\circ \Rightarrow 3a = 180^\circ \Rightarrow a = 60^\circ$$

So, the angles are $(60-d)^\circ$, 60°, $(60+d)^\circ$. Clearly, $(60-d)^\circ$ is the least angle and $(60+d)^\circ$ is the greatest angle.

$$\text{Greatest angle} = (60+d)^\circ = \left\{(60+d)\frac{\pi}{180}\right\}^c$$

It is given that:

$$\frac{\text{Number of degrees in the least angle}}{\text{Number of radians in the greatest angle}} = \frac{60}{\pi}$$

$$\Rightarrow \quad \frac{(60-d)}{(60+d)\frac{\pi}{180}} = \frac{60}{\pi} \Rightarrow 3(60-d) = (60+d) \Rightarrow 120 = 4d \Rightarrow d = 30.$$

Hence, measures of the angles are $(60-30)^\circ$, 60°, $(60+30)^\circ$ i.e. 30°, 60°, 90°.

EXAMPLE 9 *The angles of a triangle are in A.P. The number of grades in the least, is to the number of radians in the greatest as 40 : π. Find the angles in degrees.*

SOLUTION Let measures of the angles of the triangle in degrees be $(a-d)^\circ, a^\circ$ and $(a+d)^\circ$. Then,

$$(a-d)+a+(a+d) = 180 \Rightarrow 3a = 180 \Rightarrow a = 60$$

So, measures of the angles are $(60-d)^\circ$, 60° and $(60+d)^\circ$. Clearly, measure of the least angle is $(60^\circ - d)^\circ$ and that of the greatest angle is $(60+d)^\circ$.

Now,

$$\text{Measure of the least angle} = (60-d)^\circ = \left\{(60-d)\times\frac{100}{90}\right\}^g = \left\{(60-d)\times\frac{10}{9}\right\}^g \qquad [\because 90^\circ = 100^g]$$

$$\text{Measure of the greatest angle} = (60+d)^\circ = \left\{(60+d)\times\frac{\pi}{180}\right\}^c$$

It is given that:

$$\frac{\text{Number of grades in the least angle}}{\text{Number of radians in the greatest angle}} = \frac{40}{\pi}$$

$$\Rightarrow \quad \frac{(60-d)\times\frac{10}{9}}{(60+d)\frac{\pi}{180}} = \frac{40}{\pi} \Rightarrow \frac{600-10d}{9}\times\frac{180}{(60+d)\pi} = \frac{40}{\pi}$$

$$\Rightarrow \quad 600-10d = 120+2d \Rightarrow 12d = 480 \Rightarrow d = 40.$$

Hence, measures the angles of the triangle are 20°, 60° and 100°.

EXAMPLE 10 *Express the angular measurement of the angle of a regular decagon in degrees, grades and radians.*

SOLUTION We know that the angle of an n sided regular polygon is equal to $\left(\frac{2n-4}{n}\right)$ right angles. Let θ be the angle of a regular decagon. Then,

$$\theta = \left(\frac{2\times 10-4}{10}\right) = \frac{8}{5}\text{ right angles} = \left(\frac{8}{5}\times 90\right)^\circ = 144^\circ \qquad [\because 1 \text{ right angle} = 90^\circ]$$

$$\text{Again, } \theta = \frac{8}{5}\text{ right angles} = \left(\frac{8}{5}\times 100\right)^g = 160^g \qquad [\because 1 \text{ right angle} = 100^g]$$

$$\text{And, } \quad \theta = \frac{8}{5}\text{ right angles} = \left(\frac{8}{5}\times\frac{\pi}{2}\right)^c = \left(\frac{4\pi}{5}\right)^c \qquad \left[\because 1 \text{ right angle} = \frac{\pi^c}{2}\right]$$

EXAMPLE 11 *If the arcs of same length in two circles subtend angles of 60° and 75° at their centres. Find the ratio of their radii.*

SOLUTION Let r_1 and r_2 be the radii of the given circles and let their arcs of same length s subtend angles of 60° and 75° at their centres.

