CHAPTER 9

VALUES OF TRIGONOMETRIC FUNCTIONS AT MULTIPLES AND SUBMULTIPLES OF AN ANGLE

9.1 INTRODUCTION

In this chapter, we will introduce the formulae expressing the values of trigonometric functions at multiples of x i.e. $2x$, $3x$, $4x$...etc in terms of the values at x. We shall also develop formulae expressing the values of trigonometric functions at x in terms of the values at sub multiples of x i.e. $\frac{x}{2}, \frac{x}{3}, \frac{x}{4}$...etc.

9.2 VALUES OF TRIGONOMETRIC FUNCTIONS AT $2x$ IN TERMS OF VALUES AT x

THEOREM 1 *For the values of angle x for which the two sides are meaningful prove that:*

(i) $\sin 2x = 2 \sin x \cos x$ (ii) $\cos 2x = \cos^2 x - \sin^2 x$

(iii) $\cos 2x = 2\cos^2 x - 1$ or, $1 + \cos 2x = 2\cos^2 x$

(iv) $\cos 2x = 1 - 2\sin^2 x$ or, $1 - \cos 2x = 2\sin^2 x$

(v) $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$ (vi) $\sin 2x = \dfrac{2\tan x}{1 + \tan^2 x}$ (vii) $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$

PROOF (i) We know that: $\sin(x + y) = \sin x \cos y + \cos x \sin y$

$\therefore \quad \sin 2x = \sin x \cos x + \cos x \sin x$ [Replacing y by x]

$\Rightarrow \quad \sin 2x = 2 \sin x \cos x$

(ii) We know that: $\cos(x + y) = \cos x \cos y - \sin x \sin y$

$\therefore \quad \cos 2x = \cos x \cos x - \sin x \sin x$ [Replacing y by x]

$\Rightarrow \quad \cos 2x = \cos^2 x - \sin^2 x$

(iii) We know that:

$$\cos 2x = \cos^2 x - \sin^2 x \Rightarrow \cos 2x = \cos^2 x - (1 - \cos^2 x) \Rightarrow \cos 2x = 2\cos^2 x - 1$$

Again, $\cos 2x = 2\cos^2 x - 1 \Rightarrow 1 + \cos 2x = 2\cos^2 x$

(iv) We know that:

$$\cos 2x = \cos^2 x - \sin^2 x \Rightarrow \cos 2x = (1 - \sin^2 x) - \sin^2 x \Rightarrow \cos 2x = 1 - 2\sin^2 x$$

Again, $\cos 2x = 1 - 2\sin^2 x \Rightarrow 1 - \cos 2x = 2\sin^2 x$

(v) We know that: $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

$\therefore \quad \tan 2x = \dfrac{\tan x + \tan x}{1 - \tan x \tan x}$ [Replacing y by x]

$$\Rightarrow \quad \tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

(vi) We know that: $\sin 2x = 2\sin x \cos x$

$$\therefore \quad \sin 2x = \frac{2\sin x\cos x}{1} \Rightarrow \sin 2x = \frac{2\sin x\cos x}{\sin^2 x + \cos^2 x} \qquad [\because \sin^2 x + \cos^2 x = 1]$$

$$\Rightarrow \quad \sin 2x = \frac{\dfrac{2\sin x\cos x}{\cos^2 x}}{\dfrac{\sin^2 x+\cos^2 x}{\cos^2 x}}, \; x \neq (2n+1)\frac{\pi}{2}, n \in Z. \qquad \left[\begin{array}{l}\text{Dividing Numerator and}\\ \text{Denominator by } \cos^2 x\end{array}\right]$$

$$\Rightarrow \quad \sin 2x = \frac{\dfrac{2\sin x}{\cos x}}{\dfrac{\sin^2 x}{\cos^2 x}+\dfrac{\cos^2 x}{\cos^2 x}} \Rightarrow \sin 2x = \frac{2\tan x}{1+\tan^2 x}$$

(vii) We know that: $\cos 2x = \cos^2 x - \sin^2 x$

$$\therefore \quad \cos 2x = \frac{\cos^2 x - \sin^2 x}{1} = \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x} \qquad [\text{Replacing 1 by } \cos^2 x + \sin^2 x]$$

$$\Rightarrow \quad \cos 2x = \frac{\dfrac{\cos^2 x - \sin^2 x}{\cos^2 x}}{\dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}}, \; x \neq (2n+1)\frac{\pi}{2}, n \in Z. \qquad \left[\begin{array}{l}\text{Dividing Numerator and}\\ \text{Denominator by } \cos^2 x\end{array}\right]$$

$$\Rightarrow \quad \cos 2x = \frac{\dfrac{\cos^2 x}{\cos^2 x} - \dfrac{\sin^2 x}{\cos^2 x}}{\dfrac{\cos^2 x}{\cos^2 x} + \dfrac{\sin^2 x}{\cos^2 x}} \Rightarrow \cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$$

REMARK *In the above formulae it should be noted that the angle on the RHS is half of the angle on LHS.*

$$\therefore \quad \sin\frac{2\pi}{3} = 2\sin\frac{\pi}{3}\cos\frac{\pi}{3}, \; \cos\frac{2\pi}{3} = \cos^2\frac{\pi}{3} - \sin^2\frac{\pi}{3}, \; \tan\frac{\pi}{3} = \frac{2\tan\frac{\pi}{6}}{1-\tan^2\frac{\pi}{6}} \text{ etc.}$$

9.2.1 VALUES OF TRIGONOMETRIC FUNCTIONS AT x IN TERMS OF THE VALUES AT $\frac{x}{2}$

The relations in section 9.2 are true for all values of the variable x for which the two sides are meaningful. Replacing x by $x/2$ in the above relations, we obtain the following relations:

(i) $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ (ii) $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$

(iii) $\cos x = 2\cos^2\frac{x}{2} - 1$ or, $1+\cos x = 2\cos^2\frac{x}{2}$

(iv) $\cos x = 1 - 2\sin^2\frac{x}{2}$ or, $1-\cos x = 2\sin^2\frac{x}{2}$

(v) $\tan x = \dfrac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$ (vi) $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ (vii) $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

9.2.2 VALUES OF TRIGONOMETRIC FUNCTIONS AT $\frac{x}{2}$ IN TERMS OF $\cos x$

We have,

$$\cos x = 2\cos^2\frac{x}{2} - 1 \Rightarrow 2\cos^2\frac{x}{2} = 1 + \cos x \Rightarrow \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

The sign on the right hand side depends upon the quadrant in which angle $\frac{x}{2}$ lies.

Also,

$$\cos x = 1 - 2\sin^2\frac{x}{2} \Rightarrow 2\sin^2\frac{x}{2} = 1 - \cos x \Rightarrow \sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

The sign on the right hand side depends upon the quadrant in which angle $\frac{x}{2}$ lies.

Now, $$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \pm\frac{\sqrt{\frac{1-\cos x}{2}}}{\sqrt{\frac{1+\cos x}{2}}} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$

The sign on right hand side depends upon the quadrant in which angle $\frac{x}{2}$ lies.

REMARK *These relations are very useful to find the trigonometric ratios of the angles* $\frac{\pi}{8}, \frac{\pi}{24}, \frac{\pi}{16}$ *etc.*

ILLUSTRATIVE EXAMPLES

BASIC

Type I ON FINDING THE VALUES OF $\sin 2x$, $\cos 2x$, $\tan 2x$ ETC WHEN VALUES OF $\sin x$ OR $\cos x$ OR $\tan x$ ARE GIVEN

EXAMPLE 1 *If* $\sin x = \frac{3}{5}$, *where* $0 < x < \frac{\pi}{2}$, *find the values of* $\sin 2x$, $\cos 2x$, $\tan 2x$ *and* $\sin 4x$.

