CHAPTER 7

VALUES OF TRIGONOMETRIC FUNCTIONS AT SUM OR DIFFERENCE OF ANGLES

7.1 INTRODUCTION

In this chapter, we shall derive formulae which will express the values of trigonometric functions at the sum or difference of two real numbers (or angles) in terms of the values of trigonometric functions at individual numbers (or anlages).

7.2 VALUES OF TRIGONOMETRIC FUNCTIONS AT THE SUM OR DIFFERENCE

7.2.1 COSINE OF THE DIFFERENCE AND SUM OF TWO NUMBERS

THEOREM *For all values of A and B, prove that*

(i) $\cos(A-B) = \cos A \cos B + \sin A \sin B$ (ii) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

PROOF (i) Let $X'OX$ and YOY' be the coordinate axes. Consider a unit circle with O as the centre.

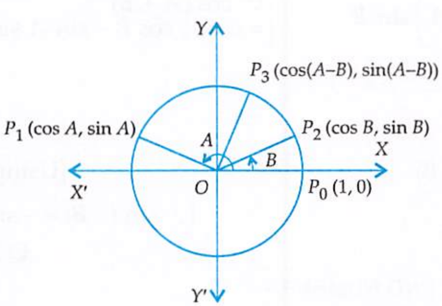

Fig. 7.1

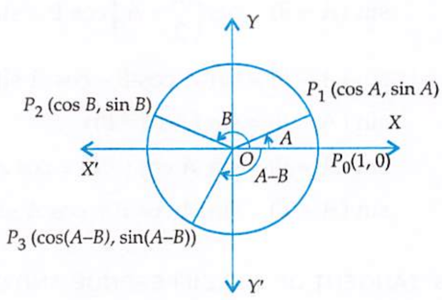

Fig. 7.2

Let P_1, P_2 and P_3 be three points on the circles such that $\angle XOP_1 = A$, $\angle XOP_2 = B$ and $\angle XOP_3 = A - B$ (see Fig. 7.1, 7.2). As we have seen in Section 5.2 that the terminal side of any angle intersects the circle with centre at O and unit radius at a point whose coordinates are respectively the cosine and sine of the angle. Therefore, coordinates of P_1, P_2 and P_3 are $(\cos A, \sin A)$, $(\cos B, \sin B)$ and $(\cos(A-B), \sin(A-B))$ respectively.

We know that equal chords of a circle make equal angles at its centre and chords $P_0 P_3$ and $P_1 P_2$ subtend equal angles at O. Therefore,

$\therefore$ Chord $P_0 P_3$ = Chord $P_1 P_2$

$$\Rightarrow \quad \sqrt{\{\cos(A-B)-1\}^2 + \{\sin(A-B)-0\}^2} = \sqrt{(\cos B - \cos A)^2 + (\sin B - \sin A)^2}$$

$$\Rightarrow \quad \{\cos(A-B)-1\}^2 + \sin^2(A-B) = (\cos B - \cos A)^2 + (\sin B - \sin A)^2$$

$\Rightarrow \quad \cos^2(A-B) - 2\cos(A-B) + 1 + \sin^2(A-B) = \cos^2 B + \cos^2 A - 2\cos A \cos B$
$+ \sin^2 B + \sin^2 A - 2\sin A \sin B$

$\Rightarrow \quad 2 - 2\cos(A-B) = 2 - 2\cos A \cos B - 2\sin A \sin B$

$\Rightarrow \quad \cos(A-B) = \cos A \cos B + \sin A \sin B$

Hence, $\cos(A-B) = \cos A \cos B + \sin A \sin B$

(ii) Clearly, $\cos(A+B) = \cos(A-(-B))$

$\Rightarrow \quad \cos(A+B) = \cos A \cos(-B) + \sin A \sin(-B)$ [Using (i)]

$\Rightarrow \quad \cos(A+B) = \cos A \cos B + \sin A \sin B$ $[\because \cos(-B) = \cos B, \sin(-B) = -\sin B]$

Hence, $\cos(A+B) = \cos A \cos B - \sin A \sin B$

Q.E.D.

REMARK *This method of proof of the above formulae is true for all values of A and B whether positive, zero or negative.*

7.2.2 SINE OF THE DIFFERENCE AND SUM OF TWO NUMBERS

THEOREM *For all values of A and B, prove that*

(i) $\sin(A-B) = \sin A \cos B - \cos A \sin B$ (ii) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

PROOF (i) $\sin(A-B) = \cos\left\{\frac{\pi}{2} - (A-B)\right\}$ $\left[\because \cos\left(\frac{\pi}{2} - x\right) = \sin x\right]$

$\Rightarrow \quad \sin(A-B) = \cos\left\{\left(\frac{\pi}{2} - A\right) + B\right\}$

$\Rightarrow \quad \sin(A-B) = \cos\left(\frac{\pi}{2} - A\right)\cos B - \sin\left(\frac{\pi}{2} - A\right)\sin B$ $\left[\begin{array}{l}\because \cos(A+B) \\ = \cos A \cos B - \sin A \sin B\end{array}\right]$

$\Rightarrow \quad \sin(A-B) = \sin A \cos B - \cos A \sin B$

(ii) $\quad \sin(A+B) = \sin(A-(-B))$

$\Rightarrow \quad \sin(A+B) = \sin A \cos(-B) - \cos A \sin(-B)$ [Using (i)]

$\Rightarrow \quad \sin(A+B) = \sin A \cos B + \cos A \sin B$ $[\because \sin(-B) = -\sin B]$

Q.E.D.

7.2.3 TANGENT OF THE DIFFERENCE AND SUM OF TWO NUMBERS

THEOREM *For those values of A and B for which both sides are defined, prove that*

(i) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ (ii) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

PROOF (i) $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)}$

$\Rightarrow \quad \tan(A+B) = \dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$

$\Rightarrow \quad \tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ [On dividing the numerator and denominator by $\cos A \cos B$]

(ii) $\quad \tan(A-B) = \tan(A+(-B))$

$\Rightarrow \quad \tan(A-B) = \dfrac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)}$ [Using (i)]

$$\Rightarrow \quad \tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$$

Q.E.D.

Similarly, it can be proved that

$$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot B + \cot A} \text{ and, } \cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

7.3 MORE USEFUL RESULTS

THEOREM *Prove that:*

(i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(iii) $\sin(A+B+C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$

(iv) $\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$

(v) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

PROOF (i)
$$\begin{aligned}\sin(A+B)\sin(A-B) &= (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B)\\ &= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B\\ &= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B\\ &= \sin^2 A - \sin^2 A \sin^2 B - \sin^2 B + \sin^2 A \sin^2 B\\ &= \sin^2 A - \sin^2 B = (1-\cos^2 A) - (1-\cos^2 B) = \cos^2 B - \cos^2 A\end{aligned}$$

(ii)
$$\begin{aligned}\cos(A+B)\cos(A-B) &= (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B)\\ &= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B\\ &= \cos^2 A(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B\\ &= \cos^2 A - \sin^2 B\\ &= (1-\sin^2 A) - (1-\cos^2 B) = \cos^2 B - \sin^2 A\end{aligned}$$

(iii)
$$\begin{aligned}&\sin(A+B+C)\\ &= \sin((A+B)+C)\\ &= \sin(A+B)\cos C + \cos(A+B)\sin C\\ &= (\sin A \cos B + \cos A \sin B)\cos C + (\cos A \cos B - \sin A \sin B)\sin C\\ &= \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C\end{aligned}$$

(iv)
$$\begin{aligned}&\cos(A+B+C)\\ &= \cos((A+B)+C)\\ &= \cos(A+B)\cos C - \sin(A+B)\sin C\\ &= (\cos A \cos B - \sin A \sin B)\cos C - (\sin A \cos B + \cos A \sin B)\sin C\\ &= \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C\end{aligned}$$

(v)
$$\tan(A+B+C) = \tan\{(A+B)+C\} = \frac{\tan(A+B) + \tan C}{1 - \tan(A+B)\tan C}$$

$$= \frac{\left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right) + \tan C}{1 - \left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right) \tan C}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

Q.E.D.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I **ON FINDING THE VALUES OF $\sin(A \pm B)$, $\cos(A \pm B)$ AND $\tan(A \pm B)$ WHEN VALUES OF ONE OF THE TRIGONOMETRIC FUNCTIONS AT A AND B ARE GIVEN**

EXAMPLE 1 *If* $\sin A = \frac{3}{5}$ *and* $\cos B = \frac{9}{41}$, $0 < A < \frac{\pi}{2}$, $0 < B < \frac{\pi}{2}$, *find the values of the following:*

(i) $\sin(A - B)$ (ii) $\sin(A + B)$ (iii) $\cos(A - B)$ (iv) $\cos(A + B)$

SOLUTION We have, $\sin A = \frac{3}{5}$ and, $\cos B = \frac{9}{41}$, where $0 < A, B < \frac{\pi}{2}$.

$\therefore \quad \cos A = \sqrt{1 - \sin^2 A}$ and, $\sin B = \sqrt{1 - \cos^2 B}$

$\Rightarrow \quad \cos A = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$ and, $\sin B = \sqrt{1 - \frac{81}{1681}} = \frac{40}{41}$

(i) $\sin(A - B) = \sin A \cos B - \cos A \sin B = \frac{3}{5} \times \frac{9}{41} - \frac{4}{5} \times \frac{40}{41} = -\frac{133}{205}$

(ii) $\sin(A + B) = \sin A \cos B + \cos A \sin B = \frac{3}{5} \times \frac{9}{41} + \frac{4}{5} \times \frac{40}{41} = \frac{187}{205}$

(iii) $\cos(A - B) = \cos A \cos B + \sin A \sin B = \frac{4}{5} \times \frac{9}{41} + \frac{3}{5} \times \frac{40}{41} = \frac{156}{205}$

(iv) $\cos(A + B) = \cos A \cos B - \sin A \sin B = \frac{4}{5} \times \frac{9}{41} - \frac{3}{5} \times \frac{40}{41} = -\frac{84}{205}$

EXAMPLE 2 *If* $\sin A = \frac{3}{5}$, $0 < A < \frac{\pi}{2}$ *and* $\cos B = \frac{-12}{13}$, $\pi < B < \frac{3\pi}{2}$, *find the following:*

(i) $\sin(A - B)$ (ii) $\cos(A + B)$ (iii) $\tan(A - B)$

SOLUTION We have, $\sin A = \frac{3}{5}$, where $0 < A < \frac{\pi}{2}$.

$\therefore \quad \cos A = +\sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$

In the I quadrant tangent function is positive. Therefore, $\tan A = \frac{\sin A}{\cos A} = \frac{3}{4}$.

It is given that: $\cos B = -\frac{12}{13}$ and $\pi < B < \frac{3\pi}{2}$.

$\therefore \quad \sin B = \pm\sqrt{1 - \cos^2 B}$

$\Rightarrow \quad \sin B = -\sqrt{1 - \cos^2 B} = -\sqrt{1 - \left(\frac{-12}{13}\right)^2} = -\frac{5}{13}$ [$\because$ Sine is negative in the III quadrant]

In the III quadrant tangent function is positive. Therefore, $\tan B = \dfrac{\sin B}{\cos B} = \dfrac{5}{12}$.

(i) $\sin(A-B) = \sin A\cos B - \cos A\sin B = \dfrac{3}{5}\times\dfrac{-12}{13} - \dfrac{4}{5}\times\dfrac{-5}{13} = \dfrac{-16}{65}$

(ii) $\cos(A+B) = \cos A\cos B - \sin A\sin B = \dfrac{4}{5}\times\dfrac{-12}{13} - \dfrac{3}{5}\times\dfrac{-5}{13} = \dfrac{-33}{65}$

(iii) $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A\tan B} = \dfrac{\frac{3}{4}-\frac{5}{12}}{1+\frac{3}{4}\times\frac{5}{12}} = \dfrac{16}{63}$

EXAMPLE 3 *If* $\cos A = \dfrac{4}{5}$, $\cos B = \dfrac{12}{13}$, $\dfrac{3\pi}{2} < A,\ B < 2\pi$, *find the values of the following:*

(i) $\cos(A+B)$ (ii) $\sin(A-B)$

SOLUTION We have, $\cos A = \dfrac{4}{5}$ and $\cos B = \dfrac{12}{13}$, where $\dfrac{3\pi}{2} < A, B < 2\pi$. It is given that A and B both lie in the IV quadrant in which $\sin A$ and $\sin B$ are negative. Therefore,

$$\sin A = -\sqrt{1-\cos^2 A} = -\sqrt{1-\frac{16}{25}} = -\frac{3}{5} \text{ and, } \sin B = -\sqrt{1-\cos^2 B} = -\sqrt{1-\frac{144}{169}} = -\frac{5}{13}$$

(i) $\cos(A+B) = \cos A\cos B - \sin A\sin B = \dfrac{4}{5}\times\dfrac{12}{13} - \left(\dfrac{-3}{5}\right)\left(\dfrac{-5}{13}\right) = \dfrac{33}{65}$

(ii) $\sin(A-B) = \sin A\cos B - \cos A\sin B = \dfrac{-3}{5}\times\dfrac{12}{13} - \dfrac{4}{5}\times\dfrac{-5}{13} = \dfrac{-16}{65}$

EXAMPLE 4 *If* $\cot\alpha = \dfrac{1}{2}$, $\sec\beta = -\dfrac{5}{3}$, *where* $\pi < \alpha < \dfrac{3\pi}{2}$ *and* $\dfrac{\pi}{2} < \beta < \pi$. *Find the value of* $\tan(\alpha+\beta)$. *State the quadrant in which* $\alpha+\beta$ *terminates.*

SOLUTION We have, $\cot\alpha = \dfrac{1}{2} \Rightarrow \tan\alpha = 2$. Since β lies in the second quadrant. Therefore, $\tan\beta$ is negative. Consequently,

$$1+\tan^2\beta = \sec^2\beta \Rightarrow \tan\beta = -\sqrt{\sec^2\beta - 1} = -\sqrt{\frac{25}{9}-1} = -\frac{4}{3}$$

$$\therefore \quad \tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{2-\frac{4}{3}}{1-2\times\frac{-4}{3}} = \frac{2}{11}$$

Now, $\pi < \alpha < \dfrac{3\pi}{2}$ and $\dfrac{\pi}{2} < \beta < \pi \Rightarrow \dfrac{3\pi}{2} < \alpha+\beta < \dfrac{5\pi}{2}$

We know that tangent function is positive in I and III quadrants.

