CHAPTER 8

# TRANSFORMATION FORMULAE

## 8.1 INTRODUCTION

In this chapter, we will establish two sets of transformation formulae: One to transform the products of two sines or two cosines or one sine and one cosine into the sum or difference of two sines or two cosines and the other to convert the sum or difference of two sines or two cosines in the product of two sines or two cosines or one sine and one cosine. These two sets of formulae are of fundamental importance and one should have thorough acquaintance with these formulae.

## 8.2 FORMULAE TO TRANSFORM THE PRODUCT INTO SUM OR DIFFERENCE

In the previous chapter we have derived the following formulae:

$$\sin A \cos B + \cos A \sin B = \sin (A + B) \quad \text{...(i)}$$

$$\sin A \cos B - \cos A \sin B = \sin (A - B) \quad \text{...(ii)}$$

$$\cos A \cos B - \sin A \sin B = \cos (A + B) \quad \text{...(iii)}$$

$$\cos A \cos B + \sin A \sin B = \cos (A - B) \quad \text{...(iv)}$$

Adding (i) and (ii), we obtain

$$2 \sin A \cos B = \sin (A + B) + \sin (A - B)$$

Subtracting (ii) from (i), we get

$$2 \cos A \sin B = \sin (A + B) - \sin (A - B)$$

Adding (iii) and (iv), we get

$$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$$

Subtracting (iii) from (iv), we get

$$2 \sin A \sin B = \cos (A - B) - \cos (A + B)$$

Thus, we obtain the following formulae :

(a) $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$ (b) $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$

(c) $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$ (d) $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$

These four formulae convert the product of two sines or two cosines or one sine and one cosine into the sum or difference of two sines or two cosines.

### ILLUSTRATIVE EXAMPLES

#### BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Convert each of the following products into the sum or difference of sines and cosines:*

(i) $2 \sin 5x \cos x$ (ii) $2 \cos 4x \cos 3x$ (iii) $2 \sin 3x \sin x$

(iv) $\sin \dfrac{5\pi}{12} \cos \dfrac{\pi}{12}$ (v) $\cos \dfrac{5\pi}{12} \cos \dfrac{\pi}{12}$

SOLUTION (i) Using $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$, we obtain

$$2 \sin 5x \cos x = \sin (5x + x) + \sin (5x - x) = \sin 6x + \sin 4x$$

(ii) Using $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$, we obtain

$2 \cos 4x \cos 3x = \cos (4x + 3x) + \cos (4x - 3x) = \cos 7x + \cos x$

(iii) Using $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$, we obtain

$2 \sin 3x \sin x = \cos (3x - x) - \cos (3x + x) = \cos 2x - \cos 4x$

(iv) $$\sin \frac{5\pi}{12} \cos \frac{\pi}{12} = \frac{1}{2}\left(2 \sin \frac{5\pi}{12} \cos \frac{\pi}{12}\right) = \frac{1}{2}\left\{\sin\left(\frac{5\pi}{12} + \frac{\pi}{12}\right) + \sin\left(\frac{5\pi}{12} - \frac{\pi}{12}\right)\right\}$$

$$= \frac{1}{2}\left(\sin \frac{\pi}{2} + \sin \frac{\pi}{3}\right)$$

(v) $$\cos \frac{5\pi}{12} \cos \frac{\pi}{12} = \frac{1}{2}\left(2 \cos \frac{5\pi}{12} \cos \frac{\pi}{12}\right) = \frac{1}{2}\left\{\cos\left(\frac{5\pi}{12} + \frac{\pi}{12}\right) + \cos\left(\frac{5\pi}{12} - \frac{\pi}{12}\right)\right\}$$

$$= \frac{1}{2}\left(\cos \frac{\pi}{2} + \cos \frac{\pi}{3}\right)$$

**EXAMPLE 2** *Prove that:* $2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} = 0.$ [NCERT]

SOLUTION $$\text{LHS} = 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13}$$

$$= \cos\left(\frac{9\pi}{13} + \frac{\pi}{13}\right) + \cos\left(\frac{9\pi}{13} - \frac{\pi}{13}\right) + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13}$$

$$= \cos \frac{10\pi}{13} + \cos \frac{8\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13}$$

$$= \cos\left(\pi - \frac{3\pi}{13}\right) + \cos\left(\pi - \frac{5\pi}{13}\right) + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13}$$

$$= -\cos \frac{3\pi}{13} - \cos \frac{5\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} = 0 = \text{RHS} \quad [\because \cos(\pi - x) = -\cos x]$$

**EXAMPLE 3** *Prove that:* $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}$

SOLUTION We have,

$$\text{LHS} = \cos 20° \cos 40° \cos 60° \cos 80° = \cos 60° (\cos 20° \cos 40°) \cos 80°$$

$$= \frac{1}{2} \times \frac{1}{2} (2 \cos 20° \cos 40°) \cos 80° \quad \left[\because \cos \frac{\pi}{3} = \frac{1}{2}\right]$$

$$= \frac{1}{4}\left[\{\cos (40° + 20°) + \cos (40° - 20°)\} \cos 80°\right] \quad \left[\begin{array}{l}\because 2 \cos A \cos B \\ = \cos (A + B) + \cos (A - B)\end{array}\right]$$

$$= \frac{1}{4}\left\{(\cos 60° + \cos 20°) \cos 80°\right\} = \frac{1}{4}\left\{\left(\frac{1}{2} + \cos 20°\right) \cos 80°\right\}$$

$$= \frac{1}{4}\left\{\frac{1}{2} \cos 80° + \cos 80° \cos 20°\right\} = \frac{1}{8}\left\{\cos 80° + 2 \cos 80° \cos 20°\right\}$$

$$= \frac{1}{8}\left[\cos 80° + \left\{\cos (80° + 20°) + \cos (80° - 20°)\right\}\right] \quad \left[\begin{array}{l}\because 2 \cos A \cos B \\ = \cos (A + B) + \cos (A - B)\end{array}\right]$$

$$= \frac{1}{8}\{\cos 80° + \cos 100° + \cos 60°\} = \frac{1}{8}\{\cos 80° + \cos (180° - 80°) + \cos 60°\}$$

$$= \frac{1}{8}\{\cos 80° - \cos 80° + \cos 60°\} \quad \left[\begin{array}{l}\because \cos (180° - x) = -\cos x \\ \therefore \cos (180° - 80°) = -\cos 80°\end{array}\right]$$

$$= \frac{1}{8}\left\{\cos 80° - \cos 80° + \frac{1}{2}\right\} = \frac{1}{8} \times \frac{1}{2} = \frac{1}{16} = \text{RHS}$$

**EXAMPLE 4** *Prove that:* $\sin 10° \sin 30° \sin 50° \sin 70° = \frac{1}{16}$.

SOLUTION
$$\text{LHS} = \sin 30° (\sin 10° \sin 50°) \sin 70° = \frac{1}{2}(\sin 50° \sin 10°) \sin 70°$$
$$= \frac{1}{2} \times \frac{1}{2}(2 \sin 50° \sin 10°) \sin 70° = \frac{1}{4}\left\{(2 \sin 50° \sin 10°) \sin 70°\right\}$$
$$= \frac{1}{4}\left[\{\cos(50°-10°) - \cos(50°+10°)\} \sin 70°\right] \quad \left[\because 2 \sin A \sin B = \cos(A-B) - \cos(A+B)\right]$$
$$= \frac{1}{4}\left\{(\cos 40° - \cos 60°) \sin 70°\right\} = \frac{1}{4}\left\{\sin 70° \cos 40° - \sin 70° \cos 60°\right\}$$
$$= \frac{1}{4}\left\{\sin 70° \cos 40° - \frac{1}{2}\sin 70°\right\} = \frac{1}{8}\left\{2 \sin 70° \cos 40° - \sin 70°\right\}$$
$$= \frac{1}{8}\left\{\sin(70°+40°) + \sin(70°-40°) - \sin 70°\right\} \quad \left[\because 2 \sin A \cos B = \sin(A+B) + \sin(A-B)\right]$$
$$= \frac{1}{8}\left\{\sin 110° + \sin 30° - \sin 70°\right\} = \frac{1}{8}\left\{\sin(180°-70°) + \sin 30° - \sin 70°\right\}$$
$$= \frac{1}{8}\left\{\sin 70° + \frac{1}{2} - \sin 70°\right\} \quad \left[\because \sin(180-x) = \sin x \therefore \sin(180°-70°) = \sin 70°\right]$$
$$= \frac{1}{8} \times \frac{1}{2} = \frac{1}{16} = \text{RHS}$$

ALITER
$$\text{LHS} = \sin 10° \sin 30° \sin 50° \sin 70°$$
$$= \sin(90°-80°) \sin(90°-60°) \sin(90°-40°) \sin(90°-20°)$$
$$= \cos 80° \cos 60° \cos 40° \cos 20°$$
$$= \cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16} = \text{RHS} \quad \text{[See Ex. 3]}$$

