CHAPTER 5

TRIGONOMETRIC FUNCTIONS

5.1 INTRODUCTION

In earlier classes, we have studied trigonometric ratios for acute angles as the ratio of the sides of a right angled triangle. In this chapter, we will extend the definitions of trigonometric ratios to any angle in terms of radian measure and study them as trigonometric functions.

5.2 TRIGONOMETRIC FUNCTIONS OF A REAL NUMBER

In the previous chapter, we have learnt that the radian measures of angles and real numbers can be considered as one and the same. In other words, every real number can be considered as the radian measure of an angle and radian measures of angles are real numbers. In fact, we have learnt that corresponding to every point P, representing a real number x, on the real line there is a point P' on the unit circle centred at the origin such that the radian measure of $\angle AOP'$ is x (see Fig. 4.13) and the radian measure of every angle determines a point on the real line representing a real number on the real line. So, let x be a real number represented by a point on the real line. Then there is a point P on the unit circle with centre at the origin of the coordinate axes such that the radian measure of $\angle AOP$ is x and so arc $AP = x$.

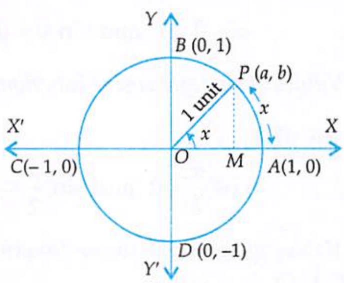

Fig. 5.1

Let the coordinates of point P be (a, b). Then, we define cosine and sine functions of radian measure (or real number) x as follows :

$$\cos x = a \quad \text{and} \quad \sin x = b$$

Thus, if x is any real number then the consine of x i.e. $\cos x$ is the x-coordinate of the point P on the unit circle such that arc $AP = x$. Similarly, sine of x i.e. $\sin x$ is the y-coordinate of point P.

REMARK 1 *In Fig. 5.1, x is the length of arc AP of the unit circle. Therefore, cos x and sin x are also known as circular functions of the real variable x.*

REMARK 2 *In Fig. 5.1, ΔOMP is a right triangle right angled at M. The trigonometric ratios $\angle MOP$ are*

$$\cos \angle MOP = \frac{OM}{OP} = \frac{a}{1} = a, \ \sin \angle MOP = \frac{PM}{OP} = \frac{b}{1} = b$$

$$\Rightarrow \quad \cos \angle AOP = a \ \text{and} \ \sin \angle AOP = b \qquad [\because \angle MOP = \angle AOP]$$

$$\Rightarrow \quad \cos \angle AOP = \cos x \ \text{and} \ \sin \angle AOP = \sin x$$

Thus, the trigonometric ratios sine and cosine of an acute angle of radian measure x are same as the corresponding trigonometric function of a real number x.

REMARK 3 *From the above definition it follows that if P is a point on the unit circle such that length of arc $AP = x$ or equivalently P is a point where the terminal side of the angle with radian measure x meets the unit circle, then the coordinates of the point P are $(\cos x, \sin x)$.*

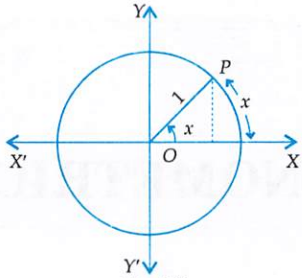

Fig. 5.2

5.2.1 VALUES OF SINE AND COSINE FUNCTIONS

Consider a unit circle with centre at the origin of the coordinate axes. Suppose the circle cuts the coordinate axes at A, B, C and D. The coordinates of these points are $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$. Clearly, $\angle AOB = \frac{\pi}{2}$, $\angle AOC = \pi$ and $\angle AOD = \frac{3\pi}{2}$.

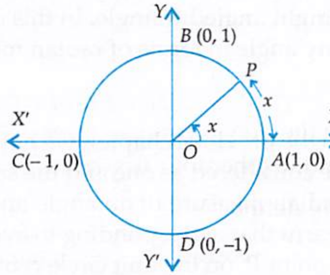

Fig. 5.3

We shall now find the values of sine and cosine functions at $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π.

Values of sine and cosine functions at $x = 0$: When $x = 0$, point P coincides with A and the terminal side OP coincides with OA. The coordinates of A are $(1, 0)$.

$\therefore \quad \cos 0 = 1$ and $\sin 0 = 0$

Values of sine and cosine functions at $x = \frac{\pi}{2}$: We observe that $\angle AOB = \frac{\pi}{2}$ and the coordinates of B are $(0, 1)$.

$\therefore \quad \cos \frac{\pi}{2} = 0$ and $\sin \frac{\pi}{2} = 1$.

Values of sine and cosine functions at $x = \pi$: Clearly, $\angle AOC = \pi$ and the coordinates of C are $(-1, 0)$.

$\therefore \quad \cos \pi = -1$ and $\sin \pi = 0$

Values of sine and cosine functions at $x = \frac{3\pi}{2}$: The coordinates of point D are $(0, -1)$ and $\angle AOC = \frac{3\pi}{2}$.

$\therefore \quad \cos \frac{3\pi}{2} = 0$ and $\sin \frac{3\pi}{2} = -1$

Values of sine and cosine functions at $x = 2\pi$: The coordinates of point A are $(1, 0)$ and one complete revolution subtends an angle of measure 2π at the centre O.

$\therefore \quad \cos 2\pi = 1$ and $\sin 2\pi = 0$

If the terminal side OP of $\angle AOP$ takes one complete revolution from the position OP, it again comes back to the same position.

$\therefore \quad \cos (2\pi + x) = \cos x$ and $\sin (2\pi + x) = \sin x$ for all $x \in R$

We also observe that if the terminal side OP of $\angle AOP$ takes any number of complete revolutions in anticlockwise or clockwise directions, it again comes back to the same position.

$\therefore \quad \cos(2n\pi + x) = \cos x$ and $\sin(2n\pi + x) = \sin x$ for all $x \in R$ and $n \in Z$.

It is evident from Fig. 5.2 that

$\sin 0 = 0, \sin \pi = 0, \sin 2\pi = 0, \sin 3\pi = 0\ldots\ldots\ldots$

Also, $\sin(-\pi) = 0, \sin(-2\pi) = 0, \sin(-3\pi) = 0\ldots\ldots\ldots$

$\therefore \quad \sin n\pi = 0$ for all $n \in Z$

and, $\cos\frac{\pi}{2} = 0,\ \cos\frac{3\pi}{2} = 0, \cos\frac{5\pi}{2} = 0\ldots\ldots\ldots\ldots$

$\cos\left(-\frac{\pi}{2}\right) = 0,\ \cos\left(-\frac{3\pi}{2}\right) = 0, \cos\left(-\frac{5\pi}{2}\right) = 0\ldots\ldots\ldots\ldots$

$\therefore \quad \cos(2n+1)\frac{\pi}{2} = 0$, for all $n \in Z$.

Thus, $\sin x = 0 \Rightarrow x = n\pi, n \in Z$ and, $\cos x = 0 \Rightarrow x = (2n+1)\frac{\pi}{2}, n \in Z$

5.2.2 OTHER TRIGONOMETRIC FUNCTIONS

In the previous subsections, we have defined sine and cosine functions. In this section, we shall define other four trigonometric functions in terms of these two functions.

We define

$$\operatorname{cosec} x = \frac{1}{\sin x}, \text{ where } x \neq n\pi, n \in Z\ ; \sec x = \frac{1}{\cos x}, \text{ where } x \neq (2n+1)\frac{\pi}{2}, n \in Z$$

$$\tan x = \frac{\sin x}{\cos x}, \text{ where } x \neq (2n+1)\frac{\pi}{2}, n \in Z\ ;\ \cot x = \frac{\cos x}{\sin x}, \text{ where } x \neq n\pi, n \in Z$$

5.3 VALUES OF TRIGONOMETRIC FUNCTIONS

In this section, we will find the values of trigonometric functions for $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ and $\frac{\pi}{2}$. In section 5.2.1, we have learnt that $\sin 0 = 0, \cos 0 = 1, \sin\frac{\pi}{2} = 1,\ \cos\frac{\pi}{2} = 0, \sin\pi = 0,\ \cos\pi = -1,$ $\sin\frac{3\pi}{2} = -1, \sin 2\pi = 0$ and $\cos 2\pi = 1$.

We have also learnt that $\operatorname{cosec} x = \frac{1}{\sin x}$, where $n \neq n\pi, n \in Z$. Therefore, cosec 0, cosec π and cosec 2π are not defined.

$$\operatorname{cosec}\frac{\pi}{2} = \frac{1}{\sin\frac{\pi}{2}} = 1 \text{ and } \operatorname{cosec}\frac{3\pi}{2} = \frac{1}{\sin\frac{3\pi}{2}} = -1$$

Similarly, $\sec x = \frac{1}{\cos x}$ implies that

$\sec 0 = \frac{1}{\cos 0} = 1,\ \sec\pi = \frac{1}{\cos\pi} = -1, \sec 2\pi = \frac{1}{\cos 2\pi} = 1$ and, $\sec\frac{\pi}{2} = \sec\frac{3\pi}{2}$ are not defined.

$\tan 0 = \frac{\sin x}{\cos x}$ implies that

$\tan 0 = \frac{\sin 0}{\cos 0} = 0, \tan\pi = \frac{\sin\pi}{\cos\pi} = 0, \tan 2\pi = \frac{\sin 2\pi}{\cos 2\pi} = 0$ and $\tan\frac{\pi}{2}, \tan\frac{3\pi}{2}$ are undefined.

$\cot x = \frac{\cos x}{\sin x}$ implies that

$$\cot\frac{\pi}{2}=\frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}}=0,\ \cot\frac{3\pi}{2}=\frac{\cos\frac{3\pi}{2}}{\sin\frac{3\pi}{2}}=0 \text{ and } \cot 0, \cot\pi, \cot 2\pi \text{ are not defined.}$$

Let us now find the values of all trigonometric functions at $\frac{\pi}{6}, \frac{\pi}{4}$ and $\frac{\pi}{3}$.

Values of trigonometric functions at $\frac{\pi}{4}$: Consider a unit circle with centre at the origin of the coordinate axes. Let P be a point on the circle such that $\angle XOP=\frac{\pi}{4}$. Draw PM perpendicular from P on OX. In right triangle OMP, we have $\angle POM=\frac{\pi}{4}$. Therefore, $\angle OPM=\frac{\pi}{4}$. Thus, we obtain

$$\angle POM=\angle OPM=\frac{\pi}{4} \Rightarrow OM=PM$$

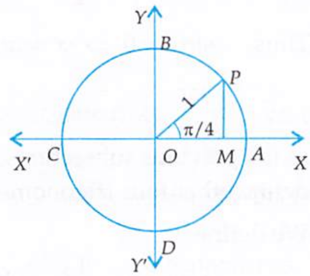

Fig. 5.4

Applying Pythagoras theorem in ΔOMP, we obtain

$$OM^2+PM^2=OP^2$$

$$\Rightarrow \quad 2\,OM^2=1 \qquad [\because OM=PM \text{ and } OP=1]$$

$$\Rightarrow \quad OM=\frac{1}{\sqrt{2}} \Rightarrow OM=PM=\frac{1}{\sqrt{2}}$$

So, the coordinates of P are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Hence, $\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$ and $\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$.

Now,

$$\operatorname{cosec} x=\frac{1}{\sin x} \Rightarrow \operatorname{cosec}\frac{\pi}{4}=\frac{1}{\sin\frac{\pi}{4}}=\sqrt{2}\,;\ \sec x=\frac{1}{\cos x} \Rightarrow \sec\frac{\pi}{4}=\frac{1}{\cos\frac{\pi}{4}}=\sqrt{2}$$

$$\tan x=\frac{\sin x}{\cos x} \Rightarrow \tan\frac{\pi}{4}=\frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}}=1\,;\ \cot x=\frac{\cos x}{\sin x} \Rightarrow \cot\frac{\pi}{4}=\frac{\cos\frac{\pi}{4}}{\cos\frac{\pi}{4}}=1$$

Values of trigonometric functions at $\frac{\pi}{3}$: Consider a unit circle with centre at the origin of the coordinate axes. Let P be a point on the circle such that $\angle XOP=\frac{\pi}{3}$. Join PA. In ΔOAP, we have $OA=OP=1$ Unit. Therefore, $\angle OPA=\angle OAP$. But, $\angle AOP=\frac{\pi}{3}$.

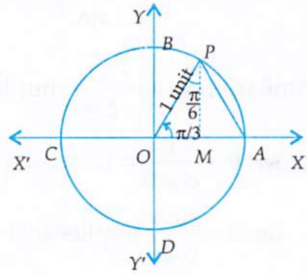

Fig. 5.5

Using angle sum property in ΔOAP, we obtain

$$\angle OAP+\angle OPA+\angle AOP=\pi \Rightarrow \frac{\pi}{3}+2\,\angle OPA=\pi \Rightarrow \angle OPA=\frac{\pi}{3}$$

Thus, in ΔOAP, we obtain: $\angle OAP=\angle OPA=\angle AOP=\frac{\pi}{3}$

So, ΔOAP is an equilateral triangle and hence perpendicular PM drawn from vertex P to the opposite side OA bisects it.

$$\therefore \quad OM=AM=\frac{1}{2}\text{ unit}$$

Applying Pythagoras theorem in ΔOMP, we obtain

$$OP^2 = OM^2 + MP^2 \Rightarrow 1^2 = \left(\frac{1}{2}\right)^2 + MP^2 \Rightarrow PM = \frac{\sqrt{3}}{2}$$

So, the coordinates of P are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. But, $\angle AOP = \frac{\pi}{3}$. So, the coordinates of P are $\left(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right)$.

$$\therefore \quad \cos\frac{\pi}{3} = \frac{1}{2} \text{ and } \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

Now, $\operatorname{cosec} x = \frac{1}{\sin x} \Rightarrow \operatorname{cosec}\frac{\pi}{3} = \frac{1}{\sin\frac{\pi}{3}} = \frac{2}{\sqrt{3}}$; $\sec x = \frac{1}{\cos x} \Rightarrow \sec\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{3}} = 2$

$$\tan x = \frac{\sin x}{\cos x} \Rightarrow \tan\frac{\pi}{3} = \frac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}} = \sqrt{3}; \ \cot x = \frac{\cos x}{\sin x} \Rightarrow \cot\frac{\pi}{3} = \frac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}} = \frac{1}{\sqrt{3}}$$

Values of trigonometric functions at $\frac{\pi}{6}$: Consider a unit circle with centre at the origin of the coordinate axes. Let P be a point on the circle such that $\angle XOP = \frac{\pi}{6}$. Draw PM perpendicular from P on OX.

In right triangle OMP right angled at M, we have $\angle POM = \frac{\pi}{6}$.

Therefore, $\angle OPM = \frac{\pi}{3}$.

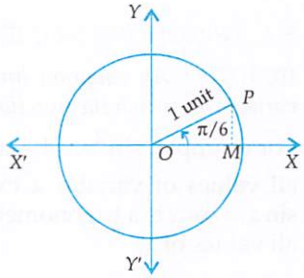

Fig. 5.6

In Fig. 5.6, we have seen that in a right triangle if the measures of angles other than the right angle are $\frac{\pi}{3}$ and $\frac{\pi}{6}$, then the sides opposite to them are of length $\frac{\sqrt{3}}{2}$ and $\frac{1}{2}$ respectively. Thus, in ΔOMP, we obtain: $OM = \frac{\sqrt{3}}{2}$ and $PM = \frac{1}{2}$

So, the coordinates of P are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. But, the coordinates of P are $\left(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\right)$.

$$\therefore \quad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \text{ and } \sin\frac{\pi}{6} = \frac{1}{2}$$

Now,

$$\operatorname{cosec} x = \frac{1}{\sin x} \Rightarrow \operatorname{cosec}\frac{\pi}{6} = \frac{1}{\sin\frac{\pi}{6}} = 2; \ \sec x = \frac{1}{\cos x} \Rightarrow \sec\frac{\pi}{6} = \frac{1}{\cos\frac{\pi}{6}} = \frac{2}{\sqrt{3}}$$

$$\tan x = \frac{\sin x}{\cos x} \Rightarrow \tan\frac{\pi}{6} = \frac{\sin\frac{\pi}{6}}{\sin\frac{\pi}{6}} = \frac{1}{\sqrt{3}}; \ \cot x = \frac{\cos x}{\sin x} \Rightarrow \cot\frac{\pi}{6} = \frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}} = \sqrt{3}$$

In the above discussion, we have obtained the values of various trigonometric functions for $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π. These values are listed below for ready reference.

Angle / Trigonometric Function	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	0	not defined	0
cosec	not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	not defined	-1	not defined
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined	-1	not defined	1
cot	not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	not defined	0	not defined

5.4 TRIGONOMETRIC IDENTITIES

IDENTITY *An equation involving trigonometric functions which is true for all these values of the variable for which the functions are defined is called a trigonometric identity.*

For example, $\sin^2 x = 1 - \cos^2 x$, $\sec^2 x - 1 = \tan^2 x$ are trigonometric identities as they hold for all values of variable x except those values for which $\sec x$ and $\tan x$ are not defined. But, $\sin x = \cos x$ is a trigonometric equation not a trigonometric identity because it does not hold for all values of x.

5.4.1 FUNDAMENTAL TRIGONOMETRIC IDENTITIES

In this section, we shall state and prove three fundamental trigonometric identities as a theorem.

THEOREM *Prove that:*

(i) $\cos^2 x + \sin^2 x = 1$ *for all* $x \in R$

(ii) $1 + \tan^2 x = \sec^2 x$ *for all* $x \in R - \left\{(2n-1)\frac{\pi}{2} : n \in Z\right\}$

(iii) $1 + \cot^2 x = \text{cosec}^2 x$ *for all* $x \in R - \{n\pi : n \in Z\}$

PROOF (i) Consider a unit circle with centre at the origin O of coordinates axes. Let $P(a, b)$ be a point on the circle such that arc $AP = x$. Then, $\angle AOP = x$. Using the definition of trigonometric functions $\cos x$ and $\sin x$, we obtain: $a = \cos x$ and $b = \sin x$

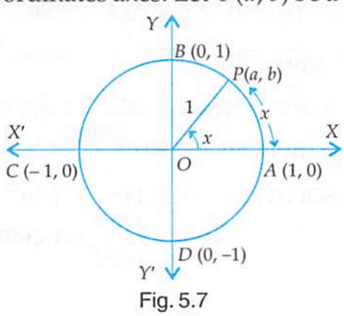

Fig. 5.7

Now,

$$OP = 1$$

$$\Rightarrow \quad \sqrt{(a-0)^2 + (b-0)^2} = 1$$

$$\Rightarrow \quad a^2 + b^2 = 1 \Rightarrow \cos^2 x + \sin^2 x = 1$$

(ii) If $x \neq (2n-1)\frac{\pi}{2}$, then $\cos x \neq 0$. So, dividing the identity $\cos^2 x + \sin^2 x = 1$ throughout by $\cos^2 x$, we obtain

$$\frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \quad \text{for all } x \neq (2n-1)\frac{\pi}{2}$$

$$\Rightarrow \quad \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \Rightarrow 1 + \tan^2 x = \sec^2 x \quad \text{for all } x \neq (2n-1)\frac{\pi}{2}$$

(iii) If $x \neq n\pi$, then $\sin x \neq 0$. So, dividing both sides of the identity $\cos^2 x + \sin^2 x = 1$ by $\sin^2 x$, we obtain

$$\frac{\cos^2 x + \sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \quad \text{for all } x \neq n\pi$$

$$\Rightarrow \quad \frac{\cos^2 x}{\sin^2 x} + \frac{\sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \Rightarrow \cot^2 x + 1 = \operatorname{cosec}^2 x \Rightarrow 1 + \cot^2 x = \operatorname{cosec}^2 x \text{ for all } x \neq n\pi$$

Q.E.D

REMARK 1 *The identity* $1 + \tan^2 x = \sec^2 x$ *is also written in the following forms :*

$$\sec^2 x - 1 = \tan^2 x \quad \text{and} \quad \sec^2 x - \tan^2 x = 1$$

REMARK 2 *The identity* $1 + \cot^2 x = \operatorname{cosec}^2 x$ *is also written in the following forms :*

$$\operatorname{cosec}^2 x - 1 = \cot^2 x \quad \text{and,} \quad \operatorname{cosec}^2 x - \cot^2 x = 1$$

REMARK 3 *We have,* $\sec^2 x - \tan^2 x = 1$

$$\Rightarrow (\sec x + \tan x)(\sec x - \tan x) = 1 \Rightarrow \sec x - \tan x = \frac{1}{\sec x + \tan x} \text{ and, } \sec x + \tan x = \frac{1}{\sec x - \tan x}$$

i.e. $\sec x + \tan x$ and $\sec x - \tan x$ are reciprocal of each other.