$\therefore \quad \dfrac{3\pi}{2} < \alpha+\beta < \dfrac{5\pi}{2}$ and $\tan(\alpha+\beta) = \dfrac{2}{11} > 0 \Rightarrow \alpha+\beta$ lies in I quadrant.

Type II ON FINDING THE VALUES TRIGONOMETRIC FUNCTIONS AT MULTIPLES OF $\frac{\pi}{12}$

EXAMPLE 5 *Find the values of the following:*

(i) $\sin\dfrac{5\pi}{12}$ (ii) $\cos\dfrac{5\pi}{12}$ (iii) $\sin\dfrac{\pi}{12}$ (iv) $\cos\dfrac{\pi}{12}$

SOLUTION (i) $\sin\dfrac{5\pi}{12} = \sin\left(\dfrac{\pi}{4}+\dfrac{\pi}{6}\right) = \sin\dfrac{\pi}{4}\cos\dfrac{\pi}{6} + \cos\dfrac{\pi}{4}\sin\dfrac{\pi}{6} = \dfrac{1}{\sqrt{2}}\times\dfrac{\sqrt{3}}{2} + \dfrac{1}{\sqrt{2}}\times\dfrac{1}{2} = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

(ii) $\cos\frac{5\pi}{12} = \cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(iii) $\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(iv) $\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

EXAMPLE 6 *Find the values of the following:*

(i) $\tan\frac{\pi}{12}$ [NCERT] (ii) $\tan\frac{5\pi}{12}$ (iii) $\tan\frac{7\pi}{12}$ (iv) $\tan\frac{13\pi}{12}$ [NCERT]

SOLUTION (i) $\tan\frac{\pi}{12} = \tan\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \frac{\tan\frac{\pi}{4}-\tan\frac{\pi}{6}}{1+\tan\frac{\pi}{4}\tan\frac{\pi}{6}} = \frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

(ii) $\tan\frac{5\pi}{12} = \tan\left(\frac{\pi}{4}+\frac{\pi}{6}\right) = \frac{\tan\frac{\pi}{4}+\tan\frac{\pi}{6}}{1-\tan\frac{\pi}{4}\tan\frac{\pi}{6}} = \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

(iii) $\tan\frac{7\pi}{12} = \tan\left(\frac{\pi}{2}+\frac{\pi}{12}\right) = -\cot\frac{\pi}{12} = -\frac{1}{\tan\frac{\pi}{12}} = -\frac{\sqrt{3}+1}{\sqrt{3}-1}$ [Using (i)]

(iv) $\tan\frac{13\pi}{12} = \tan\left(\pi+\frac{\pi}{12}\right) = \tan\frac{\pi}{12} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$ [Using (i)]

EXAMPLE 7 *Prove that:* $\tan\frac{5\pi}{12} + \cot\frac{5\pi}{12} = 4.$

SOLUTION From Example 6, we obtain: $\tan\frac{5\pi}{12} = \frac{\sqrt{3}+1}{\sqrt{3}-1} \Rightarrow \cot\frac{5\pi}{12} = \frac{1}{\tan\frac{5\pi}{12}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

Substituting the values of $\tan\frac{5\pi}{12}$ and $\cot\frac{5\pi}{12}$, we obtain

$$\text{LHS} = \tan\frac{5\pi}{12} + \cot\frac{5\pi}{12} = \frac{\sqrt{3}+1}{\sqrt{3}-1} + \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}+1)^2+(\sqrt{3}-1)^2}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$= \frac{(4+2\sqrt{3})+(4-2\sqrt{3})}{3-1} = \frac{8}{2} = 4 = \text{RHS}$$

Type III **ON THE APPLICATIONS OF THE FOLLOWING FORMULAE:**

(i) $\sin A\cos B \pm \cos A\sin B = \sin(A \pm B)$ (ii) $\cos A\cos B \pm \sin A\sin B = \cos(A \mp B)$

EXAMPLE 8 *Evaluate the following:*

(i) $\sin\frac{7\pi}{12}\cos\frac{\pi}{4} - \cos\frac{7\pi}{12}\sin\frac{\pi}{4}$ (ii) $\sin\frac{\pi}{4}\cos\frac{\pi}{12} + \cos\frac{\pi}{4}\sin\frac{\pi}{12}$

(iii) $\cos\frac{2\pi}{3}\cos\frac{\pi}{4} - \sin\frac{2\pi}{3}\sin\frac{\pi}{4}$

SOLUTION (i) $\sin\frac{7\pi}{12}\cos\frac{\pi}{4} - \cos\frac{7\pi}{12}\sin\frac{\pi}{4} = \sin\left(\frac{7\pi}{12}-\frac{\pi}{4}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

(ii) $\sin\frac{\pi}{4}\cos\frac{\pi}{12}+\cos\frac{\pi}{4}\sin\frac{\pi}{12}=\sin\left(\frac{\pi}{4}+\frac{\pi}{12}\right)=\sin\frac{4\pi}{12}=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$

(iii) $\cos\frac{2\pi}{3}\cos\frac{\pi}{4}-\sin\frac{2\pi}{3}\sin\frac{\pi}{4}=\cos\left(\frac{2\pi}{3}+\frac{\pi}{4}\right)=\cos\frac{11\pi}{12}=\cos\frac{11\pi}{12}$

$$=\cos\left(\pi-\frac{\pi}{12}\right)=-\cos\frac{\pi}{12}=-\frac{\sqrt{3}+1}{2\sqrt{2}}$$ [See Ex. 5 (iv)]

EXAMPLE 9 *Prove that:* $\cos\left(\frac{\pi}{4}-A\right)\cos\left(\frac{\pi}{4}-B\right)-\sin\left(\frac{\pi}{4}-A\right)\sin\left(\frac{\pi}{4}-B\right)=\sin(A+B)$ **[NCERT]**

SOLUTION LHS $=\cos\left(\frac{\pi}{4}-A\right)\cos\left(\frac{\pi}{4}-B\right)-\sin\left(\frac{\pi}{4}-A\right)\sin\left(\frac{\pi}{4}-B\right)$

$$=\cos\left\{\left(\frac{\pi}{4}-A\right)+\left(\frac{\pi}{4}-B\right)\right\}=\cos\left\{\frac{\pi}{2}-(A+B)\right\}=\sin(A+B)=\text{RHS}$$

EXAMPLE 10 *Prove that:* $\sin(n+1)A\sin(n+2)A+\cos(n+1)A\cos(n+2)A=\cos A$ **[NCERT]**

SOLUTION LHS $=\sin(n+1)A\sin(n+2)A+\cos(n+1)A\cos(n+2)A$

$$=\cos(n+2)A\cos(n+1)A+\sin(n+2)A\sin(n+1)A$$

$$=\cos\{(n+2)A-(n+1)A\}=\cos A=\text{RHS}$$

Type IV **ON APPLICATIONS OF THE FORMULAE:**

(i) $\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$ (ii) $\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$

(iii) $\tan(A\pm B)=\dfrac{\tan A\pm\tan B}{1\mp\tan A\tan B}$

EXAMPLE 11 *Prove that:*

(i) $\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)=\sqrt{2}\cos x$ **[NCERT]**

(ii) $\cos\left(\frac{3\pi}{4}+x\right)-\cos\left(\frac{3\pi}{4}-x\right)=-\sqrt{2}\sin x$ **[NCERT]**

SOLUTION (i) We have, $\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)$

$$=\left(\cos\frac{\pi}{4}\cos x-\sin\frac{\pi}{4}\sin x\right)+\left(\cos\frac{\pi}{4}\cos x+\sin\frac{\pi}{4}\sin x\right)$$

$$=2\cos\frac{\pi}{4}\cos x=2\times\frac{1}{\sqrt{2}}\times\cos x=\sqrt{2}\cos x$$

(ii) We have, $\cos\left(\frac{3\pi}{4}+x\right)-\cos\left(\frac{3\pi}{4}-x\right)$

$$=\left(\cos\frac{3\pi}{4}\cos x-\sin\frac{3\pi}{4}\sin x\right)-\left(\cos\frac{3\pi}{4}\cos x+\sin\frac{3\pi}{4}\sin x\right)$$

$$=\left(-\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right)-\left(-\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x\right)$$

$$=-\frac{1}{\sqrt{2}}\sin x-\frac{1}{\sqrt{2}}\sin x=-\frac{2}{\sqrt{2}}\sin x=-\sqrt{2}\sin x$$

EXAMPLE 12 *Prove that:* $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}$ [NCERT]

SOLUTION $$\text{LHS} = \frac{\sin(x+y)}{\sin(x-y)} = \frac{\sin x \cos y + \cos x \sin y}{\sin x \cos y - \cos x \sin y}$$

$$= \frac{\dfrac{\sin x \cos y + \cos x \sin y}{\cos x \cos y}}{\dfrac{\sin x \cos y - \cos x \sin y}{\cos x \cos y}} \quad \left[\begin{array}{l}\text{Dividing the numerator}\\ \text{and denominator by } \cos x \cos y\end{array}\right]$$

$$= \frac{\tan x + \tan y}{\tan x - \tan y} = \text{RHS}$$

EXAMPLE 13 *Prove that:* $\dfrac{\sin(B-C)}{\cos B \cos C} + \dfrac{\sin(C-A)}{\cos C \cos A} + \dfrac{\sin(A-B)}{\cos A \cos B} = 0$

SOLUTION We have,

$$\text{LHS} = \frac{\sin(B-C)}{\cos B \cos C} + \frac{\sin(C-A)}{\cos C \cos A} + \frac{\sin(A-B)}{\cos A \cos B}$$

$$= \frac{\sin B \cos C - \cos B \sin C}{\cos B \cos C} + \frac{\sin C \cos A - \cos C \sin A}{\cos C \cos A} + \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B}$$

$$= \frac{\sin B \cos C}{\cos B \cos C} - \frac{\cos B \sin C}{\cos B \cos C} + \frac{\sin C \cos A}{\cos C \cos A} - \frac{\cos C \sin A}{\cos C \cos A} + \frac{\sin A \cos B}{\cos A \cos B} - \frac{\cos A \sin B}{\cos A \cos B}$$

$$= \tan B - \tan C + \tan C - \tan A + \tan A - \tan B = 0 = \text{RHS}$$

EXAMPLE 14 *Prove that:* $\dfrac{\tan\left(\dfrac{\pi}{4}+x\right)}{\tan\left(\dfrac{\pi}{4}-x\right)} = \left(\dfrac{1+\tan x}{1-\tan x}\right)^2$ [NCERT]

SOLUTION $$\text{LHS} = \frac{\tan\left(\dfrac{\pi}{4}+x\right)}{\tan\left(\dfrac{\pi}{4}-x\right)} = \frac{\tan\dfrac{\pi}{4} + \tan x}{1 - \tan\dfrac{\pi}{4}\tan x} \times \frac{1 + \tan\dfrac{\pi}{4}\tan x}{\tan\dfrac{\pi}{4} - \tan x}$$

$$= \frac{1+\tan x}{1-\tan x} \times \frac{1+\tan x}{1-\tan x} = \left(\frac{1+\tan x}{1-\tan x}\right)^2 = \text{RHS}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 15 *If* $\tan A - \tan B = x$ *and,* $\cot B - \cot A = y$, *prove that* $\cot(A-B) = \dfrac{1}{x} + \dfrac{1}{y}$.

[NCERT EXEMPLAR]

SOLUTION We have, $\tan A - \tan B = x$ and, $\cot B - \cot A = y$

Now,

$$\cot B - \cot A = y$$

$$\Rightarrow \quad \frac{1}{\tan B} - \frac{1}{\tan A} = y \Rightarrow \frac{\tan A - \tan B}{\tan A \tan B} = y \Rightarrow \frac{x}{\tan A \tan B} = y \Rightarrow \tan A \tan B = \frac{x}{y}$$

$$\therefore \quad \cot(A-B) = \frac{1}{\tan(A-B)} = \frac{1+\tan A \tan B}{\tan A - \tan B} = \frac{1+\frac{x}{y}}{x} = \frac{x+y}{xy} = \frac{1}{x}+\frac{1}{y}$$

ALITER We have, $\tan A - \tan B = x$ and, $\cot B - \cot A = y$

$$\Rightarrow \quad \frac{1}{\cot A} - \frac{1}{\cot B} = x \text{ and, } \cot B - \cot A = y \Rightarrow \frac{\cot B - \cot A}{\cot A \cot B} = x \text{ and, } \cot B - \cot A = y$$

$$\Rightarrow \quad \frac{y}{\cot A \cot B} = x \text{ and } \cot B - \cot A = y \Rightarrow \cot A \cot B = \frac{y}{x} \text{ and } \cot B - \cot A = y$$

$$\therefore \quad \cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A} = \frac{\frac{y}{x}+1}{y} = \frac{1}{x}+\frac{1}{y}$$

EXAMPLE 16 *If* $\tan\alpha = \frac{1}{\sqrt{x(x^2+x+1)}}$, $\tan\beta = \frac{\sqrt{x}}{\sqrt{x^2+x+1}}$ *and* $\tan\gamma = \sqrt{x^{-3}+x^{-2}+x^{-1}}$, *prove that* $\alpha+\beta=\gamma$.