SOLUTION We have, $\sin x = \frac{3}{5}$, where $0 < x < \frac{\pi}{2}$.

$$\therefore\ \cos^2 x = 1 - \sin^2 x \Rightarrow \cos x = +\sqrt{1-\sin^2 x} = \sqrt{1-\frac{9}{25}} = \frac{4}{5} \qquad \left[\because \cos x > 0 \text{ for } 0 < x < \frac{\pi}{2}\right]$$

$$\therefore\ \tan x = \frac{\sin x}{\cos x} = \frac{3}{4}$$

Thus, we obtain

$$\sin 2x = 2\sin x\cos x = 2\times\frac{3}{5}\times\frac{4}{5} = \frac{24}{25},\ \cos 2x = 1 - 2\sin^2 x = 1 - 2\times\left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} = \frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2} = \frac{\frac{6}{4}}{1-\frac{9}{16}} = \frac{24}{7} \qquad \left[\because \tan x = \frac{3}{4}\right]$$

$$\text{and, } \sin 4x = 2\sin 2x\cos 2x = 2\times\frac{24}{25}\times\frac{7}{25} = \frac{336}{625} \qquad \left[\because \sin 2x = \frac{24}{25} \text{ and } \cos 2x = \frac{7}{25}\right]$$

EXAMPLE 2 *If* $\tan\alpha = \frac{1}{7}$, $\sin\beta = \frac{1}{\sqrt{10}}$. *Prove that* $\alpha + 2\beta = \frac{\pi}{4}$, *where* $0 < \alpha < \frac{\pi}{2}$ *and* $0 < \beta < \frac{\pi}{2}$.

SOLUTION In order to prove that $\alpha + 2\beta = \frac{\pi}{4}$, it is sufficient to prove that $\tan(\alpha + 2\beta) = \tan\frac{\pi}{4} = 1$.

We have, $\sin\beta = \frac{1}{\sqrt{10}}$, where $0 < \beta < \frac{\pi}{2}$.

$$\therefore \quad \cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-\frac{1}{10}} = \frac{3}{\sqrt{10}} \text{ and, } \tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{\frac{1}{\sqrt{10}}}{\frac{3}{\sqrt{10}}} = \frac{1}{3}$$

In order to find the value of $\tan(\alpha + 2\beta)$, we require the values of $\tan\alpha$ and $\tan 2\beta$. The value of $\tan\alpha$ is given. So, let us find $\tan 2\beta$.

$$\text{Now,} \quad \tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta} \Rightarrow \tan 2\beta = \frac{2\times\frac{1}{3}}{1-\frac{1}{9}} = \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{2\times 9}{3\times 8} = \frac{3}{4}$$

Thus, we have $\tan\alpha = \frac{1}{7}$ and $\tan 2\beta = \frac{3}{4}$

$$\therefore \quad \tan(\alpha+2\beta) = \frac{\tan\alpha + \tan 2\beta}{1-\tan\alpha\tan 2\beta} = \frac{\frac{1}{7}+\frac{3}{4}}{1-\frac{1}{7}\times\frac{3}{4}} = \frac{4+21}{28-3} = 1 \Rightarrow \alpha + 2\beta = \frac{\pi}{4}$$

Type II **ON PROVING RESULTS AND IDENTITIES BASED UPON THE FOLLOWING FORMULAE:**

$$\sin 2x = 2\sin x\cos x,\ 1+\cos 2x = 2\cos^2 x,\ 1-\cos 2x = 2\sin^2 x$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2},\ 1+\cos x = 2\cos^2\frac{x}{2},\ 1-\cos x = 2\sin^2\frac{x}{2}$$

EXAMPLE 3 *Prove that :*

(i) $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$ (ii) $\dfrac{\sin 2x}{1-\cos 2x} = \cot x$

(iii) $\dfrac{1+\sin 2x+\cos 2x}{1+\sin 2x-\cos 2x} = \cot x$ (iv) $\dfrac{1+\sin x-\cos x}{1+\sin x+\cos x} = \tan\dfrac{x}{2}$

(v) $\dfrac{\cos 2x}{1+\sin 2x} = \tan\left(\dfrac{\pi}{4}-x\right)$ (vi) $\dfrac{\cos x}{1+\sin x} = \tan\left(\dfrac{\pi}{4}-\dfrac{x}{2}\right)$

SOLUTION (i) We have, $\sin 2x = 2\sin x\cos x$ and $1+\cos 2x = 2\cos^2 x$.

$$\therefore \quad \text{LHS} = \frac{\sin 2x}{1+\cos 2x} = \frac{2\sin x\cos x}{2\cos^2 x} = \tan x = \text{RHS}$$

(ii)
$$\text{LHS} = \frac{\sin 2x}{1-\cos 2x} = \frac{2\sin x\cos x}{2\sin^2 x} = \cot x = \text{RHS}$$

(iii)
$$\text{LHS} = \frac{1+\sin 2x+\cos 2x}{1+\sin 2x-\cos 2x} = \frac{(1+\cos 2x)+\sin 2x}{(1-\cos 2x)+\sin 2x}$$

$$= \frac{2\cos^2 x+2\sin x\cos x}{2\sin^2 x+2\sin x\cos x} = \frac{2\cos x(\cos x+\sin x)}{2\sin x(\cos x+\sin x)} = \frac{\cos x}{\sin x} = \cot x = \text{RHS}$$

(iv)
$$\text{LHS} = \frac{1+\sin x-\cos x}{1+\sin x+\cos x} = \frac{(1-\cos x)+\sin x}{(1+\cos x)+\sin x}$$

$$= \frac{2\sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{2\sin\frac{x}{2}\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)}{2\cos\frac{\theta}{2}\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)} = \tan\frac{x}{2} = \text{RHS}$$

(v) $$\text{LHS} = \frac{\cos 2x}{1+\sin 2x} = \frac{\sin\left(\frac{\pi}{2}-2x\right)}{1+\cos\left(\frac{\pi}{2}-2x\right)} \qquad \left[\because \cos x = \sin\left(\frac{\pi}{2}-x\right), \sin x = \cos\left(\frac{\pi}{2}-x\right)\right]$$

$$= \frac{2\sin\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-x\right)}{2\cos^2\left(\frac{\pi}{4}-x\right)} \qquad \left[\because \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} \text{ and } 1+\cos x = 2\cos^2\frac{x}{2}\right]$$

$$= \tan\left(\frac{\pi}{4}-x\right) = \text{RHS}$$

(vi) $$\text{LHS} = \frac{\cos x}{1+\sin x} = \frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)} = \frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)} = \tan\left(\frac{\pi}{4}-\frac{x}{2}\right) = \text{RHS}$$

EXAMPLE 4 *Show that:* $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8x}}} = 2\cos x,\ 0 < x < \frac{\pi}{8}$

SOLUTION $$\text{LHS} = \sqrt{2+\sqrt{2+\sqrt{2(1+\cos 8x)}}}$$

$$= \sqrt{2+\sqrt{2+\sqrt{2(2\cos^2 4x)}}} \qquad \left[\because 1+\cos 8x = 2\cos^2\frac{8x}{2} = 2\cos^2 4x\right]$$

$$= \sqrt{2+\sqrt{2+\sqrt{(4\cos^2 4x)}}} = \sqrt{2+\sqrt{2+2\cos 4x}} = \sqrt{2+\sqrt{2(1+\cos 4x)}}$$

$$= \sqrt{2+\sqrt{2(2\cos^2 2x)}} \qquad [\because 1+\cos 4x = 2\cos^2 2x]$$

$$= \sqrt{2+2\cos 2x} = \sqrt{2(1+\cos 2x)} = \sqrt{2(2\cos^2 x)} = 2\cos x = \text{RHS}$$

EXAMPLE 5 *Prove that:* $\cos 4x = 1 - 8\sin^2 x\cos^2 x$. **[NCERT]**

SOLUTION $\text{LHS} = \cos 4x = \cos 2(2x) = 1 - 2\sin^2 2x = 1 - 2(\sin 2x)^2 = 1 - 2(2\sin x\cos x)^2$

$$= 1 - 8\sin^2 x\cos^2 x = \text{RHS}$$

EXAMPLE 6 *Prove that:* $(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 4\cos^2\left(\frac{x+y}{2}\right)$.