REMARK 4 *We have,* $\operatorname{cosec}^2 x - \cot^2 x = 1$

$$\Rightarrow (\operatorname{cosec} x - \cot x)(\operatorname{cosec} x + \cot x) = 1$$

$$\Rightarrow \operatorname{cosec} x - \cot x = \frac{1}{\operatorname{cosec} x + \cot x} \text{ and } \operatorname{cosec} x + \cot x = \frac{1}{\operatorname{cosec} x - \cot x}$$

i.e. $\operatorname{cosec} x + \cot x$ and $\operatorname{cosec} x - \cot x$ are reciprocal of each other.

We shall now discuss more identities involving trigonometric functions in the following examples.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Prove the following identities:*

(i) $\sin^8 x - \cos^8 x = (\sin^2 x - \cos^2 x)(1 - 2\sin^2 x \cos^2 x)$

(ii) $\cot^4 x + \cot^2 x = \operatorname{cosec}^4 x - \operatorname{cosec}^2 x$

(iii) $2\sec^2 x - \sec^4 x - 2\operatorname{cosec}^2 x + \operatorname{cosec}^4 x = \cot^4 x - \tan^4 x$

(iv) $(\sin x + \operatorname{cosec} x)^2 + (\cos x + \sec x)^2 = \tan^2 x + \cot^2 x + 7$

SOLUTION (i) LHS $= (\sin^8 x - \cos^8 x) = (\sin^4 x)^2 - (\cos^4 x)^2$

$$= (\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)$$

$$= (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x)$$

$$= (\sin^2 x - \cos^2 x)(\sin^4 x + \cos^4 x)$$
$$= (\sin^2 x - \cos^2 x)(\sin^4 x + \cos^4 x + 2\sin^2 x\cos^2 x - 2\sin^2 x\cos^2 x)$$
$$= (\sin^2 x - \cos^2 x)\left\{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x\right\}$$
$$= (\sin^2 x - \cos^2 x)(1 - 2\sin^2 x\cos^2 x) = \text{RHS}$$

(ii) $\text{LHS} = \cot^4 x + \cot^2 x = (\cot^2 x)^2 + \cot^2 x$
$$= (\operatorname{cosec}^2 x - 1)^2 + (\operatorname{cosec}^2 x - 1) \quad [\because 1 + \cot^2 x = \operatorname{cosec}^2 x]$$
$$= \operatorname{cosec}^4 x - 2\operatorname{cosec}^2 x + 1 + \operatorname{cosec}^2 x - 1 = \operatorname{cosec}^4 x - \operatorname{cosec}^2 x = \text{RHS}$$

(iii) $\text{LHS} = 2\sec^2 x - \sec^4 x - 2\operatorname{cosec}^2 x + \operatorname{cosec}^4 x$
$$= 2\sec^2 x - (\sec^2 x)^2 - 2\operatorname{cosec}^2 x + (\operatorname{cosec}^2 x)^2$$
$$= 2(1 + \tan^2 x) - (1 + \tan^2 x)^2 - 2(1 + \cot^2 x) + (\cot^2 x + 1)^2$$
$$= 2 + 2\tan^2 x - (1 + \tan^4 x + 2\tan^2 x) - 2 - 2\cot^2 x + (\cot^4 x + 2\cot^2 x + 1)$$
$$= \cot^4 x - \tan^4 x = \text{RHS}$$

(iv) $\text{LHS} = (\sin x + \operatorname{cosec} x)^2 + (\cos x + \sec x)^2$
$$= \sin^2 x + \operatorname{cosec}^2 x + 2\sin x\operatorname{cosec} x + \cos^2 x + \sec^2 x + 2\cos x\sec x$$
$$= (\sin^2 x + \cos^2 x) + (\operatorname{cosec}^2 x + \sec^2 x) + 2 + 2$$
$$= 1 + (1 + \cot^2 x) + (1 + \tan^2 x) + 4 = \tan^2 x + \cot^2 x + 7 = \text{RHS}$$

EXAMPLE 2 *Prove the following identities:*

(i) $(1 + \cot x - \operatorname{cosec} x)(1 + \tan x + \sec x) = 2$ (ii) $\dfrac{\tan x + \sec x - 1}{\tan x - \sec x + 1} = \dfrac{1 + \sin x}{\cos x}$

[NCERT EXEMPLAR]

SOLUTION (i) $\text{LHS} = (1 + \cot x - \operatorname{cosec} x)(1 + \tan x + \sec x)$
$$= \left(1 + \frac{\cos x}{\sin x} - \frac{1}{\sin x}\right)\left(1 + \frac{\sin x}{\cos x} + \frac{1}{\cos x}\right) = \frac{(\sin x + \cos x - 1)}{\sin x}\frac{(\sin x + \cos x + 1)}{\cos x}$$
$$= \frac{(\sin x + \cos x)^2 - 1}{\sin x\cos x} = \frac{\sin^2 x + \cos^2 x + 2\sin x\cos x - 1}{\sin x\cos x}$$
$$= \frac{2\sin x\cos x}{\sin x\cos x} = 2 = \text{RHS}$$

(ii)
$$\text{LHS} = \frac{\tan x + \sec x - 1}{\tan x - \sec x + 1} = \frac{(\tan x + \sec x) - (\sec^2 x - \tan^2 x)}{\tan x - \sec x + 1} \quad [\because \sec^2 x - \tan^2 x = 1]$$
$$= \frac{(\sec x + \tan x)\{1 - (\sec x - \tan x)\}}{\tan x - \sec x + 1} = \frac{(\sec x + \tan x)(\tan x - \sec x + 1)}{\tan x - \sec x + 1}$$
$$= \sec x + \tan x = \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \frac{1 + \sin x}{\cos x} = \text{RHS}$$

EXAMPLE 3 *If* $\tan x + \sin x = m$ *and* $\tan x - \sin x = n$, *show that* $m^2 - n^2 = 4\sqrt{mn}$.

[NCERT EXEMPLAR]

SOLUTION We have, $\tan x + \sin x = m$ and $\tan x - \sin x = n$.

$$\therefore \quad m^2 - n^2 = (\tan x + \sin x)^2 - (\tan x - \sin x)^2 = 4\tan x\sin x \quad \text{...(i)}$$

and, $4\sqrt{mn} = 4\sqrt{(\tan x + \sin x)(\tan x - \sin x)} = 4\sqrt{\tan^2 x - \sin^2 x}$

$$= 4\sqrt{\frac{\sin^2 x}{\cos^2 x} - \sin^2 x} = 4\sqrt{\frac{\sin^2 x - \sin^2 x\cos^2 x}{\cos^2 x}} = 4\sqrt{\frac{\sin^2 x(1-\cos^2 x)}{\cos^2 x}}$$

$$= 4\sqrt{\frac{\sin^4 x}{\cos^2 x}} = 4\frac{\sin^2 x}{\cos x} = 4\tan x \sin x \qquad \text{...(ii)}$$

From (i) and (ii), we obtain: $m^2 - n^2 = 4\sqrt{mn}$.

EXAMPLE 4 *If* $\cos x + \sin x = \sqrt{2}\cos x$, *prove that* $\cos x - \sin x = \sqrt{2}\sin x$.

SOLUTION We have,

$$\cos x + \sin x = \sqrt{2}\cos x$$

$\Rightarrow \quad (\cos x + \sin x)^2 = (\sqrt{2}\cos x)^2 \Rightarrow \cos^2 x + \sin^2 x + 2\sin x\cos x = 2\cos^2 x$

$\Rightarrow \quad \cos^2 x - \sin^2 x = 2\sin x\cos x \Rightarrow (\cos x + \sin x)(\cos x - \sin x) = 2\sin x\cos x$

$\Rightarrow \quad \cos x - \sin x = \dfrac{2\sin x\cos x}{\cos x + \sin x} \Rightarrow \cos x - \sin x = \dfrac{2\sin x\cos x}{\sqrt{2}\cos x} \quad [\because \cos x + \sin x = \sqrt{2}\cos x]$

$\Rightarrow \quad \cos x - \sin x = \sqrt{2}\sin x$

ALITER We know that

$$(\cos x + \sin x)^2 + (\cos x - \sin x)^2 = 2$$

$\Rightarrow \quad (\sqrt{2}\cos x)^2 + (\cos x - \sin x)^2 = 2 \quad [\because \cos x + \sin x = \sqrt{2}\cos x]$

$\Rightarrow \quad (\cos x - \sin x)^2 = 2 - 2\cos^2 x \Rightarrow (\cos x - \sin x)^2 = 2\sin^2 x \Rightarrow \cos x - \sin x = \sqrt{2}\sin x.$

EXAMPLE 5 *If* $a\cos x + b\sin x = m$ *and* $a\sin x - b\cos x = n$, *prove that* $a^2 + b^2 = m^2 + n^2$.

[NCERT EXEMPLAR]

SOLUTION We have, $m = a\cos x + b\sin x$ and $n = a\sin x - b\cos x$.

$\therefore \quad m^2 + n^2 = (a\cos x + b\sin x)^2 + (a\sin x - b\cos x)^2$

$$= (a^2\cos^2 x + b^2\sin^2 x + 2ab\sin x\cos x) + (a^2\sin^2 x + b^2\cos^2 x - 2ab\sin x\cos x)$$

$$= a^2(\cos^2 x + \sin^2 x) + b^2(\sin^2 x + \cos^2 x) = a^2 + b^2$$

EXAMPLE 6 *If* $a\cos x - b\sin x = c$, *show that* $a\sin x + b\cos x = \pm\sqrt{a^2 + b^2 - c^2}$.

SOLUTION Clearly,

$$(a\cos x - b\sin x)^2 + (a\sin x + b\cos x)^2$$

$$= a^2(\cos^2 x + \sin^2 x) + b^2(\sin^2 x + \cos^2 x) - 2ab\sin x\cos x + 2ab\sin x\cos x = a^2 + b^2$$

$\therefore \quad (a\sin x + b\cos x)^2 = a^2 + b^2 - (a\cos x - b\sin x)^2$

$\Rightarrow \quad (a\sin x + b\cos x)^2 = a^2 + b^2 - c^2 \quad [\because a\cos x - b\sin x = c]$

$\Rightarrow \quad a\sin x + b\cos x = \pm\sqrt{a^2 + b^2 - c^2}$

EXAMPLE 7 *If* $\sec x + \tan x = p$, *obtain the values of* $\sec x$, $\tan x$ *and* $\sin x$ *in terms of* p.

SOLUTION We know that : $\sec x + \tan x$ and $\sec x - \tan x$ are reciprocal of each other.

$\therefore \quad \sec x + \tan x = p \Rightarrow \sec x - \tan x = \dfrac{1}{p}$

$\Rightarrow \quad (\sec x + \tan x) + (\sec x - \tan x) = p + \dfrac{1}{p}$ and, $(\sec x + \tan x) - (\sec x - \tan x) = p - \dfrac{1}{p}$

$$\Rightarrow \quad 2\sec x = p + \frac{1}{p} \text{ and, } 2\tan x = p - \frac{1}{p} \Rightarrow \sec x = \frac{p^2+1}{2p} \text{ and, } \tan x = \frac{p^2-1}{2p}$$

$$\therefore \quad \sin x = \frac{\tan x}{\sec x} \Rightarrow \sin x = \frac{p^2-1}{p^2+1}$$

EXAMPLE 8 *Prove that:* $2\sec^2 x - \sec^4 x - 2\operatorname{cosec}^2 x + \operatorname{cosec}^4 x = \dfrac{1-\tan^8 x}{\tan^4 x}$.

SOLUTION $2\sec^2 x - \sec^4 x - 2\operatorname{cosec}^2 x + \operatorname{cosec}^4 x$

$$= 2(1+\tan^2 x) - (1+\tan^2 x)^2 - 2(1+\cot^2 x) + (1+\cot^2 x)^2$$

$$= 2(1+\tan^2 x - 1 - \cot^2 x) + (1 + 2\cot^2 x + \cot^4 x) - (1 + 2\tan^2 x + \tan^4 x)$$

$$= 2(\tan^2 x - \cot^2 x) + (2\cot^2 x - 2\tan^2 x) + \cot^4 x - \tan^4 x$$

$$= \cot^4 x - \tan^4 x = \frac{1}{\tan^4 x} - \tan^4 x = \frac{1-\tan^8 x}{\tan^4 x}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 9 *Prove that:* $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) - 13 = 0$.

SOLUTION We have,

$$3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) - 13$$

$$= 3\left\{(\sin x - \cos x)^2\right\}^2 + 6(\sin x + \cos x)^2$$
$$+ 4\left\{(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x)\right\} - 13$$

$$= 3(1 - 2\sin x\cos x)^2 + 6(1 + 2\sin x\cos x) + 4(1 - 3\sin^2 x\cos^2 x) - 13$$

$$= 3(1 - 4\sin x\cos x + 4\sin^2 x\cos^2 x) + 6(1 + 2\sin x\cos x) + 4(1 - 3\sin^2 x\cos^2 x) - 13$$

$$= 3 + 6 + 4 - 13 = 0$$

EXAMPLE 10 *Given that:* $(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma) = (1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)$. *Show that one of the values of each member of this equality is* $\sin\alpha\sin\beta\sin\gamma$.

SOLUTION We have,

$$(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma) = (1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)$$

Multiplying both sides by $(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)$, we get

$$(1+\cos\alpha)^2(1+\cos\beta)^2(1+\cos\gamma)^2 = (1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)$$
$$(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)$$

$$\Rightarrow \quad (1+\cos\alpha)^2(1+\cos\beta)^2(1+\cos\gamma)^2 = (1-\cos^2\alpha)(1-\cos^2\beta)(1-\cos^2\gamma)$$

$$\Rightarrow \quad (1+\cos\alpha)^2(1+\cos\beta)^2(1+\cos\gamma)^2 = \sin^2\alpha\sin^2\beta\sin^2\gamma$$

$$\Rightarrow \quad (1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma) = \pm\sin\alpha\sin\beta\sin\gamma$$

Hence, one of the values of $(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)$ is $\sin\alpha\sin\beta\sin\gamma$.

Similarly, by multiplying both sides by $(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)$, we find that one of the values of $(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)$ is also $\sin\alpha\sin\beta\sin\gamma$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 11 *Prove that:* $\sec^2 x + \operatorname{cosec}^2 x \geq 4$.

SOLUTION
$$\sec^2 x + \operatorname{cosec}^2 x = (1 + \tan^2 x) + 1 + (\cot^2 x) = 2 + \tan^2 x + \cot^2 x$$
$$= 2 + \tan^2 x + \cot^2 x - 2 \tan x \cot x + 2 \tan x \cot x$$
$$= 2 + (\tan x - \cot x)^2 + 2$$
$$= 4 + (\tan x - \cot x)^2 \geq 4 \qquad [\because (\tan x - \cot x)^2 \geq 0]$$

EXAMPLE 12 *If* $10 \sin^4 \alpha + 15 \cos^4 \alpha = 6$, *find the value of* $27 \operatorname{cosec}^6 \alpha + 8 \sec^6 \alpha$.

SOLUTION We have,
$$10 \sin^4 \alpha + 15 \cos^4 \alpha = 6$$
$$\Rightarrow \quad 10 \sin^4 \alpha + 15 \cos^4 \alpha = 6 (\sin^2 \alpha + \cos^2 \alpha)^2$$
$$\Rightarrow \quad 10 \tan^4 \alpha + 15 = 6 (\tan^2 \alpha + 1)^2 \qquad [\text{Dividing boths sides by } \cos^4 \alpha]$$
$$\Rightarrow \quad (2 \tan^2 \alpha - 3)^2 = 0 \Rightarrow \tan^2 \alpha = \frac{3}{2}$$
$$\therefore \quad 27 \operatorname{cosec}^6 \alpha + 8 \sec^6 \alpha = 27 (1 + \cot^2 \alpha)^3 + 8 (1 + \tan^2 \alpha)^3$$
$$= 27 \left(1 + \frac{2}{3}\right)^3 + 8 \left(1 + \frac{3}{2}\right)^3 = 27 \times \frac{125}{27} + 8 \times \frac{125}{8} = 250.$$

EXAMPLE 13 *If* $\dfrac{\sin A}{\sin B} = p$ *and* $\dfrac{\cos A}{\cos B} = q$, *find* $\tan A$ *and* $\tan B$.

SOLUTION We have,
$$\frac{\sin A}{\sin B} = p \text{ and } \frac{\cos A}{\cos B} = q \Rightarrow \frac{\sin A}{\sin B} \cdot \frac{\cos B}{\cos A} = \frac{p}{q}$$
$$\Rightarrow \quad \frac{\tan A}{\tan B} = \frac{p}{q} \Rightarrow \frac{\tan A}{p} = \frac{\tan B}{q} = \lambda \text{ (say)} \Rightarrow \tan A = p\lambda \text{ and } \tan B = q\lambda \qquad \text{...(i)}$$
Now, $\sin A = p \sin B$
$$\Rightarrow \quad \frac{\tan A}{\sqrt{1 + \tan^2 A}} = p \frac{\tan B}{\sqrt{1 + \tan^2 B}}$$
$$\Rightarrow \quad \frac{p\lambda}{\sqrt{1 + p^2 \lambda^2}} = p \frac{q\lambda}{\sqrt{1 + q^2 \lambda^2}} \Rightarrow p^2 (1 + q^2 \lambda^2) = p^2 q^2 (1 + p^2 \lambda^2)$$
$$\Rightarrow \quad \lambda^2 (q^2 - p^2 q^2) = q^2 - 1 \Rightarrow \lambda^2 = \frac{q^2 - 1}{q^2 (1 - p^2)} \Rightarrow \lambda = \pm \frac{1}{q} \sqrt{\frac{q^2 - 1}{1 - p^2}}$$
$$\therefore \quad \tan A = \pm \frac{p}{q} \sqrt{\frac{q^2 - 1}{1 - p^2}} \text{ and, } \tan B = \pm \sqrt{\frac{q^2 - 1}{1 - p^2}} \qquad [\text{Using (i)}]$$

EXAMPLE 14 *If* $\tan^2 x = 1 - a^2$, *prove that* $\sec x + \tan^3 x \operatorname{cosec} x = (2 - a^2)^{3/2}$. *Also, find the values of a for which the above result holds true.*

SOLUTION We have,

$$\sec x + \tan^3 x \operatorname{cosec} x = \sec x \left\{1 + \tan^3 x \frac{\operatorname{cosec} x}{\sec x}\right\}$$

$$= \sqrt{1+\tan^2 x}\left\{1 + \tan^3 x \times \cot x\right\}$$

$$= (1+\tan^2 x)^{3/2} = (1+1-a^2)^{3/2} = (2-a^2)^{3/2} \qquad [\because \tan^2 x = 1-a^2]$$

Now,

$$\tan^2 x \geq 0 \text{ for all } x \Rightarrow 1-a^2 \geq 0 \Rightarrow a^2 - 1 \leq 0 \Rightarrow -1 \leq a \leq 1 \qquad \text{...(i)}$$

Since LHS of $\sec x + \tan^3 x \operatorname{cosec} x = (2-a^2)^{3/2}$ is real for all $x \in R$. So, RHS must also be real.

$$\therefore \quad 2 - a^2 \geq 0 \Rightarrow a^2 - 2 \leq 0 \Rightarrow -\sqrt{2} \leq a \leq \sqrt{2} \qquad \text{...(ii)}$$

From (i) and (ii), we find that the given relation holds true for all $a \in [-1, 1]$.