SOLUTION We have,

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \frac{\frac{1}{\sqrt{x(x^2+x+1)}} + \frac{\sqrt{x}}{\sqrt{x^2+x+1}}}{1 - \frac{1}{\sqrt{x(x^2+x+1)}} \times \frac{\sqrt{x}}{\sqrt{x^2+x+1}}} = \frac{(x+1)\sqrt{x(x^2+x+1)}}{x(x^2+x+1)-x}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \frac{(x+1)\sqrt{x(x^2+x+1)}}{x^2(x+1)} = \frac{\sqrt{x(x^2+x+1)}}{x^2} = \sqrt{\frac{x(x^2+x+1)}{x^4}} = \sqrt{x^{-3}+x^{-2}+x^{-1}}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \tan\gamma$$

$$\Rightarrow \quad \alpha+\beta = \gamma.$$

EXAMPLE 17 *If α and β are acute angles such that* $\tan\alpha = \frac{m}{m+1}$ *and* $\tan\beta = \frac{1}{2m+1}$, *prove that* $\alpha+\beta = \frac{\pi}{4}$.

SOLUTION We have,

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \frac{\frac{m}{m+1}+\frac{1}{2m+1}}{1-\frac{m}{m+1}\times\frac{1}{2m+1}} = \frac{2m^2+m+m+1}{2m^2+3m+1-m} = \frac{2m^2+2m+1}{2m^2+2m+1} = 1 = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \alpha+\beta = \frac{\pi}{4}$$

EXAMPLE 18 *If* $A+B=\frac{\pi}{4}$, *prove that:*

(i) $(1+\tan A)(1+\tan B) = 2$ (ii) $(\cot A - 1)(\cot B - 1) = 2$

SOLUTION (i) We have, $A+B = \frac{\pi}{4}$

$\therefore \quad \tan(A+B) = \tan\frac{\pi}{4}$

$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$

$\Rightarrow \quad \tan A + \tan B = 1 - \tan A \tan B$

$\Rightarrow \quad \tan A + \tan B + \tan A \tan B = 1 \quad \Rightarrow 1 + \tan A + \tan B + \tan A \tan B = 2$

$\Rightarrow \quad (1 + \tan A) + \tan B(1 + \tan A) = 2 \Rightarrow (1 + \tan A)(1 + \tan B) = 2$

(ii) We have, $A + B = \frac{\pi}{4}$

$\therefore \quad \tan(A+B) = \tan\frac{\pi}{4}$

$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$

$\Rightarrow \quad \tan A + \tan B = 1 - \tan A \tan B$

$\Rightarrow \quad \tan A + \tan B + \tan A \tan B = 1$

$\Rightarrow \quad \frac{\tan A + \tan B + \tan A \tan B}{\tan A \tan B} = \frac{1}{\tan A \tan B}$

$\Rightarrow \quad \cot B + \cot A + 1 = \cot A \cot B$

$\Rightarrow \quad \cot A \cot B - \cot A - \cot B = 1 \quad \Rightarrow \cot A \cot B - \cot A - \cot B + 1 = 2$

$\Rightarrow \quad \cot A(\cot B - 1) - (\cot B - 1) = 2 \Rightarrow (\cot A - 1)(\cot B - 1) = 2$

ALITER We have, $A + B = \frac{\pi}{4}$

$\therefore \quad \cot(A+B) = \cot\frac{\pi}{4}$

$\Rightarrow \quad \frac{\cot A \cot B - 1}{\cot A + \cot B} = 1 \Rightarrow \cot A + \cot B = \cot A \cot B - 1$

$\Rightarrow \quad \cot A \cot B - \cot A - \cot B = 1 \quad \Rightarrow \cot A \cot B - \cot A \cot B + 1 = 1 + 1$

$\Rightarrow \quad \cot A(\cot B - 1) - (\cot B - 1) = 2 \Rightarrow (\cot A - 1)(\cot B - 1) = 2$

EXAMPLE 19 *If* $\tan\beta = \frac{n \sin\alpha \cos\alpha}{1 - n\sin^2\alpha}$, *show that* $\tan(\alpha - \beta) = (1 - n)\tan\alpha$.

SOLUTION We have,

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$\Rightarrow \quad \tan(\alpha - \beta) = \frac{\frac{\sin\alpha}{\cos\alpha} - \frac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}}{1 + \frac{\sin\alpha}{\cos\alpha} \times \frac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}} \qquad \left[\because \tan\beta = \frac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}\right]$$

$$\Rightarrow \quad \tan(\alpha - \beta) = \frac{\sin\alpha - n\sin^3\alpha - n\sin\alpha\cos^2\alpha}{\cos\alpha(1 - n\sin^2\alpha) + n\sin^2\alpha\cos\alpha} \qquad \text{[On taking LCM]}$$

$$\Rightarrow \quad \tan(\alpha - \beta) = \frac{\sin\alpha - n\sin^3\alpha - n\sin\alpha(1 - \sin^2\alpha)}{\cos\alpha - n\sin^2\alpha\cos\alpha + n\sin^2\alpha\cos\alpha}$$

$$\Rightarrow \quad \tan(\alpha-\beta) = \frac{\sin\alpha - n\sin\alpha}{\cos\alpha} = \frac{\sin\alpha}{\cos\alpha}(1-n) = (1-n)\tan\alpha$$

EXAMPLE 20 *Prove that:*

(i) $\tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x$

(ii) $\cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1$ **[NCERT]**

SOLUTION (i) Clearly,

$$3x = 2x + x$$

$$\Rightarrow \tan 3x = \tan(2x+x)$$

$$\Rightarrow \tan 3x = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$

$$\Rightarrow \tan 3x(1 - \tan 2x \tan x) = \tan 2x + \tan x$$

$$\Rightarrow \tan 3x - \tan 3x \tan 2x \tan x = \tan 2x + \tan x \Rightarrow \tan 3x - \tan 2x - \tan x = \tan 3x \tan 2x \tan x$$

(ii) Dividing both sides by $\tan x \tan 2x \tan 3x$, we get

$$\frac{\tan 3x \tan 2x \tan x}{\tan 3x \tan 2x \tan x} = \frac{\tan 3x - \tan 2x - \tan x}{\tan 3x \tan 2x \tan x}$$

$$\Rightarrow 1 = \frac{1}{\tan 2x \tan x} - \frac{1}{\tan 3x \tan x} - \frac{1}{\tan 3x \tan 2x} \Rightarrow 1 = \cot x \cot 2x - \cot 3x \cot x - \cot 3x \cot 2x$$

Type V ON THE APPLICATIONS OF THE FORMULAE:

(i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$ (ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$

EXAMPLE 21 *Prove that:* $\dfrac{\tan(x+y)}{\cot(x-y)} = \dfrac{\sin^2 x - \sin^2 y}{\cos^2 x - \sin^2 y}$

SOLUTION $\text{LHS} = \dfrac{\tan(x+y)}{\cot(x-y)} = \dfrac{\sin(x+y)}{\cos(x+y)} \dfrac{\sin(x-y)}{\cos(x-y)} = \dfrac{\sin^2 x - \sin^2 y}{\cos^2 x - \sin^2 y} = \text{RHS}$

EXAMPLE 22 *Prove that :* $\sin^2 6x - \sin^2 4x = \sin 2x \sin 10x$ **[NCERT]**

SOLUTION We know that $\sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)$

$$\therefore \quad \sin^2 6x - \sin^2 4x = \sin(6x+4x)\sin(6x-4x) = \sin 10x \sin 2x$$

EXAMPLE 23 *Prove that :* $\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$ **[NCERT]**

SOLUTION $\text{LHS} = \cos^2 2x - \cos^2 6x$

$$= (1-\sin^2 2x) - (1-\sin^2 6x) = \sin^2 6x - \sin^2 2x = \sin(6x+2x)\sin(6x-2x)$$

$$= \sin 8x \sin 4x = \text{RHS}$$

EXAMPLE 24 *Prove that* $\dfrac{\cos^2 33^\circ - \cos^2 57^\circ}{\sin^2 \frac{21^\circ}{2} - \sin^2 \frac{69^\circ}{2}} = -\sqrt{2}$

SOLUTION $\text{LHS} = \dfrac{\cos^2 33^\circ - \cos^2 57^\circ}{\sin^2 \frac{21^\circ}{2} - \sin^2 \frac{69^\circ}{2}} = \dfrac{(1-\sin^2 33^\circ) - (1-\sin^2 57^\circ)}{\sin^2 \frac{21^\circ}{2} - \sin^2 \frac{69^\circ}{2}}$

$$= \frac{\sin^2 57^\circ - \sin^2 33^\circ}{\sin^2 \frac{21^\circ}{2} - \sin^2 \frac{69^\circ}{2}}$$

$$= \frac{\sin(57° + 33°) \sin(57° - 33°)}{\sin\left(\frac{21°}{2} + \frac{69°}{2}\right) \sin\left(\frac{21°}{2} - \frac{69°}{2}\right)}$$

$$= \frac{\sin 90° \sin 24°}{\sin 45° \sin(-24°)} = \frac{\sin 24°}{-\frac{1}{\sqrt{2}} \sin 24°} = -\sqrt{2} = \text{RHS}$$

EXAMPLE 25 *Prove that:* $\sin^2\left(\frac{\pi}{8} + \frac{x}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{x}{2}\right) = \frac{1}{\sqrt{2}} \sin x$

SOLUTION Using $\sin^2 A - \sin^2 B = \sin(A+B) \sin(A-B)$, we obtain

$$\text{LHS} = \sin^2\left(\frac{\pi}{8} + \frac{x}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{x}{2}\right)$$

$$= \sin\left\{\left(\frac{\pi}{8} + \frac{x}{2}\right) + \left(\frac{\pi}{8} - \frac{x}{2}\right)\right\} \sin\left\{\left(\frac{\pi}{8} + \frac{x}{2}\right) - \left(\frac{\pi}{8} - \frac{x}{2}\right)\right\} = \sin\frac{\pi}{4} \sin x = \frac{1}{\sqrt{2}} \sin x = \text{RHS}$$

EXAMPLE 26 *Prove that:* $\cos 2\alpha \cos 2\beta + \sin^2(\alpha - \beta) - \sin^2(\alpha + \beta) = \cos 2(\alpha + \beta)$.

SOLUTION Using $\sin^2 A - \sin^2 B = \sin(A+B) \sin(A-B)$, we obtain

$$\text{LHS} = \cos 2\alpha \cos 2\beta + \sin^2(\alpha - \beta) - \sin^2(\alpha + \beta)$$

$$= \cos 2\alpha \cos 2\beta + \sin(\alpha - \beta + \alpha + \beta) \sin(\alpha - \beta - \alpha - \beta)$$

$$= \cos 2\alpha \cos 2\beta - \sin 2\alpha \sin 2\beta$$

$$= \cos(2\alpha + 2\beta) = \text{RHS}$$

EXAMPLE 27 *Prove that:* $\sin^2 A = \cos^2(A - B) + \cos^2 B - 2\cos(A - B) \cos A \cos B$.

SOLUTION $\text{RHS} = \cos^2(A - B) + \cos^2 B - 2\cos(A - B)\cos A \cos B$

$$= \cos^2 B + \cos^2(A - B) - 2\cos(A - B)\cos A \cos B$$

$$= \cos^2 B + \cos(A - B)\{\cos(A - B) - 2\cos A \cos B\}$$

$$= \cos^2 B + \cos(A - B)\{\cos A \cos B + \sin A \sin B - 2\cos A \cos B\}$$

$$= \cos^2 B + \cos(A - B)\{\sin A \sin B - \cos A \cos B\}$$

$$= \cos^2 B - \cos(A - B)(\cos A \cos B - \sin A \sin B)$$

$$= \cos^2 B - \cos(A - B)\cos(A + B)$$

$$= \cos^2 B - (\cos^2 A - \sin^2 B) = \cos^2 B + \sin^2 B - \cos^2 A$$

$$= 1 - \cos^2 A = \sin^2 A = \text{LHS}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 28 *If* $3 \tan A \tan B = 1$, *prove that* $2\cos(A + B) = \cos(A - B)$.