SOLUTION $\text{LHS} = (\cos x + \cos y)^2 + (\sin x - \sin y)^2$

$$= (\cos^2 x + \cos^2 y + 2\cos x\cos y) + (\sin^2 x + \sin^2 y - 2\sin x\sin y)$$

$$= (\cos^2 x + \sin^2 y) + (\cos^2 x + \sin^2 y) + 2(\cos x\cos y - \sin x\sin y)$$

$$= 1 + 1 + 2\cos(x+y) = 2 + 2\cos(x+y) = 2\left\{1+\cos(x+y)\right\}$$

$$= 2\times 2\cos^2\left(\frac{x+y}{2}\right) = 4\cos^2\left(\frac{x+y}{2}\right) = \text{RHS}. \qquad \left[\because 1+\cos x = 2\cos^2\frac{x}{2}\right]$$

EXAMPLE 7 *Prove that:* $\dfrac{\sec 8x-1}{\sec 4x-1}=\dfrac{\tan 8x}{\tan 2x}$ **[NCERT EXEMPLAR]**

SOLUTION $\text{LHS} = \dfrac{\sec 8x-1}{\sec 4x-1} = \dfrac{\dfrac{1}{\cos 8x}-1}{\dfrac{1}{\cos 4x}-1} = \dfrac{1-\cos 8x}{\cos 8x}\times\dfrac{\cos 4x}{1-\cos 4x}$

$$= \frac{2\sin^2 4x}{\cos 8x}\times\frac{\cos 4x}{2\sin^2 2x} \qquad \left[\because 1-\cos 8x = 2\sin^2 4x \text{ and, } 1-\cos 4x = 2\sin^2 2x\right]$$

$$= \frac{(2\sin 4x\cos 4x)}{\cos 8x}\times\frac{\sin 4x}{2\sin^2 2x} = \left(\frac{2\sin 4x\cos 4x}{\cos 8x}\right)\times\left(\frac{2\sin 2x\cos 2x}{2\sin^2 2x}\right)$$

$$= \left(\frac{\sin 2(4x)}{\cos 8x}\right)\times\left(\frac{\cos 2x}{\sin 2x}\right) = \left(\frac{\sin 8x}{\cos 8x}\right)\times\left(\frac{\cos 2x}{\sin 2x}\right) = \tan 8x\cot 2x = \frac{\tan 8x}{\tan 2x} = \text{RHS}$$

EXAMPLE 8 *Prove that:* $\left(1+\cos\dfrac{\pi}{8}\right)\left(1+\cos\dfrac{3\pi}{8}\right)\left(1+\cos\dfrac{5\pi}{8}\right)\left(1+\cos\dfrac{7\pi}{8}\right)=\dfrac{1}{8}$.

SOLUTION We observe that **[NCERT EXEMPLAR]**

$$\cos\frac{7\pi}{8} = \cos\left(\pi-\frac{\pi}{8}\right) = -\cos\frac{\pi}{8} \text{ and, } \cos\frac{5\pi}{8} = \cos\left(\pi-\frac{3\pi}{8}\right) = -\cos\frac{3\pi}{8}$$

$\therefore$ $$\text{LHS} = \left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$$

$$= \left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{\pi}{8}\right)$$

$$= \left\{\left(1+\cos\frac{\pi}{8}\right)\left(1-\cos\frac{\pi}{8}\right)\right\}\left\{\left(1+\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{3\pi}{8}\right)\right\}$$

$$= \left(1-\cos^2\frac{\pi}{8}\right)\left(1-\cos^2\frac{3\pi}{8}\right) = \sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}$$

$$= \frac{1}{4}\left(2\sin^2\frac{\pi}{8}\right)\left(2\sin^2\frac{3\pi}{8}\right) = \frac{1}{4}\left\{\left(1-\cos\frac{\pi}{4}\right)\left(1-\cos\frac{3\pi}{4}\right)\right\} \qquad [\because 2\sin^2 x = 1-\cos 2x]$$

$$= \frac{1}{4}\left\{\left(1-\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{\sqrt{2}}\right)\right\} = \frac{1}{4}\left(1-\frac{1}{2}\right) = \frac{1}{8} = \text{RHS}$$

EXAMPLE 9 *Prove that:*

(i) $\cos^4\dfrac{\pi}{8}+\cos^4\dfrac{3\pi}{8}+\cos^4\dfrac{5\pi}{8}+\cos^4\dfrac{7\pi}{8}=\dfrac{3}{2}$ (ii) $\sin^4\dfrac{\pi}{8}+\sin^4\dfrac{3\pi}{8}+\sin^4\dfrac{5\pi}{8}+\sin^4\dfrac{7\pi}{8}=\dfrac{3}{2}$

[NCERT EXEMPLAR]

SOLUTION (i) We know that: $\cos(\pi-x) = -\cos x$ and $\dfrac{7\pi}{8} = \pi-\dfrac{\pi}{8}, \dfrac{5\pi}{8} = \pi-\dfrac{3\pi}{8}$.

$\therefore$ $$\cos\frac{7\pi}{8} = -\cos\frac{\pi}{8} \text{ and } \cos\frac{5\pi}{8} = -\cos\frac{3\pi}{8} \Rightarrow \cos^4\frac{7\pi}{8} = \cos^4\frac{\pi}{8} \text{ and } \cos^4\frac{5\pi}{8} = \cos^4\frac{3\pi}{8}$$

Now, $$\text{LHS} = \cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8} = \cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{\pi}{8}$$

$$= 2\cos^4\frac{\pi}{8}+2\cos^4\frac{3\pi}{8} = 2\left\{\left(\cos^2\frac{\pi}{8}\right)^2+\left(\cos^2\frac{3\pi}{8}\right)^2\right\}$$

$$= 2\left[\left\{\frac{1+\cos\frac{\pi}{4}}{2}\right\}^2 + \left\{\frac{1+\cos\frac{3\pi}{4}}{2}\right\}^2\right] \qquad \left[\because \frac{1+\cos 2x}{2} = \cos^2 x\right]$$

$$= \frac{2}{4}\left\{\left(1+\cos\frac{\pi}{4}\right)^2 + \left(1+\cos\frac{3\pi}{4}\right)^2\right\} = \frac{2}{4}\left\{\left(1+\frac{1}{\sqrt{2}}\right)^2 + \left(1-\frac{1}{\sqrt{2}}\right)^2\right\}$$

$$= \frac{1}{2}\left\{\left(1+\frac{1}{\sqrt{2}}\right)^2 + \left(1-\frac{1}{\sqrt{2}}\right)^2\right\} = \frac{1}{2}\left\{\left(1+\frac{1}{2}+\sqrt{2}\right) + \left(1+\frac{1}{2}-\sqrt{2}\right)\right\} = \frac{3}{2} = \text{RHS}$$

(ii) We know that $\sin(\pi - x) = \sin x$

$\therefore \quad \sin^4\frac{7\pi}{8} = \sin^4\left(\pi - \frac{\pi}{8}\right) = \sin^4\frac{\pi}{8}$ and, $\sin^4\frac{5\pi}{8} = \sin^4\left(\pi - \frac{3\pi}{8}\right) = \sin^4\frac{3\pi}{8}$

Now, $\text{LHS} = \sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{5\pi}{8} + \sin^4\frac{7\pi}{8} = \sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{\pi}{8}$

$$= 2\left\{\sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8}\right\} = 2\left\{\left(\sin^2\frac{\pi}{8}\right)^2 + \left(\sin^2\frac{3\pi}{8}\right)^2\right\}$$

$$= 2\left[\left\{\frac{1-\cos\frac{\pi}{4}}{2}\right\}^2 + \left\{\frac{1-\cos\frac{3\pi}{4}}{2}\right\}^2\right] \qquad \left[\because \frac{1-\cos 2x}{2} = \sin^2 x\right]$$

$$= \frac{2}{4}\left\{\left(1-\cos\frac{\pi}{4}\right)^2 + \left(1-\cos\frac{3\pi}{4}\right)^2\right\}$$

$$= \frac{1}{2}\left\{\left(1-\frac{1}{\sqrt{2}}\right)^2 + \left(1+\frac{1}{\sqrt{2}}\right)^2\right\} = \frac{1}{2}\left\{\left(1+\frac{1}{2}-\sqrt{2}\right) + \left(1+\frac{1}{2}+\sqrt{2}\right)\right\} = \frac{3}{2} = \text{RHS}$$

EXAMPLE 10 *Prove that:* $\tan 4x = \dfrac{4\tan x(1-\tan^2 x)}{1-6\tan^2 x + \tan^4 x}$. **[NCERT]**