**EXAMPLE 5** *Prove that:* $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$

SOLUTION
$$\text{LHS} = \sin 20° \sin 40° \sin 60° \sin 80°$$
$$= \sin 60° (\sin 20° \sin 40°) \sin 80°$$
$$= \frac{\sqrt{3}}{2} \times \frac{1}{2}(2 \sin 20° \sin 40°) \sin 80°$$
$$= \frac{\sqrt{3}}{4}\left[\left\{\cos(40°-20°) - \cos(40°+20°)\right\} \sin 80°\right] \quad \left[\because 2 \sin A \sin B = \cos(A-B) - \cos(A+B)\right]$$
$$= \frac{\sqrt{3}}{4}\left\{(\cos 20° - \cos 60°) \sin 80°\right\} = \frac{\sqrt{3}}{4}\left\{\left(\cos 20° - \frac{1}{2}\right) \sin 80°\right\}$$
$$= \frac{\sqrt{3}}{8}\left\{2 \cos 20° \sin 80° - \sin 80°\right\}$$
$$= \frac{\sqrt{3}}{8}\left\{\sin(80°+20°) + \sin(80°-20°) - \sin 80°\right\} \quad \left[\because 2 \sin A \cos B = \sin(A+B) + \sin(A-B)\right]$$
$$= \frac{\sqrt{3}}{8}\left\{\sin 100° + \sin 60° - \sin 80°\right\} = \frac{\sqrt{3}}{8}\left\{\sin(180°-80°) + \frac{\sqrt{3}}{2} - \sin 80°\right\}$$

$$= \frac{\sqrt{3}}{8}\left\{\sin 80° + \frac{\sqrt{3}}{2} - \sin 80°\right\} = \frac{\sqrt{3}}{8} \times \frac{\sqrt{3}}{2} = \frac{3}{16} = \text{RHS} \quad [\because \sin(180° - 80°) = \sin 80°]$$

**EXAMPLE 6** *Prove that:* $4\cos 12° \cos 48° \cos 72° = \cos 36°$

SOLUTION LHS $= 4\cos 12° \cos 48° \cos 72° = 2(2\cos 12° \cos 48°)\cos 72°$

$= 2(\cos 60° + \cos 36°)\cos 72° = 2\cos 60° \cos 72° + 2\cos 36° \cos 72°$

$= \cos 72° + \cos 108° + \cos 36° = \cos 72° + \cos(180° - 72°) + \cos 36°$

$= \cos 72° - \cos 72° + \cos 36° = \cos 36° = \text{RHS}$

## BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 7** *Prove that:* $\tan 20° \tan 40° \tan 80° = \tan 60°$

SOLUTION LHS $= \tan 20° \tan 40° \tan 80° = \dfrac{\sin 20° \sin 40° \sin 80°}{\cos 20° \cos 40° \cos 80°}$

$$= \frac{(2\sin 20° \sin 40°)\sin 80°}{(2\cos 20° \cos 40°)\cos 80°} = \frac{(\cos 20° - \cos 60°)\sin 80°}{(\cos 60° + \cos 20°)\cos 80°}$$

$$= \frac{\sin 80° \cos 20° - (1/2)\sin 80°}{(1/2)\cos 80° + \cos 80° \cos 20°} = \frac{2\sin 80° \cos 20° - \sin 80°}{\cos 80° + 2\cos 80° \cos 20°}$$

$$= \frac{\sin 100° + \sin 60° - \sin 80°}{\cos 80° + \cos 100° + \cos 60°} = \frac{\sin(180° - 80°) + \sin 60° - \sin 80°}{\cos 80° + \cos(180° - 80°) + \cos 60°}$$

$$= \frac{\sin 80° + \sin 60° - \sin 80°}{\cos 80° - \cos 80° + \cos 60°} = \frac{\sin 60°}{\cos 60°} = \tan 60° = \text{RHS}$$

**EXAMPLE 8** *Prove that:* $\sin A \sin\left(\dfrac{\pi}{3} - A\right)\sin\left(\dfrac{\pi}{3} + A\right) = \dfrac{1}{4}\sin 3A$

SOLUTION LHS $= \sin A \sin\left(\dfrac{\pi}{3} - A\right)\sin\left(\dfrac{\pi}{3} + A\right) = \dfrac{1}{2}\sin A\left\{2\sin\left(\dfrac{\pi}{3} - A\right)\sin\left(\dfrac{\pi}{3} + A\right)\right\}$

$$= \frac{1}{2}\sin A\left[\cos\left\{\left(\frac{\pi}{3} - A\right) - \left(\frac{\pi}{3} + A\right)\right\} - \cos\left\{\left(\frac{\pi}{3} - A\right) + \left(\frac{\pi}{3} + A\right)\right\}\right]$$

$$= \frac{1}{2}\sin A\left\{\cos(-2A) - \cos\frac{2\pi}{3}\right\} = \frac{1}{2}\sin A\left\{\cos 2A + \frac{1}{2}\right\}$$

$$= \frac{1}{2}\sin A \cos 2A + \frac{1}{4}\sin A = \frac{1}{4}(2\sin A \cos 2A) + \frac{1}{4}\sin A$$

$$= \frac{1}{4}\left\{\sin(A + 2A) + \sin(A - 2A)\right\} + \frac{1}{4}\sin A$$

$$= \frac{1}{4}\left\{\sin 3A + \sin(-A)\right\} + \frac{1}{4}\sin A = \frac{1}{4}\sin 3A - \frac{1}{4}\sin A + \frac{1}{4}\sin A = \frac{1}{4}\sin 3A = \text{RHS}$$

**EXAMPLE 9** *Prove that:* $\cos A \cos\left(\dfrac{\pi}{3} - A\right)\cos\left(\dfrac{\pi}{3} + A\right) = \dfrac{1}{4}\cos 3A.$

SOLUTION LHS $= \cos A \cos\left(\dfrac{\pi}{3} - A\right)\cos\left(\dfrac{\pi}{3} + A\right) = \dfrac{1}{2}\cos A\left\{2\cos\left(\dfrac{\pi}{3} - A\right)\cos\left(\dfrac{\pi}{3} + A\right)\right\}$

$$= \frac{1}{2}\cos A\left[\cos\left\{\left(\frac{\pi}{3} - A\right) + \left(\frac{\pi}{3} + A\right)\right\} + \cos\left\{\left(\frac{\pi}{3} - A\right) - \left(\frac{\pi}{3} + A\right)\right\}\right]$$

$$= \frac{1}{2}\cos A\left\{\cos\frac{2\pi}{3} + \cos(-2A)\right\} = \frac{1}{2}\cos A\left\{-\frac{1}{2} + \cos 2A\right\}$$

$$= -\frac{1}{4}\cos A + \frac{1}{2}\cos A\cos 2A = -\frac{1}{4}\cos A + \frac{1}{4}(2\cos 2A\cos A)$$
$$= -\frac{1}{4}\cos A + \frac{1}{4}\left\{\cos(2A+A) + \cos(2A-A)\right\}$$
$$= -\frac{1}{4}\cos A + \frac{1}{4}(\cos 3A + \cos A) = \frac{1}{4}\cos 3A = \text{RHS}.$$

### BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

**EXAMPLE 10** *Prove that:* $4\sin x\sin\left(\frac{\pi}{3}+x\right)\sin\left(\frac{2\pi}{3}+x\right) = \sin 3x.$

SOLUTION $\text{LHS} = 4\sin x\sin\left(\frac{\pi}{3}+x\right)\sin\left(\frac{2\pi}{3}+x\right) = 2\sin x\left\{2\sin\left(\frac{2\pi}{3}+x\right)\sin\left(\frac{\pi}{3}+x\right)\right\}$

$$= 2\sin x\left[\cos\left\{\left(\frac{2\pi}{3}+x\right)-\left(\frac{\pi}{3}+x\right)\right\} - \cos\left\{\left(\frac{2\pi}{3}+x\right)+\left(\frac{\pi}{3}+x\right)\right\}\right]$$
$$= 2\sin x\left\{\cos\frac{\pi}{3} - \cos(\pi+2x)\right\} = 2\sin x\left\{\frac{1}{2}+\cos 2x\right\}$$
$$= \sin x + 2\sin x\cos 2x = \sin x + \{\sin(x+2x) + \sin(x-2x)\}$$
$$= \sin x + \sin 3x + \sin(-x) = \sin x + \sin 3x - \sin x = \sin 3x = \text{RHS}$$

**EXAMPLE 11** *Show that:* $\tan\left(\frac{\pi}{3}+x\right)\tan\left(\frac{\pi}{3}-x\right) = \frac{2\cos 2x+1}{2\cos 2x-1}.$

SOLUTION $\text{LHS} = \tan\left(\frac{\pi}{3}+x\right)\tan\left(\frac{\pi}{3}-x\right) = \dfrac{\sin\left(\frac{\pi}{3}+x\right)\sin\left(\frac{\pi}{3}-x\right)}{\cos\left(\frac{\pi}{3}+x\right)\cos\left(\frac{\pi}{3}-x\right)}$

$$= \frac{2\sin\left(\frac{\pi}{3}+x\right)\sin\left(\frac{\pi}{3}-x\right)}{2\cos\left(\frac{\pi}{3}+x\right)\cos\left(\frac{\pi}{3}-x\right)}$$
$$= \frac{\cos\left\{\left(\frac{\pi}{3}+x\right)-\left(\frac{\pi}{3}-x\right)\right\} - \cos\left\{\left(\frac{\pi}{3}+x\right)+\left(\frac{\pi}{3}-x\right)\right\}}{\cos\left\{\left(\frac{\pi}{3}+x\right)+\left(\frac{\pi}{3}-x\right)\right\} + \cos\left\{\left(\frac{\pi}{3}+x\right)-\left(\frac{\pi}{3}-x\right)\right\}}$$
$$= \frac{\cos 2x - \cos\frac{2\pi}{3}}{\cos\frac{2\pi}{3} + \cos 2x} = \frac{\cos 2x + \frac{1}{2}}{-\frac{1}{2} + \cos 2x} = \frac{2\cos 2x + 1}{2\cos 2x - 1} = \text{RHS}$$

**EXAMPLE 12** *If* $\alpha + \beta = 90°$, *find the maximum and minimum values of* $\sin\alpha\sin\beta$.

SOLUTION Let $y = \sin\alpha\sin\beta$. Then,

$$y = \frac{1}{2}(2\sin\alpha\sin\beta) = \frac{1}{2}\{\cos(\alpha-\beta) - \cos(\alpha+\beta)\} = \frac{1}{2}\{\cos(\alpha-\beta) - \cos 90°\} = \frac{1}{2}\cos(\alpha-\beta)$$

We know that

$$-1 \le \cos(\alpha-\beta) \le 1 \Rightarrow \frac{-1}{2} \le \frac{1}{2}\cos(\alpha-\beta) \le \frac{1}{2} \Rightarrow \frac{-1}{2} \le y \le \frac{1}{2} \Rightarrow \frac{-1}{2} \le \sin\alpha\sin\beta \le \frac{1}{2}$$

Hence, $-\frac{1}{2}$ and $\frac{1}{2}$ are respectively the minimum and maximum values of $\sin\alpha \sin\beta$.