EXAMPLE 15 *If $a\cos^3 x + 3a\cos x \sin^2 x = m$ and $a \sin^3 x + 3a\cos^2 x \sin x = n$, then prove that: $(m+n)^{2/3} + (m-n)^{2/3} = 2a^{2/3}$.*

SOLUTION We have,

$$a\cos^3 x + 3a\cos x\sin^2 x = m \quad \text{and} \quad a\sin^3 x + 3a\cos^2 x \sin x = n$$

$$\Rightarrow \quad a\cos^3 x + 3a\cos x \sin^2 x + a\sin^3 x + 3a\cos^2 x\sin x = m+n$$

and, $$a\cos^3 x + 3a\cos x\sin^2 x - a\sin^3 x - 3a\cos^2 x \sin x = m-n$$

$$\Rightarrow \quad a(\cos x + \sin x)^3 = m+n \quad \text{and,} \quad a(\cos x - \sin x)^3 = m-n$$

$$\Rightarrow \quad \cos x + \sin x = \left(\frac{m+n}{a}\right)^{1/3} \quad \text{and,} \quad \cos x - \sin x = \left(\frac{m-n}{a}\right)^{1/3}$$

$$\Rightarrow \quad (\cos x + \sin x)^2 + (\cos x - \sin x)^2 = \left(\frac{m+n}{a}\right)^{2/3} + \left(\frac{m-n}{a}\right)^{2/3}$$

$$\Rightarrow \quad 2(\cos^2 x + \sin^2 x) = \frac{(m+n)^{2/3}}{a^{2/3}} + \frac{(m-n)^{2/3}}{a^{2/3}}$$

$$\Rightarrow \quad (m+n)^{2/3} + (m-n)^{2/3} = 2a^{2/3}$$

EXAMPLE 16 *If $2\tan^2\alpha\tan^2\beta\tan^2\gamma + \tan^2\alpha\tan^2\beta + \tan^2\beta\tan^2\gamma + \tan^2\gamma\tan^2\alpha = 1$, prove that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 1$.*

SOLUTION We have,

$$2\tan^2\alpha\tan^2\beta\tan^2\gamma + \tan^2\alpha\tan^2\beta + \tan^2\beta\tan^2\gamma + \tan^2\gamma\tan^2\alpha = 1$$

Dividing throughout by $\tan^2\alpha\tan^2\beta\tan^2\gamma$, we get

$$\Rightarrow \quad 2 + \cot^2\gamma + \cot^2\alpha + \cot^2\beta = \cot^2\alpha\cot^2\beta\cot^2\gamma$$

$$\Rightarrow \quad 2 + \operatorname{cosec}^2\gamma - 1 + \operatorname{cosec}^2\alpha - 1 + \operatorname{cosec}^2\beta - 1 = (\operatorname{cosec}^2\alpha - 1)(\operatorname{cosec}^2\beta - 1)(\operatorname{cosec}^2\gamma - 1)$$

$$\Rightarrow \quad \operatorname{cosec}^2\alpha + \operatorname{cosec}^2\beta + \operatorname{cosec}^2\gamma - 1 = \operatorname{cosec}^2\alpha\operatorname{cosec}^2\beta\operatorname{cosec}^2\gamma - \operatorname{cosec}^2\alpha\operatorname{cosec}^2\beta$$
$$- \operatorname{cosec}^2\beta\operatorname{cosec}^2\gamma - \operatorname{cosec}^2\gamma\operatorname{cosec}^2\alpha + \operatorname{cosec}^2\alpha + \operatorname{cosec}^2\beta + \operatorname{cosec}^2\gamma - 1$$

$$\Rightarrow \quad \operatorname{cosec}^2\alpha\operatorname{cosec}^2\beta\operatorname{cosec}^2\gamma = \operatorname{cosec}^2\alpha\operatorname{cosec}^2\beta + \operatorname{cosec}^2\beta\operatorname{cosec}^2\gamma + \operatorname{cosec}^2\gamma\operatorname{cosec}^2\alpha$$

$$\Rightarrow \quad 1 = \sin^2\gamma + \sin^2\alpha + \sin^2\beta \qquad [\text{Multiplying throughout by } \sin^2\alpha\sin^2\beta\sin^2\gamma]$$

$$\Rightarrow \quad \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 1$$

EXAMPLE 17 *If* $\frac{ax}{\cos\theta}+\frac{by}{\sin\theta}=a^2-b^2$ *and,* $\frac{ax\sin\theta}{\cos^2\theta}-\frac{by\cos\theta}{\sin^2\theta}=0$, *prove that* $(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$.

SOLUTION We have,

$$\frac{ax\sin\theta}{\cos^2\theta}-\frac{by\cos\theta}{\sin^2\theta}=0 \Rightarrow ax\sin^3\theta-by\cos^3\theta=0 \Rightarrow \frac{\sin^3\theta}{by}=\frac{\cos^3\theta}{ax}$$

$$\Rightarrow \left(\frac{\sin^3\theta}{by}\right)^{2/3}=\left(\frac{\cos^3\theta}{ax}\right)^{2/3} \Rightarrow \frac{\sin^2\theta}{(by)^{2/3}}=\frac{\cos^2\theta}{(ax)^{2/3}}$$

$$\Rightarrow \frac{\sin^2\theta}{(by)^{2/3}}=\frac{\cos^2\theta}{(ax)^{2/3}}=\frac{\sin^2\theta+\cos^2\theta}{(by)^{2/3}+(ax)^{2/3}} \quad \text{[Using ratio and proportions]}$$

$$\Rightarrow \frac{\sin^2\theta}{(by)^{2/3}}=\frac{\cos^2\theta}{(ax)^{2/3}}=\frac{1}{(ax)^{2/3}+(by)^{2/3}}$$

$$\Rightarrow \sin^2\theta=\frac{(by)^{2/3}}{(ax)^{2/3}+(by)^{2/3}} \text{ and, } \cos^2\theta=\frac{(ax)^{2/3}}{(ax)^{2/3}+(by)^{2/3}}$$

$$\Rightarrow \sin\theta=\frac{(by)^{1/3}}{\sqrt{(ax)^{2/3}+(by)^{2/3}}} \text{ and, } \cos\theta=\frac{(ax)^{1/3}}{\sqrt{(ax)^{2/3}+(by)^{2/3}}}$$

Substituting these values in $\frac{ax}{\cos\theta}+\frac{by}{\sin\theta}=a^2-b^2$, we get

$$(ax)^{2/3}\sqrt{(ax)^{2/3}+(by)^{2/3}}+(by)^{2/3}\sqrt{(ax)^{2/3}+(by)^{2/3}}=a^2-b^2$$

$$\Rightarrow \left\{\sqrt{(ax)^{2/3}+(by)^{2/3}}\right\}\left\{(ax)^{2/3}+(by)^{2/3}\right\}=a^2-b^2$$

$$\Rightarrow \left\{(ax)^{2/3}+(by)^{2/3}\right\}^{3/2}=a^2-b^2 \Rightarrow (ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$$

EXAMPLE 18 *If* $m^2+m'^2+2mm'\cos x=1, n^2+n'^2+2nn'\cos x=1$, *and* $mn+m'n'+(mn'+m'n)\cos x=0$, *prove that* (i) $m^2+n^2=\operatorname{cosec}^2 x$ (ii) $m'^2+n'^2=\operatorname{cosec}^2 x$.

SOLUTION (i) We have,

$$m^2+m'^2+2mm'\cos x=1 \text{ and } n^2+n'^2+2nn'\cos x=1$$

$$\Rightarrow m'^2+2mm'\cos x+m^2\cos^2 x-m^2\cos^2 x+m^2=1$$

and, $$n'^2+2nn'\cos x+n^2\cos^2 x-n^2\cos^2 x+n^2=1$$

$$\Rightarrow (m'+m\cos x)^2+m^2(1-\cos^2 x)=1 \text{ and, } (n'+n\cos x)^2+n^2(1-\cos^2 x)=1$$

$$\Rightarrow (m'+m\cos x)^2=1-m^2\sin^2 x \quad ...(i) \quad \text{and,} \quad (n'+n\cos x)^2=1-n^2\sin^2 x \quad ...(ii)$$

Now, $$(m'+m\cos x)(n'+n\cos x)=m'n'+(mn'+m'n)\cos x+mn\cos^2 x$$

$$\Rightarrow (m'+m\cos x)(n'+n\cos x)=-mn+mn\cos^2 x \quad [\because mn+m'n'+(mn'+m'n)\cos x=0]$$

$$\Rightarrow (m'+m\cos x)(n'+n\cos x)=-mn(1-\cos^2 x)$$

$$\Rightarrow (m'+m\cos x)(n'+n\cos x)=-mn\sin^2 x$$

$$\Rightarrow (m'+m\cos x)^2(n'+n\cos x)^2=m^2n^2\sin^4 x \quad \text{[On squaring both sides]}$$

$$\Rightarrow (1-m^2\sin^2 x)(1-n^2\sin^2 x)=m^2n^2\sin^4 x \quad \text{[Using (i) and (ii)]}$$

$$\Rightarrow \quad 1-(m^2+n^2)\sin^2 x+m^2 n^2 \sin^4 x = m^2 n^2 \sin^4 x$$

$$\Rightarrow \quad 1 = (m^2+n^2)\sin^2 x \Rightarrow m^2+n^2 = \text{cosec}^2 x$$

(ii) As the given relations do not alter by replacing m by m' and n by n'. Therefore, on replacing m by m' and n by n' in $m^2+n^2=\text{cosec}^2 x$, we get $m'+n'^2=\text{cosec}^2 x$.

EXAMPLE 19 *If* $\dfrac{\sin^4 x}{a}+\dfrac{\cos^4 x}{b}=\dfrac{1}{a+b}$, *prove that*

(i) $\dfrac{\sin^8 x}{a^3}+\dfrac{\cos^8 x}{b^3}=\dfrac{1}{(a+b)^3}$ (ii) $\dfrac{\sin^{4n} x}{a^{2n-1}}+\dfrac{\cos^{4n} x}{b^{2n-1}}=\dfrac{1}{(a+b)^{2n-1}}$, $n\in N$

SOLUTION We have,

$$\frac{\sin^4 x}{a}+\frac{\cos^4 x}{b}=\frac{1}{a+b}$$

$$\Rightarrow \quad (a+b)\left(\frac{\sin^4 x}{a}+\frac{\cos^4 x}{b}\right)=1$$

$$\Rightarrow \quad (a+b)\left(\frac{\sin^4 x}{a}+\frac{\cos^4 x}{b}\right)=(\sin^2 x+\cos^2 x)^2$$

$$\Rightarrow \quad \sin^4 x+\cos^4 x+\frac{b}{a}\sin^4 x+\frac{a}{b}\cos^4 x=\sin^4 x+\cos^4 x+2\sin^2 x\cos^2 x$$

$$\Rightarrow \quad \frac{b}{a}\sin^4 x+\frac{a}{b}\cos^4 x-2\sin^2 x\cos^2 x=0$$

$$\Rightarrow \quad \left\{\sqrt{\frac{b}{a}}\sin^2 x-\sqrt{\frac{a}{b}}\cos^2 x\right\}^2=0 \Rightarrow \sqrt{\frac{b}{a}}\sin^2 x=\sqrt{\frac{a}{b}}\cos^2 x \Rightarrow \tan^2 x=\frac{a}{b} \Rightarrow \frac{\sin^2 x}{\cos^2 x}=\frac{a}{b}$$

$$\Rightarrow \quad \frac{\sin^2 x}{a}=\frac{\cos^2 x}{b}$$

$$\Rightarrow \quad \frac{\sin^2 x}{a}=\frac{\cos^2 x}{b}=\frac{\sin^2 x+\cos^2 x}{a+b}$$

$$\Rightarrow \quad \frac{\sin^2 x}{a}=\frac{\cos^2 x}{b}=\frac{1}{a+b} \Rightarrow \sin^2 x=\frac{a}{a+b}, \ \cos^2 x=\frac{b}{a+b} \qquad \text{...(i)}$$

(i) $$\frac{\sin^8 x}{a^3}+\frac{\cos^8 x}{b^3}=\frac{1}{a^3}(\sin^2 x)^4+\frac{1}{b^3}(\cos^2 x)^4$$

$$=\frac{1}{a^3}\left(\frac{a}{a+b}\right)^4+\frac{1}{b^3}\left(\frac{b}{a+b}\right)^4 \qquad \text{[Using (i)]}$$

$$=\frac{a}{(a+b)^4}+\frac{b}{(a+b)^4}=\frac{a+b}{(a+b)^4}=\frac{1}{(a+b)^3}$$

(ii) $$\frac{\sin^{4n} x}{a^{2n-1}}+\frac{\cos^{4n} x}{b^{2n-1}}=\frac{(\sin^2 x)^{2n}}{a^{2n-1}}+\frac{(\cos^2 x)^{2n}}{b^{2n-1}}=\frac{1}{a^{2n-1}}\left(\frac{a}{a+b}\right)^{2n}+\frac{1}{b^{2n-1}}\left(\frac{b}{a+b}\right)^{2n}$$

$$=\frac{a}{(a+b)^{2n}}+\frac{b}{(a+b)^{2n}}=\frac{a+b}{(a+b)^{2n}}=\frac{1}{(a+b)^{2n-1}}$$

EXAMPLE 20 *If* $\frac{\cos^4\alpha}{\cos^2\beta}+\frac{\sin^4\alpha}{\sin^2\beta}=1$, *prove that*

(i) $\sin^4\alpha+\sin^4\beta=2\sin^2\alpha\sin^2\beta$ (ii) $\frac{\cos^4\beta}{\cos^2\alpha}+\frac{\sin^4\beta}{\sin^2\alpha}=1$

SOLUTION We have,

$$\frac{\cos^4\alpha}{\cos^2\beta}+\frac{\sin^4\alpha}{\sin^2\beta}=1$$

$\Rightarrow \cos^4\alpha\sin^2\beta+\sin^4\alpha\cos^2\beta=\cos^2\beta\sin^2\beta$

$\Rightarrow \cos^4\alpha(1-\cos^2\beta)+\cos^2\beta(1-\cos^2\alpha)^2=\cos^2\beta(1-\cos^2\beta)$

$\Rightarrow \cos^4\alpha-\cos^4\alpha\cos^2\beta+\cos^2\beta-2\cos^2\alpha\cos^2\beta+\cos^4\alpha\cos^2\beta=\cos^2\beta-\cos^4\beta$

$\Rightarrow \cos^4\alpha-2\cos^2\alpha\cos^2\beta+\cos^4\beta=0$

$\Rightarrow (\cos^2\alpha-\cos^2\beta)^2=0\Rightarrow\cos^2\alpha-\cos^2\beta=0\Rightarrow\cos^2\alpha=\cos^2\beta$ ...(i)

$\Rightarrow 1-\sin^2\alpha=1-\sin^2\beta\Rightarrow\sin^2\alpha=\sin^2\beta$ ... (ii)

(i) $\sin^4\alpha+\sin^4\beta=(\sin^2\alpha-\sin^2\beta)^2+2\sin^2\alpha\sin^2\beta=2\sin^2\alpha\sin^2\beta$ $[\because \sin^2\alpha=\sin^2\beta]$

(ii) $\frac{\cos^4\beta}{\cos^2\alpha}+\frac{\sin^4\beta}{\sin^2\alpha}=\frac{\cos^2\beta\cos^2\beta}{\cos^2\alpha}+\frac{\sin^2\beta\sin^2\beta}{\sin^2\alpha}$

$$=\frac{\cos^2\beta\cos^2\alpha}{\cos^2\alpha}+\frac{\sin^2\beta\sin^2\alpha}{\sin^2\alpha}=\cos^2\beta+\sin^2\beta$$ [Using (i)and (ii)]

EXAMPLE 21 *If a is any non-zero real number, show that* $\cos x$ *and* $\sin x$ *can never be equal to* $a+\frac{1}{a}$.

SOLUTION We have following cases:

Case I *When* $a>0$: In this case, we have

$$a+\frac{1}{a}=(\sqrt{a})^2+\left(\frac{1}{\sqrt{a}}\right)^2-2\times\sqrt{a}\times\frac{1}{\sqrt{a}}+2\times\sqrt{a}\times\frac{1}{\sqrt{a}}=\left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2+2\geq 2\quad\left[\because\left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2\geq 0\right]$$

Case II *When* $x<0$: Let $a=-b$. Then, $b>0$

$$\therefore\quad a+\frac{1}{a}=-b-\frac{1}{b}=-\left(b+\frac{1}{b}\right)$$

But, $b+\frac{1}{b}\geq 2\Rightarrow-\left(b+\frac{1}{b}\right)\leq-2\Rightarrow a+\frac{1}{a}\leq-2$ [From Case I]

$\therefore$ $a+\frac{1}{a}\geq 2$ for $a>0$ and, $a+\frac{1}{a}\leq-2$ for $a<0$. But, $-1\leq\sin x\leq 1$ and $-1\leq\cos x\leq 1$ for all x.

Hence, $\sin x$ and $\cos x$ cannot be equal to $a+\frac{1}{a}$ for any non-zero a.

EXAMPLE 22 *If* $A=\cos^2 x+\sin^4 x$, *prove that* $\frac{3}{4}\leq A\leq 1$ *for all values of* x.

SOLUTION We have, $A=\cos^2 x+\sin^4 x=\cos^2 x+(\sin^2 x)^2$

Now, $-1\leq\sin x\leq 1$ for all x

$\Rightarrow 0\leq\sin^2 x\leq 1$ for all x

$\Rightarrow (\sin^2 x)^2\leq\sin^2 x$ [For $0<x<1$, $x^n<x$ for all $n\in N-\{1\}$]

$\Rightarrow \cos^2 x+(\sin^2 x)^2\leq\cos^2 x+\sin^2 x$ for all x

$\Rightarrow \quad A \leq 1 \quad \text{for all } x$ ...(i)

Again,

$$A = \cos^2 x + \sin^4 x = 1 - \sin^2 x + (\sin^2 x)^2 = 1 - \frac{1}{4} + \left\{\frac{1}{4} - \sin^2 x + (\sin^2 x)^2\right\} = \frac{3}{4} + \left(\frac{1}{2} - \sin^2 x\right)^2$$

Now, $\left(\frac{1}{2} - \sin^2 x\right)^2 \geq 0$ for all $x \Rightarrow \frac{3}{4} + \left(\frac{1}{2} - \sin^2 x\right)^2 \geq \frac{3}{4}$ for all $x \Rightarrow A \geq \frac{3}{4}$ for all x ...(ii)

From (i) and (ii), we obtain $\frac{3}{4} \leq A \leq 1$ for all x.