SOLUTION $\text{LHS} = \tan 4x = \tan\left(2(2x)\right) = \dfrac{2\tan 2x}{1-\tan^2 2x}$

$$= \frac{2\left(\dfrac{2\tan x}{1-\tan^2 x}\right)}{1-\left(\dfrac{2\tan x}{1-\tan^2 x}\right)^2} = \frac{4\tan x(1-\tan^2 x)}{(1-\tan^2 x)^2 - 4\tan^2 x} = \frac{4\tan x(1-\tan^2 x)}{1-6\tan^2 x+\tan^4 x} = \text{RHS}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 11 *Prove that:*

(i) $\cos^2 x + \cos^2\left(x+\frac{2\pi}{3}\right) + \cos^2\left(x-\frac{2\pi}{3}\right) = \frac{3}{2}$ (ii) $\cos^2 x + \cos^2\left(x+\frac{\pi}{3}\right) + \cos^2\left(x-\frac{\pi}{3}\right) = \frac{3}{2}$

SOLUTION (i) $\text{LHS} = \cos^2 x + \cos^2\left(x+\frac{2\pi}{3}\right) + \cos^2\left(x-\frac{2\pi}{3}\right)$

$$= \frac{1}{2}\left\{2\cos^2 x + 2\cos^2\left(x+\frac{2\pi}{3}\right) + 2\cos^2\left(x-\frac{2\pi}{3}\right)\right\}$$

$$= \frac{1}{2}\left[1+\cos 2x + \left\{1+\cos 2\left(x+\frac{2\pi}{3}\right)\right\} + \left\{1+\cos 2\left(x-\frac{2\pi}{3}\right)\right\}\right]$$

$$= \frac{1}{2}\left[1+\cos 2x + 1 + \cos\left(2x+\frac{4\pi}{3}\right) + 1 + \cos\left(2x-\frac{4\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[3+\cos 2x + \left\{\cos\left(2x+\frac{4\pi}{3}\right) + \cos\left(2x-\frac{4\pi}{3}\right)\right\}\right]$$

$$= \frac{1}{2}\left[3+\cos 2x + 2\cos 2x \cos\frac{4\pi}{3}\right] \qquad [\because \cos(x+y)+\cos(x-y) = 2\cos x\cos y]$$

$$= \frac{1}{2}\left[3+\cos 2x + 2(\cos 2x)\left(-\frac{1}{2}\right)\right] = \frac{1}{2}(3+\cos 2x - \cos 2x) = \frac{3}{2} = \text{RHS}$$

(ii) LHS $= \cos^2 x + \cos^2\left(x+\frac{\pi}{3}\right) + \cos^2\left(x-\frac{\pi}{3}\right)$

$$= \frac{1}{2}\left\{2\cos^2 x + 2\cos^2\left(x+\frac{\pi}{3}\right) + 2\cos^2\left(x-\frac{\pi}{3}\right)\right\}$$

$$= \frac{1}{2}\left\{(1+\cos 2x) + 1 + \cos\left(2x+\frac{2\pi}{3}\right) + 1 + \cos\left(2x-\frac{2\pi}{3}\right)\right\}$$

$$= \frac{1}{2}\left[3+\cos 2x + \left\{\cos\left(2x+\frac{2\pi}{3}\right) + \cos\left(2x-\frac{2\pi}{3}\right)\right\}\right]$$

$$= \frac{1}{2}\left\{3+\cos 2x + 2\cos 2x\cos\frac{2\pi}{3}\right\} \qquad [\because \cos(x+y)+\cos(x-y) = 2\cos x\cos y]$$

$$= \frac{1}{2}\left\{3+\cos 2x + 2(\cos 2x)\times -\frac{1}{2}\right\} = \frac{1}{2}\{3+\cos 2x - \cos 2x\} = \frac{3}{2} = \text{RHS}$$

EXAMPLE 12 *Prove that:*

(i) $\dfrac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = \tan x$ [NCERT] (ii) $\dfrac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = 2\sin x$ [NCERT]

(iii) $\dfrac{\tan 5x + \tan 3x}{\tan 5x - \tan 3x} = 4\cos 2x\cos 4x$

(iv) $\sin 2x + 2\sin 4x + \sin 6x = 4\cos^2 x\sin 4x$ [NCERT]

SOLUTION (i) LHS $= \dfrac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = \dfrac{(\sin 5x + \sin x) - 2\sin 3x}{\cos 5x - \cos x}$

$$= \frac{2\sin\left(\frac{5x+x}{2}\right)\cos\left(\frac{5x-x}{2}\right) - 2\sin 3x}{-2\sin\left(\frac{5x+x}{2}\right)\sin\left(\frac{5x-x}{2}\right)} = \frac{2\sin 3x\cos 2x - 2\sin 3x}{-2\sin 3x\sin 2x}$$

$$= -\frac{2\sin 3x(1-\cos 2x)}{-2\sin 3x\sin 2x} = \frac{1-\cos 2x}{\sin 2x} = \frac{2\sin^2 x}{2\sin\cos x} = \tan x = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \frac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = \frac{2\sin\left(\frac{x-3x}{2}\right)\cos\left(\frac{x+3x}{2}\right)}{-(\cos^2 x - \sin^2 x)} = \frac{2\sin(-x)\cos 2x}{-\cos 2x} = \frac{-2\sin x\cos 2x}{-\cos 2x}$$

$$= 2\sin x = \text{RHS}$$

(iii) We have,

$$\text{L.H.S} = \frac{\tan 5x + \tan 3x}{\tan 5x - \tan 3x} = \frac{\frac{\sin 5x}{\cos 5x} + \frac{\sin 3x}{\cos 3x}}{\frac{\sin 5x}{\cos 5x} - \frac{\sin 3x}{\cos 3x}} = \frac{\sin 5x\cos 3x + \cos 5x\sin 3x}{\sin 5x\cos 3x - \cos 5x\sin 3x} = \frac{\sin(5x+3x)}{\sin(5x-3x)}$$

$$= \frac{\sin 8x}{\sin 2x} = \frac{2\sin 4x\cos 4x}{\sin 2x} = \frac{2\times(2\sin 2x\cos 2x)\cos 4x}{\sin 2x} = 4\cos 2x\cos 4x = \text{RHS}$$

(iv) $\text{LHS} = \sin 2x + 2\sin 4x + \sin 6x = (\sin 6x + \sin 2x) + 2\sin 4x$

$$= 2\sin\left(\frac{6x+2x}{2}\right)\cos\left(\frac{6x-2x}{2}\right) + 2\sin 4x = 2\sin 4x\cos 2x + 2\sin 4x$$

$$= 2\sin 4x(\cos 2x + 1) = 2\sin 4x \times 2\cos^2 x = 4\cos^2 x\sin 4x = \text{RHS}$$

EXAMPLE 13 *Show that* : $2\sin^2\beta + 4\cos(\alpha+\beta)\sin\alpha\sin\beta + \cos 2(\alpha+\beta) = \cos 2\alpha$

[NCERT EXEMPLAR]

SOLUTION $\text{LHS} = 2\sin^2\beta + 4\cos(\alpha+\beta)\sin\alpha\sin\beta + \cos 2(\alpha+\beta)$

$= 2\sin^2\beta + 2\cos(\alpha+\beta)(2\sin\alpha\sin\beta) + \cos 2(\alpha+\beta)$

$= 2\sin^2\beta + 2\cos(\alpha+\beta)\{\cos(\alpha-\beta) - \cos(\alpha+\beta)\} + \cos 2(\alpha+\beta)$

$= 2\sin^2\beta + 2\cos(\alpha+\beta)\cos(\alpha-\beta) - 2\cos^2(\alpha+\beta) + \cos 2(\alpha+\beta)$

$= 2\sin^2\beta + 2(\cos^2\alpha - \sin^2\beta) - 2\cos^2(\alpha+\beta) + 2\cos^2(\alpha+\beta) - 1$

$= 2\sin^2\beta + 2\cos^2\alpha - 2\sin^2\beta - 2\cos^2(\alpha+\beta) + 2\cos^2(\alpha+\beta) - 1 = 2\cos^2\alpha - 1$

$= \cos 2\alpha = \text{RHS}$

EXAMPLE 14 *Show that* : $\sqrt{3}\,\text{cosec}\,20° - \sec 20° = 4.$ **[NCERT EXEMPLAR]**