## EXERCISE 8.1

### BASIC

1. Express each of the following as the sum or difference of sines and cosines:
   (i) $2\sin 3x\cos x$ (ii) $2\cos 3x\sin 2x$ (iii) $2\sin 4x\sin 3x$ (iv) $2\cos 7x\cos 3x$
2. Prove that:
   (i) $2\sin\frac{5\pi}{12}\sin\frac{\pi}{12}=\frac{1}{2}$ (ii) $2\cos\frac{5\pi}{12}\cos\frac{\pi}{12}=\frac{1}{2}$ (iii) $2\sin\frac{5\pi}{12}\cos\frac{\pi}{12}=\frac{\sqrt{3}+2}{2}$
3. Show that:
   (i) $\sin 50°\cos 85°=\frac{1-\sqrt{2}\sin 35°}{2\sqrt{2}}$ (ii) $\sin 25°\cos 115°=\frac{1}{2}(\sin 140°-1)$
4. Prove that: $4\cos x\cos\left(\frac{\pi}{3}+x\right)\cos\left(\frac{\pi}{3}-x\right)=\cos 3x$
5. Prove that :
   (i) $\cos 10°\cos 30°\cos 50°\cos 70°=\frac{3}{16}$ (ii) $\cos 40°\cos 80°\cos 160°=-\frac{1}{8}$
   (iii) $\sin 20°\sin 40°\sin 80°=\frac{\sqrt{3}}{8}$ (iv) $\cos 20°\cos 40°\cos 80°=\frac{1}{8}$
   (v) $\tan 20°\tan 40°\tan 60°\tan 80°=3$ (vi) $\tan 20°\tan 30°\tan 40°\tan 80°=1$
   (vii) $\sin 10°\sin 50°\sin 60°\sin 70°=\frac{\sqrt{3}}{16}$ (viii) $\sin 20°\sin 40°\sin 60°\sin 80°=\frac{3}{16}$

### BASED ON LOTS

6. Show that :
   (i) $\sin A\sin(B-C)+\sin B\sin(C-A)+\sin C\sin(A-B)=0$
   (ii) $\sin(B-C)\cos(A-D)+\sin(C-A)\cos(B-D)+\sin(A-B)\cos(C-D)=0$

### BASED ON HOTS

7. Prove that : $\tan x\tan\left(\frac{\pi}{3}-x\right)\tan\left(\frac{\pi}{3}+x\right)=\tan 3x$.
8. If $\alpha+\beta=\frac{\pi}{2}$, show that the maximum value of $\cos\alpha\cos\beta$ is $\frac{1}{2}$.

## ANSWERS

1. (i) $\sin 4x+\sin 2x$ (ii) $\sin 5x-\sin x$ (iii) $\cos x-\cos 7x$ (iv) $\cos 10x+\cos 4x$

## 8.3 FORMULAE TO TRANSFORM THE SUM OR DIFFERENCE INTO PRODUCT

In the previous section, we have used the following formulae:

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B,\ \sin(A+B)-\sin(A-B)=2\cos A\sin B$$

$$\cos(A+B)+\cos(A-B)=2\cos A\cos B \text{ and, } \cos(A-B)-\cos(A+B)=2\sin A\sin B.$$

Let $A+B=C$ and $A-B=D$. Then, $A=\frac{C+D}{2}$ and $B=\frac{C-D}{2}$.

Substituting the values of $A, B, C$ and $D$ in the above formulae, we get

$$\sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \quad \text{...(i)}$$

$$\sin C - \sin D = 2 \sin\left(\frac{C-D}{2}\right)\cos\left(\frac{C+D}{2}\right) \quad \text{...(ii)}$$

$$\cos C + \cos D = 2 \cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \quad \text{...(iii)}$$

$$\cos D - \cos C = 2 \sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

or, $$\cos C - \cos D = -2 \sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \quad \text{...(iv)}$$

or, $$\cos C - \cos D = 2 \sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$$

These four formulae are used to convert the sum or difference of two sines or two cosines into the product of sines and cosines.

## ILLUSTRATIVE EXAMPLES

### BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Express each of the following as a product:*

(i) $\sin 4x + \sin 2x$ (ii) $\sin 6x - \sin 2x$ (iii) $\cos 4x + \cos 8x$ (iv) $\cos 6x - \cos 8x$

SOLUTION (i) $\sin 4x + \sin 2x$

$$= 2\sin\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right) \qquad \left[\because \sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\right]$$

$$= 2\sin 3x \cos x$$

(ii) $\sin 6x - \sin 2x$

$$= 2\sin\left(\frac{6x-2x}{2}\right)\cos\left(\frac{6x+2x}{2}\right) \qquad \left[\because \sin C - \sin D = 2\sin\frac{C-D}{2}\cos\frac{C+D}{2}\right]$$

$$= 2\sin 2x \cos 4x$$

(iii) $\cos 4x + \cos 8x$

$$= 2\cos\left(\frac{8x+4x}{2}\right)\cos\left(\frac{8x-4x}{2}\right) \qquad \left[\because \cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}\right]$$

$$= 2\cos 6x \cos 2x$$

(iv) $\cos 6x - \cos 8x$

$$= 2\sin\left(\frac{6x+8x}{2}\right)\sin\left(\frac{8x-6x}{2}\right) \qquad \left[\because \cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}\right]$$

$$= 2\sin 7x \sin x$$

EXAMPLE 2 *Prove that:* $\cos 18° - \sin 18° = \sqrt{2}\sin 27°$

SOLUTION LHS $= \cos 18° - \sin 18° = \cos 18° - \cos 72°$ $[\because \sin 18° = \sin(90° - 72°) = \cos 72°]$

$$= 2\sin\left(\frac{18°+72°}{2}\right)\sin\left(\frac{72°-18°}{2}\right) = 2\sin 45° \sin 27° = \sqrt{2}\sin 27° = \text{RHS}$$

EXAMPLE 3 *Prove that:*

(i) $\dfrac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A} = \tan A$ (ii) $\dfrac{\sin A + \sin 3A}{\cos A + \cos 3A} = \tan 2A$ [NCERT]

(iii) $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan\left(\dfrac{A+B}{2}\right)$ (iv) $\dfrac{\cos 7A + \cos 5A}{\sin 7A - \sin 5A} = \cot A$ [NCERT]

SOLUTION (i) $\text{LHS} = \dfrac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A} = \dfrac{2\sin\left(\dfrac{5A-3A}{2}\right)\cos\left(\dfrac{5A+3A}{2}\right)}{2\cos\left(\dfrac{5A+3A}{2}\right)\cos\left(\dfrac{5A-3A}{2}\right)} = \dfrac{2\sin A\cos 4A}{2\cos 4A\cos A}$

$= \tan A = \text{RHS}$

(ii) $\text{LHS} = \dfrac{\sin 3A + \sin A}{\cos 3A + \cos A} = \dfrac{2\sin\left(\dfrac{3A+A}{2}\right)\cos\left(\dfrac{3A-A}{2}\right)}{2\cos\left(\dfrac{3A+A}{2}\right)\cos\left(\dfrac{3A-A}{2}\right)} = \dfrac{\sin 2A\cos A}{\cos 2A\cos A} = \tan 2A = \text{RHS}$

(iii) $\text{LHS} = \dfrac{\sin A + \sin B}{\cos A + \cos B} = \dfrac{2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)}{2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)} = \tan\left(\dfrac{A+B}{2}\right) = \text{RHS}$

(iv) $\text{LHS} = \dfrac{\cos 7A + \cos 5A}{\sin 7A - \sin 5A} = \dfrac{2\cos\left(\dfrac{7A+5A}{2}\right)\cos\left(\dfrac{7A-5A}{2}\right)}{2\sin\left(\dfrac{7A-5A}{2}\right)\cos\left(\dfrac{7A+5A}{2}\right)} = \dfrac{2\cos 6A\cos A}{2\sin A\cos 6A} = \cot A = \text{RHS}$

EXAMPLE 4 *Prove that:*

(i) $\dfrac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = -\dfrac{\sin 2x}{\cos 10x}$ [NCERT] (ii) $\dfrac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$ [NCERT]

(iii) $(\sin 3x + \sin x)\sin x + (\cos 3x - \cos x)\cos x = 0$ [NCERT]