EXERCISE 5.1

BASIC

Prove the following identities (1-16)

1. $\sec^4 x - \sec^2 x = \tan^4 x + \tan^2 x$

2. $\sin^6 x + \cos^6 x = 1 - 3\sin^2 x \cos^2 x$

3. $(\operatorname{cosec} x - \sin x)(\sec x - \cos x)(\tan x + \cot x) = 1$

4. $\operatorname{cosec} x(\sec x - 1) - \cot x(1 - \cos x) = \tan x - \sin x$

5. $\dfrac{1 - \sin x \cos x}{\cos x(\sec x - \operatorname{cosec} x)} \cdot \dfrac{\sin^2 x - \cos^2 x}{\sin^3 x + \cos^3 x} = \sin x$

6. $\dfrac{\tan x}{1 - \cot x} + \dfrac{\cot x}{1 - \tan x} = (\sec x \operatorname{cosec} x + 1)$

7. $\dfrac{\sin^3 x + \cos^3 x}{\sin x + \cos x} + \dfrac{\sin^3 x - \cos^3 x}{\sin x - \cos x} = 2$

8. $(\sec x \sec y + \tan x \tan y)^2 - (\sec x \tan y + \tan x \sec y)^2 = 1$

9. $\dfrac{\cos x}{1 - \sin x} = \dfrac{1 + \cos x + \sin x}{1 + \cos x - \sin x}$

BASED ON LOTS

10. $\dfrac{\tan^3 x}{1 + \tan^2 x} + \dfrac{\cot^3 x}{1 + \cot^2 x} = \dfrac{1 - 2\sin^2 x \cos^2 x}{\sin x \cos x}$

11. $1 - \dfrac{\sin^2 x}{1 + \cot x} - \dfrac{\cos^2 x}{1 + \tan x} = \sin x \cos x$

12. $\left(\dfrac{1}{\sec^2 x - \cos^2 x} + \dfrac{1}{\operatorname{cosec}^2 x - \sin^2 x}\right)\sin^2 x \cos^2 x = \dfrac{1 - \sin^2 x \cos^2 x}{2 + \sin^2 x \cos^2 x}$

13. $(1 + \tan\alpha \tan\beta)^2 + (\tan\alpha - \tan\beta)^2 = \sec^2\alpha \sec^2\beta$

14. $\dfrac{(1 + \cot x + \tan x)(\sin x - \cos x)}{\sec^3 x - \operatorname{cosec}^3 x} = \sin^2 x \cos^2 x$

15. $\dfrac{2\sin x \cos x - \cos x}{1 - \sin x + \sin^2 x - \cos^2 x} = \cot x$

16. $\cos x(\tan x + 2)(2\tan x + 1) = 2\sec x + 5\sin x$

17. If $a = \dfrac{2\sin x}{1 + \cos x + \sin x}$, then prove that $\dfrac{1 - \cos x + \sin x}{1 + \sin x}$ is also equal to a.

[NCERT EXEMPLAR]

18. If $\sin x = \dfrac{a^2 - b^2}{a^2 + b^2}$, find the values of $\tan x$, $\sec x$ and $\operatorname{cosec} x$

19. If $\tan x = \frac{b}{a}$, then find the value of $\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}}$. [NCERT EXEMPLAR]

20. If $\tan x = \frac{a}{b}$, show that $\frac{a \sin x - b \cos x}{a \sin x + b \cos x} = \frac{a^2 - b^2}{a^2 + b^2}$.

BASED ON HOTS

21. If $\operatorname{cosec} x - \sin x = a^3$, $\sec x - \cos x = b^3$, then prove that $a^2 b^2 (a^2 + b^2) = 1$.

22. If $\cot x (1 + \sin x) = 4m$ and $\cot x (1 - \sin x) = 4n$, prove that $(m^2 - n^2)^2 = mn$.

23. If $\sin x + \cos x = m$, then prove that $\sin^6 x + \cos^6 x = \frac{4 - 3(m^2 - 1)^2}{4}$, where $m^2 \leq 2$

24. If $a = \sec x - \tan x$ and $b = \operatorname{cosec} x + \cot x$, then show that $ab + a - b + 1 = 0$.

[NCERT EXEMPLAR]

25. Prove that: $\left| \sqrt{\frac{1 - \sin x}{1 + \sin x}} + \sqrt{\frac{1 + \sin x}{1 - \sin x}} \right| = -\frac{2}{\cos x}$, where $\frac{\pi}{2} < x < \pi$

26. If $T_n = \sin^n x + \cos^n x$, prove that

(i) $\frac{T_3 - T_5}{T_1} = \frac{T_5 - T_7}{T_3}$ (ii) $2T_6 - 3T_4 + 1 = 0$ (iii) $6T_{10} - 15T_8 + 10T_6 - 1 = 0$

ANSWERS

18. $\tan x = \frac{a^2 - b^2}{2ab}$, $\sec x = \frac{a^2 + b^2}{2ab}$, $\operatorname{cosec} x = \frac{a^2 + b^2}{a^2 - b^2}$ 19. $\frac{2\cos x}{\sqrt{\cos^2 x - \sin^2 x}}$.

HINTS TO SELECTED PROBLEMS

17. We have,

$$a = \frac{2\sin x}{1 + \cos x + \sin x} = \frac{2\sin x (1 - \cos x + \sin x)}{(1 + \cos x + \sin x)(1 - \cos x + \sin x)}$$

$$\Rightarrow \quad a = \frac{2\sin x (1 - \cos x + \sin x)}{(1 + \sin x)^2 - \cos^2 x} = \frac{2\sin x (1 - \cos x + \sin x)}{1 + 2\sin x + \sin^2 x - \cos^2 x}$$

$$\Rightarrow \quad a = \frac{2\sin x (1 - \cos x + \sin x)}{2\sin x + \sin^2 x + \sin^2 x} = \frac{1 - \cos x + \sin x}{1 + \sin x}$$

19. We have, $\tan x = \frac{b}{a}$

$$\therefore \quad \sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}} = \sqrt{\frac{1 + \frac{b}{a}}{1 - \frac{b}{a}}} + \sqrt{\frac{1 - \frac{b}{a}}{1 + \frac{b}{a}}} = \sqrt{\frac{1 + \tan x}{1 - \tan x}} + \sqrt{\frac{1 - \tan x}{1 + \tan x}} = \sqrt{\frac{\cos x + \sin x}{\cos x - \sin x}} + \sqrt{\frac{\cos x - \sin x}{\cos x + \sin x}}$$

$$= \frac{\cos x + \sin x + \cos x - \sin x}{\sqrt{\cos^2 x - \sin^2 x}} = \frac{2\cos x}{\sqrt{\cos^2 x - \sin^2 x}}$$

21. We have, $\operatorname{cosec} x - \sin x = a^3$, $\sec x - \cos x = b^3$

$$\Rightarrow \quad \frac{1 - \sin^2 x}{\sin x} = a^3, \frac{1 - \cos^2 x}{\cos x} = b^3 \Rightarrow \frac{\cos^2 x}{\sin x} = a^3, \frac{\sin^2 x}{\cos x} = b^3$$

$$\Rightarrow \frac{\sin^2 x}{\cos x} \div \frac{\cos^2 x}{\sin x} = \frac{b^3}{a^3} \Rightarrow \tan^3 x = \frac{b^3}{a^3} \Rightarrow \tan x = \frac{b}{a} \Rightarrow \sin x = \frac{b}{\sqrt{a^2+b^2}} \text{ and } \cos x = \frac{a}{\sqrt{a^2+b^2}}$$

Substituting these values of sin x and cos x in $\frac{\cos^2 x}{\sin x} = a^3$, we obtain

$$\frac{a^2}{b\sqrt{(a^2+b^2)}} = a^3 \Rightarrow ab\sqrt{a^2+b^2} = 1 \Rightarrow a^2 b^2 (a^2+b^2) = 1$$

22. We have, $\cot x (1 + \sin x) = 4m$ and $\cot x (1 - \sin x) = 4n$

$$\Rightarrow \quad \cot x + \cos x = 4m \text{ and } \cot x - \cos x = 4n$$

$$\Rightarrow \quad (\cot x + \cos x)^2 - (\cot x - \cos x)^2 = 16m^2 - 16n^2 \text{ and } (\cot x + \cos x)(\cot x - \cos x) = 16\,mn$$

$$\Rightarrow \quad 4 \cot x \cos x = 16(m^2 - n^2) \text{ and } \cot^2 x - \cos^2 x = 16\,mn$$

$$\Rightarrow \quad \frac{\cos^2 x}{\sin x} = 4(m^2 - n^2) \text{ and } \frac{\cos^4 x}{\sin^2 x} = 16\,mn$$

$$\Rightarrow \quad \frac{\cos^4 x}{\sin^2 x} = 16(m^2 - n^2)^2 \text{ and } \frac{\cos^4 x}{\sin^2 x} = 16\,mn \Rightarrow 16(m^2 - n^2)^2 = 16\,mn \Rightarrow (m^2 - n^2)^2 = mn$$

24. We have, $a = \sec x - \tan x$ and $b = \operatorname{cosec} x + \cot x \Rightarrow a = \frac{1 - \sin x}{\cos x}$ and $b = \frac{1 + \cos x}{\sin x}$

$$\therefore \quad ab + a - b + 1 = \frac{(1 - \sin x)(1 + \cos x)}{\sin x \cos x} + \frac{1 - \sin x}{\cos x} - \frac{1 + \cos x}{\sin x} + 1$$

$$= \frac{1 - \sin x + \cos x - \sin x \cos x + \sin x - \sin^2 x - \cos x - \cos^2 x + \sin x \cos x}{\sin x \cos x}$$

$$= \frac{1 - (\cos^2 x + \sin^2 x)}{\sin x \cos x} = \frac{1 - 1}{\sin x \cos x} = 0$$

5.5 SIGNS OF TRIGONOMETRIC FUNCTIONS

Consider a unit circle with centre at the origin O of the coordinate axes. Clearly, this circle meets the coordinates axes at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$. Let $P(a, b)$ be a point on the circle such that length of arc $AP = x$ or equivalently, let $P(a, b)$ be the point where the terminal side of the angle $\angle AOP$ with radian measure x meets the unit circle. Then,

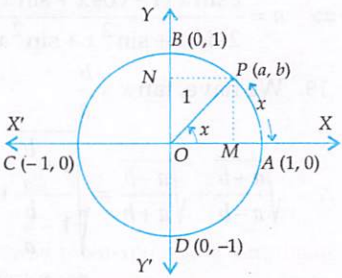

Fig. 5.8

(i) $\cos x = a$ for all $x \in R$

(ii) $\sin x = b$ for all $x \in R$

(iii) $\tan x = \frac{\sin x}{\cos x} = \frac{b}{a}$ for all $x \neq (2n+1)\frac{\pi}{2}$, $n \in Z$

(iv) $\cot x = \frac{\cos x}{\sin x} = \frac{a}{b}$ for all $x \neq n\pi$, $n \in Z$

(v) $\sec x = \frac{1}{a}$ for all $x \neq (2n+1)\frac{\pi}{2}$, $n \in Z$ (vi) $\operatorname{cosec} x = \frac{1}{b}$ for all $x \neq n\pi$, $n \in Z$

We observe that as the point $P(a, b)$ moves on the unit circle, point M moves between C and A and N moves between D and B. Consequently, $OM = a$ varies between -1 and 1 and $ON = PM = b$ also varies between -1 and 1 i.e. $-1 \le a \le 1$ and $-1 \le b \le 1$. Therefore, $-1 \le \cos x \le 1$ and $-1 \le \sin x \le 1$ for all x.

Also,

$a > 0, b > 0$ in I quadrant ; $a < 0, b > 0$ in II quadrant

$a < 0, b < 0$ in III quadrant ; $a > 0, b < 0$ in IV quadrant.

Thus, the signs of trigonometric functions in various quadrants are as discussed below :

In the first quadrant : We have, $a > 0$ and $b > 0$

$\therefore$ $\cos x = a > 0$, $\sin x = b > 0$, $\tan x = \frac{b}{a} > 0$, $\cot x = \frac{a}{b} > 0$, $\sec x = \frac{1}{a} > 0$ and $\text{cosec } x = \frac{1}{b} > 0$

Consequently, all the six trigonometric functions are positive in the first quadrant.

In the second quadrant : We have, $a < 0$ and $b > 0$

$\therefore$ $\cos x = a < 0$, $\sin x = b > 0$, $\tan x = \frac{b}{a} < 0$, $\cot x = \frac{a}{b} < 0$, $\sec x = \frac{1}{a} < 0$ and $\text{cosec } x = \frac{1}{b} > 0$

Consequently, in the second quadrant $\sin x$ and $\text{cosec } x$ both are positive and all other trigonometric functions are negative.

In the third quadrant : We have, $a < 0$ and $b < 0$.

$\therefore$ $\cos x = a < 0$, $\sin x = b < 0$, $\tan x = \frac{b}{a} > 0$, $\cot x = \frac{a}{b} > 0$, $\sec x = \frac{1}{a} < 0$ and $\text{cosec } x = \frac{1}{b} < 0$

Thus, in the third quadrant $\tan x$ and $\cot x$ are positive and all other trigonometric functions are negative.

In the fourth quadrant : We have, $a > 0$ and $b < 0$.

$\therefore$ $\cos x = a > 0$, $\sin x = b < 0$, $\tan x = \frac{b}{a} < 0$, $\cot x = \frac{a}{b} < 0$, $\sec x = \frac{1}{a} > 0$ and $\text{cosec } x = \frac{1}{b} < 0$

Thus, in the fourth quadrant $\cos x$ and $\sec x$ both are positive and all other trigonometric functions are negative.

The signs of trigonometric functions in different quadrants can be summarised as under :

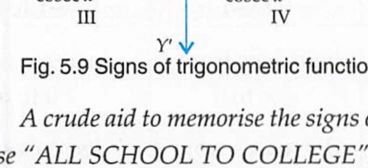

Fig. 5.9 Signs of trigonometric functions

SIMPLE RULE TO REMEMBER *A crude aid to memorise the signs of trigonometrical ratios in different quadrants is the four-word phrase "ALL SCHOOL TO COLLEGE". The first letter of the first word in this phrase is 'A'. This may be taken to indicate that all trigonometric ratios are positive in the first quadrant. The first letter of the second word is 'S'. This indicates that sine and its reciprocal are positive in the second quadrant. The first letter of third word is 'T'. This may be taken as to indicate that tangent and its reciprocal are positive in the third quadrant. The first letter of the fourth word in the phrase is 'C' which may be taken as to indicate that only cosine and its reciprocal are positive in the fourth quadrant.*

5.6 VARIATIONS IN VALUES OF TRIGONOMETRIC FUNCTIONS IN DIFFERENT QUADRANTS

Consider a unit circle centred at the origin O of the coordinate axes. The circle cuts the coordinates axes at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$. Let $P(a, b)$ be a point on the circle whose equation is $x^2 + y^2 = 1$ such that arc $AP = x$ or equivalently radian measure of $\angle AOP$ is x. Then, $a = \cos x$ and $b = \sin x$.

It is evident from Fig. 5.10 that

$$-1 \le a \le 1 \text{ and } -1 \le b \le 1 \Rightarrow -1 \le \cos x \le 1 \text{ and} -1 \le \sin x \le 1 \text{ for all } x.$$

Further, we observe that in the first quadrant, as x increases from 0 to $\frac{\pi}{2}$, b increases from 0 to 1 and as x increases from $\frac{\pi}{2}$ to π, b decreases from 1 to 0. But, $b = \sin x$. Thus, in the first quadrant $\sin x$ increases from 0 to 1 and in the second quadrant it decreases from 1 to 0. In the third quadrant as x increases from π to $\frac{3\pi}{2}$ the values of b decrease from 0 to –1 and in the fourth quadrant as x increases from $\frac{3\pi}{2}$ to 2π the values b of increase from –1 to 0. Thus, in the third quadrant as x increases from π to $\frac{3\pi}{2}$, $\sin x$ decreases from 0 to –1 and finally in the fourth quadrant as x increases from $\frac{3\pi}{2}$ to 2π, $\sin x$ increases from –1 to 0.

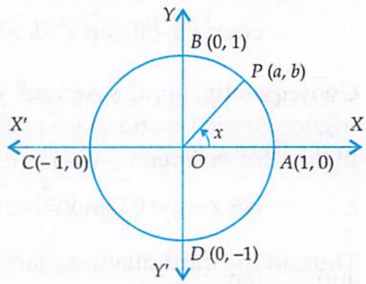

Fig. 5.10

Similarly, we can observe the variations in the values of other trigonometric functions. The following table exhibits the same.

Trigonometric function	*I quadrant*	*II quadrant*	*III quadrant*	*IV quadrant*
sine	increase from 0 to 1	decreases from 1 to 0	decreases from 0 to –1	increases from –1 to 0
cosine	decreases from 1 to 0	decreases from 0 to –1	increases from –1 to 0	increases from 0 to 1
tangent	increases from 0 to ∞	increases from $-\infty$ to 0	increases from 0 to ∞	increases from $-\infty$ to 0
cotangent	decreases from ∞ to 0	decreases from 0 to $-\infty$	decreases from ∞ to 0	decreases from 0 to $-\infty$
secant	increases from 1 to ∞	increases from $-\infty$ to –1	decreases from –1 to $-\infty$	decreases from ∞ to 1
cosecant	decreases from ∞ to 1	increases from 1 to ∞	increases from $-\infty$ to –1	decreases from –1 to $-\infty$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find* $\sin x$ and $\tan x$, *if* $\cos x = -\frac{12}{13}$ *and x lies in the third quadrant.*

SOLUTION We know that: $\cos^2 x + \sin^2 x = 1 \Rightarrow \sin x = \pm\sqrt{1-\cos^2 x}$

In third quadrant $\sin x$ is negative.

$$\therefore \quad \sin x = -\sqrt{1-\cos^2 x} \Rightarrow \sin x = -\sqrt{1-\left(-\frac{12}{13}\right)^2} = -\frac{5}{13}$$

$$\text{and,} \quad \tan x = \frac{\sin x}{\cos x} \Rightarrow \tan x = -\frac{5}{13} \times \frac{13}{-12} = \frac{5}{12}$$

EXAMPLE 2 *Find the values of* $\cos x$ *and* $\tan x$, *if* $\sin x = -\frac{3}{5}$ *and* $\pi < x < \frac{3\pi}{2}$.

SOLUTION We know that: $\cos^2 x + \sin^2 x = 1 \Rightarrow \cos x = \pm\sqrt{1-\sin^2 x}$

In the third quadrant $\cos x$ is negative and $\tan x$ is positive.

$$\therefore \quad \cos x = -\sqrt{1-\sin^2 x} \Rightarrow \cos x = -\sqrt{1-\frac{9}{25}} = -\frac{4}{5}$$

$$\text{and,} \quad \tan x = \frac{\sin x}{\cos x} \Rightarrow \tan x = -\frac{3}{5} \times -\frac{5}{4} = \frac{3}{4}$$

EXAMPLE 3 *Find all other trigonometrical ratios, if* $\sin x = -\frac{2\sqrt{6}}{5}$ *and x lies in quadrant III.*

SOLUTION We know that: $\cos^2 x + \sin^2 x = 1 \Rightarrow \cos x = \pm\sqrt{1-\sin^2 x}$

In the third quadrant $\cos x$ is negative.

$$\therefore \quad \cos x = -\sqrt{1-\sin^2 x} \Rightarrow \cos x = -\sqrt{1-\frac{24}{25}} = -\frac{1}{5}$$

In the third quadrant $\tan x$ is positive.

$$\therefore \quad \tan x = \frac{\sin x}{\cos x} \Rightarrow \tan x = -\frac{2\sqrt{6}}{5} \times -\frac{5}{1} = 2\sqrt{6}$$

$$\text{Now,} \quad \operatorname{cosec} x = \frac{1}{\sin x} \Rightarrow \operatorname{cosec} x = -\frac{5}{2\sqrt{6}}; \ \sec x = \frac{1}{\cos x} \Rightarrow \sec x = -5$$

$$\text{and,} \quad \cot x = \frac{1}{\tan x} \Rightarrow \cot x = \frac{1}{2\sqrt{6}}.$$

EXAMPLE 4 *If* $\cos x = -\frac{1}{2}$ *and* $\pi < x < \frac{3\pi}{2}$, *find the value of* $4\tan^2 x - 3\operatorname{cosec}^2 x$.