SOLUTION $\text{LHS} = \sqrt{3}\,\text{cosec}\,20° - \sec 20°$

$$= \frac{\sqrt{3}}{\sin 20°} - \frac{1}{\cos 20°} = \frac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20°\cos 20°}$$

$$= \frac{2\left\{\frac{\sqrt{3}}{2}\cos 20° - \frac{1}{2}\sin 20°\right\}}{\sin 20°\cos 20°} = \frac{2(\sin 60°\cos 20° - \cos 60°\sin 20°)}{\sin 20°\cos 20°}$$

$$= \frac{2\sin(60° - 20°)}{\sin 20°\cos 20°} = \frac{2\sin 40°}{\sin 20°\cos 20°} = \frac{4\sin 40°}{2\sin 20°\cos 20°} = \frac{4\sin 40°}{\sin 40°} = 4 = \text{RHS}$$

Type III ON FINDING THE VALUES OF $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ AND $\tan\frac{x}{2}$ WHEN VALUES OF $\sin x$ OR $\cos x$ OR $\tan x$ ARE GIVEN

EXAMPLE 15 If $0 \le x \le 2\pi$, find $\sin\frac{x}{2}$, $\cos\frac{x}{2}$, and $\tan\frac{x}{2}$, when:

(i) $\tan x = -\frac{4}{3}$, x lies in quadrant II (ii) $\cos x = -\frac{1}{3}$, x lies in quadrant III

(iii) $\sin x = -\frac{1}{2}$, x lies in quadrant IV.

SOLUTION (i) It is given that x lies in the second quadrant in which $\cos x$ is negative.

$$\therefore \quad \cos x = -\frac{1}{\sqrt{1+\tan^2 x}} = -\frac{1}{\sqrt{1+\frac{16}{9}}} = -\frac{3}{5}$$

Again x lies in the second quadrant.

i.e. $\frac{\pi}{2} < x < \pi \Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2} \Rightarrow \frac{x}{2}$ lies in first quadrant $\Rightarrow \sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$ are positive

$$\therefore \quad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}} \Rightarrow \cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}} = \sqrt{\frac{1-3/5}{2}} = \frac{1}{\sqrt{5}}$$

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}} \Rightarrow \sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} = \sqrt{\frac{1+3/5}{2}} = \frac{2}{\sqrt{5}}$$

$$\text{and,} \quad \tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} \Rightarrow \tan\frac{x}{2} = \sqrt{\frac{1-\cos x}{1+\cos x}} = \sqrt{\frac{1+3/5}{1-3/5}} = 2$$

(ii) It is given that x lies in the third quadrant.

i.e. $\pi < x < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4} \Rightarrow \frac{x}{2}$ lies in IInd quadrant $\Rightarrow \cos\frac{x}{2} < 0$, $\sin\frac{x}{2} > 0$ and $\tan\frac{x}{2} < 0$

$$\therefore \quad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\Rightarrow \quad \cos\frac{x}{2} = -\sqrt{\frac{1+\cos x}{2}} = -\sqrt{\frac{1-1/3}{2}} = -\frac{1}{\sqrt{3}} \qquad \left[\because \cos\frac{x}{2} \text{ is } -\text{ve and } \cos x = -\frac{1}{3}\right]$$

$$\text{and,} \quad \sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\Rightarrow \quad \sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} = \sqrt{\frac{1+1/3}{2}} = \sqrt{\frac{2}{3}} \qquad \left[\because \sin\frac{x}{2} > 0 \text{ and } \cos x = \frac{1}{3}\right]$$

$$\text{and,} \quad \tan\frac{x}{2} = \frac{\sin x/2}{\cos x/2} = \sqrt{\frac{2}{3}} \times -\sqrt{3} = -\sqrt{2}$$

(iii) It is given that x lies in the fourth quadrant in which $\cos x$ is positive.

$$\therefore \quad \sin x = -\frac{1}{2} \Rightarrow \cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$$

Again, x lies in the fourth quadrant

i.e. $\frac{3\pi}{2} < x < 2\pi \Rightarrow \frac{3\pi}{4} < \frac{x}{2} < \pi \Rightarrow \frac{x}{2}$ lies in IInd quadrant $\Rightarrow \cos\frac{x}{2} < 0$, $\sin\frac{x}{2} > 0$ and $\tan\frac{x}{2} < 0$

$$\therefore \quad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\Rightarrow \quad \cos\frac{x}{2} = -\sqrt{\frac{1+\cos x}{2}} = -\sqrt{\frac{1+\sqrt{3}/2}{2}} = -\frac{\sqrt{2+\sqrt{3}}}{2} \qquad \left[\because \cos\frac{x}{2} < 0 \text{ and } \cos x = \frac{\sqrt{3}}{2}\right]$$

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\Rightarrow \quad \sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} = \sqrt{\frac{1-\sqrt{3}/2}{2}} = \frac{\sqrt{2-\sqrt{3}}}{2} \qquad \left[\because \sin\frac{x}{2} > 0 \text{ and } \cos x = \frac{\sqrt{3}}{2}\right]$$

and, $\tan\frac{x}{2} = \frac{\sin(x/2)}{\cos(x/2)} = \frac{\sqrt{2-\sqrt{3}}}{2} \times \frac{-2}{\sqrt{2+\sqrt{3}}} = -\sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}$

EXAMPLE 16 *If* $\tan x = \frac{3}{4}$, $\pi < x < \frac{3\pi}{2}$, *find the values of* $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$. **[NCERT]**

SOLUTION It is given that $\tan x = \frac{3}{4}$ and, $\pi < x < \frac{3\pi}{2}$.

$\therefore \quad \cos x = -\frac{1}{\sqrt{1+\tan^2 x}} = -\frac{1}{\sqrt{1+\frac{9}{16}}} = -\frac{4}{5}$ $\qquad \left[\because \pi < x < \frac{3\pi}{2} \therefore \cos x \text{ is negative}\right]$

Again, $\pi < x < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4} \Rightarrow \cos x < 0$ and $\sin x > 0$

$\therefore \quad \cos\frac{x}{2} = -\sqrt{\frac{1+\cos x}{2}} = -\sqrt{\frac{1-\frac{4}{5}}{2}} = -\frac{1}{\sqrt{10}}$ and, $\sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} = \sqrt{\frac{1+\frac{4}{5}}{2}} = \frac{3}{\sqrt{10}}$

$\therefore \quad \tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{3/\sqrt{10}}{-1/\sqrt{10}} = -3$

Hence, $\cos\frac{x}{2} = -\frac{1}{\sqrt{10}}$, $\sin\frac{x}{2} = \frac{3}{\sqrt{10}}$ and $\tan\frac{x}{2} = -3$

Type IV ON FINDING THE VALUES TRIGONOMETRICAL FUNCTIONS FOR $\frac{\pi}{24}, \frac{\pi}{16}, \frac{\pi}{8}$

Formulae: $\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$, $\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$, $\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$

EXAMPLE 17 *Find the values of*

(i) $\cos\frac{\pi}{8}$ (ii) $\sin\frac{\pi}{8}$ (iii) $\tan\frac{\pi}{8}$ **[NCERT EXEMPLAR]** (iv) $\sin\frac{\pi}{24}$ (v) $\cos\frac{\pi}{24}$

SOLUTION (i) We know that $\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$. Putting $x = \frac{\pi}{4}$, we get

$$\cos\frac{\pi}{8} = \sqrt{\frac{1+\cos\pi/4}{2}} = \sqrt{\frac{1+1/\sqrt{2}}{2}} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}} \qquad \left[\because \cos\frac{\pi}{8} \text{ is positive}\right]$$

(ii) We have, $\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$. Putting $x = \frac{\pi}{4}$, we get

$$\sin\frac{\pi}{8} = \sqrt{\frac{1-\cos\pi/4}{2}} = \sqrt{\frac{1-1/\sqrt{2}}{2}} = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} \qquad \left[\because \sin\frac{\pi}{8} \text{ is positive}\right]$$

(iii) We have, $\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$. Putting $x = \frac{\pi}{4}$, we get

$$\tan\frac{\pi}{8} = \sqrt{\frac{1-\cos\pi/4}{1+\cos\pi/4}} = \sqrt{\frac{1-1/\sqrt{2}}{1+1/\sqrt{2}}} = \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}} = \sqrt{\frac{(\sqrt{2}-1)^2}{(\sqrt{2}+1)(\sqrt{2}-1)}} = \sqrt{2}-1$$