SOLUTION (i) $\text{LHS} = \dfrac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = \dfrac{-2\sin\left(\dfrac{9x+5x}{2}\right)\sin\left(\dfrac{9x-5x}{2}\right)}{2\sin\left(\dfrac{17x-3x}{2}\right)\cos\left(\dfrac{17x+3x}{2}\right)} = \dfrac{-2\sin 7x\sin 2x}{2\sin 7x\cos 10x}$

$= \dfrac{-\sin 2x}{\cos 10x} = \text{RHS}$

(ii) $\text{LHS} = \dfrac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \dfrac{2\sin\left(\dfrac{5x+3x}{2}\right)\cos\left(\dfrac{5x-3x}{2}\right)}{2\cos\left(\dfrac{5x+3x}{2}\right)\cos\left(\dfrac{5x-3x}{2}\right)} = \dfrac{2\sin 4x\cos x}{2\cos 4x\cos x} = \tan 4x = \text{RHS}$

(iii) $\text{LHS} = (\sin 3x + \sin x)\sin x + (\cos 3x - \cos x)\cos x$

$= \left\{2\sin\left(\dfrac{3x+x}{2}\right)\cos\left(\dfrac{3x-x}{2}\right)\right\}\sin x + \left\{-2\sin\left(\dfrac{3x+x}{2}\right)\sin\left(\dfrac{3x-x}{2}\right)\right\}\cos x$

$= 2\sin 2x\cos x\sin x - 2\sin 2x\sin x\cos x = 0 = \text{RHS}$

EXAMPLE 5 *Prove that:* $\cot 4x(\sin 5x + \sin 3x) = \cot x(\sin 5x - \sin 3x)$ [NCERT]

SOLUTION $\text{LHS} = \cot 4x(\sin 5x + \sin 3x) = \cot 4x \times 2\sin\left(\dfrac{5x+3x}{2}\right)\cos\left(\dfrac{5x-3x}{2}\right)$

$= \dfrac{\cos 4x}{\sin 4x} \times 2\sin 4x\cos x = 2\cos 4x\cos x$ ...(i)

$$\text{RHS} = \cot x(\sin 5x - \sin 3x) = \cot x \times 2\sin\left(\frac{5x-3x}{2}\right)\cos\left(\frac{5x+3x}{2}\right)$$

$$= \frac{\cos x}{\sin x} \times 2\sin x\cos 4x = 2\cos 4x\cos x \qquad \text{...(ii)}$$

From (i) and (ii), we obtain: LHS = RHS.

EXAMPLE 6 *Prove that:* $\sin x + \sin 3x + \sin 5x + \sin 7x = 4\cos x\cos 2x\sin 4x$ [NCERT]

SOLUTION $\text{LHS} = \sin x + \sin 3x + \sin 5x + \sin 7x$

$$= (\sin 7x + \sin x) + (\sin 5x + \sin 3x)$$

$$= 2\sin\left(\frac{7x+x}{2}\right)\cos\left(\frac{7x-x}{2}\right) + 2\sin\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)$$

$$= 2\sin 4x\cos 3x + 2\sin 4x\cos x = 2\sin 4x(\cos 3x + \cos x)$$

$$= 2\sin 4x \times 2\cos\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right)$$

$$= 2\sin 4x \times 2\cos 2x\cos x = 4\cos x\cos 2x\sin 4x = \text{RHS}$$

EXAMPLE 7 *Prove that:* $1 + \cos 2x + \cos 4x + \cos 6x = 4\cos x\cos 2x\cos 3x$

SOLUTION $\text{LHS} = 1 + \cos 2x + \cos 4x + \cos 6x = (\cos 0x + \cos 2x) + (\cos 4x + \cos 6x)$

$$= 2\cos x\cos x + 2\cos 5x\cos x = 2\cos x(\cos x + \cos 5x)$$

$$= 2\cos x(2\cos 3x\cos 2x) = 4\cos x\cos 2x\cos 3x = \text{RHS}$$

### BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 8 *Prove that:*

(i) $(\sin 3A + \sin A)\sin A + (\cos 3A - \cos A)\cos A = 0$

(ii) $\cos 2x\cos\frac{x}{2} - \cos 3x\cos\frac{9x}{2} = \sin 5x\sin\frac{5x}{2}$ [NCERT]

(iii) $\sin\alpha + \sin\left(\alpha + \frac{2\pi}{3}\right) + \sin\left(\alpha + \frac{4\pi}{3}\right) = 0$

SOLUTION (i) $\text{LHS} = (\sin 3A + \sin A)\sin A + (\cos 3A - \cos A)\cos A$

$$= \left\{2\sin\left(\frac{3A+A}{2}\right)\cos\left(\frac{3A-A}{2}\right)\right\}\sin A + \left\{-2\sin\left(\frac{3A+A}{2}\right)\sin\left(\frac{3A-A}{2}\right)\right\}\cos A$$

$$= 2\sin 2A\cos A\sin A - 2\sin 2A\sin A\cos A = 0 = \text{RHS}$$

(ii) $\text{LHS} = \cos 2x\cos\frac{x}{2} - \cos 3x\cos\frac{9x}{2} = \frac{1}{2}\left\{2\cos 2x\cos\frac{x}{2} - 2\cos 3x\cos\frac{9x}{2}\right\}$

$$= \frac{1}{2}\left[\cos\left\{\left(2x+\frac{x}{2}\right) + \cos\left(2x - \frac{x}{2}\right)\right\} - \left\{\cos\left(3x + \frac{9x}{2}\right) + \cos\left(\frac{9x}{2} - 3x\right)\right\}\right]$$

[Using: $2\cos A\cos B = \cos(A+B) + \cos(A-B)$]

$$= \frac{1}{2}\left\{\cos\frac{5x}{2} + \cos\frac{3x}{2} - \cos\frac{15x}{2} - \cos\frac{3x}{2}\right\} = \frac{1}{2}\left\{\cos\frac{5x}{2} - \cos\frac{15x}{2}\right\}$$

$$= \frac{1}{2}\left\{2\sin\left(\frac{\frac{5x}{2} + \frac{15x}{2}}{2}\right)\sin\left(\frac{\frac{15x}{2} - \frac{5x}{2}}{2}\right)\right\} \quad \left[\because \cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}\right]$$

$$= \sin 5x\sin\frac{5x}{2} = \text{RHS}$$

(iii) LHS $= \sin\alpha + \sin\left(\alpha + \frac{2\pi}{3}\right) + \sin\left(\alpha + \frac{4\pi}{3}\right) = \sin\alpha + \left\{\sin\left(\alpha + \frac{2\pi}{3}\right) + \sin\left(\alpha + \frac{4\pi}{3}\right)\right\}$

$$= \sin\alpha + \left\{2\sin\left(\frac{\alpha + \frac{2\pi}{3} + \alpha + \frac{4\pi}{3}}{2}\right)\cos\left(\frac{\alpha + \frac{4\pi}{3} - \alpha - \frac{2\pi}{3}}{2}\right)\right\}$$

$$= \sin\alpha + 2\sin(\alpha+\pi)\cos\frac{\pi}{3} = \sin\alpha + 2(-\sin\alpha)\left(\frac{1}{2}\right) = \sin\alpha - \sin\alpha = 0 = \text{RHS}$$

**EXAMPLE 9** *Prove that:*

(i) $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\cos^2\left(\frac{\alpha-\beta}{2}\right)$ **[NCERT]**

(ii) $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\left(\frac{\alpha-\beta}{2}\right)$ **[NCERT]**

(iii) $\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$

SOLUTION (i) LHS $= (\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2$

$$= \left\{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right\}^2 + \left\{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right\}^2$$

$$= 4\cos^2\left(\frac{\alpha+\beta}{2}\right)\cos^2\left(\frac{\alpha-\beta}{2}\right) + 4\sin^2\left(\frac{\alpha+\beta}{2}\right)\cos^2\left(\frac{\alpha-\beta}{2}\right)$$

$$= 4\cos^2\left(\frac{\alpha-\beta}{2}\right)\left\{\cos^2\left(\frac{\alpha+\beta}{2}\right) + \sin^2\left(\frac{\alpha+\beta}{2}\right)\right\} = 4\cos^2\left(\frac{\alpha-\beta}{2}\right) = \text{RHS}$$

(ii) LHS $= (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$

$$= \left\{-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right\}^2 + \left\{2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)\right\}^2$$

$$= 4\sin^2\left(\frac{\alpha+\beta}{2}\right)\sin^2\left(\frac{\alpha-\beta}{2}\right) + 4\sin^2\left(\frac{\alpha-\beta}{2}\right)\cos^2\left(\frac{\alpha+\beta}{2}\right)$$

$$= 4\sin^2\left(\frac{\alpha-\beta}{2}\right)\left\{\sin^2\left(\frac{\alpha+\beta}{2}\right) + \cos^2\left(\frac{\alpha+\beta}{2}\right)\right\} = 4\sin^2\left(\frac{\alpha-\beta}{2}\right) = \text{RHS}$$

(iii) LHS $= \cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = (\cos\alpha + \cos\beta) + [\cos\gamma + \cos(\alpha+\beta+\gamma)]$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + 2\cos\left(\frac{\alpha+\beta+\gamma+\gamma}{2}\right)\cos\left(\frac{\alpha+\beta+\gamma-\gamma}{2}\right)$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)\right\}$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{2\cos\left(\frac{\frac{\alpha-\beta}{2} + \frac{\alpha+\beta+2\gamma}{2}}{2}\right)\cos\left(\frac{\frac{\alpha+\beta+2\gamma}{2} - \frac{\alpha-\beta}{2}}{2}\right)\right\}$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{2\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\right\}$$

$$= 4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\gamma+\alpha}{2}\right) = \text{RHS}$$