SOLUTION It is given that x lies in the third quadrant. Therefore, $\sin x$ is negative and $\tan x$ is positive. Thus,

$$\sin x = \pm\sqrt{1-\cos^2 x} \Rightarrow \sin x = -\sqrt{1-\frac{1}{4}} = -\frac{\sqrt{3}}{2} \Rightarrow \operatorname{cosec} x = \frac{-2}{\sqrt{3}}$$

And, $\tan x = \dfrac{\sin x}{\cos x} = \dfrac{-\sqrt{3}/2}{-1/2} = \sqrt{3}.$

Hence, $4\tan^2 x - 3\operatorname{cosec}^2 x = 4 \times 3 - 3 \times \dfrac{4}{3} = 8.$

EXAMPLE 5 *If* $\sec x = \sqrt{2}$ *and* $\dfrac{3\pi}{2} < x < 2\pi$, *find the value of* $\dfrac{1 + \tan x + \operatorname{cosec} x}{1 + \cot x - \operatorname{cosec} x}$.

SOLUTION We have, $\sec x = \sqrt{2}$. Therefore, $\cos x = \dfrac{1}{\sec x} \Rightarrow \cos x = \dfrac{1}{\sqrt{2}}$

It is given that x lies in the fourth quadrant in which $\sin x$ is negative.

$$\therefore \quad \sin x = -\sqrt{1-\cos^2 x} = -\sqrt{1-\frac{1}{2}} = -\frac{1}{\sqrt{2}}, \quad \operatorname{cosec} x = \frac{1}{\sin x} \Rightarrow \operatorname{cosec} x = -\sqrt{2}$$

$$\text{and,} \quad \tan x = \frac{\sin x}{\cos x} \Rightarrow \tan x = -\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{1} = -1 \Rightarrow \cot x = -1$$

$$\therefore \quad \frac{1 + \tan x + \operatorname{cosec} x}{1 + \cot x - \operatorname{cosec} x} = \frac{1 - 1 - \sqrt{2}}{1 - 1 + \sqrt{2}} = -1$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 6 *Prove that :* $\sqrt{\dfrac{1-\sin x}{1+\sin x}} = \begin{cases} \sec x - \tan x, & \text{if } -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ -\sec x + \tan x, & \text{if } \dfrac{\pi}{2} < x < \dfrac{3\pi}{2} \end{cases}$

SOLUTION We have,

$$\sqrt{\frac{1-\sin x}{1+\sin x}} = \sqrt{\frac{(1-\sin x)^2}{1-\sin^2 x}} = \frac{1-\sin x}{\sqrt{\cos^2 x}} = \frac{1-\sin x}{|\cos x|} = \begin{cases} \dfrac{1-\sin x}{\cos x}, & \text{if } -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ \dfrac{1-\sin x}{-\cos x}, & \text{if } \dfrac{\pi}{2} < x < \dfrac{3\pi}{2} \end{cases}$$

$$= \begin{cases} \sec x - \tan x, & \text{if } -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ -\sec x + \tan x, & \text{if } \dfrac{\pi}{2} < x < \dfrac{3\pi}{2} \end{cases}$$

EXAMPLE 7 *Prove that :* $\sqrt{\dfrac{1+\cos x}{1-\cos x}} = \begin{cases} \operatorname{cosec} x + \cot x, & \text{if } 0 < x < \pi \\ -\operatorname{cosec} x - \cot x, & \text{if } \pi < x < 2\pi \end{cases}$

SOLUTION We have,

$$\sqrt{\frac{1+\cos x}{1-\cos x}} = \sqrt{\frac{(1+\cos x)^2}{1-\cos^2 x}} = \frac{1+\cos x}{\sqrt{\sin^2 x}} = \frac{1+\cos x}{|\sin x|} = \begin{cases} \dfrac{1+\cos x}{\sin x}, & \text{if } 0 < x < \pi \\ \dfrac{1+\cos x}{-\sin x}, & \text{if } \pi < x < 2\pi \end{cases}$$

$$= \begin{cases} \operatorname{cosec} x + \cot x, & \text{if } 0 < x < \pi \\ -\operatorname{cosec} x - \cot x, & \text{if } \pi < x < 2\pi \end{cases}$$

EXAMPLE 8 *Prove that :* $\sqrt{\dfrac{1-\sin x}{1+\sin x}} + \sqrt{\dfrac{1+\sin x}{1-\sin x}} = \begin{cases} \dfrac{2}{\cos x}, & \text{if } 0 \le x < \dfrac{\pi}{2} \\ -\dfrac{2}{\cos x}, & \text{if } \dfrac{\pi}{2} < x \le \pi \end{cases}$

SOLUTION We have,

$$\sqrt{\frac{1-\sin x}{1+\sin x}}+\sqrt{\frac{1+\sin x}{1-\sin x}}=\frac{(1-\sin x)+(1+\sin x)}{\sqrt{1-\sin^2 x}}=\frac{2}{\sqrt{\cos^2 x}}=\frac{2}{|\cos x|} \quad [\because \sqrt{x^2}=|x|]$$

$$=\begin{cases}\dfrac{2}{\cos x}, \text{ if } 0 \le x < \dfrac{\pi}{2} \\ -\dfrac{2}{\cos x}, \text{ if } \dfrac{\pi}{2} < x \le \pi\end{cases}$$

EXERCISE 5.2

BASIC

1. Find the values of the other five trigonometric functions in each of the following:
   (i) $\cot x=\frac{12}{5}$, x in quadrant III (ii) $\cos x=-\frac{1}{2}$, x in quadrant II
   (iii) $\tan x=\frac{3}{4}$, x in quadrant III (iv) $\sin x=\frac{3}{5}$, x in quadrant I
2. If $\sin x=\frac{12}{13}$ and x lies in the second quadrant, find the value of $\sec x+\tan x$.
3. If $\sin x=\frac{3}{5}$, $\tan y=\frac{1}{2}$ and $\frac{\pi}{2}<x<\pi<y<\frac{3\pi}{2}$, find the value of $8\tan x-\sqrt{5}\sec y$.

BASED ON LOTS

4. If $\sin x+\cos x=0$ and x lies in the fourth quadrant, find $\sin x$ and $\cos x$.
5. If $\cos x=-\frac{3}{5}$ and $\pi<x<\frac{3\pi}{2}$, find the values of other five trigonometric functions and hence evaluate $\frac{\operatorname{cosec} x+\cot x}{\sec x-\tan x}$.

ANSWERS

1. (i) $\sin x=-\frac{5}{13}$, $\cos x=-\frac{12}{13}$, $\tan x=\frac{5}{12}$, $\operatorname{cosec} x=-\frac{13}{5}$, $\sec x=-\frac{13}{12}$
   (ii) $\sin x=\frac{\sqrt{3}}{2}$, $\tan x=-\sqrt{3}$, $\operatorname{cosec} x=\frac{2}{\sqrt{3}}$, $\cot x=\frac{-1}{\sqrt{3}}$, $\sec x=-2$
   (iii) $\sin x=-\frac{3}{5}$, $\cos x=-\frac{4}{5}$, $\operatorname{cosec} x=-\frac{5}{3}$, $\sec x=-\frac{5}{4}$, $\cot x=\frac{4}{3}$
   (iv) $\cos x=\frac{4}{5}$, $\tan x=\frac{3}{4}$, $\sec x=\frac{5}{4}$, $\cot x=\frac{4}{3}$, $\operatorname{cosec} x=\frac{5}{3}$

2. -5 3. $-\frac{7}{2}$ 4. $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ 5. $\frac{1}{6}$

HINTS TO SELECTED PROBLEM

4. We have, $\sin x+\cos x=0 \Rightarrow \sin x=-\cos x \Rightarrow \tan x=-1$
   $\therefore \sec^2 x=1+\tan^2 x \Rightarrow \sec^2 x=1+(-1)^2=2 \Rightarrow \sec x=\sqrt{2} \Rightarrow \cos x=\frac{1}{\sqrt{2}}$

5.7 VALUES OF TRIGONOMETRIC FUNCTIONS AT ALLIED ANGLES

Two angles are said to be allied when their sum or difference is either zero or a multiple of $\frac{\pi}{2}$.

The angles allied to x are $-x$, $\frac{\pi}{2}\pm x$, $\pi\pm x$, $\frac{3\pi}{2}\pm x$, $2\pi\pm x$ etc. In this section, we will express the values of trigonometric functions at allied angles of an angle x in terms of the values at x.

5.7.1 VALUES OF TRIGONOMETRIC FUNCTIONS AT $-x$

Consider a circle of unit radius centred at the origin of coordinate axes which cuts the coordinates axes at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$. Let $P(a, b)$ be a point on the circle such that arc $AP = x$ so that the measure of $\angle AOP$ is x. Then, $a = \cos x$ and $b = \sin x$. Let Q be the image of P in x-axis. Then, $\angle AOQ = -x$ and the coordinates of Q are $(a, -b)$.

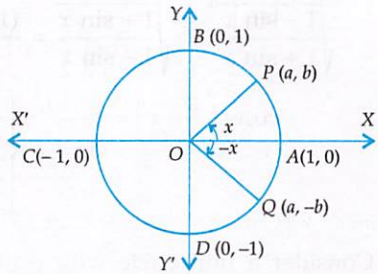

Fig. 5.11

$\therefore \quad a = \cos(-x)$ and $-b = \sin(-x)$

$\Rightarrow \quad \cos x = \cos(-x)$ and $-\sin x = \sin(-x)$ $\quad [\because a = \cos x, b = \sin x]$

$\Rightarrow \quad \cos(-x) = \cos x$ and $\sin(-x) = -\sin x$

$$\therefore \quad \tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\tan x, \ \cot(-x) = \frac{\cos(-x)}{\sin(-x)} = \frac{\cos x}{-\sin x} = -\frac{\cos x}{\sin x} = -\cot x$$

$$\sec(-x) = \frac{1}{\cos(-x)} = \frac{1}{\cos x} = \sec x \ \text{ and, } \ \operatorname{cosec}(-x) = \frac{1}{\sin(-x)} = -\frac{1}{\sin x} = -\operatorname{cosec} x$$

5.7.2 VALUES OF TRIGONOMETRIC FUNCTIONS AT $\left(\frac{\pi}{2} - x\right)$

Consider a unit circle centred at the origin of coordinates axes which cuts the coordinate axes at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$. Let $P(a, b)$ be a point on the circle such that are $AP = x$ consequently $\angle AOP = x$ and hence $a = \cos x$ and $b = \sin x$. Let Q be a point on the circle such that arc $AQ = \frac{\pi}{2} - x$ and so $\angle AOQ = \frac{\pi}{2} - x$. Consequently, measure of $\angle BOQ$ is x.

Draw perpendiculars PM and QN from P and Q respectively on OX. In triangle ONQ, we have $\angle QON = \frac{\pi}{2} - x$, $\angle ONQ = \frac{\pi}{2}$. Therefore, by using angle sum property, we obtain $\angle OQN = x$.

In triangles OMP and ONQ, we have

$\angle OMP = \angle ONQ$, $\angle POM = \angle OQN$ and, $OP = OQ$

So, by using RHS criterion of congruence, we obtain

$\Delta OMP \cong \Delta QNO$

$\Rightarrow \quad OM = QN$ and $ON = PM$

$\Rightarrow \quad QN = a$ and $ON = b$

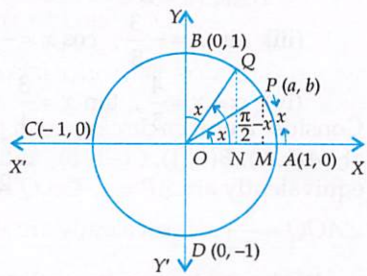

Fig. 5.12

So, the coordinates of Q are (b, a). Since $\angle QON = \frac{\pi}{2} - x$

$\therefore \quad b = \cos\left(\frac{\pi}{2} - x\right)$ and $a = \sin\left(\frac{\pi}{2} - x\right)$

$\Rightarrow \quad \sin x = \cos\left(\frac{\pi}{2} - x\right)$ and $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ $\quad [\because a = \cos x$ and $b = \sin x]$

$\Rightarrow \quad \cos\left(\frac{\pi}{2} - x\right) = \sin x$ and $\sin\left(\frac{\pi}{2} - x\right) = \cos x$

$$\therefore \quad \tan\left(\frac{\pi}{2}-x\right)=\frac{\sin\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right)}=\frac{\cos x}{\sin x}=\cot x,\ \cot\left(\frac{\pi}{2}-x\right)=\frac{\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)}=\frac{\sin x}{\cos x}=\tan x$$

$$\operatorname{cosec}\left(\frac{\pi}{2}-x\right)=\frac{1}{\sin\left(\frac{\pi}{2}-x\right)}=\frac{1}{\cos x}=\sec x,\ \sec\left(\frac{\pi}{2}-x\right)=\frac{1}{\cos\left(\frac{\pi}{2}-x\right)}=\frac{1}{\sin x}=\operatorname{cosec} x$$

5.7.3 VALUES OF TRIGONOMETRIC FUNCTIONS AT $(\pi - x)$

Consider a unit circle with centre at the origin O of the coordinate axes and cutting the coordinate axes at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$. Let $P(a, b)$ be a point on the circle such that arc $AP = x$ consequently, the measure of $\angle AOP$ is x. Further, let Q be a point on the unit circle such that $\angle AOQ = \pi - x$ and so measure of arc $AQ = \pi - x$.

$\therefore \quad$ arc $QC = \pi - (\pi - x) = x \Rightarrow \angle QOC = x$ $\qquad$ [$\because$ arc $AC = \pi$]

In triangles OMP and ONQ, we have

$\angle POM = \angle QON, OP = OQ = 1$ unit and $\angle OMP = \angle ONQ$

So, $\quad \Delta OMP \cong \Delta ONQ$

$\Rightarrow \quad OM = ON$ and $PM = QN$

$\Rightarrow \quad ON = a$ and $QN = b$

So, the coordinates of Q are $(-a, b)$. Since, $\angle QOM = (\pi - x)$.

$\therefore \quad -a = \cos(\pi - x)$ and $b = \sin(\pi - x)$

$\Rightarrow \quad \cos(\pi - x) = -\cos x$ and $\sin(\pi - x) = \sin x$

[$\because a = \cos x, b = \sin x$]

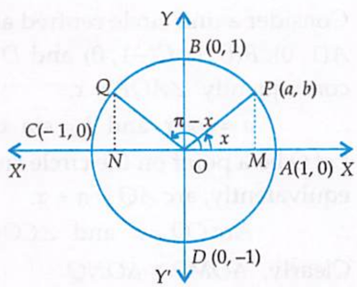

Fig. 5.13

Thus, we obtain

$\cos(\pi - x) = -\cos x$ and $\sin(\pi - x) = \sin x$

$$\therefore \quad \tan(\pi-x)=\frac{\sin(\pi-x)}{\cos(\pi-x)}=\frac{\sin x}{-\cos x}=-\frac{\sin x}{\cos x}=-\tan x,$$

$$\cot(\pi-x)=\frac{\cos(\pi-x)}{\sin(\pi-x)}=\frac{-\cos x}{\sin x}=-\frac{\cos x}{\sin x}=-\cot x.$$

$$\operatorname{cosec}(\pi-x)=\frac{1}{\sin(\pi-x)}=\frac{1}{\sin x}=\operatorname{cosec} x,\ \sec(\pi-x)=\frac{1}{\cos(\pi-x)}=\frac{1}{-\cos x}=-\sec x$$

5.7.4 VALUES OF TRIGONOMETRIC FUNCTIONS AT $\left(\frac{\pi}{2}+x\right)$

Consider a unit circle centred at the origin of the coordinate axes which cuts the coordinate axes at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $(0, -1)$. Let $P(a, b)$ a point on the circle such that $\angle AOP = x$ equivalently arc $AP = x$. Let Q be a point on the circle such that $\angle AOQ = \frac{\pi}{2} + x$ equivalently arc $AQ = \frac{\pi}{2} + x$.

$$\therefore \quad \text{arc } QC = \pi - \left(\frac{\pi}{2}+x\right) = \frac{\pi}{2} - x. \Rightarrow \angle QON = \frac{\pi}{2} - x.$$

In triangles OMP and OQN, we have

$\angle POM = \angle OQN$

$\angle OMP = \angle ONQ$ and, $OP = OQ$

$\therefore \quad \Delta POM \cong \Delta OQN$

$\Rightarrow \quad OM = QN$ and $PM = ON$

$\Rightarrow \quad a = QN$ and $b = ON \Rightarrow ON = b$ and $QN = a$

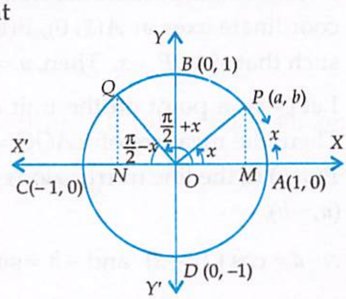

Fig. 5.14

So, the coordinates of Q are $(-b, a)$. Since $\angle QOM = \left(\frac{\pi}{2}+x\right)$.

$\therefore\ -b = \cos\left(\frac{\pi}{2}+x\right)$ and $a = \sin\left(\frac{\pi}{2}+x\right)$

$\Rightarrow \cos\left(\frac{\pi}{2}+x\right) = -\sin x$ and $\sin\left(\frac{\pi}{2}+x\right) = \cos x$ $\qquad [\because a = \cos x, b = \sin x]$

$$\therefore\ \tan\left(\frac{\pi}{2}+x\right) = \frac{\sin\left(\frac{\pi}{2}+x\right)}{\cos\left(\frac{\pi}{2}+x\right)} = \frac{\cos x}{-\sin x} = -\cot x,\ \cot\left(\frac{\pi}{2}+x\right) = \frac{\cos\left(\frac{\pi}{2}+x\right)}{\sin\left(\frac{\pi}{2}+x\right)} = \frac{-\sin x}{\cos x} = -\tan x$$

$$\operatorname{cosec}\left(\frac{\pi}{2}+x\right) = \frac{1}{\sin\left(\frac{\pi}{2}+x\right)} = \frac{1}{\cos x} = \sec x,\ \sec\left(\frac{\pi}{2}+x\right) = \frac{1}{\cos\left(\frac{\pi}{2}+x\right)} = \frac{1}{-\sin x} = -\operatorname{cosec} x$$

5.7.5 VALUES OF TRIGONOMETRIC FUNCTIONS AT $(\pi + x)$

Consider a unit circle centred at the origin of the coordinate axes cutting the coordinate axes at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$. Let $P(a, b)$ a point on the circle such that arc $AP = x$ consequently $\angle AOP = x$.

$\therefore\quad a = \cos x$ and $b = \sin x$.

Let Q be a point on the circle such that $\angle AOQ = \pi + x$ or equivalently, arc $AQ = \pi + x$.

$\therefore\quad$ Arc $CQ = x$ and $\angle COQ = x$

Clearly, $\Delta OMP \cong \Delta ONQ$

$\Rightarrow\quad OM = ON$ and $PM = QN \Rightarrow a = ON$ and $b = QN$

$\Rightarrow\quad ON = a$ and $QN = b$

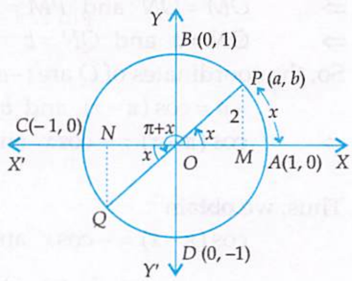

Fig. 5.15

So, the coordinates of Q are $(-a, -b)$. Since $\angle QOM = (\pi + x)$.