(iv) We observe that

$$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Putting $x=\frac{\pi}{12}$ in $\sin\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{2}}$, we get

$$\sin\frac{\pi}{24}=\sqrt{\frac{1-\cos\frac{\pi}{12}}{2}}=\sqrt{\frac{1-\frac{\sqrt{3}+1}{2\sqrt{2}}}{2}}=\sqrt{\frac{2\sqrt{2}-\sqrt{3}-1}{4\sqrt{2}}}=\sqrt{\frac{4-\sqrt{6}-\sqrt{2}}{8}}=\frac{\sqrt{4-\sqrt{6}-\sqrt{2}}}{2\sqrt{2}}$$

(v) Putting $x=\frac{\pi}{12}$ in $\cos\frac{x}{2}=\pm\sqrt{\frac{1+\cos x}{2}}$, we get

$$\cos\frac{\pi}{24}=\sqrt{\frac{1+\cos\frac{\pi}{12}}{2}} \qquad \left[\because \cos\frac{\pi}{24}=\cos 7\frac{1}{2}^\circ \text{ is positive}\right]$$

$$\Rightarrow \quad \cos\frac{\pi}{24}=\sqrt{\frac{1+\frac{\sqrt{3}+1}{2\sqrt{2}}}{2}}=\sqrt{\frac{2\sqrt{2}+\sqrt{3}+1}{4\sqrt{2}}}=\sqrt{\frac{4+\sqrt{6}+\sqrt{2}}{8}}=\frac{\sqrt{4+\sqrt{6}+\sqrt{2}}}{2\sqrt{2}}$$

EXAMPLE 18 *Prove that:*

(i) $\cot\frac{\pi}{24}=\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$ (ii) $\tan\frac{\pi}{16}=\sqrt{4+2\sqrt{2}}-(\sqrt{2}+1)$

(iii) $\tan 142\frac{1}{2}^\circ=2+\sqrt{2}-\sqrt{3}-\sqrt{6}$

SOLUTION (i) LHS $=\cot\frac{\pi}{24}$

$$=\frac{\cos\frac{\pi}{24}}{\sin\frac{\pi}{24}}=\frac{2\cos\frac{\pi}{24}\cos\frac{\pi}{24}}{2\sin\frac{\pi}{24}\cos\frac{\pi}{24}}=\frac{2\cos^2\frac{\pi}{24}}{2\sin\frac{\pi}{24}\cos\frac{\pi}{24}}=\frac{1+\cos\frac{\pi}{12}}{\sin\frac{\pi}{12}}$$

$$=\frac{1+\cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right)}=\frac{1+\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}=\frac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1}=\frac{(2\sqrt{2}+\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$=\frac{2\sqrt{6}+2\sqrt{2}+3+\sqrt{3}+\sqrt{3}+1}{3-1}=\frac{2\sqrt{6}+2\sqrt{2}+2\sqrt{3}+4}{2}$$

$$=\sqrt{2}+\sqrt{3}+2+\sqrt{6}=\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}=\text{RHS}$$

(ii) $$\text{LHS}=\tan\frac{\pi}{16}=\frac{\sin\frac{\pi}{16}}{\cos\frac{\pi}{16}}=\frac{\sin\frac{\pi}{16}}{\cos\frac{\pi}{16}}\times\frac{2\sin\frac{\pi}{16}}{2\sin\frac{\pi}{16}}=\frac{2\sin^2\frac{\pi}{16}}{2\sin\frac{\pi}{16}\cos\frac{\pi}{16}}=\frac{1-\cos\frac{\pi}{8}}{\sin\frac{\pi}{8}}$$

$$=\frac{1-\sqrt{\frac{1+\cos\frac{\pi}{4}}{2}}}{\sqrt{\frac{1-\cos\frac{\pi}{4}}{2}}}=\frac{\sqrt{2}-\sqrt{1+\cos\frac{\pi}{4}}}{\sqrt{1-\cos\frac{\pi}{4}}} \qquad \left[\because \cos\frac{x}{2}=\sqrt{\frac{1+\cos x}{2}} \text{ and } \sin\frac{x}{2}=\sqrt{\frac{1-\cos x}{2}}\right]$$

$$= \frac{\sqrt{2}-\sqrt{1+\frac{1}{\sqrt{2}}}}{\sqrt{1-\frac{1}{\sqrt{2}}}} = \frac{\sqrt{2}-\sqrt{\frac{\sqrt{2}+1}{\sqrt{2}}}}{\sqrt{\frac{\sqrt{2}-1}{\sqrt{2}}}} = \frac{\sqrt{2\sqrt{2}}-\sqrt{\sqrt{2}+1}}{\sqrt{\sqrt{2}-1}}$$

$$= \frac{\sqrt{2\sqrt{2}}-\sqrt{\sqrt{2}+1}}{\sqrt{\sqrt{2}-1}} \times \frac{\sqrt{\sqrt{2}+1}}{\sqrt{\sqrt{2}+1}} = \frac{\sqrt{2\sqrt{2}}\,\sqrt{\sqrt{2}+1}-\sqrt{(\sqrt{2}+1)^2}}{\sqrt{(\sqrt{2}+1)(\sqrt{2}-1)}}$$

$$= \frac{\sqrt{2\sqrt{2}\,(\sqrt{2}+1)}-(\sqrt{2}+1)}{2-1} = \sqrt{4+2\sqrt{2}}-(\sqrt{2}+1) = \text{RHS}$$

(iii) $$\text{LHS} = \tan 142\frac{1}{2}^\circ = \tan\left(180^\circ - 37\frac{1}{2}^\circ\right) = \tan 37\frac{1}{2}^\circ = -\tan\frac{5\pi}{24}$$

$$= -\frac{\sin\frac{5\pi}{24}}{\cos\frac{5\pi}{24}} = -\frac{2\sin^2\frac{5\pi}{24}}{2\sin\frac{5\pi}{24}\cos\frac{5\pi}{24}} = -\frac{1-\cos\frac{5\pi}{12}}{\sin\frac{5\pi}{12}} = -\frac{1-\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right)}$$

$$= -\frac{1-\left(\cos\frac{\pi}{4}\cos\frac{\pi}{6}-\sin\frac{\pi}{4}\sin\frac{\pi}{6}\right)}{\sin\frac{\pi}{4}\cos\frac{\pi}{6}+\cos\frac{\pi}{4}\sin\frac{\pi}{6}} = \left(\frac{1-\frac{\sqrt{3}-1}{2\sqrt{2}}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}\right) = -\left(\frac{2\sqrt{2}-\sqrt{3}+1}{\sqrt{3}+1}\right)$$

$$= -\frac{(2\sqrt{2}-\sqrt{3}+1)}{(\sqrt{3}+1)} \times \frac{(\sqrt{3}-1)}{\sqrt{3}-1} = -\left\{\frac{(2\sqrt{2}-\sqrt{3}+1)(\sqrt{3}-1)}{3-1}\right\}$$

$$= -\left\{\frac{2\sqrt{2}\,(\sqrt{3}-1)-(\sqrt{3}-1)^2}{2}\right\} = -\left\{\frac{2\sqrt{2}\,(\sqrt{3}-1)-(3+1-2\sqrt{3})}{2}\right\}$$

$$= -\left\{\sqrt{2}\,(\sqrt{3}-1)-(2-\sqrt{3})\right\} = 2+\sqrt{2}-\sqrt{3}-\sqrt{6} = \text{RHS}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 19 Prove that: $\cos A \cos 2A \cos 2^2 A \cos 2^3 A \ldots \cos 2^{n-1} A = \dfrac{\sin 2^n A}{2^n \sin A}$