Now, $60° = \left(60 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c$ and, $75° = \left(75 \times \frac{\pi}{180}\right)^c = \left(\frac{5\pi}{12}\right)^c$

$\therefore \quad \frac{\pi}{3} = \frac{s}{r_1}$, and $\frac{5\pi}{12} = \frac{s}{r_2}$ $\quad \left[\because \theta = \left(\frac{s}{r}\right)^c\right]$

$\Rightarrow \quad \frac{\pi}{3} r_1 = s$, and $\frac{5\pi}{12} r_2 = s \Rightarrow \frac{\pi}{3} r_1 = \frac{5\pi}{12} r_2 \Rightarrow 4r_1 = 5r_2 \Rightarrow r_1 : r_2 = 5 : 4$

Hence, $r_1 : r_2 = 5 : 4$

EXAMPLE 12 *A circular wire of radius 3 cm is cut and bent so as to lie along the circumference of a hoop whose radius is 48 cm. Find in degrees the angle which is subtended at the centre of the hoop.*

[NCERT EXEMPLAR]

SOLUTION It is given that the radius of the circular wire is 3 cm.

$\therefore$ Length of the circular wire $= 2\pi \times 3 = 6\pi$ cm $\quad [\because$ Circumference $= 2\pi r]$

Radius of the hoop = 48 cm.

Let θ be the angle subtended by the wire at the centre of the hoop. Then,

$$\theta = \frac{\text{arc}}{\text{radius}} \Rightarrow \theta = \left(\frac{6\pi}{48}\right)^c = \left(\frac{\pi}{8}\right)^c = \left(\frac{\pi}{8} \times \frac{180}{\pi}\right)^\circ = 22°\,30'$$

EXAMPLE 13 *The perimeter of a certain sector of a circle is equal to the length of the arc of a semi-circle having the same radius. Express the angle of the sector in degrees, minutes and seconds.*

SOLUTION Let r be the radius of the circle and θ be the sector angle. Then,

Perimeter of the sector $= 2r + r\theta$, Length of the arc of a semi-circle of radius $r = \pi r$

It is given that

$$2r + r\theta = \pi r \Rightarrow 2 + \theta = \pi$$

$$\Rightarrow \quad \theta = (\pi - 2) \text{ radians} = \left\{(\pi - 2) \times \frac{180}{\pi}\right\}^\circ = 180° - \left(\frac{360}{\pi}\right)^\circ = 180° - 114°\,32'\,44'' = 65°\,27'\,16''$$

EXAMPLE 14 *The minute hand of a watch is 1.5 cm long. How far does its tip move in 40 minutes?*

(Use $\pi = 3.14$)

SOLUTION In 60 minutes, the minute hand of a watch completes one rotation i.e., it rotates through 360°.

$\therefore$ Angle traced by the minute hand in 1 minute $= \left(\frac{360}{60}\right)^\circ = 6°$

$\Rightarrow$ Angle traced by the minute hand in 40 minutes $= (40 \times 6)° = 240° = \left(240 \times \frac{\pi}{180}\right)^c = \left(\frac{4\pi}{3}\right)^c$

Now,

$$\theta = \frac{\text{arc}}{\text{radius}} \Rightarrow \frac{4\pi}{3} = \frac{\text{arc}}{1.5} \Rightarrow \text{arc} = \left(\frac{4\pi}{3} \times 1.5\right) \text{cm} = 2\pi \text{ cm} = 2 \times 3.14 \text{ cm} = 6.28 \text{ cm}$$

Hence, the tip of the minute hand travels 6.28 cm in 40 minutes.

EXAMPLE 15 *Find the angle between the minute hand of a clock and the hour hand when the time is 7 : 20 AM.*

SOLUTION We know that the hour hand completes one rotation in 12 hours while the minute hand completes one rotation in 60 minutes.