SOLUTION We have,

$$3 \tan A \tan B = 1 \Rightarrow \frac{3 \sin A \sin B}{\cos A \cos B} = 1 \Rightarrow \frac{\cos A \cos B}{\sin A \sin B} = \frac{3}{1}$$

$$\Rightarrow \frac{\cos A \cos B + \sin A \sin B}{\cos A \cos B - \sin A \sin B} = \frac{3+1}{3-1} \quad \text{[Applying componendo-dividendo]}$$

$$\Rightarrow \frac{\cos(A - B)}{\cos(A + B)} = 2 \Rightarrow 2\cos(A + B) = \cos(A - B)$$

EXAMPLE 29 *If* $\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha)=-\frac{3}{2}$, *prove that*

$$\cos\alpha+\cos\beta+\cos\gamma = \sin\alpha+\sin\beta+\sin\gamma = 0$$

SOLUTION We have,

$$\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha) = -\frac{3}{2}$$

$$\Rightarrow \quad 2\cos\alpha\cos\beta+2\sin\alpha\sin\beta+2\cos\beta\cos\gamma+2\sin\beta\sin\gamma + 2\cos\gamma\cos\alpha+2\sin\gamma\sin\alpha=-3$$

$$\Rightarrow \quad (2\cos\alpha\cos\beta+2\cos\beta\cos\gamma+2\cos\gamma\cos\alpha)+(2\sin\alpha\sin\beta+2\sin\beta\sin\gamma + 2\sin\gamma\sin\alpha)+3=0$$

$$\Rightarrow \quad (2\cos\alpha\cos\beta+2\cos\beta\cos\gamma+2\cos\gamma\cos\alpha)+(2\sin\alpha\sin\beta+2\sin\beta\sin\gamma + 2\sin\gamma\sin\alpha+(\cos^2\alpha+\sin^2\alpha)+(\cos^2\beta+\sin^2\beta)+(\cos^2\gamma+\sin^2\gamma)=0$$

$$\Rightarrow \quad (\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta+2\cos\beta\cos\gamma+2\cos\gamma\cos\alpha) + (\sin^2\alpha+\sin^2\beta+\sin^2\gamma+2\sin\alpha\sin\beta+2\sin\beta\sin\gamma+2\sin\gamma\sin\alpha)=0$$

$$\Rightarrow \quad (\cos\alpha+\cos\beta+\cos\gamma)^2+(\sin\alpha+\sin\beta+\sin\gamma)^2 = 0$$

$$\Rightarrow \quad \cos\alpha+\cos\beta+\cos\gamma = 0 \text{ and } \sin\alpha+\sin\beta+\sin\gamma = 0$$

EXAMPLE 30 *If* $\sin B = 3\sin(2A+B)$, *prove that* $2\tan A+\tan(A+B)=0$.

SOLUTION We have,

$$\sin B = 3\sin(2A+B)$$

$$\Rightarrow \quad \frac{\sin(2A+B)}{\sin B} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{\sin\{(A+B)+A\}}{\sin\{(A+B)-A\}} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{\sin\{A+B)+A\}+\sin\{(A+B)-A\}}{\sin\{(A+B)+A\}-\sin\{(A+B)-A\}} = \frac{1+3}{1-3} \qquad \text{[Using componendo – dividendo]}$$

$$\Rightarrow \quad \frac{\{\sin(A+B)\cos A+\cos(A+B)\sin A\}+\{\sin(A+B)\cos A-\cos(A+B)\sin A\}}{\{\sin(A+B)\cos A+\cos(A+B)\sin A\} - \{\sin(A+B)\cos A-\cos(A+B)\sin A\}} = \frac{1+3}{1-3}$$

$$\Rightarrow \quad \frac{2\sin(A+B)\cos A}{2\cos(A+B)\sin A} = -2$$

$$\Rightarrow \quad \tan(A+B)\cot A=-2 \Rightarrow \tan(A+B)=-2\tan A \Rightarrow 2\tan A+\tan(A+B)=0$$

EXAMPLE 31 *If* $2\tan\beta+\cot\beta=\tan\alpha$, *prove that* $\cot\beta=2\tan(\alpha-\beta)$.

SOLUTION Clearly,

$$2\tan(\alpha-\beta) = 2\left\{\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\right\}$$

$$= 2\left\{\frac{2\tan\beta+\cot\beta-\tan\beta}{1+(2\tan\beta+\cot\beta)\tan\beta}\right\} \qquad [\text{Using}: \tan\alpha=2\tan\beta+\cot\beta]$$

$$= 2\left\{\frac{\tan\beta+\cot\beta}{1+2\tan^2\beta+1}\right\}$$

$$= \frac{2(\tan\beta+\cot\beta)}{2+2\tan^2\beta} = \frac{2\left\{\tan\beta+\frac{1}{\tan\beta}\right\}}{2(1+\tan^2\beta)} = \frac{1}{\tan\beta} = \cot\beta$$

Hence, $\cot\beta = 2\tan(\alpha-\beta)$.

EXAMPLE 32 *If* $\cos(\alpha+\beta)\sin(\gamma+\delta)=\cos(\alpha-\beta)\sin(\gamma-\delta)$, *prove that* $\cot\alpha\cot\beta\cot\gamma=\cot\delta$.

SOLUTION We have,

$$\cos(\alpha+\beta)\sin(\gamma+\delta) = \cos(\alpha-\beta)\sin(\gamma-\delta)$$

$$\Rightarrow \frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} = \frac{\sin(\gamma+\delta)}{\sin(\gamma-\delta)}$$

$$\Rightarrow \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{\cos(\alpha-\beta)-\cos(\alpha+\beta)} = \frac{\sin(\gamma+\delta)+\sin(\gamma-\delta)}{\sin(\gamma+\delta)-\sin(\gamma-\delta)} \quad \text{[Using componendo – dividendo]}$$

$$\Rightarrow \frac{2\cos\alpha\cos\beta}{2\sin\alpha\sin\beta} = \frac{2\sin\gamma\cos\delta}{2\cos\gamma\sin\delta} \Rightarrow \cot\alpha\cot\beta = \tan\gamma\cot\delta \Rightarrow \cot\alpha\cot\beta\cot\gamma = \cot\delta$$

EXAMPLE 33 *Prove that:* $\dfrac{\sin(x+\theta)}{\sin(x+\phi)}=\cos(\theta-\phi)+\cot(x+\phi)\sin(\theta-\phi)$.

SOLUTION

$$\text{LHS} = \frac{\sin(x+\theta)}{\sin(x+\phi)} = \frac{\sin\{(x+\phi)+(\theta-\phi)\}}{\sin(x+\phi)}$$

$$= \frac{\sin(x+\phi)\cos(\theta-\phi)+\cos(x+\phi)\sin(\theta-\phi)}{\sin(x+\phi)}$$

$$= \cos(\theta-\phi)+\cot(x+\phi)\sin(\theta-\phi) = \text{RHS}$$

EXAMPLE 34 *If* $\cos(\alpha+\beta)=\dfrac{4}{5}$, $\sin(\alpha-\beta)=\dfrac{5}{13}$ *and* α, β *lie between 0 and* $\dfrac{\pi}{4}$, *prove that* $\tan 2\alpha=\dfrac{56}{33}$ **[NCERT EXEMPLAR]**

SOLUTION It is given that α, β lie between 0 and $\pi/4$. Therefore, $-\pi/4<\alpha-\beta<\pi/4$ and $0<\alpha+\beta<\pi/2$. So, $\cos(\alpha-\beta)$ and $\sin(\alpha+\beta)$ are positive.

Now, $\sin(\alpha+\beta) = \sqrt{1-\cos^2(\alpha+\beta)} \Rightarrow \sin(\alpha+\beta) = \sqrt{1-\dfrac{16}{25}} = \dfrac{3}{5}$

and, $\cos(\alpha-\beta) = \sqrt{1-\sin^2(\alpha-\beta)} \Rightarrow \cos(\alpha-\beta) = \sqrt{1-\dfrac{25}{169}} = \dfrac{12}{13}$

$\therefore \quad \tan(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \dfrac{3/5}{4/5} = \dfrac{3}{4}$ and, $\tan(\alpha-\beta) = \dfrac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)} = \dfrac{5}{12}$

Now,

$$\tan 2\alpha = \tan\{(\alpha+\beta)+(\alpha-\beta)\} = \frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\tan(\alpha-\beta)} = \frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\times\frac{5}{12}} = \frac{56}{33}$$

EXAMPLE 35 *Prove that:* $\tan 70° = \tan 20° + 2\tan 50°$.

SOLUTION We have,

$$\tan(x-y) = \frac{\tan x-\tan y}{1+\tan x\tan y} \Rightarrow \tan x-\tan y = \tan(x-y)(1+\tan x\tan y)$$

Replacing x by 70° and y by 20°, we get

$$\tan 70°-\tan 20° = \tan(70°-20°)(1+\tan 70°\tan 20°)$$

$$= \tan 50°\ (1+\tan 70°\cot 70°) = 2\tan 50°$$

EXAMPLE 36 *If* $\tan(\alpha+x)=n\tan(\alpha-x)$, *show that:* $(n+1)\sin 2x=(n-1)\sin 2\alpha$.

SOLUTION We have,

$$\tan(\alpha+x) = n\tan(\alpha-x)$$

$$\Rightarrow \quad \frac{\tan(\alpha+x)}{\tan(\alpha-x)} = \frac{n}{1}$$

$$\Rightarrow \quad \frac{\tan(\alpha+x)+\tan(\alpha-x)}{\tan(\alpha+x)-\tan(\alpha-x)} = \frac{n+1}{n-1} \qquad \text{[Applying componendo-dividendo]}$$

$$\Rightarrow \quad \frac{\sin(\alpha+x)\cos(\alpha-x)+\cos(\alpha+x)\sin(\alpha-x)}{\sin(\alpha+x)\cos(\alpha-x)-\cos(\alpha+x)\sin(\alpha-x)} = \frac{n+1}{n-1}$$

$$\Rightarrow \quad \frac{\sin(\alpha+x)+(\alpha-x)}{\sin(\alpha+x)-(\alpha-x)} = \frac{n+1}{n-1} \Rightarrow \frac{\sin 2\alpha}{\sin 2x} = \frac{n+1}{n-1} \Rightarrow (n+1)\sin 2x = (n-1)\sin 2\alpha$$

EXAMPLE 37 *Prove that:* $\cot x \cot 2x + \cot 2x \cot 3x + 2 = \cot x(\cot x - \cot 3x)$.

SOLUTION LHS $= \cot x \cot 2x + \cot 2x \cot 3x + 2 = (\cot x \cot 2x + 1) + (\cot 2x \cot 3x + 1)$

$$= \left(\frac{\cos x \cos 2x}{\sin x \sin 2x}+1\right)+\left(\frac{\cos 2x \cos 3x}{\sin 2x \sin 3x}+1\right)$$

$$= \left(\frac{\cos 2x \cos x + \sin 2x \sin x}{\sin x \sin 2x}\right)+\left(\frac{\cos 3x \cos 2x + \sin 3x \sin 2x}{\sin 2x \sin 3x}\right)$$

$$= \frac{\cos(2x-x)}{\sin x \sin 2x}+\frac{\cos(3x-2x)}{\sin 2x \sin 3x} = \frac{\cos x}{\sin x \sin 2x}+\frac{\cos x}{\sin 2x \sin 3x}$$

$$= \cos x\left\{\frac{1}{\sin x \sin 2x}+\frac{1}{\sin 2x \sin 3x}\right\} = \frac{\cos x}{\sin x}\left\{\frac{\sin x}{\sin x \sin 2x}+\frac{\sin x}{\sin 2x \sin 3x}\right\}$$

$$= \cot x\left\{\frac{\sin(2x-x)}{\sin x \sin 2x}+\frac{\sin(3x-2x)}{\sin 2x \sin 3x}\right\}$$

$$= \cot x\left\{\frac{\sin 2x \cos x - \cos 2x \sin x}{\sin x \sin 2x}+\frac{\sin 3x \cos 2x - \cos 3x \sin 2x}{\sin 2x \sin 3x}\right\}$$

$$= \cot x\,\{\cot x - \cot 2x + \cot 2x - \cot 3x\} = \cot x(\cot x - \cot 3x) = \text{RHS}$$

EXAMPLE 38 *If* $\tan(\pi\cos x) = \cot(\pi \sin x)$, *prove that* $\cos\left(x-\frac{\pi}{4}\right) = \pm\frac{1}{2\sqrt{2}}$.

SOLUTION We have,

$$\tan(\pi\cos x) = \cot(\pi\sin x)$$

$$\Rightarrow \quad \frac{\sin(\pi\cos x)}{\cos(\pi\cos x)} = \frac{\cos(\pi\sin x)}{\sin(\pi\sin x)}$$

$$\Rightarrow \quad \sin(\pi\cos x)\sin(\pi\sin x) = \cos(\pi\sin x)\cos(\pi\cos x)$$

$$\Rightarrow \quad \cos(\pi\cos x)\cos(\pi\sin x) - \sin(\pi\cos x)\sin(\pi\sin x) = 0$$

$$\Rightarrow \quad \cos(\pi\cos x + \pi\sin x) = 0$$

$$\Rightarrow \quad \pi\cos x + \pi\sin x = \pm\frac{\pi}{2} \qquad \left[\because \cos\left(\pm\frac{\pi}{2}\right)=0\right]$$

$$\Rightarrow \quad \cos x + \sin x = \pm\frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x = \pm\frac{1}{2\sqrt{2}} \qquad \text{[Multiplying both sides by } 1/\sqrt{2}\text{]}$$

$$\Rightarrow \quad \cos x\cos\frac{\pi}{4} + \sin x \sin\frac{\pi}{4} = \pm\frac{1}{2\sqrt{2}} \Rightarrow \cos\left(x-\frac{\pi}{4}\right) = \pm\frac{1}{2\sqrt{2}}$$

EXAMPLE 39 *If* $a\tan\alpha + b\tan\beta = (a+b)\tan\left(\frac{\alpha+\beta}{2}\right)$, *where* $\alpha \neq \beta$, *prove that* $a\cos\beta = b\cos\alpha$.