SOLUTION $\text{LHS} = \cos A \cos 2A \cos 2^2 A \cos 2^3 A \ldots \cos 2^{n-1} A$

$$= \frac{1}{2\sin A}\left\{(2\sin A\cos A)\cos 2A\cos 2^2 A\cos 2^3 A\ldots\cos 2^{n-1} A\right\}$$

$$= \frac{1}{2\sin A}\left\{\sin 2A\cos 2A\cos 2^2 A\cos 2^3 A\ldots\cos 2^{n-1} A\right\}$$

$$= \frac{1}{2^2\sin A}\left\{(2\sin 2A\cos 2A)\cos 2^2 A\cos 2^3 A\ldots\cos 2^{n-1} A\right\}$$

$$= \frac{1}{2^2\sin A}\left\{\sin 2(2A)\cos 2^2 A\cos 2^3 A\ldots\cos 2^{n-1} A\right\}$$

$$= \frac{1}{2^3\sin A}\left\{(2\sin 2^2 A\cos 2^2 A)\cos 2^3 A\ldots\cos 2^{n-1} A\right\}$$

$$= \frac{1}{2^3 \sin A}\left\{\sin(2\times 2^2 A)\cos 2^3 A \ldots \cos 2^{n-1} A\right\}$$

$$= \frac{1}{2^3 \sin A}\left\{\sin 2^3 A \cos 2^3 A \cos 2^4 A \ldots \cos 2^{n-1} A\right\}$$

$$= \ldots\ldots = \frac{1}{2^{n-1}\sin A}\left\{\sin 2^{n-1} A \cos 2^{n-1} A\right\} = \frac{1}{2^n \sin A}\left\{2\sin 2^{n-1} A \cos 2^{n-1} A\right\}$$

$$= \frac{1}{2^n \sin A}\sin(2\times 2^{n-1} A) = \frac{1}{2^n \sin A}\sin 2^n A = \text{RHS}$$

NOTE *The result proved in the above example may be used as a standard formula.*

Type V PROBLEMS BASED ON THE FORMULA

$$\cos A \cos 2A \cos 2^2 A \cos 2^3 A \ldots \cos 2^{n-1} A = \frac{\sin 2^n A}{2^n \sin A}$$

EXAMPLE 20 *Prove that:* $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}$

SOLUTION LHS $= \cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{2}(\cos 20° \cos 40° \cos 80°)$

$$= \frac{1}{2}\left(\cos A \cos 2A \cos 2^2 A\right), \text{ where } A = 20°$$

$$= \frac{1}{2}\left(\frac{\sin 2^3 A}{2^3 \sin A}\right) = \frac{1}{2^4}\frac{\sin 8A}{\sin A}$$

$$= \frac{1}{2^4}\frac{\sin 160°}{\sin 20°} = \frac{1}{2^4}\frac{\sin(180° - 20°)}{\sin 20°} = \frac{1}{2^4}\frac{\sin 20°}{\sin 20°} = \frac{1}{2^4} = \frac{1}{16} = \text{RHS}$$

EXAMPLE 21 *Prove that:* $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} = -\frac{1}{8}$.

SOLUTION $\text{LHS} = \cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} = \cos A \cos 2A \cos 2^2 A$, where $A = \frac{\pi}{7}$

$$= \frac{\sin 2^3 A}{2^3 \sin A} = \frac{\sin\frac{8\pi}{7}}{8\sin\frac{\pi}{7}}\ \frac{\sin\left(\pi + \frac{\pi}{7}\right)}{8\sin\frac{\pi}{7}} = -\frac{\sin\frac{\pi}{7}}{8\sin\frac{\pi}{7}} = -\frac{1}{8} = \text{RHS}$$

EXAMPLE 22 *If* $\theta = \frac{\pi}{2^n + 1}$, *prove that:* $2^n \cos\theta \cos 2\theta \cos 2^2\theta \ldots \cos 2^{n-1}\theta = 1$.

SOLUTION We have, $\theta = \frac{\pi}{2^n + 1} \Rightarrow 2^n\theta + \theta = \pi \Rightarrow 2^n\theta = \pi - \theta$

$$\therefore\ 2^n \cos\theta \cos 2\theta \cos 2^2\theta \ldots \cos 2^{n-1}\theta = 2^n\left\{\frac{\sin 2^n\theta}{2^n \sin\theta}\right\} = \frac{\sin 2^n\theta}{\sin\theta} = \frac{\sin(\pi - \theta)}{\sin\theta} = 1 \quad [\because\ 2^n\theta = \pi - \theta]$$

EXAMPLE 23 *Prove that:* $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = \frac{1}{8}$

SOLUTION LHS $= \cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = \cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\left(\pi - \frac{4\pi}{7}\right)$

$$= -\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} = -\left(-\frac{1}{8}\right) = \frac{1}{8} = \text{RHS}$$

[See Example 21]

EXAMPLE 24 *Prove that:* $\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{14\pi}{15}=\frac{1}{16}$

SOLUTION $\text{LHS}=\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\left(\pi-\frac{\pi}{15}\right)$

$$=\left(\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\right)\left(-\cos\frac{\pi}{15}\right) \qquad [\because \cos(\pi-\theta)=-\cos\theta]$$

$$=-\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}$$

$$=-\cos A\cos 2A\cos 2^2A\cos 2^3A,\quad \text{where } A=\frac{\pi}{15}$$

$$=-\frac{\sin 2^4A}{2^4\sin A}=-\frac{\sin 16A}{2^4\sin A}=-\frac{\sin\frac{16\pi}{15}}{16\sin\frac{\pi}{15}}=-\frac{\sin\left(\pi+\frac{\pi}{15}\right)}{16\sin\frac{\pi}{15}}=\frac{\sin\frac{\pi}{15}}{16\sin\frac{\pi}{15}}=\frac{1}{16}=\text{RHS}$$

EXAMPLE 25 *Prove that:* $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}=\frac{1}{64}$

SOLUTION We know that $A+B=\pi \Rightarrow \sin A=\sin(\pi-B)=\sin B$

$\therefore \quad \frac{\pi}{14}+\frac{13\pi}{14}=\pi,\ \frac{3\pi}{14}+\frac{11\pi}{14}=\pi,\ \frac{5\pi}{14}+\frac{9\pi}{14}=\pi$

$\Rightarrow \quad \sin\frac{\pi}{14}=\sin\frac{13\pi}{14},\ \sin\frac{3\pi}{14}=\sin\frac{11\pi}{14},\ \sin\frac{5\pi}{14}=\sin\frac{9\pi}{14}$

Now, $\text{LHS}=\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$

$$=\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{3\pi}{14}\sin\frac{\pi}{14}$$

$$=\left\{\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\right\}^2\times\sin\frac{7\pi}{14}=\left\{\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\right\}^2\times 1$$

$$=\left\{\cos\left(\frac{\pi}{2}-\frac{\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{3\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{14}\right)\right\}^2=\left\{\cos\frac{6\pi}{14}\cos\frac{4\pi}{14}\cos\frac{2\pi}{14}\right\}^2$$

$$=\left\{\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\right\}^2=\left(\frac{1}{8}\right)^2=\frac{1}{64} \qquad \text{[See Example 23]}$$

EXAMPLE 26 *Find the value of* $\sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}$.

SOLUTION $\sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}=\cos\left(\frac{\pi}{2}-\frac{\pi}{18}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{18}\right)\cos\left(\frac{\pi}{2}-\frac{7\pi}{18}\right)$

$$=\cos\frac{4\pi}{9}\cos\frac{2\pi}{9}\cos\frac{\pi}{9}=\frac{\sin\left(2^3\times\frac{\pi}{9}\right)}{2^3\sin\frac{\pi}{9}}=\frac{\sin\frac{8\pi}{9}}{8\sin\frac{\pi}{9}}=\frac{\sin\left(\pi-\frac{\pi}{9}\right)}{8\sin\frac{\pi}{9}}=\frac{\sin\frac{\pi}{9}}{8\sin\frac{\pi}{9}}=\frac{1}{8}$$

EXAMPLE 27 *Prove that:* $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}=\frac{1}{128}$