**EXAMPLE 10** *Prove that:* $\dfrac{\cos 4x+\cos 3x+\cos 2x}{\sin 4x+\sin 3x+\sin 2x} = \cot 3x$ **[NCERT]**

SOLUTION LHS $= \dfrac{\cos 4x+\cos 3x+\cos 2x}{\sin 4x+\sin 3x+\sin 2x} = \dfrac{(\cos 4x+\cos 2x)+\cos 3x}{(\sin 4x+\sin 2x)+\sin 3x}$

$$= \frac{2\cos\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right)+\cos 3x}{2\sin\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right)+\sin 3x}$$

$$= \frac{2\cos 3x\cos x+\cos 3x}{2\sin 3x\cos x+\sin 3x} = \frac{\cos 3x\,(2\cos x+1)}{\sin 3x\,(2\cos x+1)} = \frac{\cos 3x}{\sin 3x} = \cot 3x = \text{RHS}$$

**EXAMPLE 11** *Prove that:* $\dfrac{\sin A+\sin 3A+\sin 5A+\sin 7A}{\cos A+\cos 3A+\cos 5A+\cos 7A} = \tan 4A$

SOLUTION LHS $= \dfrac{(\sin 7A+\sin A)+(\sin 5A+\sin 3A)}{(\cos 7A+\cos A)+(\cos 5A+\cos 3A)}$

$$= \frac{2\sin\left(\frac{7A+A}{2}\right)\cos\left(\frac{7A-A}{2}\right)+2\sin\left(\frac{5A+3A}{2}\right)\cos\left(\frac{5A-3A}{2}\right)}{2\cos\left(\frac{7A+A}{2}\right)\cos\left(\frac{7A-A}{2}\right)+2\cos\left(\frac{5A+3A}{2}\right)\cos\left(\frac{5A-3A}{2}\right)}$$

$$= \frac{\sin 4A\cos 3A+\sin 4A\cos A}{\cos 4A\cos 3A+\cos 4A\cos A} = \frac{\sin 4A\,(\cos 3A+\cos A)}{\cos 4A\,(\cos 3A+\cos A)} = \tan 4A = \text{RHS}$$

**EXAMPLE 12** *Prove that:* $\dfrac{\cos 8A\cos 5A-\cos 12A\cos 9A}{\sin 8A\cos 5A+\cos 12A\sin 9A} = \tan 4A$

SOLUTION LHS $= \dfrac{2\cos 8A\cos 5A-2\cos 12A\cos 9A}{2\sin 8A\cos 5A+2\cos 12A\sin 9A}$

$$= \frac{\{\cos(8A+5A)+\cos(8A-5A)\}-\{\cos(12A+9A)+\cos(12A-9A)\}}{\{\sin(8A+5A)+\sin(8A-5A)\}+\{\sin(9A+12A)+\sin(9A-12A)\}}$$

$$= \frac{\{\cos 13A+\cos 3A\}-\{\cos 21A+\cos 3A\}}{\{\sin 13A+\sin 3A\}+\{\sin 21A+\sin(-3A)\}}$$

$$= \frac{(\cos 13A+\cos 3A)-(\cos 21A+\cos 3A)}{(\sin 13A+\sin 3A)+(\sin 21A-\sin 3A)} = \frac{\cos 13A-\cos 21A}{\sin 13A+\sin 21A}$$

$$= \frac{2\sin\left(\frac{13A+21A}{2}\right)\sin\left(\frac{21A-13A}{2}\right)}{2\sin\left(\frac{3A+21A}{2}\right)\cos\left(\frac{21A-13A}{2}\right)} = \frac{\sin 17A\sin 4A}{\sin 17A\cos 4A} = \tan 4A = \text{RHS}$$

**EXAMPLE 13** *Prove that:* $\dfrac{\cos 2A\cos 3A-\cos 2A\cos 7A+\cos A\cos 10A}{\sin 4A\sin 3A-\sin 2A\sin 5A+\sin 4A\sin 7A} = \cot 6A\cot 5A$

SOLUTION LHS $= \dfrac{2\cos 3A\cos 2A - 2\cos 7A\cos 2A + 2\cos 10A\cos A}{2\sin 4A\sin 3A - 2\sin 5A\sin 2A + 2\sin 7A\sin 4A}$

$$= \frac{(\cos 5A + \cos A) - (\cos 9A + \cos 5A) + (\cos 11A + \cos 9A)}{(\cos A - \cos 7A) - (\cos 3A - \cos 7A) + (\cos 3A - \cos 11A)}$$

$$= \frac{\cos A + \cos 11A}{\cos A - \cos 11A} = \frac{2\cos\left(\dfrac{11A + A}{2}\right)\cos\left(\dfrac{11A - A}{2}\right)}{2\sin\left(\dfrac{A + 11A}{2}\right)\sin\left(\dfrac{11A - A}{2}\right)}$$

$$= \frac{\cos 6A\cos 5A}{\sin 6A\sin 5A} = \cot 6A\cot 5A = \text{RHS}$$

**EXAMPLE 14** *Prove that:* $\dfrac{\sin(A - C) + 2\sin A + \sin(A + C)}{\sin(B - C) + 2\sin B + \sin(B + C)} = \dfrac{\sin A}{\sin B}$

SOLUTION LHS $= \dfrac{\sin(A - C) + \sin(A + C) + 2\sin A}{\sin(B - C) + \sin(B + C) + 2\sin B}$

$$= \frac{2\sin\left(\dfrac{A - C + A + C}{2}\right)\cos\left(\dfrac{A + C - A + C}{2}\right) + 2\sin A}{2\sin\left(\dfrac{B + C + B - C}{2}\right)\cos\left(\dfrac{B + C - B + C}{2}\right) + 2\sin B}$$

$$= \frac{2\sin A\cos C + 2\sin A}{2\sin B\cos C + 2\sin B} = \frac{2\sin A(\cos C + 1)}{2\sin B(\cos C + 1)} = \frac{\sin A}{\sin B} = \text{RHS}$$

## BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

**EXAMPLE 15** *If* $\sin x = n\sin(x + 2\alpha)$, *prove that* $\tan(x + \alpha) = \dfrac{1 + n}{1 - n}\tan\alpha$.

SOLUTION We have,

$$\sin x = n\sin(x + 2\alpha)$$

$$\Rightarrow \frac{\sin(x + 2\alpha)}{\sin x} = \frac{1}{n} \Rightarrow \frac{\sin(x + 2\alpha) + \sin x}{\sin(x + 2\alpha) - \sin x} = \frac{1 + n}{1 - n} \quad \text{[Applying componendo-dividendo]}$$

$$\Rightarrow \frac{2\sin(x + \alpha)\cos\alpha}{2\sin\alpha\cos(x + \alpha)} = \frac{1 + n}{1 - n} \Rightarrow \tan(x + \alpha) = \frac{1 + n}{1 - n}\tan\alpha$$

**EXAMPLE 16** *Prove that:*

$$\left(\frac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\frac{\sin A + \sin B}{\cos A - \cos B}\right)^n = \begin{cases} 2\cot^n\left(\dfrac{A - B}{2}\right), & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

SOLUTION LHS $= \left(\dfrac{2\cos\dfrac{A + B}{2}\cos\dfrac{A - B}{2}}{2\sin\dfrac{A - B}{2}\cos\dfrac{A + B}{2}}\right)^n + \left(\dfrac{2\sin\dfrac{A + B}{2}\cos\dfrac{A - B}{2}}{-2\sin\dfrac{A + B}{2}\sin\dfrac{A - B}{2}}\right)^n$

$$= \left\{\cot\left(\frac{A - B}{2}\right)\right\}^n + \left\{-\cot\left(\frac{A - B}{2}\right)\right\}^n = \cot^n\left(\frac{A - B}{2}\right) + (-1)^n\cot^n\left(\frac{A - B}{2}\right)$$

$$= \cot^n\left(\frac{A - B}{2}\right)\left\{1 + (-1)^n\right\} = \begin{cases} 2\cot^n\left(\dfrac{A - B}{2}\right), & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

**EXAMPLE 17** *If three angles A, B and C are in A.P., prove that:* $\cot B = \dfrac{\sin A - \sin C}{\cos C - \cos A}$.