$\therefore$ Angle traced by the hour hand in 12 hours = 360°

$\Rightarrow$ Angle traced by the hour hand in 7 hrs 20 min. i.e $\frac{22}{3}$ hrs $= \left(\frac{360}{12} \times \frac{22}{3}\right)^\circ = 220°$.

Also, the angle traced by the minute hand in 60 min = 360°.

$\Rightarrow$ Angle traced by the minute hand in 20 min $= \left(\frac{360}{60} \times 20\right)^\circ = 120°$

Hence, the required angle between two hands $= 220° - 120° = 100°$.

EXAMPLE 16 *Find in degrees and radians the angle between the hour hand and the minute-hand of a clock at half past three.*

SOLUTION The angle traced by the hour hand in 12 hours = 360°

$\therefore$ The angle traced by the hour hand in 3 hrs 30 min. i.e. $\frac{7}{2}$ hrs $= \left(\frac{360}{12} \times \frac{7}{2}\right)^\circ = 105°$

The angle traced by the minute hand in 60 min = 360°

$\Rightarrow$ The angle traced by the minute hand in 30 min $= \left(\frac{360}{60} \times 30\right) = 180°$

Hence, the required angle between two hands $= 180° - 105° = 75° = \left(75 \times \frac{\pi}{180}\right) = \frac{5\pi}{12}$ radians.

BASED ON HOTS

EXAMPLE 17 *The moon's distance from the earth is 360,000 kms and its diameter subtends an angle of 31′ at the eye of the observer. Find the diameter of the moon.*

SOLUTION Let AB be the diameter of the moon and let E be the eye of the observer. Since the distance between the earth and the moon is quite large, so we take diameter AB as arc AB. Let d be the diameter of the moon. Then, d = arc AB.

We have,

$$\theta = 31' = \left(\frac{31}{60}\right)^\circ = \left(\frac{31}{60} \times \frac{\pi}{180}\right)^c \text{ and, } r = 360000 \text{ kms}$$

$$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}}$$

$$\Rightarrow \quad \frac{31}{60} \times \frac{\pi}{180} = \frac{d}{360000}$$

$$\Rightarrow \quad d = \left(\frac{31}{60} \times \frac{\pi}{180} \times 360000\right) \text{km} = 3247.62 \text{ kms}$$

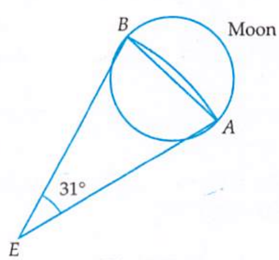


Fig. 4.16

Hence, the diameter of the moon is 3247.62 km.

EXAMPLE 18 *If the angular diameter of the moon be 30′, how far from the eye a coin of diameter 2.2 cm be kept to hide the moon ?*

SOLUTION Suppose the coin is kept at a distance r from the eye to hide the moon completely. Let E be the eye of the observer and let AB be the diameter of the coin. Then, arc AB = diameter $AB = 2.2$ cm.

We have, $\theta = 30' = \left(\frac{30}{60}\right)^{\circ} = \left(\frac{1}{2} \times \frac{\pi}{180}\right)^{c} = \left(\frac{\pi}{360}\right)^{c}$

$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}}$

$\Rightarrow \quad \frac{\pi}{360} = \frac{2.2}{r}$

$\Rightarrow \quad r = \frac{2.2 \times 360}{\pi}$ cm $\Rightarrow r = \frac{2.2 \times 360 \times 7}{22} = 252$ cm.