$\therefore\quad -a = \cos(\pi + x)$ and $-b = \sin(\pi + x) \Rightarrow$

$\cos(\pi + x) = -\cos x$ and $\sin(\pi + x) = -\sin x$

$$\therefore\quad \tan(\pi + x) = \frac{\sin(\pi + x)}{\cos(\pi + x)} = \frac{-\sin x}{-\cos x} = \tan x,\ \cot(\pi + x) = \frac{\cos(\pi + x)}{\sin(\pi + x)} = \frac{-\cos x}{-\sin x} = \cot x$$

$$\operatorname{cosec}(\pi + x) = \frac{1}{\sin(\pi + x)} = \frac{1}{-\sin x} = -\operatorname{cosec} x,\ \sec(\pi + x) = \frac{1}{\cos(\pi + x)} = \frac{1}{-\cos x} = -\sec x$$

5.7.6 VALUES OF TRIGONOMETRIC FUNCTIONS AT $(2\pi - x)$

Consider a unit circle centred at the origin of the coordinate axes. Suppose the circle cuts the coordinate axes at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$. Let $P(a, b)$ a point on the unit circle such that $\angle AOP = x$. Then, $a = \cos x$ and $b = \sin x$.

Let Q be a point on the unit circle such that $\angle AOQ = 2\pi - x$. Then, the measure of $\angle AOQ = x$. Consequently, Q is image of $P(a, b)$ in the line mirror along OX. So, the coordinates of Q are $(a, -b)$.

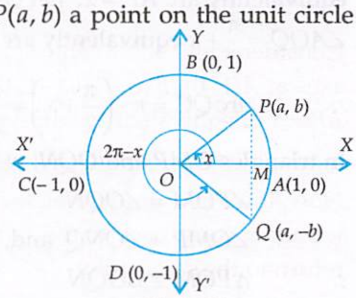

Fig. 5.16

$\therefore\ a = \cos(2\pi - x)$ and $-b = \sin(2\pi - x)$

$\Rightarrow \cos x = \cos(2\pi - x)$ and $-\sin x = \sin(2\pi + x)$

$[\because a = \cos x, b = \sin x]$

$\Rightarrow \cos(2\pi - x) = \cos x$ and $\sin(2\pi - x) = -\sin x$

$\therefore\quad \tan(2\pi - x) = \dfrac{\sin(2\pi - x)}{\cos(2\pi - x)} = \dfrac{-\sin x}{\cos x} = -\tan x,\ \cot(2\pi - x) = \dfrac{\cos(2\pi - x)}{\sin(2\pi - x)} = \dfrac{\cos x}{-\sin x} = -\cot x$

$\operatorname{cosec}(2\pi - x) = \dfrac{1}{\sin(2\pi - x)} = \dfrac{1}{\sin x} = \operatorname{cosec} x,\ \sec(2\pi - x) = \dfrac{1}{\cos(2\pi - x)} = \dfrac{1}{\cos x} = \sec x$

5.7.7 VALUES OF TRIGONOMETRIC FUNCTIONS AT $(2\pi + x)$

Consider a unit circle centred at the origin of the coordinates axes. The circle cuts the coordinate axes at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$. Let P be a point on the circle such that arc AP or equivalently $\angle AOP = x$. Therefore, $a = \cos x$ and $b = \sin x$.

The circumference of the unit circle is 2π. Therefore, if we begin from P and travel distance 2π along the circle, we return to the same point P.

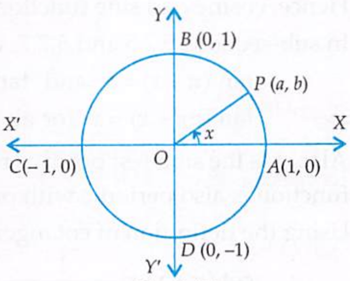

Fig. 5.17

$\therefore\quad a = \cos(2\pi + x)$ and $b = \sin(2\pi + x)$

$\Rightarrow\quad \cos x = \cos(2\pi + x)$ and $\sin x = \sin(2\pi + x)$

$\Rightarrow\quad \cos(2\pi + x) = \cos x$ and $\sin(2\pi + x)$

$\therefore\quad \tan(2\pi + x) = \dfrac{\sin(2\pi + x)}{\cos(2\pi + x)} = \dfrac{\sin x}{\cos x} = \tan x,$

$\cot(2\pi + x) = \dfrac{\cos(2\pi + x)}{\sin(2\pi + x)} = \dfrac{\cos x}{\sin x} = \cot x$

$\operatorname{cosec}(2\pi + x) = \dfrac{1}{\sin(2\pi + x)} = \dfrac{1}{\sin x} = \operatorname{cosec} x,\ \sec(2\pi + x) = \dfrac{1}{\cos(2\pi + x)} = \dfrac{1}{\cos x} = \sec x$

5.8 PERIODIC FUNCTIONS

A function $f(x)$ is said to be a periodic function, if there exits a positive real number T such that $f(x + T) = f(x)$ for all x.

If T is the smallest positive real number such that $f(x + T) = f(x)$ for all x, then T is called the fundamental period of $f(x)$.

We observe that if $f(x)$ is periodic with period T, then

$f(x + 2T) = f((x + T) + T) = f(x + T) = f(x),\ f(x + 3T) = f\left((x + 2T) + T\right) = f(x + 2T) = f(x)$ and so on.

In general, $f(x + nT) = f(x)$ for all x and $n \in N$.

5.8.1 PERIODICITY OF TRIGONOMETRIC FUNCTIONS

Consider a unit circle centred at the origin of the coordinate axes. The circle cuts the coordinate axes at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$. Let $P(a, b)$ be a point on the circle such that arc $AP = x$ or equivalently $\angle AOP = x$.

$\therefore\quad a = \cos x$ and $b = \sin x$.

Now, if we take one complete revolution from the point P along the circumference of the circle, we again come back to the same point P. In other words, if x increases or decreases by 2π, we return to the same point.

$\therefore\quad a = \cos(2\pi + x)$ and $b = \sin(2\pi + x)$

$\Rightarrow\quad \cos x = \cos(2\pi + x)$ and $\sin x = \sin(2\pi + x)$

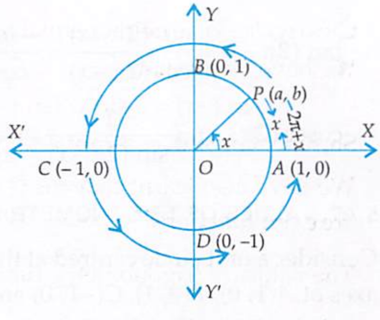

Fig. 5.18

Also, $a = \cos(-2\pi + x)$ and $b = \sin(-2\pi + x)$

$\therefore$ $\cos(-2\pi + x) = \cos x$ and $\sin(-2\pi + x) = \sin x$

We also observe that if x increases or decreases by any integral multiple of 2π, we come back to the same point P.

$\therefore$ $\cos(2n\pi + x) = \cos x$ and $\sin(2n\pi + x) = \sin x, n \in Z$

Thus, cosine and sine functions are periodic functions.

It is evident from the above discussion that 2π is the smallest positive number such that

$$\cos(2\pi + x) = \cos x \text{ and } \sin(2\pi + x) = \sin x \text{ for all } x$$

Hence, cosine and sine functions are periodic functions with period 2π.

In sub-sections 5.7.5 and 5.7.7, we have learnt that

$$\tan(\pi + x) = x \text{ and } \tan(2\pi + x) = x \text{ for all } x \in R$$

$$\Rightarrow \quad \tan(n\pi + x) = x \text{ for all } x \in R \text{ and } n \in Z$$

Also, π is the smallest positive real number such that $\tan(\pi + x) = \tan x$ for all $x \in R$. So, tangent function is also periodic with period π.

Using the definition of cotangent, cosecants and secant functions, we obtain

$$\cot(\pi + x) = \frac{1}{\tan(\pi + x)} = \frac{1}{\tan x} = \cot x \text{ for all } x\ (\neq n\pi) \in R$$

$$\operatorname{cosec}(2\pi + x) = \frac{1}{\sin(2\pi + x)} = \frac{1}{\sin x} = \operatorname{cosec} x \text{ for all } x\ (\neq n\pi) \in R$$

$$\sec(2\pi + x) = \frac{1}{\cos(2\pi + x)} = \frac{1}{\cos x} = \sec x \text{ for all } x \left(\neq (2n+1)\frac{\pi}{2}\right) \in R$$

Thus, consecant and secant functions are periodic with period 2π and cotangent is periodic with period π.

5.9 EVEN AND ODD FUNCTIONS

EVEN FUNCTION *A function $f(x)$ is said to be an even function, if $f(-x) = f(x)$ for all x in its domain.*

ODD FUNCTION *A function $f(x)$ is said to be an odd function, if $f(-x) = -f(x)$ for all x in its domain.*

ILLUSTRATION 1 *Determine whether the following functions are even or odd or neither :*

(i) $f(x) = x^3 + x$ (ii) $g(x) = 3x^2 + 1$ (iii) $h(x) = x^2 + x + 4$

SOLUTION (i) We have, $f(x) = x^3 + x$

$\therefore$ $f(-x) = (-x)^3 + (-x) = -x^3 - x = -(x^3 + x) = -f(x)$ for all $x \in R$.

So, $f(x)$ is an odd function.

(ii) We have, $g(x) = 3x^2 + 1$

$\therefore$ $g(-x) = 3(-x)^2 + 1 = 3x^2 + 1 = g(x)$ for all $x \in R$

So, $g(x)$ is an even function.

(iii) We have, $h(x) = x^2 + x + 4$

$\therefore$ $h(-x) = (-x)^2 + (-x) + 4 = x^2 - x + 4$

Clearly, $h(-x)$ is neither equal to $h(x)$ nor to $-h(x)$. So, $h(x)$ is neither even nor odd function.
We have learnt that

$$\sin(-x) = -\sin x,\ \tan(-x) = -\tan x,\ \text{cosec}(-x) = -\text{cosec}\, x \text{ and } \cot(-x) = -\cot x$$

So, sine, tangent, cosecant and cotangent functions are odd functions.

We have also learnt that $\cos(-x) = \cos x$ and $\sec(-x) = \sec x$. So, cosine and secant functions are even functions.

The values of trigonometric functions at $-x$, $\frac{\pi}{2} \pm x$, $\pi \pm x$, $\frac{3\pi}{2} \pm x$ and $2\pi \pm x$ are given in terms of values at x in the following tabular form for ready reference.

Trigonometric Function / *Point /Angle*	sin	cos	tan	cot	cosec	sec
$-x$	$-\sin x$	$\cos x$	$-\tan x$	$-\cot x$	$-\text{cosec}\, x$	$\sec x$
$\frac{\pi}{2}-x$	$\cos x$	$\sin x$	$\cot x$	$\tan x$	$\sec x$	$\text{cosec}\, x$
$\frac{\pi}{2}+x$	$\cos x$	$-\sin x$	$-\cot x$	$-\tan x$	$\sec x$	$-\text{cosec}\, x$
$\pi-x$	$\sin x$	$-\cos x$	$-\tan x$	$-\cot x$	$\text{cosec}\, x$	$-\sec x$
$\pi+x$	$-\sin x$	$-\cos x$	$\tan x$	$\cot x$	$-\text{cosec}\, x$	$-\sec x$
$\frac{3\pi}{2}-x$	$-\cos x$	$-\sin x$	$\cot x$	$\tan x$	$-\sec x$	$-\text{cosec}\, x$
$\frac{3\pi}{2}+x$	$-\cos x$	$\sin x$	$-\cot x$	$-\tan x$	$-\sec x$	$\text{cosec}\, x$
$2\pi-x$	$-\sin x$	$\cos x$	$-\tan x$	$-\cot x$	$-\text{cosec}\, x$	$\sec x$
$2\pi+x$	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\text{cosec}\, x$	$\sec x$

From the above table we observe that the values of sine function at $\frac{\pi}{2} \pm x = 1 \times \frac{\pi}{2} \pm x$ and $\frac{3\pi}{2} \pm x = 3 \times \frac{\pi}{2} \pm x$ are $\cos x$ or $-\cos x$ depending upon the quadrant in which the terminating ray of the angle lies. Also, sine function is periodic with period 2π. So, the values of sine function at $2n\pi + \frac{\pi}{2} \pm x$ and $2n\pi + \frac{3\pi}{2} \pm x$ are $\cos x$ or $-\cos x$. But, $2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}$ and $2n\pi + \frac{3\pi}{2} = (4n+3)\frac{\pi}{2}$ are odd multiples of $\frac{\pi}{2}$. Therefore, the value of sine function at $(2n-1)\frac{\pi}{2} \pm x$ is $\cos x$ or $-\cos x$ depending upon the position of the terminating ray of the angle. We also note from the table that the values of sine function at $\pi \pm x = 2 \times \frac{\pi}{2} \pm x$ and $2\pi \pm x = 4 \times \frac{\pi}{2} \pm x$ are sin or $-\sin x$ depending upon the position of the terminating ray of the angle. The periodicity of the sine function gives that the values of sine function at $n\pi \pm x = 2n \times \frac{\pi}{2} \pm x$ are $\sin x$ or $-\sin x$ depending upon the quadrant in which the terminating ray of the angle lies.

We also note that at point expressible in the form $(2n-1)\frac{\pi}{2} \pm x$ the values of cosine, tangent, cotangent, secant and cosecant functions are $\pm \sin x$, $\pm \cot x$, $\pm \tan x$, $\pm \text{cosec}\, x$ and $\pm \sec x$ respectively. At a point expressible in the form $2n\left(\frac{\pi}{2}\right) \pm x$ the values of cosine, tangent, cotangent, secant and cosecant functions are $\pm \cos x$, $\pm \tan x$, $\pm \cot x$, $\pm \sec x$ and $\pm \text{cosec}\, x$ respectively.

The above discussion suggests us the following algorithm to find the value of a trigonometric function at a point.

ALGORITHM

Step I *Obtain the point x at which the value of a trigonometric function is to be determined.*

Step II *Check whether x is positive or negative. If x is negative, say $x = -y$, then write $f(x) = -f(y)$, if f is an odd function. Otherwise, write $f(x) = f(y)$. Here, f is the given trigonometric function.*

Step III *Express the positive value of x in step II, in the form $x = \frac{n\pi}{2} \pm \alpha$, where $\alpha \in \left(0, \frac{\pi}{2}\right)$.*

Step IV *Determine the quadrant in which the terminating ray of the angle x lies and determine the sign of the trigonometric function in that quadrant.*

Step V *If n is step III is an odd positive integer, then* $\sin x = \pm \cos \alpha$, $\cos x = \pm \sin \alpha$, $\sec x = \pm \text{cosec}\, \alpha$, $\text{cosec}\, x = \pm \sec \alpha$, *where the sign on RHS of these values will be the sign obtained in step IV.*

If n in step III is an even positive integer, then $\sin x = \pm \sin \alpha$, $\cos x = \pm \cos \alpha$, $\tan x = \pm \tan \alpha$, $\cot x = \pm \cot \alpha$, $\text{cosec}\, x = \pm \text{cosec}\, \alpha$, $\sec x = \pm \sec \alpha$, *where the sign on RHS of these values will be the sign obtained in step IV.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate the following :*

(i) $\sin \frac{7\pi}{4}$ (ii) $\cos \frac{7\pi}{6}$ (iii) $\cos\left(-\frac{8\pi}{3}\right)$ (iv) $\sin\left(-\frac{25\pi}{4}\right)$

SOLUTION (i) Clearly, $\sin \frac{7\pi}{4} = \sin\left(3 \times \frac{\pi}{2} + \frac{\pi}{4}\right)$. Since $3 \times \frac{\pi}{2} + \frac{\pi}{4}$ lies in the IVth quadrant in which sine function is negative and 3 is an odd integer.

$$\therefore \quad \sin \frac{7\pi}{4} = \sin\left(3 \times \frac{\pi}{2} + \frac{\pi}{4}\right) = -\cos \frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

(ii) Clearly, $\cos \frac{7\pi}{6} = \cos\left(2 \times \frac{\pi}{2} + \frac{\pi}{6}\right)$. Since $\frac{7\pi}{6}$ is in the III quadrant in which cosine function is negative. Also the multiple of $\frac{\pi}{2}$ is even.

$$\therefore \quad \cos \frac{7\pi}{6} = \cos\left(2 \times \frac{\pi}{2} + \frac{\pi}{6}\right) = -\cos \frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$

(iii) We know that cosine is an even function. Therefore, $\cos\left(-\frac{8\pi}{3}\right) = \cos \frac{8\pi}{3}$.

Also, $\frac{8\pi}{3} = 5 \times \frac{\pi}{2} + \frac{\pi}{6}$. So, $\frac{8\pi}{3}$ is in the II quadrant in which cosine function is negative. Also, the multiple of $\frac{\pi}{2}$ is odd.

$$\therefore \quad \cos \frac{8\pi}{3} = \cos\left(5 \times \frac{\pi}{2} + \frac{\pi}{6}\right) = -\sin \frac{\pi}{6} = -\frac{1}{2}. \text{ Hence, } \cos\left(-\frac{8\pi}{3}\right) = \cos \frac{8\pi}{3} = -\frac{1}{2}$$

(iv) The sine function is an odd function. Therefore, $\sin\left(-\frac{25\pi}{4}\right) = -\sin\frac{25\pi}{4}$.

Now, $\frac{25\pi}{4} = \left(12\times\frac{\pi}{2}+\frac{\pi}{4}\right) \Rightarrow \frac{25\pi}{4}$ lies in the I quadrant and multiple of $\frac{\pi}{2}$ in this expression is even.

$\therefore \quad \sin\frac{25\pi}{4} = \sin\left(12\times\frac{\pi}{2}+\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$. Hence, $\sin\left(-\frac{25\pi}{4}\right) = -\sin\frac{25\pi}{4} = -\frac{1}{\sqrt{2}}$.

EXAMPLE 2 *Evaluate the following :*

(i) cosec 390° (ii) cot 570° (iii) tan 480° (iv) cos 270°

(v) $\tan\frac{19\pi}{3}$ (vi) $\sin\left(-\frac{11\pi}{3}\right)$ (vii) $\cot\left(-\frac{15\pi}{4}\right)$

SOLUTION (i) We have, $390° = \frac{13\pi}{6} = 4\times\frac{\pi}{2}+\frac{\pi}{6}$. This shows that $\frac{13\pi}{6}$ is in I quadrant in which cosecant function is positive and the multiple of $\frac{\pi}{2}$ is even.

$$\therefore \quad \text{cosec } 390° = \text{cosec}\frac{13\pi}{6} = \text{cosec}\left(4\times\frac{\pi}{2}+\frac{\pi}{6}\right) = \text{cosec}\frac{\pi}{6} = 2.$$

(ii) We have, $570° = \frac{19\pi}{6} = 6\times\frac{\pi}{2}+\frac{\pi}{6}$. It shows that $\frac{19\pi}{6}$ is in the IIIrd quadrant in which cotangent function is positive and the multiple of $\frac{\pi}{2}$ is even.

$$\therefore \quad \cot 570° = \cot\frac{19\pi}{6} = \cot\left(6\times\frac{\pi}{2}+\frac{\pi}{6}\right) = \cot\frac{\pi}{6} = \sqrt{3}$$

(iii) We have, $480° = \frac{8\pi}{3} = 5\times\frac{\pi}{2}+\frac{\pi}{6}$. Clearly, $\frac{8\pi}{3}$ is in the IInd quadrant in which tangent function is negative and the multiple of $\frac{\pi}{2}$ is odd.

$$\therefore \quad \tan 480° = \tan\frac{8\pi}{3} = \tan\left(5\times\frac{\pi}{2}+\frac{\pi}{6}\right) = -\cot\frac{\pi}{6} = -\sqrt{3}$$

(iv) We have, $270° = \frac{3\pi}{2} = \left(3\times\frac{\pi}{2}+0\right)$. Clearly, 270° is in the negative direction of y-axis i.e. on the boundary line of II and III quadrant. Also, the multiple of $\frac{\pi}{2}$ is an odd integer.

$$\therefore \quad \cos 270° = \cos\left(3\times\frac{\pi}{2}+0\right) = \pm\sin 0 = 0$$

ALITER We know that $\cos(2n-1)\frac{\pi}{2} = 0$. Therefore, $\cos\frac{3\pi}{2} = 0$.