SOLUTION We have,

$$a \tan \alpha + b \tan \beta = (a+b) \tan \left(\frac{\alpha+\beta}{2}\right)$$

$$\Rightarrow \quad a\left\{\tan \alpha - \tan \left(\frac{\alpha+\beta}{2}\right)\right\} = b\left\{\tan \left(\frac{\alpha+\beta}{2}\right) - \tan \beta\right\}$$

$$\Rightarrow \quad \frac{a \sin \left(\alpha - \frac{\alpha+\beta}{2}\right)}{\cos \alpha \cos \left(\frac{\alpha+\beta}{2}\right)} = \frac{b \sin \left(\frac{\alpha+\beta}{2} - \beta\right)}{\cos \left(\frac{\alpha+\beta}{2}\right) \cos \beta} \qquad \left[\because \tan A - \tan B = \frac{\sin (A-B)}{\cos A \cos B}\right]$$

$$\Rightarrow \quad \frac{a \sin \left(\frac{\alpha-\beta}{2}\right)}{\cos \alpha} = \frac{b \sin \left(\frac{\alpha-\beta}{2}\right)}{\cos \beta} \Rightarrow a \cos \beta = b \cos \alpha \qquad \left[\because \alpha \neq \beta \therefore \sin\left(\frac{\alpha-\beta}{2}\right) \neq 0\right]$$

EXAMPLE 40 *If* $\sin \alpha + \sin \beta = a$ *and* $\cos \alpha + \cos \beta = b$, *show that*

(i) $\cos(\alpha+\beta) = \dfrac{b^2-a^2}{b^2+a^2}$ (ii) $\sin(\alpha+\beta) = \dfrac{2ab}{a^2+b^2}$

SOLUTION (i) We have,

$$b^2 + a^2 = (\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2$$

$$\Rightarrow b^2 + a^2 = (\cos^2 \alpha + \sin^2 \alpha) + (\cos^2 \beta + \sin^2 \beta) + 2(\cos \alpha \cos \beta + \sin \alpha \sin \beta)$$

$$\Rightarrow b^2 + a^2 = 1 + 1 + 2\cos(\alpha-\beta) = 2 + 2\cos(\alpha-\beta) \qquad \text{...(i)}$$

and, $b^2 - a^2 = (\cos \alpha + \cos \beta)^2 - (\sin \alpha + \sin \beta)^2$

$$\Rightarrow b^2 - a^2 = \cos^2 \alpha + \cos^2 \beta - \sin^2 \alpha - \sin^2 \beta + 2(\cos \alpha \cos \beta - \sin \alpha \sin \beta)$$

$$\Rightarrow b^2 - a^2 = (\cos^2 \alpha - \sin^2 \beta) + (\cos^2 \beta - \sin^2 \alpha) + 2\cos(\alpha+\beta)$$

$$\Rightarrow b^2 - a^2 = \cos(\alpha+\beta)\cos(\alpha-\beta) + \cos(\beta+\alpha)\cos(\beta-\alpha) + 2\cos(\alpha+\beta)$$

$$\Rightarrow b^2 - a^2 = 2\cos(\alpha+\beta)\cos(\alpha-\beta) + 2\cos(\alpha+\beta) \quad [\because \cos(\beta-\alpha) = \cos\{-(\alpha-\beta)\} = \cos(\alpha-\beta)]$$

$$\Rightarrow b^2 - a^2 = \cos(\alpha+\beta)\{2\cos(\alpha-\beta) + 2\}$$

$$\Rightarrow b^2 - a^2 = \cos(\alpha+\beta)(b^2+a^2) \qquad \text{[Using (i)]}$$

Thus, $b^2 - a^2 = (b^2+a^2)\cos(\alpha+\beta) \Rightarrow \cos(\alpha+\beta) = \dfrac{b^2-a^2}{b^2+a^2}$

(ii) $\sin(\alpha+\beta) = \sqrt{1-\cos^2(\alpha+\beta)}$

$$\Rightarrow \sin(\alpha+\beta) = \sqrt{1 - \left(\frac{b^2-a^2}{b^2+a^2}\right)^2} = \sqrt{\frac{4a^2b^2}{(a^2+b^2)^2}} = \frac{2ab}{b^2+a^2}$$

EXAMPLE 41 *If* α *and* β *are the solutions of the equation* $a \tan x + b \sec x = c$, *then show that* $\tan(\alpha+\beta) = \dfrac{2ac}{a^2-c^2}$. **[NCERT EXEMPLAR]**

SOLUTION We have,

$$a \tan x + b \sec x = c \qquad \text{...(i)}$$

$\Rightarrow \quad c - a\tan x = b\sec x$

$\Rightarrow \quad (c - a\tan x)^2 = b^2\sec^2 x$

$\Rightarrow \quad c^2 + a^2\tan^2 x - 2ac\tan x = b^2(1+\tan^2 x) \Rightarrow \tan^2 x\,(a^2-b^2) - 2ac\tan x + (c^2-b^2) = 0$ ...(ii)

It is given that α and β are the solutions of equation (i). Therefore, $\tan\alpha$ and $\tan\beta$ are roots of equation (ii).

$\therefore \quad \tan\alpha + \tan\beta = \dfrac{2ac}{a^2-b^2}$ and, $\tan\alpha\tan\beta = \dfrac{c^2-b^2}{a^2-b^2}$

Hence, $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \dfrac{\dfrac{2ac}{(a^2-b^2)}}{1-\dfrac{c^2-b^2}{a^2-b^2}} = \dfrac{2ac}{a^2-c^2}$

EXAMPLE 42 *If α and β are the solutions of $a\cos x + b\sin x = c$, then show that*

(i) $\cos(\alpha+\beta) = \dfrac{a^2-b^2}{a^2+b^2}$ (ii) $\cos(\alpha-\beta) = \dfrac{2c^2-(a^2+b^2)}{a^2+b^2}$ [NCERT EXEMPLAR]

SOLUTION We have,

$a\cos x + b\sin x = c$ ...(i)

$\Rightarrow \quad a\cos x = c - b\sin x$

$\Rightarrow \quad a^2\cos^2 x = (c - b\sin x)^2$

$\Rightarrow \quad a^2(1-\sin^2 x) = c^2 - 2bc\sin x + b^2\sin^2 x \Rightarrow (a^2+b^2)\sin^2 x - 2bc\sin x + (c^2-a^2) = 0$ ...(ii)

Since α, β are roots of equation (i). Therefore, $\sin\alpha$ and $\sin\beta$ are roots of equation (ii).

$\therefore \quad \sin\alpha\sin\beta = \dfrac{c^2-a^2}{a^2+b^2}$ ...(iii)

Again, $a\cos x + b\sin x = c$

$\Rightarrow \quad b\sin x = c - a\cos x$

$\Rightarrow \quad b^2\sin^2 x = (c - a\cos x)^2$

$\Rightarrow \quad b^2(1-\cos^2 x) = (c-a\cos x)^2 \Rightarrow (a^2+b^2)\cos^2 x - 2ac\cos x + c^2 - b^2 = 0$ ...(iv)

It is given that α, β are the roots of equation (i). So, $\cos\alpha$, $\cos\beta$ are the roots of equation (iv).

$\therefore \quad \cos\alpha\cos\beta = \dfrac{c^2-b^2}{a^2+b^2}$ ...(v)

Using (iii) and (v), we obtain

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{c^2-b^2}{a^2+b^2} - \frac{c^2-a^2}{a^2+b^2} = \frac{a^2-b^2}{a^2+b^2}$$

and, $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \dfrac{c^2-b^2}{a^2+b^2} + \dfrac{c^2-a^2}{a^2+b^2} = \dfrac{2c^2-(a^2+b^2)}{a^2+b^2}$

EXAMPLE 43 *Prove that:* $\dfrac{\sin x}{\cos 3x} + \dfrac{\sin 3x}{\cos 9x} + \dfrac{\sin 9x}{\cos 27x} = \dfrac{1}{2}(\tan 27x - \tan x)$.

SOLUTION We have,

$\dfrac{\sin x}{\cos 3x} + \dfrac{\sin 3x}{\cos 9x} + \dfrac{\sin 9x}{\cos 27x}$

$$= \frac{1}{2}\left\{\frac{2\sin x\cos x}{\cos 3x\cos x} + \frac{2\sin 3x\cos 3x}{\cos 9x\cos 3x} + \frac{2\sin 9x\cos 9x}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin(x+x)}{\cos 3x\cos x} + \frac{\sin(3x+3x)}{\cos 9x\cos 3x} + \frac{\sin(9x+9x)}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin 2x}{\cos 3x\cos x} + \frac{\sin 6x}{\cos 9x\cos 3x} + \frac{\sin 18x}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin(3x-x)}{\cos 3x\cos x} + \frac{\sin(9x-3x)}{\cos 9x\cos 3x} + \frac{\sin(27x-9x)}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin 3x\cos x - \cos 3x\sin x}{\cos 3x\cos x} + \frac{\sin 9x\cos 3x - \cos 9x\sin 3x}{\cos 9x\cos 3x} + \frac{\sin 27x\cos 9x - \cos 27x\sin 9x}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin 3x\cos x}{\cos 3x\cos x} - \frac{\cos 3x\sin x}{\cos 3x\cos x} + \frac{\sin 9x\cos 3x}{\cos 9x\cos 3x} - \frac{\cos 9x\sin 3x}{\cos 9x\cos 3x} + \frac{\sin 27x\cos 9x}{\cos 27x\cos 9x} - \frac{\cos 27x\sin 9x}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\{(\tan 3x - \tan x) + (\tan 9x - \tan 3x) + (\tan 27x - \tan 9x)\} = \frac{1}{2}(\tan 27x - \tan x)$$

EXERCISE 7.1

BASIC

1. If $\sin A = \frac{4}{5}$ and $\cos B = \frac{5}{13}$, where $0 < A,\ B < \frac{\pi}{2}$, find the values of the following:
   (i) $\sin(A+B)$ (ii) $\cos(A+B)$ (iii) $\sin(A-B)$ (iv) $\cos(A-B)$
2. (a) If $\sin A = \frac{12}{13}$ and $\sin B = \frac{4}{5}$, where $\frac{\pi}{2} < A < \pi$ and $0 < B < \frac{\pi}{2}$, find the following:
   (i) $\sin(A+B)$ (ii) $\cos(A+B)$
   (b) If $\sin A = \frac{3}{5}$, $\cos B = -\frac{12}{13}$, where A and B both lie in second quadrant, find the value of $\sin(A+B)$. [NCERT]
3. If $\cos A = -\frac{24}{25}$ and $\cos B = \frac{3}{5}$, where $\pi < A < \frac{3\pi}{2}$ and $\frac{3\pi}{2} < B < 2\pi$, find the following:
   (i) $\sin(A+B)$ (ii) $\cos(A+B)$
4. If $\tan A = \frac{3}{4}$, $\cos B = \frac{9}{41}$, where $\pi < A < \frac{3\pi}{2}$ and $0 < B < \frac{\pi}{2}$, find $\tan(A+B)$.
5. If $\sin A = \frac{1}{2}$, $\cos B = \frac{12}{13}$, where $\frac{\pi}{2} < A < \pi$ and $\frac{3\pi}{2} < B < 2\pi$, find $\tan(A-B)$.
6. If $\sin A = \frac{1}{2}$, $\cos B = \frac{\sqrt{3}}{2}$,, where $\frac{\pi}{2} < A < \pi$ and $0 < B < \frac{\pi}{2}$, find the following :
   (i) $\tan(A+B)$ (ii) $\tan(A-B)$
7. Evaluate the following:
   (i) $\sin 78^\circ\cos 18^\circ - \cos 78^\circ\sin 18^\circ$ (ii) $\cos 47^\circ\cos 13^\circ - \sin 47^\circ\sin 13^\circ$
   (iii) $\sin 36^\circ\cos 9^\circ + \cos 36^\circ\sin 9^\circ$ (iv) $\cos 80^\circ\cos 20^\circ + \sin 80^\circ\sin 20^\circ$
8. If $\cos A = -\frac{12}{13}$ and $\cot B = \frac{24}{7}$, where A lies in the second quadrant and B in the third quadrant, find the values of the following :
   (i) $\sin(A+B)$ (ii) $\cos(A+B)$ (iii) $\tan(A+B)$

9. Prove that: $\cos\frac{7\pi}{12}+\cos\frac{\pi}{12} = \sin\frac{5\pi}{12}-\sin\frac{\pi}{12}$

10. Prove that: $\frac{\tan A+\tan B}{\tan A-\tan B} = \frac{\sin(A+B)}{\sin(A-B)}$

11. Prove that:
(i) $\frac{\cos 11°+\sin 11°}{\cos 11°-\sin 11°}=\tan 56°$ (ii) $\frac{\cos 9°+\sin 9°}{\cos 9°-\sin 9°}=\tan 54°$ (iii) $\frac{\cos 8°-\sin 8°}{\cos 8°+\sin 8°}=\tan 37°$