SOLUTION RHS $= \dfrac{\sin A - \sin C}{\sin A + \sin C} = \dfrac{2\sin\dfrac{A-C}{2}\cos\dfrac{A+C}{2}}{2\sin\dfrac{A+C}{2}\sin\dfrac{A-C}{2}}$

$= \cot\left(\dfrac{A+C}{2}\right) = \cot B = \text{LHS}$ [$\because$ $A, B, C$ are in A.P. $\therefore$ $2B = A + C$]

**EXAMPLE 18** *If* $\sin\theta + \sin\phi = \sqrt{3}(\cos\phi - \cos\theta)$, *prove that* $\sin 3\theta + \sin 3\phi = 0$

SOLUTION We have, $\sin\theta + \sin\phi = \sqrt{3}(\cos\phi - \cos\theta)$ **[NCERT EXEMPLAR]**

$\Rightarrow \quad 2\sin\dfrac{\theta+\phi}{2}\cos\dfrac{\theta-\phi}{2} = 2\sqrt{3}\sin\dfrac{\theta+\phi}{2}\sin\dfrac{\theta-\phi}{2}$

$\Rightarrow \quad \left\{\cos\dfrac{\theta-\phi}{2} - \sqrt{3}\sin\dfrac{\theta-\phi}{2}\right\}\sin\left(\dfrac{\theta+\phi}{2}\right) = 0$

$\Rightarrow \quad \sin\left(\dfrac{\theta+\phi}{2}\right) = 0$ or, $\cos\left(\dfrac{\theta-\phi}{2}\right) - \sqrt{3}\sin\left(\dfrac{\theta-\phi}{2}\right) = 0$

$\Rightarrow \quad \sin\left(\dfrac{\theta+\phi}{2}\right) = 0$ or, $\tan\left(\dfrac{\theta-\phi}{2}\right) = \dfrac{1}{\sqrt{3}} = \tan\dfrac{\pi}{6}$

$\Rightarrow \quad \dfrac{\theta+\phi}{2} = 0$ or, $\dfrac{\theta-\phi}{2} = \dfrac{\pi}{6} \Rightarrow \theta = -\phi$ or, $\theta - \phi = \dfrac{\pi}{3}$

Case I When $\theta = -\phi$: In this case, we obtain

$\sin 3\theta + \sin 3\phi = \sin 3(-\phi) + \sin 3\phi = -\sin 3\phi + \sin 3\phi = 0$

Case II When $\theta - \phi = \dfrac{\pi}{3}$: In this case, we obtain

$\theta - \phi = \dfrac{\pi}{3} \Rightarrow 3\theta - 3\phi = \pi \Rightarrow 3\theta = \pi + 3\phi$

$\therefore \quad \sin 3\theta + \sin 3\phi = \sin(\pi + 3\phi) + \sin 3\phi = -\sin 3\phi + \sin 3\phi = 0$

**EXAMPLE 19** *If* $\dfrac{\sin(x+\alpha)}{\cos(x-\alpha)} = \dfrac{1-m}{1+m}$, *prove that* $\tan\left(\dfrac{\pi}{4} - x\right)\tan\left(\dfrac{\pi}{4} - \alpha\right) = m$.

SOLUTION We have,

$\dfrac{\sin(x+\alpha)}{\cos(x-\alpha)} = \dfrac{1-m}{1+m}$

$\Rightarrow \quad \dfrac{\sin(x+\alpha) + \cos(x-\alpha)}{\sin(x+\alpha) - \cos(x-\alpha)} = \dfrac{2}{-2m}$ [Using componendo-dividendo]

$\Rightarrow \quad \dfrac{\sin(x+\alpha) + \sin\left\{\dfrac{\pi}{2} - (x-\alpha)\right\}}{\sin(x+\alpha) - \sin\left\{\dfrac{\pi}{2} - (x-\alpha)\right\}} = -\dfrac{1}{m}$

$$\Rightarrow \quad \frac{2\sin\left(\dfrac{x+\alpha+\frac{\pi}{2}-x+\alpha}{2}\right)\cos\left(\dfrac{x+\alpha-\frac{\pi}{2}+x-\alpha}{2}\right)}{2\sin\left(\dfrac{x+\alpha-\frac{\pi}{2}+x-\alpha}{2}\right)\cos\left(\dfrac{x+\alpha+\frac{\pi}{2}-x+\alpha}{2}\right)} = -\frac{1}{m}$$

$$\Rightarrow \quad \frac{\sin\left(\frac{\pi}{4}+\alpha\right)\cos\left(-\frac{\pi}{4}+x\right)}{\sin\left(-\frac{\pi}{4}+x\right)\cos\left(\frac{\pi}{4}+\alpha\right)} = -\frac{1}{m} \Rightarrow \frac{\sin\left(\frac{\pi}{4}+\alpha\right)}{\cos\left(\frac{\pi}{4}+\alpha\right)}\cdot\frac{\cos\left(\frac{\pi}{4}-x\right)}{\sin\left(\frac{\pi}{4}-x\right)} = \frac{1}{m}$$

$$\Rightarrow \quad \tan\left(\frac{\pi}{4}+\alpha\right)\cot\left(\frac{\pi}{4}-x\right) = \frac{1}{m} \Rightarrow m = \cot\left(\frac{\pi}{4}+\alpha\right)\tan\left(\frac{\pi}{4}-x\right)$$

$$\Rightarrow \quad m = \tan\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}+\alpha\right)\right\}\tan\left(\frac{\pi}{4}-x\right) \Rightarrow m = \tan\left(\frac{\pi}{4}-\alpha\right)\tan\left(\frac{\pi}{4}-x\right)$$

**EXAMPLE 20** *If* $a\sin x = b\sin\left(x+\frac{2\pi}{3}\right) = c\sin\left(x+\frac{4\pi}{3}\right)$, *prove that* $ab+bc+ca=0$.

SOLUTION We have ,

$$a\sin x = b\sin\left(x+\frac{2\pi}{3}\right) = c\sin\left(x+\frac{4\pi}{3}\right) = \lambda\,(\text{say})$$

$$\Rightarrow \quad \frac{\lambda}{a} = \sin x,\ \frac{\lambda}{b} = \sin\left(x+\frac{2\pi}{3}\right) \text{ and } \frac{\lambda}{c} = \sin\left(x+\frac{4\pi}{3}\right)$$

$$\therefore \quad \frac{\lambda}{a}+\frac{\lambda}{b}+\frac{\lambda}{c} = \sin x + \sin\left(x+\frac{2\pi}{3}\right)+\sin\left(x+\frac{4\pi}{3}\right) = \left\{\sin\left(x+\frac{4\pi}{3}\right)+\sin x\right\}+\sin\left(x+\frac{2\pi}{3}\right)$$

$$\Rightarrow \quad \frac{\lambda}{a}+\frac{\lambda}{b}+\frac{\lambda}{c} = 2\sin\left(x+\frac{2\pi}{3}\right)\cos\frac{2\pi}{3}+\sin\left(x+\frac{2\pi}{3}\right) = -\sin\left(x+\frac{2\pi}{3}\right)+\sin\left(x+\frac{2\pi}{3}\right)=0$$

$$\Rightarrow \quad \lambda\left(\frac{bc+ca+ab}{abc}\right) = 0 \Rightarrow ab+bc+ca=0$$

**EXAMPLE 21** *If* $\sin(y+z-x)$, $\sin(z+x-y)$, $\sin(x+y-z)$ *are in A.P., prove that* $\tan x$, $\tan y$, $\tan z$ *are also in A.P.*

SOLUTION It is given that $\sin(y+z-x)$, $\sin(z+x-y)$ and $\sin(x+y-z)$ are in A.P.

$\therefore \quad \sin(z+x-y)-\sin(y+z-x) = \sin(x+y-z)-\sin(z+x-y)$

$\Rightarrow \quad 2\sin(x-y)\cos z = 2\sin(y-z)\cos x$

$\Rightarrow \quad \sin(x-y)\cos z = \sin(y-z)\cos x$

$\Rightarrow \quad \sin x\cos y\cos z - \cos x\sin y\cos z = \sin y\cos z\cos x - \cos y\sin z\cos x$

$\Rightarrow \quad 2\sin y\cos x\cos z = \sin x\cos y\cos z + \cos x\cos y\sin z$

$\Rightarrow \quad 2\tan y = \tan x + \tan z$ [Dividing throughout by $\cos x\cos y\cos z$]

$\Rightarrow \quad \tan x, \tan y, \tan z$ are in A.P.

**EXAMPLE 22** *If* $\frac{\tan(x+\alpha)}{a} = \frac{\tan(x+\beta)}{b} = \frac{\tan(x+\gamma)}{c}$, *prove that*

$$\frac{a+b}{a-b}\sin^2(\alpha-\beta)+\frac{b+c}{b-c}\sin^2(\beta-\gamma)+\frac{c+a}{c-a}\sin^2(\gamma-\alpha)=0$$

SOLUTION We have,

$$\frac{\tan(x+\alpha)}{a} = \frac{\tan(x+\beta)}{b}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{\tan(x+\alpha)}{\tan(x+\beta)}$$

$$\Rightarrow \quad \frac{a+b}{a-b} = \frac{\tan(x+\alpha)+\tan(x+\beta)}{\tan(x+\alpha)-\tan(x+\beta)} \qquad \text{[Applying Componendo-dividendo]}$$

$$\Rightarrow \quad \frac{a+b}{a-b} = \frac{\sin(2x+\alpha+\beta)}{\sin(\alpha-\beta)} \qquad \left[\because \frac{\tan A+\tan B}{\tan A-\tan B} = \frac{\sin(A+B)}{\sin(A-B)}\right]$$

$$\Rightarrow \quad \frac{a+b}{a-b}\sin^2(\alpha-\beta) = \sin(2x+\alpha+\beta)\sin(\alpha-\beta) = \frac{1}{2}\left\{2\sin(2x+\alpha+\beta)\sin(\alpha-\beta)\right\}$$

$$\Rightarrow \quad \frac{a+b}{a-b}\sin^2(\alpha-\beta) = \frac{1}{2}\left\{\cos(2x+2\beta) - \cos(2x+2\alpha)\right\}$$