(v) We have, $\frac{19\pi}{3} = \left(12\times\frac{\pi}{2}+\frac{\pi}{3}\right)$. Clearly, this angle lies in I quadrant in which tangent function is positive and the multiple of $\frac{\pi}{2}$ is even.

$$\therefore \quad \tan\frac{19\pi}{3} = \tan\left(12\times\frac{\pi}{2}+\frac{\pi}{3}\right) = \tan\frac{\pi}{3} = \sqrt{3}$$

(vi) We have, $\frac{11\pi}{3}=7\times\frac{\pi}{2}+\frac{\pi}{6}$. This angle lies in the IV quadrant in which sine function is negative and the multiple of $\frac{\pi}{2}$ is odd.

$\therefore\ \sin\frac{11\pi}{3}=\sin\left(7\times\frac{\pi}{2}\times\frac{\pi}{6}\right)=-\cos\frac{\pi}{6}=-\frac{\sqrt{3}}{2}$. Hence, $\sin\left(-\frac{11\pi}{3}\right)=-\sin\frac{11\pi}{3}=-\left(-\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{3}}{2}$

(vii) We have, $\frac{15\pi}{4}=\left(7\times\frac{\pi}{2}+\frac{\pi}{4}\right)$. This means that $\frac{15\pi}{4}$ is in the IVth quadrant in which cotangent function is negative and the multiple of $\frac{\pi}{2}$ is odd.

$\therefore\quad \cot\frac{15\pi}{4}=\cot\left(7\times\frac{\pi}{2}+\frac{\pi}{4}\right)=-\tan\frac{\pi}{4}=-1$. Hence, $\cot\left(-\frac{15\pi}{4}\right)=-\cot\left(\frac{15\pi}{4}\right)=-(-1)=1.$

EXAMPLE 3 *Prove that:* $\cos 510°\cos 330°+\sin 390°\cos 120°=-1.$

SOLUTION LHS $=\cos 510°\cos 330°+\sin 390°\cos 120°=\cos\frac{17\pi}{6}\cos\frac{11\pi}{6}+\sin\frac{13\pi}{6}\cos\frac{2\pi}{3}$

$$=\cos\left(5\times\frac{\pi}{2}+\frac{\pi}{3}\right)\cos\left(3\times\frac{\pi}{2}+\frac{\pi}{3}\right)+\sin\left(4\times\frac{\pi}{2}+\frac{\pi}{6}\right)\cos\left(1\times\frac{\pi}{2}+\frac{\pi}{6}\right)$$

$$=\left(-\sin\frac{\pi}{3}\right)\left(\sin\frac{\pi}{3}\right)+\left(\sin\frac{\pi}{6}\right)\left(-\sin\frac{\pi}{6}\right)$$

$$=-\frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2}+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)=-\frac{3}{4}-\frac{1}{4}=-1=\text{RHS}$$

EXAMPLE 4 *Prove that :* $\sin(-420°)(\cos 390°)+\cos(-660°)(\sin 330°)=-1.$

SOLUTION We know that $\sin(-x)=-\sin x$ and $\cos(-x)=\cos x$

$\therefore\quad$ LHS $=\sin(-420°)(\cos 390°)+\cos(-660°)(\sin 330°)$

$$=-\sin 420°\cos 390°+\cos 660°\sin 330°=-\sin\frac{7\pi}{3}\cos\frac{13\pi}{6}+\cos\frac{11\pi}{3}\sin\frac{11\pi}{6}$$

$$=-\sin\left(4\times\frac{\pi}{2}+\frac{\pi}{3}\right)\cos\left(4\times\frac{\pi}{2}+\frac{\pi}{6}\right)+\cos\left(7\times\frac{\pi}{2}+\frac{\pi}{6}\right)\sin\left(3\times\frac{\pi}{2}+\frac{\pi}{3}\right)$$

$$=-\sin\frac{\pi}{3}\cos\frac{\pi}{6}+\left(\sin\frac{\pi}{6}\right)\left(-\cos\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2}+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)=-\frac{3}{4}-\frac{1}{4}=-1=\text{RHS}.$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 5 *Prove that:*

(i) $\sin^2\frac{\pi}{6}+\cos^2\frac{\pi}{3}-\tan^2\frac{\pi}{4}=-\frac{1}{2}$ (ii) $2\sin^2\frac{\pi}{6}+\text{cosec}^2\frac{7\pi}{6}\cos^2\frac{\pi}{3}=\frac{3}{2}$

(iii) $\cot^2\frac{\pi}{6}+\text{cosec}\frac{5\pi}{6}+3\tan^2\frac{\pi}{6}=6$ (iv) $2\sin^2\frac{3\pi}{4}+2\cos^2\frac{\pi}{4}+2\sec^2\frac{\pi}{3}=10$

SOLUTION (i) LHS $=\sin^2\frac{\pi}{6}+\cos^2\frac{\pi}{3}-\tan^2\frac{\pi}{4}$

$$=\left(\sin\frac{\pi}{6}\right)^2+\left(\cos\frac{\pi}{3}\right)^2-\left(\tan\frac{\pi}{4}\right)^2=\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2-(1)^2=\frac{1}{4}+\frac{1}{4}-1=-\frac{1}{2}$$

(ii) LHS $=2\sin^2\frac{\pi}{6}+\text{cosec}^2\frac{7\pi}{6}\cos^2\frac{\pi}{3}=2\left(\sin\frac{\pi}{6}\right)^2+\left(\text{cosec}\frac{7\pi}{6}\right)^2\left(\cos\frac{\pi}{3}\right)^2$

$$= 2\left(\sin\frac{\pi}{6}\right)^2 + \left\{\text{cosec}\left(\pi+\frac{\pi}{6}\right)\right\}^2\left(\cos\frac{\pi}{3}\right)^2$$

$$= 2\left(\sin\frac{\pi}{6}\right)^2 + \left\{-\text{cosec}\frac{\pi}{6}\right\}^2\left(\cos\frac{\pi}{3}\right)^2 \qquad [\because \text{cosec}(\pi+x) = -\text{cosec}\, x]$$

$$= 2\left(\frac{1}{2}\right)^2 + (-2)^2\times\left(\frac{1}{2}\right)^2 = \frac{1}{2}+1 = \frac{3}{2} = \text{RHS}$$

(iii) $\text{LHS} = \cot^2\frac{\pi}{6} + \text{cosec}\frac{5\pi}{6} + 3\tan^2\frac{\pi}{6} = \left(\cot\frac{\pi}{6}\right)^2 + \text{cosec}\left(\pi-\frac{\pi}{6}\right) + 3\left(\tan\frac{\pi}{6}\right)^2$

$$= \left(\cot\frac{\pi}{6}\right)^2 + \text{cosec}\frac{\pi}{6} + 3\left(\tan\frac{\pi}{6}\right)^2 = (\sqrt{3})^2 + 2 + 3\left(\frac{1}{\sqrt{3}}\right)^2 = 3+2+1 = 6 = \text{RHS}$$

(iv) $\text{LHS} = 2\sin^2\frac{3\pi}{4} + 2\cos^2\frac{\pi}{4} + 2\sec^2\frac{\pi}{3} = 2\left(\sin\frac{3\pi}{4}\right)^2 + 2\left(\cos\frac{\pi}{4}\right)^2 + 2\left(\sec\frac{\pi}{3}\right)^2$

$$= 2\left(\sin\frac{\pi}{4}\right)^2 + 2\left(\cos\frac{\pi}{4}\right)^2 + 2\left(\sec\frac{\pi}{3}\right)^2 \qquad \left[\because \sin\frac{3\pi}{4} = \sin\left(\pi-\frac{\pi}{4}\right) = \sin\frac{\pi}{4}\right]$$

$$= 2\left(\frac{1}{\sqrt{2}}\right)^2 + 2\left(\frac{1}{\sqrt{2}}\right)^2 + 2(2)^2 = 1+1+8 = 10 = \text{RHS}$$

EXAMPLE 6 *Prove that:* $\dfrac{\cos\left(\frac{\pi}{2}+x\right)\sec(-x)\tan(\pi-x)}{\sec(2\pi-x)\sin(\pi+x)\cot\left(\frac{\pi}{2}-x\right)} = -1.$

SOLUTION We have,

$$\text{LHS} = \frac{\cos\left(\frac{\pi}{2}+x\right)\sec(-x)\tan(\pi-x)}{\sec(2\pi-x)\sin(\pi+x)\cot\left(\frac{\pi}{2}-x\right)} = \frac{(-\sin x)(\sec x)(-\tan x)}{(\sec x)(-\sin x)(\tan x)} = -1 = \text{RHS}$$

EXAMPLE 7 *Prove that :*

(i) $\dfrac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\frac{\pi}{2}+x\right)} = \cot^2 x$

(ii) $\cos\left(\frac{3\pi}{2}+x\right)\cos(2\pi+x)\left\{\cot\left(\frac{3\pi}{2}-x\right)+\cot(2\pi+x)\right\} = 1$

SOLUTION (i) $\text{LHS} = \dfrac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\frac{\pi}{2}+x\right)} = \dfrac{(-\cos x)\times(\cos x)}{(\sin x)(-\sin x)} = \dfrac{-\cos^2 x}{-\sin^2 x} = \cot^2 x = \text{RHS}$

(ii) We know that

$$\cos\left(\frac{3\pi}{2}+x\right) = \sin x,\ \cos(2\pi+x) = \cos x,\ \cot\left(\frac{3\pi}{2}-x\right) = \tan x \text{ and } \cot(2\pi+x) = \cot x$$

$$\therefore\ \text{LHS} = \cos\left(\frac{3\pi}{2}+x\right)\cos(2\pi+x)\left\{\cot\left(\frac{3\pi}{2}-x\right)+\cot(2\pi+x)\right\}$$

$$= (\sin x)(\cos x)(\tan x + \cot x) = \sin x\cos x\left\{\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}\right\}$$

$$= \sin x \cos x \left\{ \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} \right\} = \sin x \cos x \times \frac{1}{\sin x \cos x} = 1$$

EXAMPLE 8 *If A, B, C, D are angles of a cyclic quadrilateral, prove that* $\cos A + \cos B + \cos C + \cos D = 0$.

SOLUTION We know that the opposite angles of a cyclic quadrilateral are supplementary i.e. $A + C = \pi$ and $B + D = \pi$. Therefore, $A = \pi - C$ and $B = \pi - D$.

$\therefore \quad \cos A = \cos(\pi - C) = -\cos C$ and, $\cos B = \cos(\pi - D) = -\cos D$

Hence, $\cos A + \cos B + \cos C + \cos D = -\cos C - \cos D + \cos C + \cos D = 0$

EXAMPLE 9 *In any quadrilateral ABCD, prove that*

(i) $\sin(A+B) + \sin(C+D) = 0$ (ii) $\cos(A+B) = \cos(C+D)$

SOLUTION (i) $A + B + C + D = 2\pi$

$\Rightarrow \quad A + B = 2\pi - (C + D)$

$\Rightarrow \quad \sin(A+B) = \sin\{2\pi - (C+D)\}$

$\Rightarrow \quad \sin(A+B) = -\sin(C+D) \Rightarrow \sin(A+B) + \sin(C+D) = 0$ $[\because \sin(2\pi - x) = -\sin x]$

(ii) $\quad A + B + C + D = 2\pi$

$\Rightarrow \quad A + B = 2\pi - (C + D)$

$\Rightarrow \quad \cos(A+B) = \cos\{2\pi - (C+D)\} \Rightarrow \cos(A+B) = \cos(C+D)$ $[\because \cos(2\pi - x) = \cos x]$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 10 *Find the value of the expression*

$$3\left\{\sin^4\left(\frac{3\pi}{2} - x\right) + \sin^4(3\pi + x)\right\} - 2\left\{\sin^6\left(\frac{\pi}{2} + x\right) + \sin^6(5\pi - x)\right\}$$ [NCERT EXEMPLAR]

SOLUTION The given expression is

$$3\left\{\sin^4\left(\frac{3\pi}{2} - x\right) + \sin^4(3\pi + x)\right\} - 2\left\{\sin^6\left(\frac{\pi}{2} + x\right) + \sin^6(5\pi - x)\right\}$$

$$= 3\left\{(-\cos x)^4 + (-\sin x)^4\right\} - 2\left\{(\cos x)^6 + (\sin x)^6\right\} = 3(\cos^4 x + \sin^4 x) - 2(\cos^6 x + \sin^6 x)$$

$$= 3\left\{(\cos^2 x + \sin^2 x)^2 - 2\sin^2 x \cos^2 x\right\} - 2\left\{(\cos^2 x + \sin^2 x)^3 - 3\cos^2 x \sin^2 x(\cos^2 x + \sin^2 x)\right\}$$

$$= 3(1 - 2\sin^2 x\cos^2 x) - 2(1 - 3\cos^2 x \sin^2 x) = 3 - 6\sin^2 x\cos^2 x - 2 + 6\sin^2 x \cos^2 x = 1.$$

EXERCISE 5.3

BASIC

1. Find the values of the following trigonometric ratios:

(i) $\sin\frac{5\pi}{3}$ (ii) $\sin 17\pi$ (iii) $\tan\frac{11\pi}{6}$ (iv) $\cos\left(-\frac{25\pi}{4}\right)$

(v) $\tan\frac{7\pi}{4}$ (vi) $\sin\frac{17\pi}{6}$ (vii) $\cos\frac{19\pi}{6}$ (viii) $\sin\left(-\frac{11\pi}{6}\right)$

(ix) $\text{cosec}\left(-\frac{20\pi}{3}\right)$ (x) $\tan\left(-\frac{13\pi}{4}\right)$ (xi) $\cos\frac{19\pi}{4}$ (xii) $\sin\frac{41\pi}{4}$

(xiii) $\cos\frac{39\pi}{4}$ (xiv) $\sin\frac{151\pi}{6}$

2. Prove that:

(i) $\tan 225° \cot 405° + \tan 765° \cot 675° = 0$

(ii) $\sin \frac{8\pi}{3} \cos \frac{23\pi}{6} + \cos \frac{13\pi}{3} \sin \frac{35\pi}{6} = \frac{1}{2}$

(iii) $\cos 24° + \cos 55° + \cos 125° + \cos 204° + \cos 300° = \frac{1}{2}$

(iv) $\tan(-225°) \cot(-405°) - \tan(-765°) \cot(675°) = 0$

(v) $\cos 570° \sin 510° + \sin(-330°) \cos(-390°) = 0$

(vi) $\tan \frac{11\pi}{3} - 2 \sin \frac{4\pi}{6} - \frac{3}{4} \operatorname{cosec}^2 \frac{\pi}{4} + 4 \cos^2 \frac{17\pi}{6} = \frac{3 - 4\sqrt{3}}{2}$

(vii) $3 \sin \frac{\pi}{6} \sec \frac{\pi}{3} - 4 \sin \frac{5\pi}{6} \cot \frac{\pi}{4} = 1$

3. Prove that:

(i) $\dfrac{\cos(2\pi + x) \operatorname{cosec}(2\pi + x) \tan(\pi/2 + x)}{\sec(\pi/2 + x) \cos x \cot(\pi + x)} = 1$

(ii) $\dfrac{\operatorname{cosec}(90° + x) + \cot(450° + x)}{\operatorname{cosec}(90° - x) + \tan(180° - x)} + \dfrac{\tan(180° + x) + \sec(180° - x)}{\tan(360° + x) - \sec(-x)} = 2$

(iii) $\dfrac{\sin(\pi + x) \cos\left(\frac{\pi}{2} + x\right) \tan\left(\frac{3\pi}{2} - x\right) \cot(2\pi - x)}{\sin(2\pi - x) \cos(2\pi + x) \operatorname{cosec}(-x) \sin\left(\frac{3\pi}{2} - x\right)} = 1$

(iv) $\left\{1 + \cot x - \sec\left(\frac{\pi}{2} + x\right)\right\} \left\{1 + \cot x + \sec\left(\frac{\pi}{2} + x\right)\right\} = 2 \cot x$

(v) $\dfrac{\tan\left(\frac{\pi}{2} - x\right) \sec(\pi - x) \sin(-x)}{\sin(\pi + x) \cot(2\pi - x) \operatorname{cosec}\left(\frac{\pi}{2} - x\right)} = 1$

BASED ON LOTS

4. Prove that : $\sin^2 \frac{\pi}{18} + \sin^2 \frac{\pi}{9} + \sin^2 \frac{7\pi}{18} + \sin^2 \frac{4\pi}{9} = 2$

5. Prove that : $\sec\left(\frac{3\pi}{2} - x\right) \sec\left(x - \frac{5\pi}{2}\right) + \tan\left(\frac{5\pi}{2} + x\right) \tan\left(x - \frac{3\pi}{2}\right) = -1.$

6. In a ΔABC, prove that :

(i) $\cos(A + B) + \cos C = 0$ (ii) $\cos\left(\frac{A + B}{2}\right) = \sin \frac{C}{2}$ (iii) $\tan \frac{A + B}{2} = \cot \frac{C}{2}$

7. If A, B, C, D be the angles of a cyclic quadrilateral, taken in order, prove that :
$\cos(180° - A) + \cos(180° + B) + \cos(180° + C) - \sin(90° + D) = 0$

8. Find x from the following equations :

(i) $\operatorname{cosec}\left(\frac{\pi}{2} + \theta\right) + x \cos \theta \cot\left(\frac{\pi}{2} + \theta\right) = \sin\left(\frac{\pi}{2} + \theta\right)$

(ii) $x \cot\left(\frac{\pi}{2} + \theta\right) + \tan\left(\frac{\pi}{2} + \theta\right) \sin \theta + \operatorname{cosec}\left(\frac{\pi}{2} + \theta\right) = 0$

9. Prove that:

(i) $\tan 4\pi - \cos \frac{3\pi}{2} - \sin \frac{5\pi}{6} \cos \frac{2\pi}{3} = \frac{1}{4}$ (ii) $\sin \frac{13\pi}{3} \sin \frac{8\pi}{3} + \cos \frac{2\pi}{3} \sin \frac{5\pi}{6} = \frac{1}{2}$

(iii) $\sin\frac{13\pi}{3}\sin\frac{2\pi}{3}+\cos\frac{4\pi}{3}\sin\frac{13\pi}{6}=\frac{1}{2}$ (iv) $\sin\frac{10\pi}{3}\cos\frac{13\pi}{6}+\cos\frac{8\pi}{3}\sin\frac{5\pi}{6}=-1$

(v) $\tan\frac{5\pi}{4}\cot\frac{9\pi}{4}+\tan\frac{17\pi}{4}\cot\frac{15\pi}{4}=0$

ANSWERS

1. (i) $-\frac{\sqrt{3}}{2}$ (ii) 0 (iii) $-\frac{1}{\sqrt{3}}$ (iv) $\frac{1}{\sqrt{2}}$ (v) -1 (vi) $\frac{1}{2}$
(vii) $-\frac{\sqrt{3}}{2}$ (viii) $\frac{1}{2}$ (ix) $-\frac{2}{\sqrt{3}}$ (x) -1 (xi) $-\frac{1}{\sqrt{2}}$ (xii) $\frac{1}{\sqrt{2}}$
(xiii) $1/\sqrt{2}$ (xiv) $-1/2$

8. (i) $\tan\theta$ (ii) $\sin\theta$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. The value of $\frac{\sin 70^\circ}{\sin 110^\circ}$ is ....
2. The value of $\frac{\cos 50^\circ}{\cos 130^\circ}$ is ....
3. The values of $f(x)=2\sin\sqrt{x^2+x+1}$ lie in the interval ....
4. If $\sin x+\cos x=a$, then $\sin x-\cos x=$ ....
5. If $-\frac{\pi}{2}<x<\frac{\pi}{2}$, then $\sqrt{\frac{1-\sin x}{1+\sin x}}$ is equal to ....
6. If $\sec x+\tan x=\sqrt{3}$, then $\sec x-\tan x=$ ....
7. If $\sec x-\tan x=\frac{2}{3}$, the $\tan x=$ ....
8. If $\operatorname{cosec} x+\cot x=\alpha$, then $\sin x=$ ....
9. If $\operatorname{cosec} x+\cot x=\frac{11}{2}$, then the value of $\tan x$ is ....
10. If $\sin x=\frac{-24}{25}$, then the value of $\tan x$ is ....
11. If $\sin x+\operatorname{cosec} x=2$, then $\sin^2 x+\operatorname{cosec}^2 x=$ ....
12. The value of $\tan x+\cot(\pi+x)+\cot\left(\frac{\pi}{2}+x\right)+\cot(2\pi-x)$ is ....
13. The value of $3(\sin x-\cos x)^4+6(\sin x+\cos x)^2+4(\sin^6 x+\cos^6 x)$ is ....
14. Given $x>0$, the value of $f(x)=-3\cos\sqrt{3+x+x^2}$ lie in the interval ....
15. If $\sin x+\cos x=a$, then $\sin^6 x+\cos^6 x=$ ....
16. The value of $\cos 1^\circ\cos 2^\circ\cos 3^\circ\ldots\ldots\cos 179^\circ$ is ....
17. The value of $\tan 1^\circ\tan 2^\circ\tan 3^\circ\ldots\ldots\tan 89^\circ$ is ....
18. If $\frac{\pi}{2}<x<\pi$ and $\sqrt{\frac{1+\sin x}{1-\sin x}}=k\sec x$, then $k=$ ....
19. If $\frac{\pi}{2}<x<\pi$ and $\sqrt{\frac{1+\sin x}{1-\sin x}}+\sqrt{\frac{1-\sin x}{1+\sin x}}=k\sec x$, then $k=$ ....