12. Prove that:
(i) $\sin\left(\frac{\pi}{3}-x\right)\cos\left(\frac{\pi}{6}+x\right)+\cos\left(\frac{\pi}{3}-x\right)\sin\left(\frac{\pi}{6}+x\right)=1$
(ii) $\sin\left(\frac{4\pi}{9}+7\right)\cos\left(\frac{\pi}{9}+7\right)-\cos\left(\frac{4\pi}{9}+7\right)\sin\left(\frac{\pi}{9}+7\right)=\frac{\sqrt{3}}{2}$
(iii) $\sin\left(\frac{3\pi}{8}-5\right)\cos\left(\frac{\pi}{8}+5\right)+\cos\left(\frac{3\pi}{8}-5\right)\sin\left(\frac{\pi}{8}+5\right)=1$

13. Prove that: $\frac{\tan 69°+\tan 66°}{1-\tan 69°\tan 66°}=-1$

BASED ON LOTS

14. (i) If $\tan A=\frac{5}{6}$ and $\tan B=\frac{1}{11}$, prove that $A+B=\frac{\pi}{4}$
(ii) If $\tan A=\frac{m}{m-1}$ and $\tan B=\frac{1}{2m-1}$, then prove that $A-B=\frac{\pi}{4}$

15. Prove that:
(i) $\cos^2\frac{\pi}{4}-\sin^2\frac{\pi}{12}=\frac{\sqrt{3}}{4}$ (ii) $\sin^2(n+1)A-\sin^2 nA=\sin(2n+1)A\sin A$

16. Prove that:
(i) $\frac{\sin(A+B)+\sin(A-B)}{\cos(A+B)+\cos(A-B)}=\tan A$
(ii) $\frac{\sin(A-B)}{\cos A\cos B}+\frac{\sin(B-C)}{\cos B\cos C}+\frac{\sin(C-A)}{\cos C\cos A}=0$
(iii) $\frac{\sin(A-B)}{\sin A\sin B}+\frac{\sin(B-C)}{\sin B\sin C}+\frac{\sin(C-A)}{\sin C\sin A}=0$
(iv) $\sin^2 B=\sin^2 A+\sin^2(A-B)-2\sin A\cos B\sin(A-B)$
(v) $\cos^2 A+\cos^2 B-2\cos A\cos B\cos(A+B)=\sin^2(A+B)$
(vi) $\frac{\tan(A+B)}{\cot(A-B)}=\frac{\tan^2 A-\tan^2 B}{1-\tan^2 A\tan^2 B}$

17. Prove that:
(i) $\tan 8x-\tan 6x-\tan 2x=\tan 8x\tan 6x\tan 2x$
(ii) $\tan\frac{\pi}{12}+\tan\frac{\pi}{6}+\tan\frac{\pi}{12}\tan\frac{\pi}{6}=1$
(iii) $\tan 36°+\tan 9°+\tan 36°\tan 9°=1$
(iv) $\tan 13x-\tan 9x-\tan 4x=\tan 13x\tan 9x\tan 4x$

18. Prove that: $\dfrac{\tan^2 2x - \tan^2 x}{1 - \tan^2 2x \tan^2 x} = \tan 3x \tan x$

19. (i) If $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{a+b}{a-b}$, show that $\dfrac{\tan x}{\tan y} = \dfrac{a}{b}$. [NCERT]

(ii) If $\cos(\theta+\phi) = m\cos(\theta-\phi)$, then prove that $\tan\theta = \dfrac{1-m}{1+m}\cot\phi$. [NCERT EXEMPLAR]

20. If $\tan A = x \tan B$, prove that $\dfrac{\sin(A-B)}{\sin(A+B)} = \dfrac{x-1}{x+1}$.

21. If $\tan(A+B) = x$ and $\tan(A-B) = y$, find the values of $\tan 2A$ and $\tan 2B$. [NCERT EXEMPLAR]

22. If $\cos A + \sin B = m$ and $\sin A + \cos B = n$, prove that $2\sin(A+B) = m^2 + n^2 - 2$.

23. If $\tan A + \tan B = a$ and $\cot A + \cot B = b$, prove that: $\cot(A+B) = \dfrac{1}{a} - \dfrac{1}{b}$.

BASED ON HOTS

24. If x lies in the first quadrant and $\cos x = \dfrac{8}{17}$, then prove that

$$\cos\left(\frac{\pi}{6}+x\right) + \cos\left(\frac{\pi}{4}-x\right) + \cos\left(\frac{2\pi}{3}-x\right) = \frac{23}{17}\left(\frac{\sqrt{3}-1}{2} + \frac{1}{\sqrt{2}}\right)$$ [NCERT EXEMPLAR]

25. If $\tan x + \tan\left(x+\dfrac{\pi}{3}\right) + \tan\left(x+\dfrac{2\pi}{3}\right) = 3$, then prove that $\dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} = 1$.

26. If $\sin(\alpha+\beta) = 1$ and $\sin(\alpha-\beta) = \dfrac{1}{2}$, where $0 \le \alpha, \beta \le \dfrac{\pi}{2}$, then find the values of $\tan(\alpha+2\beta)$ and $\tan(2\alpha+\beta)$.

27. If α, β are two different values of x lying between 0 and 2π which satisfy the equation $6\cos x + 8\sin x = 9$, find the value of $\sin(\alpha+\beta)$.

28. If $\sin\alpha + \sin\beta = a$ and $\cos\alpha + \cos\beta = b$, show that

(i) $\sin(\alpha+\beta) = \dfrac{2ab}{a^2+b^2}$ (ii) $\cos(\alpha+\beta) = \dfrac{b^2-a^2}{b^2+a^2}$

29. Prove that :

(i) $\dfrac{1}{\sin(x-a)\sin(x-b)} = \dfrac{\cot(x-a) - \cot(x-b)}{\sin(a-b)}$

(ii) $\dfrac{1}{\sin(x-a)\cos(x-b)} = \dfrac{\cot(x-a) + \tan(x-b)}{\cos(a-b)}$

(iii) $\dfrac{1}{\cos(x-a)\cos(x-b)} = \dfrac{\tan(x-b) - \tan(x-a)}{\sin(a-b)}$

30. If $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$, prove that $1 + \cot\alpha\tan\beta = 0$.

31. If $\tan\alpha = x+1$, $\tan\beta = x-1$, show that $2\cot(\alpha-\beta) = x^2$.

32. If angle θ is divided into two parts such that the tangent of one part is λ times the tangent of other, and ϕ is their difference, then show that $\sin\theta = \dfrac{\lambda+1}{\lambda-1}\sin\phi$. [NCERT EXEMPLAR]

33. If $\tan x = \dfrac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$, then show that $\sin\alpha + \cos\alpha = \sqrt{2}\cos x$. [NCERT EXEMPLAR]

34. If α and β are two solutions of the equation $a\tan x + b\sec x = c$, then find the values of $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$.

ANSWERS

1. (i) $\frac{56}{65}$ (ii) $\frac{-33}{65}$ (iii) $\frac{-16}{65}$ (iv) $\frac{63}{65}$

2. (a) (i) $\frac{16}{65}$ (ii) $-\frac{63}{65}$ (b) $\frac{-56}{65}$

3. (i) $\frac{3}{5}$ (ii) $-\frac{4}{5}$ 4. $-\frac{187}{84}$ 5. $\frac{5\sqrt{3}-12}{5+12\sqrt{3}}$

6. (i) 0 (ii) $-\sqrt{3}$

7. (i) $\frac{\sqrt{3}}{2}$ (ii) $\frac{1}{2}$ (iii) $\frac{1}{\sqrt{2}}$ (iv) $\frac{1}{2}$

8. (i) $\frac{-36}{325}$ (ii) $\frac{323}{325}$ (iii) $-\frac{36}{323}$ 18. $\frac{x+y}{1-xy}, \frac{x-y}{1+xy}$

26. $-\sqrt{3}, -\frac{1}{\sqrt{3}}$ 27. $\frac{24}{25}$ 34. $\sin(\alpha+\beta) = -\frac{2ac}{a^2+c^2}$, $\cos(\alpha+\beta) = \frac{c^2-a^2}{c^2+a^2}$

HINTS TO SELECTED PROBLEMS

9. $\text{LHS} = \cos\left(\frac{\pi}{2}+\frac{\pi}{12}\right) + \cos\left(\frac{\pi}{2}-\frac{5\pi}{12}\right) = -\sin\frac{\pi}{12} + \sin\frac{5\pi}{12} = \text{RHS}$

11. $$\text{LHS} = \frac{\dfrac{\cos 11° + \sin 11°}{\cos 11°}}{\dfrac{\cos 11° - \sin 11°}{\cos 11°}} = \frac{1+\tan 11°}{1-\tan 11°} = \frac{\tan 45° + \tan 11°}{1 - \tan 45° \tan 11°} = \tan(45°+11°) = \tan 56° = \text{RHS}$$

19. (i) $\frac{\sin(x+y)}{\sin(x-y)} = \frac{a+b}{a-b}$

$\Rightarrow \frac{\sin(x+y)+\sin(x-y)}{\sin(x+y)-\sin(x-y)} = \frac{(a+b)+(a-b)}{(a+b)-(a-b)}$ [Applying componendo and dividendo]

$\Rightarrow \frac{2\sin x\cos y}{2\cos x\sin y} = \frac{2a}{2b} \Rightarrow \frac{\tan x}{\tan y} = \frac{a}{b}$

(ii) $\cos(\theta+\phi) = m\cos(\theta-\phi)$

$\Rightarrow \frac{\cos(\theta-\phi)}{\cos(\theta+\phi)} = \frac{1}{m}$

$\Rightarrow \frac{\cos(\theta-\phi)+\cos(\theta+\phi)}{\cos(\theta-\phi)-\cos(\theta+\phi)} = \frac{1+m}{1-m}$ [Applying componendo and dividendo]

$\Rightarrow \frac{2\cos\theta\cos\phi}{2\sin\theta\sin\phi} = \frac{1+m}{1-m} \Rightarrow \frac{\cot\phi}{\tan\theta} = \frac{1+m}{1-m} \Rightarrow \tan\theta = \frac{1-m}{1+m}\cot\phi$

29. (i) We have,

$$\frac{1}{\sin(x-a)\sin(x-b)} = \frac{1}{\sin(a-b)}\left\{\frac{\sin(a-b)}{\sin(x-a)\sin(x-b)}\right\}$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin\{(x-b)-(x-a)\}}{\sin(x-a)\sin(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin(x-b)\cos(x-a)}{\sin(x-a)\sin(x-b)}-\frac{\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}\right] = \frac{1}{\sin(a-b)}[\cot(x-a)-\cot(x-b)]$$

Similarly, we can prove other two parts.

32. Let α and β be two parts of angle θ. Then, $\alpha+\beta=\theta$. It is given that $\alpha-\beta=\phi$ and $\tan\alpha=\lambda\tan\beta$.

Now, $\tan\alpha=\lambda\tan\beta$

$$\Rightarrow \quad \frac{\tan\alpha}{\tan\beta}=\frac{\lambda}{1}$$

$$\Rightarrow \quad \frac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}=\frac{\lambda+1}{\lambda-1} \qquad \text{[Applying componendo and dividendo]}$$

$$\Rightarrow \quad \frac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}{\dfrac{\sin\alpha}{\cos\alpha}-\dfrac{\sin\beta}{\cos\beta}}=\frac{\lambda+1}{\lambda-1} \Rightarrow \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\cos\beta-\cos\alpha\sin\beta}=\frac{\lambda+1}{\lambda-1}$$

$$\Rightarrow \quad \frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\frac{\lambda+1}{\lambda-1} \Rightarrow \frac{\sin\theta}{\sin\phi}=\frac{\lambda+1}{\lambda-1} \Rightarrow \sin\theta=\frac{\lambda+1}{\lambda-1}\sin\phi.$$

33. $$\tan x=\frac{\sin\alpha-\cos\alpha}{\sin\alpha+\cos\alpha}=\frac{\dfrac{\sin\alpha-\cos x}{\cos\alpha}}{\dfrac{\sin\alpha+\cos\alpha}{\cos\alpha}} \qquad \text{[Dividing numerator and denominator by } \cos\alpha]$$

$$\Rightarrow \quad \tan x=\frac{\tan\alpha-1}{\tan\alpha+1} \Rightarrow \tan x=\tan\left(\alpha-\frac{\pi}{4}\right) \Rightarrow x=\alpha-\frac{\pi}{4} \Rightarrow \alpha=x+\frac{\pi}{4}$$

$$\therefore \quad \sin\alpha+\cos\alpha=\sin\left(x+\frac{\pi}{4}\right)+\cos\left(x+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}(\sin x+\cos x)+\frac{1}{\sqrt{2}}(\cos x-\sin x)=\sqrt{2}\cos x$$

ALITER $\tan x=\dfrac{\sin\alpha-\cos\alpha}{\sin\alpha+\cos\alpha}$

$$\Rightarrow \quad \sec x=\sqrt{1+\tan^2 x}=\sqrt{1+\left(\frac{\sin\alpha-\cos\alpha}{\sin\alpha+\cos\alpha}\right)^2}$$

$$\Rightarrow \quad \frac{1}{\cos x}=\sqrt{\frac{(\sin\alpha+\cos\alpha)^2+(\sin\alpha-\cos\alpha)^2}{(\sin\alpha+\cos\alpha)^2}} \Rightarrow \frac{1}{\cos x}=\sqrt{\frac{2}{(\sin\alpha+\cos\alpha)^2}}$$

$$\Rightarrow \quad \frac{1}{\cos x}=\frac{\sqrt{2}}{\sin\alpha+\cos\alpha} \Rightarrow \sin\alpha+\cos\alpha=\sqrt{2}\cos x$$

7.4 MAXIMUM AND MINIMUM VALUES OF TRIGONOMETRICAL EXPRESSIONS

We have learnt that for those values of x for which trigonometrical functions are defined, we have

$-1 \leq \sin x \leq 1, -1 \leq \cos x \leq 1, -\infty < \tan x < \infty$ $\text{cosec}\, x \geq 1$ or $\text{cosec}\, x \leq -1$, $\sec x \geq 1$ or $\sec x \leq -1$ and, $-\infty < \cot x < \infty$

In this section, we will find the maximum and minimum values of trigonometrical expressions of the form $a \cos x + b \sin x$ for varying values of x.