Similarly, we obtain

$$\frac{b+c}{b-c}\sin^2(\beta-\gamma) = \frac{1}{2}\left\{\cos(2x+2\gamma) - \cos(2x+2\beta)\right\}$$

and, $$\frac{c+a}{c-a}\sin^2(\gamma-\alpha) = \frac{1}{2}\left\{\cos(2x+2\alpha) - \cos(2x+2\gamma)\right\}$$

$$\therefore \quad \frac{a+b}{a-b}\sin^2(\alpha-\beta) + \frac{b+c}{b-c}\sin^2(\beta-\gamma) + \frac{c+a}{c-a}\sin^2(\gamma-\alpha)$$

$$= \frac{1}{2}\left\{\cos(2x+2\beta) - \cos(2x+2\alpha) + \cos(2x+2\gamma) - \cos(2x+2\beta) + \cos(2x+2\alpha) - \cos(2x+2\gamma)\right\}$$

$$= \frac{1}{2}\times 0 = 0$$

**EXAMPLE 23** *Prove that :* $\dfrac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{\cos 5x + 5\cos 3x + 10\cos x} = 2\cos x$

SOLUTION $\cos 6x + 6\cos 4x + 15\cos 2x + 10$

$= (\cos 6x + \cos 4x) + (5\cos 4x + 5\cos 2x) + (10\cos 2x + 10)$

$= (\cos 6x + \cos 4x) + 5(\cos 4x + \cos 2x) + 10(\cos 2x + \cos 0x)$

$= 2\cos 5x\cos x + 5\times 2\cos 3x\cos x + 10\times 2\cos x\cos x = 2\cos x(\cos 5x + 5\cos 3x + 10\cos x)$

$$\therefore \quad \text{LHS} = \frac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{\cos 5x + 5\cos 3x + 10\cos x} = \frac{2\cos x(\cos 5x + 5\cos 3x + 10\cos x)}{\cos 5x + 5\cos 3x + 10\cos x}$$

$$= 2\cos x = \text{RHS}$$

## EXERCISE 8.2

### BASIC

1. Express each of the following as the product of sines and cosines:

(i) $\sin 12x + \sin 4x$ (ii) $\sin 5x - \sin x$ (iii) $\cos 12x + \cos 8x$

(iv) $\cos 12x - \cos 4x$ (v) $\sin 2x + \cos 4x$

2. Prove that:

(i) $\sin 38^\circ + \sin 22^\circ = \sin 82^\circ$ (ii) $\cos 100^\circ + \cos 20^\circ = \cos 40^\circ$

(iii) $\sin 50^\circ + \sin 10^\circ = \cos 20^\circ$ (iv) $\sin 23^\circ + \sin 37^\circ = \cos 7^\circ$

(v) $\sin 105° + \cos 105° = \cos 45°$

(vi) $\sin 40° + \sin 20° = \cos 10°$

3. Prove that:

(i) $\cos 55° + \cos 65° + \cos 175° = 0$

(ii) $\sin 50° - \sin 70° + \sin 10° = 0$

(iii) $\cos 80° + \cos 40° - \cos 20° = 0$

(iv) $\cos 20° + \cos 100° + \cos 140° = 0$

(v) $\sin \dfrac{5\pi}{18} - \cos \dfrac{4\pi}{9} = \sqrt{3} \sin \dfrac{\pi}{9}$

(vi) $\cos \dfrac{\pi}{12} - \sin \dfrac{\pi}{12} = \dfrac{1}{\sqrt{2}}$

(vii) $\sin 80° - \cos 70° = \cos 50°$

(viii) $\sin 51° + \cos 81° = \cos 21°$

4. Prove that:

(i) $\cos\left(\dfrac{3\pi}{4} + x\right) - \cos\left(\dfrac{3\pi}{4} - x\right) = -\sqrt{2} \sin x$ [NCERT]

(ii) $\cos\left(\dfrac{\pi}{4} + x\right) + \cos\left(\dfrac{\pi}{4} - x\right) = \sqrt{2} \cos x$ [NCERT]

5. Prove that :

(i) $\sin 65° + \cos 65° = \sqrt{2} \cos 20°$

(ii) $\sin 47° + \cos 77° = \cos 17°$

6. Prove that :

(i) $\cos 3A + \cos 5A + \cos 7A + \cos 15A = 4 \cos 4A \cos 5A \cos 6A$

(ii) $\cos A + \cos 3A + \cos 5A + \cos 7A = 4 \cos A \cos 2A \cos 4A$

(iii) $\sin A + \sin 2A + \sin 4A + \sin 5A = 4 \cos \dfrac{A}{2} \cos \dfrac{3A}{2} \sin 3A$

(iv) $\sin 3A + \sin 2A - \sin A = 4 \sin A \cos \dfrac{A}{2} \cos \dfrac{3A}{2}$

(v) $\cos 20° \cos 100° + \cos 100° \cos 140° - \cos 140° \cos 200° = -\dfrac{3}{4}$

(vi) $\sin \dfrac{x}{2} \sin \dfrac{7x}{2} + \sin \dfrac{3x}{2} \sin \dfrac{11x}{2} = \sin 2x \sin 5x.$

(vii) $\cos x \cos \dfrac{x}{2} - \cos 3x \cos \dfrac{9x}{2} = \sin 4x \sin \dfrac{7x}{2}.$ [NCERT EXEMPLAR]

7. Prove that:

(i) $\dfrac{\sin A + \sin 3A}{\cos A - \cos 3A} = \cot A$

(ii) $\dfrac{\sin 9A - \sin 7A}{\cos 7A - \cos 9A} = \cot 8A$

(iii) $\dfrac{\sin A - \sin B}{\cos A + \cos B} = \tan \dfrac{A - B}{2}$

(iv) $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \tan\left(\dfrac{A+B}{2}\right) \cot\left(\dfrac{A-B}{2}\right)$

(v) $\dfrac{\cos A + \cos B}{\cos B - \cos A} = \cot\left(\dfrac{A+B}{2}\right) \cot\left(\dfrac{A-B}{2}\right)$

**BASED ON LOTS**

8. Prove that:

(i) $\dfrac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \tan 3A$

(ii) $\dfrac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A} = \dfrac{\cos 5A}{\cos 3A}$

(iii) $\dfrac{\cos 4A + \cos 3A + \cos 2A}{\sin 4A + \sin 3A + \sin 2A} = \cot 3A$ [NCERT]

(iv) $\dfrac{\sin 5A \cos 2A - \sin 6A \cos A}{\sin A \sin 2A - \cos 2A \cos 3A} = \tan A$

(v) $\dfrac{\sin 11A \sin A + \sin 7A \sin 3A}{\cos 11A \sin A + \cos 7A \sin 3A} = \tan 8A$

(vi) $\dfrac{\sin 3A \cos 4A - \sin A \cos 2A}{\sin 4A \sin A + \cos 6A \cos A} = \tan 2A$

(vii) $\dfrac{\sin A \sin 2A + \sin 3A \sin 6A}{\sin A \cos 2A + \sin 3A \cos 6A} = \tan 5A$

(viii) $\dfrac{\sin A + 2 \sin 3A + \sin 5A}{\sin 3A + 2 \sin 5A + \sin 7A} = \dfrac{\sin 3A}{\sin 5A}$

(ix) $\dfrac{\sin (\theta + \phi) - 2 \sin \theta + \sin (\theta - \phi)}{\cos (\theta + \phi) - 2 \cos \theta + \cos (\theta - \phi)} = \tan \theta$

(x) $\dfrac{\sin 3A + \sin 5A + \sin 7A + \sin 9A}{\cos 3A + \cos 5A + \cos 7A + \cos 9A} = \tan 6A$

(xi) $\dfrac{\sin 5A - \sin 7A + \sin 8A - \sin 4A}{\cos 4A + \cos 7A - \cos 5A - \cos 8A} = \cot 6A$

9. Prove that:

(i) $\sin \alpha + \sin \beta + \sin \gamma - \sin (\alpha + \beta + \gamma) = 4 \sin \left(\dfrac{\alpha + \beta}{2}\right) \sin \left(\dfrac{\beta + \gamma}{2}\right) \sin \left(\dfrac{\gamma + \alpha}{2}\right)$

(ii) $\cos (A + B + C) + \cos (A - B + C) + \cos (A + B - C) + \cos (-A + B + C) = 4 \cos A \cos B \cos C$

**BASED ON HOTS**

10. If $\cos A + \cos B = \dfrac{1}{2}$ and $\sin A + \sin B = \dfrac{1}{4}$, prove that: $\tan \left(\dfrac{A + B}{2}\right) = \dfrac{1}{2}$

11. If $\operatorname{cosec} A + \sec A = \operatorname{cosec} B + \sec B$, prove that: $\tan A \tan B = \cot \dfrac{A + B}{2}$

12. If $\sin 2A = \lambda \sin 2B$, prove that : $\dfrac{\tan (A + B)}{\tan (A - B)} = \dfrac{\lambda + 1}{\lambda - 1}$

13. Prove that:

(i) $\dfrac{\cos (A + B + C) + \cos (-A + B + C) + \cos (A - B + C) + \cos (A + B - C)}{\sin (A + B + C) + \sin (-A + B + C) + \sin (A - B + C) - \sin (A + B - C)} = \cot C$

(ii) $\sin (B - C) \cos (A - D) + \sin (C - A) \cos (B - D) + \sin (A - B) \cos (C - D) = 0$

14. If $\dfrac{\cos (A - B)}{\cos (A + B)} + \dfrac{\cos (C + D)}{\cos (C - D)} = 0$, prove that $\tan A \tan B \tan C \tan D = -1$

15. If $\cos (\alpha + \beta) \sin (\gamma + \delta) = \cos (\alpha - \beta) \sin (\gamma - \delta)$, prove that $\cot \alpha \cot \beta \cot \gamma = \cot \delta$