20. If $\pi < x < 2\pi$ and $\sqrt{\dfrac{1+\cos x}{1-\cos x}} + \sqrt{\dfrac{1-\cos x}{1+\cos x}} = k \operatorname{cosec} x$, then $k =$ ....

21. The minimum value of $9\tan^2\theta + 4\cot^2\theta$ is ....

22. If $\sec x = m$ and $\tan x = n$, then $\dfrac{1}{m}\left\{(m+n) + \dfrac{1}{m+n}\right\}$ is equal to ....

23. If $\cos^2 x + \sin x + 1 = 0$, and $0 < x < 2\pi$ then $x =$ ....

24. If $\sin x = \dfrac{2t}{1+t^2}$ and x lies in the second quadrant, then $\cos x =$ ....

25. If $\sec x = t + \dfrac{1}{4t}$, then the value of $\sec x + \tan x$ is ....

26. If $\tan x + \cot x = 4$, then $\tan^4 x + \cot^4 x =$ ....

27. If $\dfrac{3\pi}{4} < x < \pi$, then $\sqrt{\operatorname{cosec}^2 x + 2\cot x}$ is equal to ....

ANSWERS

1. 1 2. -1 3. $[-2, 2]$ 4. $\sqrt{2-a^2}$ 5. $\sec x - \tan x$ 6. $\dfrac{1}{\sqrt{3}}$ 7. $\dfrac{5}{12}$

8. $\dfrac{2\alpha}{\alpha^2+1}$ 9. $\dfrac{44}{117}$ 10. $-\dfrac{24}{7}$ 11. 2 12. 0 13. 13 14. $[-3, 3]$

15. $\dfrac{1}{4}[4-3(a^2-1)^2]$ 16. 0 17. 1 18. -1 19. -2 20. -2 21. 12

22. 2 23. $\dfrac{3\pi}{2}$ 24. $\dfrac{|1-t^2|}{1+t^2}$ 25. $2t$ 26. 194 27. $-1-\cot x$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the maximum and minimum values of $\cos(\cos x)$.
2. Write the maximum and minimum values of $\sin(\sin x)$.
3. Write the maximum value of $\sin(\cos x)$.
4. If $\sin x = \cos^2 x$, then write the value of $\cos^2 x(1+\cos^2 x)$.
5. If $\sin x + \operatorname{cosec} x = 2$, then write the value of $\sin^n x + \operatorname{cosec}^n x$.
6. If $\sin x + \sin^2 x = 1$, then write the value of $\cos^{12} x + 3\cos^{10} x + 3\cos^8 x + \cos^6 x$.
7. If $\sin x + \sin^2 x = 1$, then write the value of $\cos^8 x + 2\cos^6 x + \cos^4 x$.
8. If $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 3$, then write the value of $\cos\theta_1 + \cos\theta_2 + \cos\theta_3$.
9. Write the value of $\sin 10° + \sin 20° + \sin 30° + \ldots + \sin 360°$.
10. A circular wire of radius 15 cm is cut and bent so as to lie along the circumference of a loop of radius 120 cm. Write the measure of the angle subtended by it at the centre of the loop.
11. Write the value of $2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1$.
12. Write the value of $\cos 1° + \cos 2° + \cos 3° + \ldots + \cos 180°$.
13. If $\cot(\alpha+\beta) = 0$, then write the value of $\sin(\alpha + 2\beta)$.

14. If $\tan A + \cot A = 4$, then write the value of $\tan^4 A + \cot^4 A$.

15. Write the least value of $\cos^2 x + \sec^2 x$.

16. If $x = \sin^{14} x + \cos^{20} x$, then write the smallest interval in which the value of x lie.

17. If $3 \sin x + 5 \cos x = 5$, then write the value of $5 \sin x - 3 \cos x$.

ANSWERS

1. 1, cos 1 2. sin 1, – sin 1 3. sin 1 4. 1 5. 2 6. 1 7. 1 8. 0
9. 0 10. 45° 11. 0 12. –1 13. $\sin \alpha$ or $\cos \beta$ 14. 194 15. 2 16. (0, 1]
17. 3 or – 3

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If $\tan x = x - \frac{1}{4x}$, then $\sec x - \tan x$ is equal to

(a) $-2x, \frac{1}{2x}$ (b) $-\frac{1}{2x}, 2x$ (c) $2x$ (d) $2x, \frac{1}{2x}$

2. If $\sec x = x + \frac{1}{4x}$, then $\sec x + \tan x =$

(a) $x, \frac{1}{x}$ (b) $2x, \frac{1}{2x}$ (c) $-2x, \frac{1}{2x}$ (d) $-\frac{1}{x}, x$

3. If $\frac{\pi}{2} < x < \frac{3\pi}{2}$, then $\sqrt{\frac{1 - \sin x}{1 + \sin x}}$ is equal to

(a) $\sec x - \tan x$ (b) $\sec x + \tan x$ (c) $\tan x - \sec x$ (d) none of these

4. If $\pi < x < 2\pi$, then $\sqrt{\frac{1 + \cos x}{1 - \cos x}}$ is equal to

(a) $\operatorname{cosec} x + \cot x$ (b) $\operatorname{cosec} x - \cot x$ (c) $-\operatorname{cosec} x + \cot x$ (d) $-\operatorname{cosec} x - \cot x$

5. If $0 < x < \frac{\pi}{2}$, and if $\frac{y+1}{1-y} = \sqrt{\frac{1 + \sin x}{1 - \sin x}}$, then y is equal to

(a) $\cot \frac{x}{2}$ (b) $\tan \frac{x}{2}$ (c) $\cot \frac{x}{2} + \tan \frac{x}{2}$ (d) $\cot \frac{x}{2} - \tan \frac{x}{2}$

6. If $\frac{\pi}{2} < x < \pi$, then $\sqrt{\frac{1 - \sin x}{1 + \sin x}} + \sqrt{\frac{1 + \sin x}{1 - \sin x}}$ *is equal to*

(a) $2 \sec x$ (b) $-2 \sec x$ (c) $\sec x$ (d) $-\sec x$

7. If $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$ and $z = r \cos \theta$, then $x^2 + y^2 + z^2$ is independent of

(a) θ, ϕ (b) r, θ (c) r, ϕ (d) r

8. If $\tan x + \sec x = \sqrt{3}$, $0 < x < \pi$, then x is equal to

(a) $\frac{5\pi}{6}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$

9. If $\tan x = -\frac{1}{\sqrt{5}}$ and x lies in the IV quadrant, then the value of $\cos x$ is

(a) $\frac{\sqrt{5}}{\sqrt{6}}$ (b) $\frac{2}{\sqrt{6}}$ (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{6}}$

10. If $\frac{3\pi}{4} < \alpha < \pi$, then $\sqrt{2 \cot \alpha + \frac{1}{\sin^2 \alpha}}$ is equal to

(a) $1 - \cot \alpha$ (b) $1 + \cot \alpha$ (c) $-1 + \cot \alpha$ (d) $-1 - \cot \alpha$

11. $\sin^6 A + \cos^6 A + 3 \sin^2 A \cos^2 A =$

(a) 0 (b) 1 (c) 2 (d) 3

12. If $\operatorname{cosec} x - \cot x = \frac{1}{2}$, $0 < x < \frac{\pi}{2}$, then $\cos x$ is equal to

(a) $\frac{5}{3}$ (b) $\frac{3}{5}$ (c) $-\frac{3}{5}$ (d) $-\frac{5}{3}$

13. If $\operatorname{cosec} x + \cot x = \frac{11}{2}$, then $\tan x =$

(a) $\frac{21}{22}$ (b) $\frac{15}{16}$ (c) $\frac{44}{117}$ (d) $\frac{117}{44}$

14. $\sec^2 x = \frac{4xy}{(x+y)^2}$ is true if and only if

(a) $x + y \neq 0$ (b) $x = y, x \neq 0$ (c) $x = y$ (d) $x \neq 0, y \neq 0$

15. If x is an acute angle and $\tan x = \frac{1}{\sqrt{7}}$, then the value of $\frac{\operatorname{cosec}^2 x - \sec^2 x}{\operatorname{cosec}^2 x + \sec^2 x}$ is

(a) 3/4 (b) 1/2 (c) 2 (d) 5/4

16. The value of $\sin^2 5° + \sin^2 10° + \sin^2 15° + \ldots + \sin^2 85° + \sin^2 90°$ is

(a) 7 (b) 8 (c) 9.5 (d) 10

17. $\sin^2 \frac{\pi}{18} + \sin^2 \frac{\pi}{9} + \sin^2 \frac{7\pi}{18} + \sin^2 \frac{4\pi}{9} =$

(a) 1 (b) 4 (c) 2 (d) 0

18. If $\tan A + \cot A = 4$, then $\tan^4 A + \cot^4 A$ is equal to

(a) 110 (b) 191 (c) 80 (d) 194

19. If $x \sin 45° \cos^2 60° = \frac{\tan^2 60° \operatorname{cosec} 30°}{\sec 45° \cot^{2°} 30°}$, then $x =$

(a) 2 (b) 4 (c) 8 (d) 16

20. If A lies in second quadrant and $3 \tan A + 4 = 0$, then the value of $2 \cot A - 5 \cos A + \sin A$ is equal to

(a) –53/10 (b) 23/10 (c) 37/10 (d) 7/10

[NCERT EXEMPLAR]

21. If $\operatorname{cosec} x + \cot x = \frac{11}{2}$, then $\tan x =$

(a) 21/22 (b) 15/16 (c) 44/117 (d) 117/43

22. If $\tan \theta + \sec \theta = e^x$, then $\cos \theta$ equals

(a) $\frac{e^x + e^{-x}}{2}$ (b) $\frac{2}{e^x + e^{-x}}$ (c) $\frac{e^x - e^{-x}}{2}$ (d) $\frac{e^x - e^{-x}}{e^x + e^{-x}}$

23. If $\sec x + \tan x = k$, $\cos x =$

(a) $\frac{k^2+1}{2k}$ (b) $\frac{2k}{k^2+1}$ (c) $\frac{k}{k^2+1}$ (d) $\frac{k}{k^2-1}$

24. If $f(x) = \cos^2 x + \sec^2 x$, then

(a) $f(x) < 1$ (b) $f(x) = 1$ (c) $1 < f(x) < 2$ (d) $f(x) \geq 2$

[NCERT EXEMPLAR]

25. Which of the following is incorrect?

(a) $\sin x = -1/5$ (b) $\cos x = 1$ (c) $\sec x = 1/2$ (d) $\tan x = 20$

26. The value of $\cos 1° \cos 2° \cos 3° \ldots \cos 179°$ is

(a) $1/\sqrt{2}$ (b) 0 (c) 1 (d) -1

[NCERT EXEMPLAR]

27. The value of $\tan 1° \tan 2° \tan 3° \ldots \tan 89°$ is

(a) 0 (b) 1 (c) 1/2 (d) not defined

[NCERT EXEMPLAR]

28. Which of the following is correct?

(a) $\sin 1° > \sin 1$ (b) $\sin 1° < \sin 1$ (c) $\sin 1° = \sin 1$ (d) $\sin 1° = \frac{\pi}{180} \sin 1$

[NCERT EXEMPLAR]

29. If $\tan\theta = -\frac{4}{3}$, then $\sin\theta$ is equal to

(a) $-\frac{4}{5}$ but not $\frac{4}{5}$ (b) $-\frac{4}{5}$ or $\frac{4}{5}$ (c) $\frac{4}{5}$ but not $-\frac{4}{5}$ (d) none of these

[NCERT EXEMPLAR]

30. If $\sin\theta$ and $\cos\theta$ are the roots of the equation $ax^2 - bx + c = 0$, then a, b and c satisfy the relation

(a) $a^2 + b^2 + 2ac = 0$ (b) $a^2 - b^2 + 2ac = 0$ (c) $a^2 + c^2 + 2ab = 0$ (d) $a^2 - b^2 - 2ac = 0$

[NCERT EXEMPLAR]

31. If $\sin\theta + \text{cosec}\,\theta = 2$, then $\sin^2\theta + \text{cosec}^2\theta$ is equal to

(a) 1 (b) 4 (c) 2 (d) none of these

32. Which of the following is incorrect?

(a) $\sin\theta = -\frac{1}{5}$ (b) $\cos\theta = 1$ (c) $\sec\theta = \frac{1}{2}$ (d) $\tan\theta = 20$

[NCERT EXEMPLAR]

33. If for real values of x, $\cos\theta = x + \frac{1}{x}$, then

(a) θ is an acute angle (b) θ is a right angle

(c) θ is an obtuse angle (d) No value of θ is possible

ANSWERS

1. (a)	2. (b)	3. (c)	4. (d)	5. (b)	6. (b)	7. (a)	8. (c)
9. (a)	10. (d)	11. (b)	12. (b)	13. (c)	14. (b)	15. (a)	16. (c)
17. (c)	18. (d)	19. (c)	20. (b)	21. (c)	22. (b)	23. (b)	24. (d)
25. (c)	26. (b)	27. (b)	28. (b)	29. (b)	30. (b)	31. (c)	32. (c)
33. (d)							

SUMMARY

1. Following are some of the fundamental trigonometric identities:

(i) $\sin x = \dfrac{1}{\operatorname{cosec} x}$ or, $\operatorname{cosec} x = \dfrac{1}{\sin x}$

(ii) $\cos x = \dfrac{1}{\sec x}$ or, $\sec x = \dfrac{1}{\cos x}$ (iii) $\cot x = \dfrac{1}{\tan x}$ or, $\tan x = \dfrac{1}{\cot x}$

(iv) $\tan x = \dfrac{\sin x}{\cos x}$ or, $\cot x = \dfrac{\cos x}{\sin x}$ (v) $\sin^2 x + \cos^2 x = 1$

(vi) $1 + \tan^2 x = \sec^2 x$ or, $\sec x - \tan x = \dfrac{1}{\sec x + \tan x}$

(vii) $1 + \cot^2 x = \operatorname{cosec}^2 x$ or, $\operatorname{cosec} x - \cot x = \dfrac{1}{\operatorname{cosec} x + \cot x}$

2. (i) $\sin(-x) = -\sin x$ or, $\operatorname{cosec}(-x) = -\operatorname{cosec} x$

(ii) $\cos(-x) = \cos x$ or, $\sec(-x) = \sec x$

(iii) $\tan(-x) = -\tan x$ or, $\cot(-x) = -\cot x$

(iv) $\sin\left(\dfrac{\pi}{2}-x\right) = \cos x$, $\cos\left(\dfrac{\pi}{2}-x\right) = \sin x$, $\tan\left(\dfrac{\pi}{2}-x\right) = \cot x$, $\sec\left(\dfrac{\pi}{2}-x\right) = \operatorname{cosec} x$

$\operatorname{cosec}\left(\dfrac{\pi}{2}-x\right) = \sec x$, $\cot\left(\dfrac{\pi}{2}-x\right) = \tan x$

(v) $\sin\left(\dfrac{\pi}{2}+x\right) = \cos x$, $\cos\left(\dfrac{\pi}{2}+x\right) = -\sin x$, $\tan\left(\dfrac{\pi}{2}+x\right) = -\cot x$, $\cot\left(\dfrac{\pi}{2}+x\right) = -\tan x$,

$\sec\left(\dfrac{\pi}{2}+x\right) = -\operatorname{cosec} x$, $\operatorname{cosec}\left(\dfrac{\pi}{2}+x\right) = \sec x$

(vi) $\sin(\pi-x) = \sin x$, $\cos(\pi-x) = -\cos x$, $\tan(\pi-x) = -\tan x$, $\cot(\pi-x) = -\cot x$

$\sec(\pi-x) = -\sec x$, $\operatorname{cosec}(\pi-x) = \operatorname{cosec} x$

(vii) $\sin\left(\dfrac{3\pi}{2}-x\right) = -\cos x$, $\cos\left(\dfrac{3\pi}{2}-x\right) = -\sin x$,

$\tan\left(\dfrac{3\pi}{2}-x\right) = \cot x$, $\cot\left(\dfrac{3\pi}{2}-x\right) = \tan x$

$\operatorname{cosec}\left(\dfrac{3\pi}{2}-x\right) = -\sec x$, $\sec\left(\dfrac{3\pi}{2}-x\right) = -\operatorname{cosec} x$

(viii) $\sin\left(\dfrac{3\pi}{2}+x\right) = -\cos x$, $\cos\left(\dfrac{3\pi}{2}+x\right) = \sin x$,

$\tan\left(\dfrac{3\pi}{2}+x\right) = -\cot x$, $\cot\left(\dfrac{3\pi}{2}+x\right) = -\tan x$

$\operatorname{cosec}\left(\dfrac{3\pi}{2}+x\right) = -\sec x$, $\sec\left(\dfrac{3\pi}{2}+x\right) = \operatorname{cosec} x$

(ix) $\sin(2\pi-x) = -\sin x$, $\cos(2\pi-x) = \cos x$,

$\tan(2\pi-x) = -\tan x$, $\operatorname{cosec}(2\pi-x) = -\operatorname{cosec} x$

$\sec(2\pi-x) = \sec x$, $\cot(2\pi-x) = -\cot x$

(x) Sine and Cosine functions and their reciprocals i.e. Cosecant and Secant functions are periodic functions with period 2π. Tangent and Cotangent functions are periodic with period π.

(xi) Cosine and secant functions are even functions and all other trigonometric functions are odd functions.