Let $f(x) = a \cos x + b \sin x$ Further, let $a = r \sin \alpha$ and $b = r \cos \alpha$. This assumption is construct a right angled triangle with a and b as two sides and $r = \sqrt{a^2 + b^2}$ as hypotenuse.

Then, $\quad a^2 + b^2 = r^2 \sin^2 \alpha + r^2 \cos^2 \alpha$ and, $\dfrac{a}{b} = \dfrac{r \sin \alpha}{r \cos \alpha}$

$\Rightarrow \quad a^2 + b^2 = r^2 (\sin^2 \alpha + \cos^2 \alpha)$ and, $\dfrac{a}{b} = \tan \alpha \Rightarrow r = \sqrt{a^2 + b^2}$ and, $\tan \alpha = \dfrac{a}{b}$

Substituting the values of a and b in $f(x)$, we obtain

$$f(x) = r \sin \alpha \cos x + r \cos \alpha \sin x = r (\sin \alpha \cos x + \cos \alpha \sin x) = r \sin (\alpha + x)$$

We know that

$$-1 \leq \sin (\alpha + x) \leq 1 \quad \text{for all } x$$

$\Rightarrow \quad -r \leq r \sin (\alpha + x) \leq r \quad$ for all x [Multiplying throughout by r]

$\Rightarrow \quad -\sqrt{a^2 + b^2} \leq f(x) \leq \sqrt{a^2 + b^2}$ for all x

$\Rightarrow \quad -\sqrt{a^2 + b^2} \leq a \cos x + b \sin x \leq \sqrt{a^2 + b^2}$ for all x

Hence, maximum and minimum values of $a \cos x + b \sin x$ are $\sqrt{a^2 + b^2}$ and $-\sqrt{a^2 + b^2}$ respectively.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the maximum and minimum values of* $7 \cos x + 24 \sin x$.

SOLUTION We know that the maximum and minimum values of $a \cos x + b \sin x$ are $\sqrt{a^2 + b^2}$ and $-\sqrt{a^2 + b^2}$ respectively. Hence, the maximum and minimum values of $7 \cos x + 24 \sin x$ are $\sqrt{7^2 + 24^2} = 25$ and $-\sqrt{7^2 + 24^2} = -25$ respectively.

EXAMPLE 2 *Find the maximum and minimum values of the following expressions:*

(i) $3 \cos x + 5 \sin \left(x - \dfrac{\pi}{6} \right)$ (ii) $4 \sin x - 3 \cos x + 7$

SOLUTION (i) Let $f(x) = 3 \cos x + 5 \sin \left(x - \dfrac{\pi}{6} \right)$. Then,

$$f(x) = 3 \cos x + 5 \left\{ \sin x \cos \frac{\pi}{6} - \cos x \sin \frac{\pi}{6} \right\} = 3 \cos x + \frac{5\sqrt{3}}{2} \sin x - \frac{5}{2} \cos x$$

$$\Rightarrow \quad f(x) = \frac{1}{2} \cos x + \frac{5\sqrt{3}}{2} \sin x$$

$$\because \quad -\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2} \le \frac{1}{2}\cos x+\frac{5\sqrt{3}}{2}\sin x \le \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2} \text{ for all } x.$$

$$\therefore \quad -\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2} \le f(x) \le \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2} \text{ for all } x$$

$$\Rightarrow \quad -\sqrt{\frac{1}{4}+\frac{75}{4}} \le 3\cos x+5\sin\left(x-\frac{\pi}{6}\right) \le \sqrt{\frac{1}{4}+\frac{75}{4}} \text{ for all } x$$

$$\Rightarrow \quad -\sqrt{19} \le 3\cos x+5\sin\left(x-\frac{\pi}{6}\right) \le \sqrt{19} \quad \text{for all } x$$

Hence, $-\sqrt{19}$ and $\sqrt{19}$ are respectively the minimum and the maximum values of $3\cos x+5\sin\left(x-\frac{\pi}{6}\right)$.

(ii) Let $f(x) = 4\sin x - 3\cos x + 7$

We know that

$$-\sqrt{4^2+(-3)^2} \le 4\sin x - 3\cos x \le \sqrt{4^2+(-3)^2} \text{ for all } x$$

$$\Rightarrow \quad -5 \le 4\sin x - 3\cos x \le 5 \text{ for all } x$$

$$\Rightarrow \quad -5+7 \le 4\sin x - 3\cos x + 7 \le 5+7 \text{ for all } x$$

$$\Rightarrow \quad 2 \le f(x) \le 12 \text{ for all } x$$

Hence, minimum and maximum values of $4\sin x - 3\cos x + 7$ are 2 and 12 respectively.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 3 *Prove that* $5\cos x + 3\cos\left(x+\frac{\pi}{3}\right) + 3$ *lies between* -4 *and* 10.

SOLUTION Let $f(x) = 5\cos x + 3\cos\left(x+\frac{\pi}{3}\right) + 3$. Then,

$$f(x) = 5\cos x + 3\left(\cos x\cos\frac{\pi}{3} - \sin x\sin\frac{\pi}{3}\right) + 3 = 5\cos x + \frac{3}{2}\cos x - \frac{3\sqrt{3}}{2}\sin x + 3$$

$$\Rightarrow \quad f(x) = \frac{13}{2}\cos x - \frac{3\sqrt{3}}{2}\sin x + 3 \quad \text{...(i)}$$

$$\because \quad -\sqrt{\left(\frac{13}{2}\right)^2+\left(\frac{3\sqrt{3}}{2}\right)^2} \le \frac{13}{2}\cos x - \frac{3\sqrt{3}}{2}\sin x \le \sqrt{\left(\frac{13}{2}\right)^2+\left(\frac{3\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow \quad -7 \le \frac{13}{2}\cos x - \frac{3\sqrt{3}}{2}\sin x \le 7 \text{ for all } x$$

$$\Rightarrow \quad -7+3 \le \frac{13}{2}\cos x - \frac{3\sqrt{3}}{2}\sin x + 3 \le 7+3 \text{ for all } x$$

$$\Rightarrow \quad -4 \le 5\cos x + 3\cos\left(x+\frac{\pi}{3}\right) + 3 \le 10 \text{ for all } x \quad \text{[Using (i)]}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 4 *Find a and b such that the following inequality holds good for all x :*

$$a \le 3\cos x + 5\sin\left(x-\frac{\pi}{6}\right) \le b.$$

Also, find the greatest and least values of a and b respectively.

SOLUTION Let $f(x) = 3\cos x + 5\sin\left(x - \frac{\pi}{6}\right)$. Then,

$$f(x) = 3\cos x + 5\left(\sin x \cos\frac{\pi}{6} - \cos x \sin\frac{\pi}{6}\right) = 3\cos x + 5\left(\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x\right)$$

$$\Rightarrow \quad f(x) = \frac{1}{2}\cos x + \frac{5\sqrt{3}}{2}\sin x$$

$$\because \quad \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{5\sqrt{3}}{2}\right)^2} \le \frac{1}{2}\cos x + \frac{5\sqrt{3}}{2}\sin x \le \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{5\sqrt{3}}{2}\right)^2} \quad \text{for all } x$$

$$\Rightarrow \quad -\sqrt{19} \le 3\cos x + 5\sin\left(x - \frac{\pi}{6}\right) \le \sqrt{19} \quad \text{for all } x$$

Hence, $a \le -\sqrt{19}$ and $b \ge \sqrt{19}$. The greatest value of a is $-\sqrt{19}$ and the least value of b is $\sqrt{19}$.

7.5 TO EXPRESS $a\cos x + b\sin x$ IN THE FORM $r\sin(x \pm \alpha)$ OR $r\cos(x \pm \alpha)$

Sometimes we need to express trigonometrical expressions of the form $a\cos x + b\sin x$ in terms of sine or cosine of single term. We may use the following algorithm to do so.

ALGORITHM

Step I *Multiply and divide* $f(x) = a\cos x + b\sin x$ *by* $\sqrt{a^2+b^2}$ *to get*

$$f(x) = \sqrt{a^2+b^2}\left\{\frac{a}{\sqrt{a^2+b^2}}\cos x + \frac{b}{\sqrt{a^2+b^2}}\sin x\right\}$$

Step II *In order to express* $f(x)$ *in terms of sine of some term, replace* $\frac{a}{\sqrt{a^2+b^2}}$ *i.e. coefficient of* $\cos x$ *by* $\sin\alpha$ *and* $\frac{b}{\sqrt{a^2+b^2}}$ *i.e. coefficient of* $\sin x$ *by* $\cos\alpha$. *This gives the following :*

$$f(x) = \sqrt{a^2+b^2}\,\{\sin\alpha\cos x + \cos\alpha\sin x\} = \sqrt{a^2+b^2}\,\sin(x+\alpha)$$

To express $f(x)$ *in terms of cosine of some term, replace coefficient of* $\cos x$ *i.e.* $\frac{a}{\sqrt{a^2+b^2}}$ *by* $\cos\alpha$ *and coefficient of* $\sin x$ *i.e.* $\frac{b}{\sqrt{a^2+b^2}}$ *by* $\sin\alpha$. *This gives the following:*

$$f(x) = \sqrt{a^2+b^2}\,(\cos\alpha\cos x + \sin\alpha\sin x) = \sqrt{a^2+b^2}\,\cos(x-\alpha).$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Reduce* $\sqrt{3}\sin x + \cos x$ *as a single term consisting (i) sine only (ii) cosine only.*

SOLUTION Let $f(x) = \sqrt{3}\sin x + \cos x$. Then,

$$f(x) = \sqrt{3}\sin x + \cos x$$

$$\Rightarrow f(x) = 2\left\{\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x\right\} \qquad \left[\text{Multiplying and dividing by } \sqrt{(\sqrt{3})^2 + 1^2} \text{ i.e. by } 2\right]$$

$$\Rightarrow f(x) = 2\left\{\sin x\cos\frac{\pi}{6} + \cos x\sin\frac{\pi}{6}\right\} = 2\sin\left(x + \frac{\pi}{6}\right) \qquad \left[\because \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} \text{ and } \frac{1}{2} = \sin\frac{\pi}{6}\right]$$

Again,

$$f(x) = 2\left\{\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x\right\} = 2\left\{\sin\frac{\pi}{3}\sin x + \cos\frac{\pi}{3}\cos x\right\}$$

$$\Rightarrow f(x) = 2\left\{\cos x\cos\frac{\pi}{3} + \sin x\sin\frac{\pi}{3}\right\} = 2\cos\left(x - \frac{\pi}{3}\right).$$

EXAMPLE 2 *Express* $3\cos x - 4\sin x$ *as sines and cosines of a single expression.*

SOLUTION Let $f(x) = 3\cos x - 4\sin x$. Multiplying and dividing by $\sqrt{3^2 + (-4)^2}$ i.e. by 5, we get

$$f(x) = \sqrt{3^2 + (-4)^2}\left\{\frac{3}{\sqrt{3^2 + (-4)^2}}\cos x - \frac{4}{\sqrt{3^2 + (-4)^2}}\sin x\right\}$$

$$\Rightarrow f(x) = 5\left(\frac{3}{5}\cos x - \frac{4}{5}\sin x\right)$$

$$\Rightarrow f(x) = 5(\sin\alpha\cos x - \cos\alpha\sin x), \text{ where } \sin\alpha = \frac{3}{5} \text{ and } \cos\alpha = \frac{4}{5}$$

$$\Rightarrow f(x) = 5\sin(\alpha - x), \text{ where } \tan\alpha = \frac{3}{4}$$

Again,

$$f(x) = 5\left(\frac{3}{5}\cos x - \frac{4}{5}\sin x\right)$$

$$\Rightarrow f(x) = 5(\cos\alpha\cos x - \sin\alpha\sin x), \text{ where } \cos\alpha = \frac{3}{5} \text{ and } \sin\alpha = \frac{4}{5}$$

$$\Rightarrow f(x) = 5\cos(\alpha + x), \text{ where } \tan\alpha = \frac{4}{3}$$

EXAMPLE 3 *Find the sign of the expression* $\sin 100° + \cos 100°$.