16. If $y \sin \phi = x \sin (2\theta + \phi)$, prove that $(x + y) \cot (\theta + \phi) = (y - x) \cot \theta$

17. If $\cos(A + B) \sin(C - D) = \cos(A - B) \sin (C + D)$, prove that $\tan A \tan B \tan C + \tan D = 0$

18. If $x \cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right)$, prove that $xy+yz+zx=0$.

[NCERT EXEMPLAR]

19. If $m\sin\theta = n\sin(\theta+2\alpha)$, prove that $\tan(\theta+\alpha)\cot\alpha = \frac{m+n}{m-n}$. [NCERT EXEMPLAR]

## ANSWERS

1. (i) $2\sin 8x\cos 4x$ (ii) $2\sin 2x\cos 3x$ (iii) $2\cos 10x\cos 2x$

(iv) $-2\sin 8x\sin 4x$ (v) $2\cos\left(\frac{\pi}{4}+x\right)\cos\left(\frac{\pi}{4}-3x\right)$

## HINTS TO SELECTED PROBLEMS

18. Let $x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right) = \lambda$

$$\Rightarrow \quad \frac{\lambda}{x} = \cos\theta,\ \frac{\lambda}{y} = \cos\left(\theta+\frac{2\pi}{3}\right) \text{ and } \frac{\lambda}{z} = \cos\left(\theta+\frac{4\pi}{3}\right)$$

$$\therefore \quad \frac{\lambda}{x}+\frac{\lambda}{y}+\frac{\lambda}{z} = \cos\theta+\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{4\pi}{3}\right)$$

$$\Rightarrow \quad \lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \cos\theta+\left\{\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{4\pi}{3}\right)\right\}$$

$$\Rightarrow \quad \lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \cos\theta+2\cos(\theta+\pi)\cos\frac{\pi}{3} = \cos\theta-2\times\frac{1}{2}\cos\theta = 0$$

$$\Rightarrow \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 0 \Rightarrow \frac{xy+yz+zx}{xyz} = 0 \Rightarrow xy+yz+zx = 0.$$

19. We have, $m\sin\theta = n\sin(\theta+2\alpha)$

$$\Rightarrow \quad \frac{\sin(\theta+2\alpha)}{\sin\theta} = \frac{m}{n}$$

$$\Rightarrow \quad \frac{\sin(\theta+2\alpha)+\sin\theta}{\sin(\theta+2\alpha)-\sin\theta} = \frac{m+n}{m-n} \qquad \text{[Applying componendo and dividendo]}$$

$$\Rightarrow \quad \frac{2\sin(\theta+\alpha)\cos\alpha}{2\sin\alpha\cos(\theta+\alpha)} = \frac{m+n}{m-n} \Rightarrow \frac{\tan(\theta+\alpha)}{\tan\alpha} = \frac{m+n}{m-n} \Rightarrow \tan(\theta+\alpha)\cot\alpha = \frac{m+n}{m-n}$$

## FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. The value of $\frac{\sin 70°+\cos 40°}{\cos 70°+\sin 40°}$ is .................................... .
2. If $\tan(A+B) = p$, $\tan(A-B) = q$, then the value of $\tan 2A$ in terms of $p$ and $q$ is .................... .
3. If $1+\cos 2x+\cos 4x+\cos 6x = k\cos x\cos 2x\cos 3x$, then $k =$ .................................... .
4. The value of $\sin 50°-\sin 70°+\sin 10°$ is .................................... .
5. The value of $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}$ is .................................... .

## ANSWERS

1. $\sqrt{3}$ 2. $\frac{p+q}{1-pq}$ 3. 4 4. 0 5. 0

## VERY SHORT ANSWER QUESTIONS (VSAQs)

*Answer each of the following questions in one word or one sentence or as per exact requirement of the question:*

1. If $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = \lambda\cos^2\left(\frac{\alpha-\beta}{2}\right)$, write the value of $\lambda$.
2. Write the value of $\sin\frac{\pi}{12}\sin\frac{5\pi}{12}$.
3. If $\sin A + \sin B = \alpha$ and $\cos A + \cos B = \beta$, then write the value of $\tan\left(\frac{A+B}{2}\right)$.
4. If $\cos A = m\cos B$, then write the value of $\cot\frac{A+B}{2}\cot\frac{A-B}{2}$.
5. Write the value of the expression $\frac{1-4\sin 10^\circ \sin 70^\circ}{2\sin 10^\circ}$.
6. If $A + B = \frac{\pi}{3}$ and $\cos A + \cos B = 1$, then find the value of $\cos\frac{A-B}{2}$.
7. Write the value of $\sin\frac{\pi}{15}\sin\frac{4\pi}{15}\sin\frac{3\pi}{10}$.
8. If $\sin 2A = \lambda\sin 2B$, then write the value of $\frac{\lambda+1}{\lambda-1}$.
9. Write the value of $\frac{\sin A + \sin 3A}{\cos A + \cos 3A}$.
10. If $\cos(A+B)\sin(C-D) = \cos(A-B)\sin(C+D)$, then write the value $\tan A\tan B\tan C$.

## ANSWERS

1. 4  2. $\frac{1}{4}$  3. $\frac{\alpha}{\beta}$  4. $\frac{1+m}{1-m}$  5. 1  6. $\frac{1}{\sqrt{3}}$  7. $\frac{1}{8}$
8. $\frac{\tan(A+B)}{\tan(A-B)}$  9. $\tan 2A$  10. $-\tan D$

## MULTIPLE CHOICE QUESTIONS (MCQs)

*Mark the correct alternative in each of the following:*

1. $\cos 40^\circ + \cos 80^\circ + \cos 160^\circ + \cos 240^\circ =$
   (a) 0  (b) 1  (c) 1/2  (d) −1/2
2. $\sin 163^\circ\cos 347^\circ + \sin 73^\circ\sin 167^\circ =$
   (a) 0  (b) 1/2  (c) 1  (d) none of these
3. If $\sin 2\theta + \sin 2\phi = \frac{1}{2}$ and $\cos 2\theta + \cos 2\phi = \frac{3}{2}$, then $\cos^2(\theta-\phi) =$
   (a) 3/8  (b) 5/8  (c) 3/4  (d) 5/4
4. The value of $\cos 52^\circ + \cos 68^\circ + \cos 172^\circ$ is
   (a) 0  (b) 1  (c) 2  (d) 3/2
5. The value of $\sin 78^\circ - \sin 66^\circ - \sin 42^\circ + \sin 6^\circ$ is
   (a) 1/2  (b) −1/2  (c) −1  (d) none of these

6. If $\sin\alpha + \sin\beta = a$ and $\cos\alpha - \cos\beta = b$, then $\tan\dfrac{\alpha-\beta}{2} =$

(a) $-\dfrac{a}{b}$ (b) $-\dfrac{b}{a}$ (c) $\sqrt{a^2+b^2}$ (d) none of these

7. $\cos 35^\circ + \cos 85^\circ + \cos 155^\circ =$

(a) 0 (b) $\dfrac{1}{\sqrt{3}}$ (c) $\dfrac{1}{\sqrt{2}}$ (d) $\cos 275^\circ$

8. The value of $\sin 50^\circ - \sin 70^\circ + \sin 10^\circ$ is equal to

(a) 1 (b) 0 (c) 1/2 (d) 2

[NCERT EXEMPLAR]

9. $\sin 47^\circ + \sin 61^\circ - \sin 11^\circ - \sin 25^\circ$ is equal to

(a) $\sin 36^\circ$ (b) $\cos 36^\circ$ (c) $\sin 7^\circ$ (d) $\cos 7^\circ$

10. If $\cos A = m\cos B$, then $\cot\dfrac{A+B}{2}\cot\dfrac{B-A}{2} =$

(a) $\dfrac{m-1}{m+1}$ (b) $\dfrac{m+2}{m-2}$ (c) $\dfrac{m+1}{m-1}$ (d) none of these

11. If $A, B, C$ are in A.P., then $\dfrac{\sin A - \sin C}{\cos C - \cos A} =$

(a) $\tan B$ (b) $\cot B$ (c) $\tan 2B$ (d) none of these

12. If $\sin(B+C-A)$, $\sin(C+A-B)$, $\sin(A+B-C)$ are in A.P., then $\cot A$, $\cot B$, $\cot C$ are in

(a) GP (b) HP (c) AP (d) none of these

13. If $\sin x + \sin y = \sqrt{3}(\cos y - \cos x)$, then $\sin 3x + \sin 3y =$

(a) $2\sin 3x$ (b) 0 (c) 1 (d) none of these

14. The value of $\sin\dfrac{\pi}{18} + \sin\dfrac{\pi}{9} + \sin\dfrac{2\pi}{9} + \sin\dfrac{5\pi}{18}$ is given by

(a) $\sin\dfrac{7\pi}{18} + \sin\dfrac{4\pi}{9}$ (b) 1 (c) $\cos\dfrac{\pi}{6} + \cos\dfrac{3\pi}{7}$ (d) $\cos\dfrac{\pi}{9} + \sin\dfrac{\pi}{9}$

## ANSWERS

1. (d) 2. (b) 3. (b) 4. (a) 5. (b) 6. (b) 7. (a) 8. (b)
9. (d) 10. (c) 11. (b) 12. (b) 13. (b) 14. (a)

## SUMMARY

1. (i) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$ (ii) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(iii) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$ (iv) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

2. (i) $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$ (ii) $\sin C - \sin D = 2\sin\dfrac{C-D}{2}\cos\dfrac{C+D}{2}$

(iii) $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$ (iv) $\cos C - \cos D = -2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$