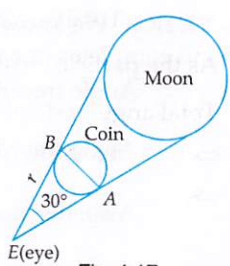


Fig. 4.17

EXAMPLE 19 *Assuming that a person of normal sight can read print at such a distance that the letters subtend an angle of 5′ at his eye, find what is the height of the letters that he can read at a distance of 12 metres.*

SOLUTION Let h be the required height in metres. Here h can be considered as the arc of a circle of radius 12 m, which subtends an angle of 5′ at its centre.

Here, $\theta = 5' = \left(\frac{5}{60}\right)^{\circ} = \left(\frac{1}{12} \times \frac{\pi}{180}\right)^{c}$ and, $r = 12$ m.

$\therefore \ \theta = \frac{\text{arc}}{\text{radius}} \Rightarrow \frac{\pi}{12 \times 180} = \frac{h}{12} \Rightarrow h = \left(\frac{\pi}{180}\right)$ metre $= 1.7$ cm.

EXAMPLE 20 *For each natural number k, let C_k denote the circle with radius k centimetres and centre at the origin. On the circle C_k, a particle moves k centimetres in the counter-clockwise direction. After completing its motion on C_k, the particle moves on C_{k+1} in the radial direction. The motion of the particle continues in this manner. The particle starts at (1, 0). If the particle crosses the positive direction of the x-axis for the first time on the circle C_n, then find the value of n.*

SOLUTION The path of the particle is shown by bold line segments and arcs. It is given that on the circle C_k of radius k centimetres the particle moves k centimeters. Therefore, angular displacement on kth circle is given by

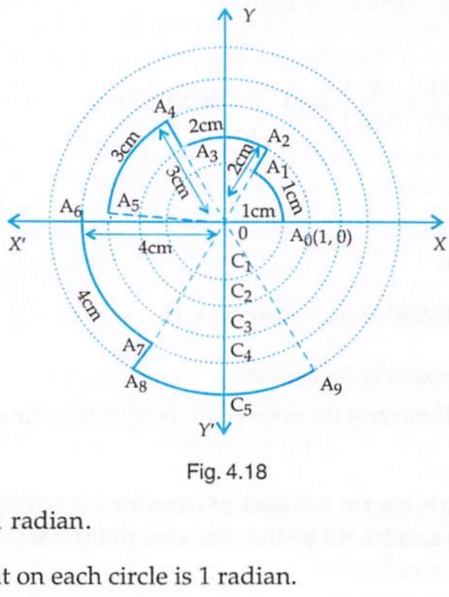


Fig. 4.18

$$\theta = \frac{k}{k} \text{ radian} = 1 \text{ radian}.$$

Thus, angular displacement on each circle is 1 radian.

If the particle crosses the x-axis for the first time on circle C_n then

Total angular displacement $= n$ radians.

As the particle crosses the positive direction of the x-axis for the first time on the n^{th} circle C_n.

Total angular displacement $> 2\pi$ radians

$\Rightarrow \quad n > 2\pi$

$\Rightarrow \quad n = 7$ $\quad$ [$\because$ n is the natural number such that $n > 2\pi$]

EXERCISE 4.1

BASIC

1. Find the degree measure corresponding to the following radian measures (Use $\pi = 22/7$:
   (i) $\frac{9\pi}{5}$ (ii) $-\frac{5\pi}{6}$ (iii) $\left(\frac{18\pi}{5}\right)^c$ (iv) $(-3)^c$ (v) 11^c (vi) 1^c
2. Find the radian measure corresponding to the following degree measures:
   (i) $300°$ (ii) $35°$ (iii) $-56°$ (iv) $135°$ (v) $-300°$
   (vi) $7° 30'$ (vii) $125° 30'$ (viii) $-47° 30'$
3. The difference between the two acute angles of a right-angled triangle is $\frac{2\pi}{5}$ radians. Express the angles in degrees.
4. One angle of a triangle is $\frac{2}{3}x$ grades and another is $\frac{3}{2}x$ degrees while the third is $\frac{\pi x}{75}$ radians. Express all the angles in degrees.
5. A rail road curve is to be laid out on a circle. What radius should be used if the track is to change direction by $25°$ in a distance of 40 metres?
6. Find the angle in radians through which a pendulum swings, if its length is 75 cm and the tip describes an arc of length (i) 10 cm (ii) 15 cm (iii) 21 cm.
7. Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm (Use $\pi = 22/7$).