SOLUTION $\sin 100° + \cos 100°$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin 100° + \frac{1}{\sqrt{2}}\cos 100°\right) = \sqrt{2}(\cos 45°\sin 100° + \sin 45°\cos 100°)$$

$= \sqrt{2}\sin(100° + 45°) = \sqrt{2}\sin 145°$, which is a positive real number. [$\because$ $\sin 145°$ is positive]

EXERCISE 7.2

BASIC

1. Find the maximum and minimum values of each of the following trigonometrical expressions:
   (i) $12\sin x - 5\cos x$ (ii) $12\cos x + 5\sin x + 4$
   (iii) $5\cos x + 3\sin\left(\frac{\pi}{6} - x\right) + 4$ (iv) $\sin x - \cos x + 1$

2. Reduce each of the following expressions to the sine and cosine of a single expression:
   (i) $\sqrt{3}\sin x - \cos x$ (ii) $\cos x - \sin x$ (iii) $24\cos x + 7\sin x$

BASED ON HOTS

3. Show that $\sin 100° - \sin 10°$ is positive.
4. Prove that $(2\sqrt{3} + 3)\sin x + 2\sqrt{3}\cos x$ lies between $-(2\sqrt{3} + \sqrt{15})$ and $(2\sqrt{3} + \sqrt{15})$.

ANSWERS

1. | | Minimum | Maximum | | Minimum | Maximum |
|---|---|---|---|---|---|
| (i) | -13 | 13 | (ii) | -9 | 17 |
| (iii) | -3 | 11 | (iv) | $1-\sqrt{2}$ | $1+\sqrt{2}$ |

2. (i) $2\sin\left(x-\frac{\pi}{6}\right), -2\cos\left(\frac{\pi}{3}+x\right)$ (ii) $\sqrt{2}\sin\left(\frac{\pi}{4}-x\right), \sqrt{2}\cos\left(\frac{\pi}{4}+x\right)$

(iii) $25\sin(\alpha+x)$, where $\tan\alpha=\frac{24}{7}$, $25\cos(x-\alpha)$, where $\tan\alpha=\frac{7}{24}$

HINTS TO SELECTED PROBLEMS

4. $(2\sqrt{3}+3)\sin x+2\sqrt{3}\cos x=(2\sqrt{3}\sin x+\sqrt{3}\cos x)+(\sqrt{3}\cos x+3\sin x)$

Now, $-\sqrt{15}\le 2\sqrt{3}\sin x+\sqrt{3}\cos x\le\sqrt{15}$ and, $-\sqrt{12}\le\sqrt{3}\cos x+3\sin x\le\sqrt{12}$

Adding the two inequalities, we obtain

$\therefore\quad -\sqrt{15}-\sqrt{12}\le 2\sqrt{3}\sin x+\sqrt{3}\cos x+\sqrt{3}\cos x+3\sin x\le\sqrt{15}+\sqrt{12}$

$\Rightarrow\quad -(2\sqrt{3}+\sqrt{15})\le(2\sqrt{3}+3)\sin x+2\sqrt{3}\cos x\le(2\sqrt{3}+\sqrt{15})$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. The maximum value of $3\cos x+4\sin x+5$ is ....
2. The minimum value of $4\cos x-3\sin x+7$ is ....
3. If $\sin\theta+\cos\theta=1$, then the value of $\sin 2\theta$ is ....
4. If $A-B=\frac{\pi}{4}$, then $(1+\tan A)(1-\tan B)=$ ....
5. If $A+B=\frac{\pi}{4}$, then $(1+\tan A)(1+\tan B)=$ ....
6. If $\cos(A-B)=\frac{3}{5}$ and $\tan A\tan B=2$, then $\sin A\sin B=$ ....
7. If $\frac{x}{\cos\theta}=\frac{y}{\cos\left(\theta-\frac{2\pi}{3}\right)}=\frac{z}{\cos\left(\theta+\frac{2\pi}{3}\right)}$, then $x+y+z=$ ....
8. The value of $\cot\left(\frac{\pi}{4}+x\right)\cot\left(\frac{\pi}{4}-x\right)$ is ....
9. If $\sin x\cos y=\frac{1}{4}$ and $3\tan x=4\tan y$, then $\sin(x-y)$ is equal to ....
10. If $\cos^2\left(\frac{\pi}{6}+x\right)-\sin^2\left(\frac{\pi}{6}-x\right)=k\cos 2x$ then $k=$ ....
11. If $\tan x=\frac{1}{2}$ and $\tan y=\frac{1}{3}$, then the value of $x+y$ is ....
12. The value of $\tan 5x\tan 3x\tan 2x-\tan 5x+\tan 3x+\tan 2x$ is ....
13. The value of $\cos\frac{\pi}{12}-\sin\frac{\pi}{2}$ is ....

ANSWERS

1. 10 2. 2 3. 0 4. 2 5. 2 6. $\frac{2}{5}$ 7. 0 8. 1
9. $\frac{1}{16}$ 10. $\frac{1}{2}$ 11. $\frac{\pi}{4}$ 12. 0 13. $\frac{1}{\sqrt{2}}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If $\alpha+\beta-\gamma=\pi$, and $\sin^2\alpha+\sin^2\beta-\sin^2\gamma=\lambda\sin\alpha\sin\beta\cos\gamma$, then write the value of λ.
2. If $x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right)$, then write the value of $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$.
3. Write the maximum and minimum values of $3\cos x+4\sin x+5$.
4. Write the maximum value of $12\sin x-9\sin^2 x$.
5. If $12\sin x-9\sin^2 x$ attains its maximum value at $x=\alpha$, then write the value of $\sin\alpha$.
6. Write the interval in which the values of $5\cos x+3\cos\left(x+\frac{\pi}{3}\right)+3$ lie.
7. If $\tan(A+B) = p$ and $\tan(A-B) = q$, then write the value of $\tan 2B$.
8. If $\frac{\cos(x-y)}{\cos(x+y)} = \frac{m}{n}$, then write the value of $\tan x\tan y$.
9. If $a = b\cos\frac{2\pi}{3} = c\cos\frac{4\pi}{3}$, then write the value of $ab+bc+ca$.
10. If $A+B = C$, then write the value of $\tan A\tan B\tan C$.
11. If $\sin\alpha-\sin\beta = a$ and $\cos\alpha+\cos\beta = b$, then write the value of $\cos(\alpha+\beta)$.
12. If $\tan\alpha = \frac{1}{1+2^{-x}}$ and $\tan\beta = \frac{1}{1+2^{x+1}}$, then write the value of $\alpha+\beta$ lying in the interval $(0, \pi/2)$.

ANSWERS

1. 2 2. 0 3. Maximum = 10, Minimum = 0 4. 4 5. $\frac{2}{3}$
6. $[-4, 10]$ 7. $\frac{p-q}{1+pq}$ 8. $\frac{m-n}{m+n}$ 9. 0 10. $\tan C-\tan A-\tan B$
11. $\frac{a^2+b^2-2}{2}$ 12. $\frac{\pi}{4}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The value of $\sin^2\frac{5\pi}{12}-\sin^2\frac{\pi}{12}$ is
   (a) $1/2$ (b) $\sqrt{3}/2$ (c) 1 (d) 0
2. If $A+B+C=\pi$, then $\sec A(\cos B\cos C-\sin B\sin C)$ is equal to
   (a) 0 (b) -1 (c) 1 (d) none of these

3. $\tan 20° + \tan 40° + \sqrt{3} \tan 20° \tan 40°$ is equal to

(a) $\frac{\sqrt{3}}{4}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\sqrt{3}$ (d) 1

4. If $\tan A = \frac{a}{a+1}$ and $\tan B = \frac{1}{2a+1}$, then the value of $A + B$ is

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

5. If $3 \sin x + 4 \cos x = 5$, then $4 \sin x - 3 \cos x =$

(a) 0 (b) 5 (c) 1 (d) none of these

6. If in a ΔABC, $\tan A + \tan B + \tan C = 6$, then $\cot A \cot B \cot C =$

(a) 6 (b) 1 (c) 1/6 (d) none of these

7. $\tan 3A - \tan 2A - \tan A$ is equal to

(a) $\tan 3A \tan 2A \tan A$ (b) $-\tan 3A \tan 2A \tan A$

(c) $\tan A \tan 2A - \tan 2A \tan 3A - \tan 3A \tan A$ (d) none of these

8. If $A + B + C = \pi$, then $\frac{\tan A + \tan B + \tan C}{\tan A \ \tan B \ \tan C}$ is equal to

(a) $\tan A \tan B \tan C$ (b) 0 (c) 1 (d) none of these

9. If $\cos P = \frac{1}{7}$ and $\cos Q = \frac{13}{14}$, where P and Q both are acute angles. Then, the value of $P - Q$ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{5\pi}{12}$

10. If $\cot(\alpha + \beta) = 0$, then $\sin(\alpha + 2\beta)$ is equal to

(a) $\sin \alpha$ (b) $\cos 2\beta$ (c) $\cos \alpha$ (d) $\sin 2\alpha$

11. $\frac{\cos 10° + \sin 10°}{\cos 10° - \sin 10°}$ is equal to

(a) $\tan 55°$ (b) $\cot 55°$ (c) $-\tan 35°$ (d) $-\cot 35°$

12. The value of $\cos^2\left(\frac{\pi}{6} + x\right) - \sin^2\left(\frac{\pi}{6} - x\right)$ is

(a) $\frac{1}{2} \cos 2x$ (b) 0 (c) $-\frac{1}{2} \cos 2x$ (d) $\frac{1}{2}$

13. If $\tan \theta_1 \tan \theta_2 = k$, then $\frac{\cos(\theta_1 - \theta_2)}{\cos(\theta_1 + \theta_2)} =$

(a) $\frac{1+k}{1-k}$ (b) $\frac{1-k}{1+k}$ (c) $\frac{k+1}{k-1}$ (d) $\frac{k-1}{k+1}$

14. If $\sin(\pi \cos x) = \cos(\pi \sin x)$, then $\sin 2x =$

(a) $\pm\frac{3}{4}$ (b) $\pm\frac{4}{3}$ (c) $\pm\frac{1}{3}$ (d) none of these

15. If $\tan \theta = \frac{1}{2}$ and $\tan \phi = \frac{1}{3}$, then the value of $\theta + \phi$ is

(a) $\frac{\pi}{6}$ (b) π (c) 0 (d) $\frac{\pi}{4}$

[NCERT EXEMPLAR]

16. The value of $\cos(36° - A) \cos(36° + A) + \cos(54° + A) \cos(54° - A)$ is

(a) $\sin 2A$ (b) $\cos 2A$ (c) $\cos 3A$ (d) $\sin 3A$

17. If $\tan(\pi/4 + x) + \tan(\pi/4 - x) = a$, then $\tan^2(\pi/4 + x) + \tan^2(\pi/4 - x) =$

(a) $a^2 + 1$ (b) $a^2 + 2$ (c) $a^2 - 2$ (d) none of these

18. If $\tan(A - B) = 1$, $\sec(A + B) = \frac{2}{\sqrt{3}}$, then the smallest positive value of B is

(a) $\frac{25\pi}{24}$ (b) $\frac{19\pi}{24}$ (c) $\frac{13\pi}{24}$ (d) $\frac{11\pi}{24}$

19. If $A - B = \pi/4$, then $(1 + \tan A)(1 - \tan B)$ is equal to

(a) 2 (b) 1 (c) 0 (d) 3

20. The maximum value of $\sin^2\left(\frac{2\pi}{3} + x\right) + \sin^2\left(\frac{2\pi}{3} - x\right)$ is

(a) 1/2 (b) 3/2 (c) 1/4 (d) 3/4

21. If $\cos(A - B) = \frac{3}{5}$ and $\tan A \tan B = 2$, then

(a) $\cos A \cos B = \frac{1}{5}$ (b) $\cos A \cos B = -\frac{1}{5}$

(c) $\sin A \sin B = -\frac{1}{5}$ (d) $\sin A \sin B = -\frac{1}{5}$

22. If $\tan 69° + \tan 66° - \tan 69° \tan 66° = 2k$, then $k =$

(a) -1 (b) $1/2$ (c) $-1/2$ (d) none of these

23. If $\alpha + \beta = \frac{\pi}{4}$, then the value of $(1 + \tan\alpha)(1 + \tan\beta)$ is

(a) 1 (b) 2 (c) -2 (d) not defined

[NCERT EXEMPLAR]

24. The value of $\sin\left(\frac{\pi}{4} + \theta\right) - \cos\left(\frac{\pi}{4} - \theta\right)$ is

(a) $2\cos\theta$ (b) $2\sin\theta$ (c) 1 (d) 0

[NCERT EXEMPLAR]

25. The minimum value of $3\cos x + 4\sin x + 8$ is

(a) 5 (b) 9 (c) 7 (d) 3

[NCERT EXEMPLAR]

ANSWERS

1. (b)	2. (b)	3. (c)	4. (d)	5. (a)	6. (c)	7. (a)	8. (c)
9. (b)	10. (a)	11. (a)	12. (a)	13. (a)	14. (a)	15. (d)	16. (b)
17. (c)	18. (b)	19. (a)	20. (b)	21. (a)	22. (c)	23. (b)	24. (d)
25. (d)							

SUMMARY

1. (i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

(ii) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

(iii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(iv) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(vi) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(vii) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

(viii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$

2. (i) $\sin(A+B+C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$

(ii) $\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$

(iii) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

3. (i) If $A + B = \pi$, then $\sin A = \sin B$, $\cos A = -\cos B$ and $\tan A = -\tan B$

(ii) If $A + B = 2\pi$, then $\sin A = -\sin B$, $\cos A = \cos B$ and $\tan A = -\tan B$