BASED ON LOTS

8. The angle in one regular polygon is to that in another as 3 : 2 and the number of sides in first is twice that in the second. Determine the number of sides of two polygons.
9. The angles of a triangle are in A.P. such that the greatest is 5 times the least. Find the angles in radians.
10. The number of sides of two regular polygons are as 5 : 4 and the difference between their angles is $9°$. Find the number of sides of the polygons.
11. If the arcs of the same length in two circles subtend angles $65°$ and $110°$ at the centre, find the ratio of their radii.
12. Find the length which at a distance of 5280 m will subtend an angle of $1'$ at the eye.
13. A wheel makes 360 revolutions per minute. Through how many radians does it turn in 1 second?
14. A railway train is travelling on a circular curve of 1500 metres radius at the rate of 66 km/hr. Through what angle has it turned in 10 seconds?
15. The radius of a circle is 30 cm. Find the length of an arc of this circle, if the length of the chord of the arc is 30 cm.
16. The angles of a triangle are in *A.P.* and the number of degrees in the least angle is to the number of degrees in the mean angle as 1 : 120. Find the angles in radians.
17. Find the magnitude, in radians and degrees, of the interior angle of a regular
    (i) pentagon (ii) octagon (iii) heptagon (iv) duodecagon.

18. The angles of a quadrilateral are in A.P. and the greatest angle is 120°. Express the angles in radians.

BASED ON HOTS

19. Find the distance from the eye at which a coin of 2 cm diameter should be held so as to conceal the full moon whose angular diameter is 31′.

20. Find the diameter of the sun in km supposing that it subtends an angle of 32′ at the eye of an observer. Given that the distance of the sun is 151.92×10^6 km.

ANSWERS

1. (i) 324° (ii) −150° (iii) 648° (iv) −171° 49′ 5″
(v) 630° (vi) 57° 16′ 21″

2. (i) $\frac{5\pi}{3}$ (ii) $\frac{7\pi}{36}$ (iii) $-\frac{14\pi}{45}$ (iv) $\frac{3\pi}{4}$
(v) $-\frac{5\pi}{3}$ (vi) $\frac{\pi}{24}$ (vii) $\frac{251\pi}{360}$ (viii) $-\frac{19\pi}{72}$

3. 81°, 9° **4.** 24°, 60°, 96° **5.** 91.64 m

6. $\left(\frac{2}{15}\right)^c$ (ii) $\left(\frac{1}{5}\right)^c$ (iii) $\left(\frac{7}{25}\right)^c$ **7.** 12°, 36′ **8.** 8, 4

9. $\frac{\pi}{9}, \frac{\pi}{3}, \frac{5\pi}{9}$ **10.** 10, 8 **11.** 22 : 13 **12.** 1.5365

13. 12π **14.** $\left(\frac{11}{90}\right)^c$ **15.** 10π cm **16.** $\frac{\pi}{360}, \frac{\pi}{3}, \frac{239\pi}{360}$

17. (i) $\left(\frac{3\pi}{5}\right)^c$; 108° (ii) $\left(\frac{3\pi}{4}\right)^c$; 135° (iii) $\left(\frac{5\pi}{7}\right)^c$, 128° 34′ 17″ (iv) $\left(\frac{5\pi}{6}\right)^c$; 150°

18. $\frac{\pi}{3}, \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{2\pi}{3}$ **19.** 2.217m **20.** 1.4147×10^6 km

HINTS TO SELECTED PROBLEMS

4. $$\left(\frac{2}{3}x\right)^g = \left(\frac{2}{3}x \times \frac{90}{100}\right)^\circ = \left(\frac{3x}{5}\right)^\circ \text{ and, } \left(\frac{\pi x}{75}\right)^c = \left(\frac{\pi}{75} \times \frac{180}{\pi}\right)^c = \left(\frac{12x}{5}\right)^\circ$$

$$\therefore \quad \left(\frac{3}{5}x\right)^\circ + \left(\frac{3}{2}x\right)^\circ + \left(\frac{12x}{5}\right)^\circ = 180^\circ \Rightarrow x = 40^\circ$$

5. Here, $\theta = 25^\circ = \left(25 \times \frac{\pi}{180}\right)^c$ and arc = 40 meters.

17. A heptagon has seven sides and the number of sides of a dudecagon is twelve .

18. Let the measures of angles in degrees be $a - 3d, a - d, a + d, a + 3d$. Then,
Sum of the angles = $360^\circ \Rightarrow 4a = 360^\circ \Rightarrow a = 90^\circ$.
Also, Greatest angle = $120^\circ \Rightarrow a + 3d = 120^\circ \Rightarrow d = 10^\circ$.

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If D, G and R denote respectively the number of degrees, grades and radians in an angle, then
   (a) $\frac{D}{100} = \frac{G}{90} = \frac{2R}{\pi}$ (b) $\frac{D}{90} = \frac{G}{100} = \frac{R}{\pi}$
   (c) $\frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi}$ (d) $\frac{D}{90} = \frac{G}{100} = \frac{R}{2\pi}$
2. If the angles of a triangle are in A.P., then the measure of one of the angles in radians is
   (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$
3. The angle between the minute and hour hands of a clock at 8:30 is
   (a) 80° (b) 75° (c) 60° (d) 105°
4. At 3:40, the hour and minute hands of a clock are inclined at
   (a) $\frac{2\pi^c}{3}$ (b) $\frac{7\pi^c}{12}$ (c) $\frac{13\pi^c}{18}$ (d) $\frac{3\pi^c}{4}$
5. If the arcs of the same length in two circles subtend angles 65° and 110° at the centre, then the ratio of the radii of the circles is
   (a) 22 : 13 (b) 11 : 13 (c) 22 : 15 (d) 21 : 13
6. If OP makes 4 revolutions in one second, the angular velocity in radians per second is
   (a) π (b) 2π (c) 4π (d) 8π
7. A circular wire of radius 7 cm is cut and bent again into an arc of a circle of radius 12 cm. The angle subtended by the arc at the centre is
   (a) 50° (b) 210° (c) 100° (d) 60° (e) 195°
8. The radius of the circle whose arc of length 15 π cm makes an angle of $3\pi/4$ radian at the centre is
   (a) 10 cm (b) 20 cm (c) $11\frac{1}{4}$ cm (d) $22\frac{1}{2}$ cm

ANSWERS

1. (c) 2. (b) 3. (b) 4. (c) 5. (a) 6. (d) 7. (b) 8. (b)

SUMMARY

1. The measure of an angle is the amount of rotation from the initial side to the terminal side.
2. The sense of an angle is positive or negative according as the initial side rotates in anti-clockwise or clockwise direction to get the terminal side.
3. Three systems of measuring angles are:
   (i) Sexagesimal system (ii) Centesimal system (iii) Circular system

   In sexagesimal system:

   1 right angle = 90 degrees (= 90°)

   1° = 60 minutes (= 60′)

   1′ = 60 seconds (= 60″)

In centesimal system:

1 right angle = 100 grades $(=100^g)$

$1^g = 100$ minutes $(= 100')$

$1' = 100$ seconds $(=100'')$

In circular system, the unit of measurement is radian. One radian is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.

π radians $= 180°$

4. The relation between three systems of measurement of an angle is

$$\frac{D}{90°} = \frac{G}{100} = \frac{2R}{\pi}$$