SOLUTION $\text{LHS}=\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}$

$$=\left\{\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15}\right\}\left\{\cos\frac{3\pi}{15}\cos\frac{6\pi}{15}\right\}\cos\frac{5\pi}{15}$$

$$= \frac{1}{2}\left\{\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15}\right\}\left\{\cos\frac{3\pi}{15}\cos\frac{6\pi}{15}\right\} \qquad \left[\because \cos\frac{5\pi}{15} = \cos\frac{\pi}{3} = \frac{1}{2}\right]$$

$$= -\frac{1}{2}\left\{\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\right\}\left\{\cos\frac{3\pi}{15}\cos\frac{6\pi}{15}\right\}\left[\because \cos\frac{7\pi}{15} = \cos\left(\pi - \frac{8\pi}{15}\right) = -\cos\frac{8\pi}{15}\right]$$

$$= -\frac{1}{2}\left\{\frac{\sin\frac{2^4\pi}{15}}{2^4\sin\frac{\pi}{15}}\right\}\times\left\{\frac{\sin\left(2^2\times\frac{3\pi}{15}\right)}{2^2\sin\frac{3\pi}{15}}\right\} = -\frac{1}{2}\left\{\frac{\sin\frac{16\pi}{15}}{16\sin\frac{\pi}{15}}\right\}\times\left\{\frac{\sin\frac{12\pi}{15}}{4\sin\frac{3\pi}{15}}\right\}$$

$$= -\frac{1}{2}\left\{\frac{\sin\left(\pi+\frac{\pi}{15}\right)}{16\sin\frac{\pi}{15}}\right\}\times\left\{\frac{\sin\left(\pi-\frac{3\pi}{15}\right)}{4\sin\frac{3\pi}{15}}\right\} = -\frac{1}{2}\left\{\frac{-\sin\frac{\pi}{15}}{16\sin\frac{\pi}{15}}\right\}\times\left\{\frac{\sin\frac{3\pi}{15}}{4\sin\frac{3\pi}{15}}\right\}$$

$$= -\frac{1}{2}\times-\frac{1}{16}\times\frac{1}{4} = \frac{1}{128} = \text{RHS}$$

EXAMPLE 28 *Prove that:* $(1+\sec 2x)(1+\sec 4x)(1+\sec 8x)\dots(1+\sec 2^n x) = \tan 2^n x \cot x, n \in N.$

SOLUTION LHS $= (1+\sec 2x)(1+\sec 4x)(1+\sec 8x)\dots(1+\sec 2^n x)$

$$= \frac{(1+\cos 2x)(1+\cos 4x)(1+\cos 8x)\dots(1+\cos 2^n x)}{\cos 2x \cos 4x \cos 8x \dots \cos 2^n x}$$

$$= \frac{(2\cos^2 x)(2\cos^2 2x)(2\cos^2 2^2 x)\dots(2\cos^2 2^{n-1}x)}{\cos 2x \cos 2^2 x \cos 2^3 x \dots \cos 2^n x}$$

$$= \frac{2^n \cos x}{\cos 2^n x}\left\{\cos x \cos 2x \cos 2^2 x \dots \cos 2^{n-1}x\right\} = \frac{2^n \cos x}{\cos 2^n x}\left\{\frac{\sin 2^n x}{2^n \sin x}\right\}$$

$$= \tan 2^n x \cot x = \text{RHS}$$

Type VI MISCELLANEOUS PROBLEMS BASED UPON FOLLOWING FORMULAE

$\sin 2x = 2\sin x\cos x$, $\cos 2x = \cos^2 x - \sin^2 x$, $\cos 2x = 2\cos^2 x - 1$,

$\cos 2x = 1 - 2\sin^2 x$, $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$, $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$, $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$

EXAMPLE 29 *If* $\tan^2 x = 2\tan^2\phi + 1$, *prove that* $\cos 2x + \sin^2\phi = 0$.

SOLUTION Using $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$, we obtain

$$\therefore \quad \cos 2x + \sin^2\phi = \frac{1-\tan^2 x}{1+\tan^2 x} + \sin^2\phi = \frac{1-(2\tan^2\phi+1)}{1+(2\tan^2\phi+1)} + \sin^2\phi \; [\because \tan^2 x = 2\tan^2\phi + 1]$$

$$= \frac{-2\tan^2\phi}{2+2\tan^2\phi} + \sin^2\phi = \frac{-\tan^2\phi}{\sec^2\phi} + \sin^2\phi = -\sin^2\phi + \sin^2\phi = 0$$

EXAMPLE 30 *Prove that:*

(i) $\dfrac{1+\cos 4x}{\cot x - \tan x} = \dfrac{1}{2}\sin 4x$ (ii) $\dfrac{\cos 5x + \cos 4x}{1 - 2\cos 3x} = -\cos 2x - \cos x$

(iii) $\dfrac{\cos 7x - \cos 8x}{1 + 2\cos 5x} = \cos 2x - \cos 3x$

SOLUTION (i) We have,

$$\text{LHS} = \frac{1+\cos 4x}{\cot x - \tan x} = \frac{2\cos^2 2x \times \cos x \sin x}{\cos^2 x - \sin^2 x} = \frac{2\cos^2 2x \times 2\sin x \cos x}{2\cos 2x}$$

$$= \cos 2x \sin 2x = \frac{1}{2}(2\sin 2x \cos 2x) = \frac{1}{2}\sin 4x = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} = \frac{\sin 3x(\cos 5x + \cos 4x)}{\sin 3x(1 - 2\cos 3x)} \quad [\text{Multiplying and dividing by } \sin 3x]$$

$$= \frac{\left\{2\sin\frac{3x}{2}\cos\frac{3x}{2}\right\}\left\{2\cos\frac{9x}{2}\cos\frac{x}{2}\right\}}{\sin 3x - 2\sin 3x\cos 3x} = \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{\sin 3x - \sin 6x}$$

$$= \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{2\sin\left(\frac{3x-6x}{2}\right)\cos\left(\frac{3x+6x}{2}\right)} = \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{2\sin\left(-\frac{3x}{2}\right)\cos\frac{9x}{2}}$$

$$= \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{-2\sin\frac{3x}{2}\cos\frac{9x}{2}} = -2\cos\frac{3x}{2}\cos\frac{x}{2} = -(\cos 2x + \cos x) = \text{RHS}$$

(iii) We have,

$$\text{LHS} = \frac{\cos 7x - \cos 8x}{1 + 2\cos 5x}$$

$$= \frac{2\sin\frac{5x}{2}(\cos 7x - \cos 8x)}{2\sin\frac{5x}{2}(1 + 2\cos 5x)} \quad \left[\text{Multiplying numerator and denominator by } 2\sin\frac{5x}{2}\right]$$

$$= \frac{2\sin\frac{5x}{2}\cos 7x - 2\sin\frac{5x}{2}\cos 8x}{2\sin\frac{5x}{2} + 4\sin\frac{5x}{2}\cos 5x} = \frac{2\sin\frac{5x}{2}\cos 7x - 2\sin\frac{5x}{2}\cos 8x}{2\left\{\sin\frac{5x}{2} + 2\sin\frac{5x}{2}\cos 5x\right\}}$$

$$= \frac{\left(\sin\frac{19x}{2} - \sin\frac{9x}{2}\right) - \left(\sin\frac{21x}{2} - \sin\frac{11x}{2}\right)}{2\left\{\sin\frac{5x}{2} + \sin\frac{15x}{2} - \sin\frac{5x}{2}\right\}}$$

$$= \frac{\left(\sin\frac{19x}{2} + \sin\frac{11x}{2}\right) - \left(\sin\frac{9x}{2} + \sin\frac{21x}{2}\right)}{2\sin\frac{15x}{2}}$$

$$= \frac{2\sin\left(\frac{\frac{19x}{2}+\frac{11x}{2}}{2}\right)\cos\left(\frac{\frac{19x}{2}-\frac{11x}{2}}{2}\right) - 2\sin\left(\frac{\frac{9x}{2}+\frac{21x}{2}}{2}\right)\cos\left(\frac{\frac{9x}{2}-\frac{21x}{2}}{2}\right)}{2\sin\frac{15x}{2}}$$

$$= \frac{2\sin\frac{15x}{2}\cos 2x - 2\sin\frac{15x}{2}\cos(-3x)}{2\sin\frac{15x}{2}} = \cos 2x - \cos 3x = \text{RHS}$$

EXAMPLE 31 *If* $\tan\alpha = \frac{p}{q}$, *where* $\alpha = 6\beta$, α *being an acute angle, prove that :*

$$\frac{1}{2}\left\{p\operatorname{cosec} 2\beta - q\sec 2\beta\right\} = \sqrt{p^2+q^2}.$$

SOLUTION We have, $\tan\alpha = \frac{p}{q}$. Therefore, $\sin\alpha = \frac{p}{\sqrt{p^2+q^2}}$ and, $\cos\alpha = \frac{q}{\sqrt{p^2+q^2}}$

Now,

$$\text{LHS} = \frac{1}{2}\left\{p\operatorname{cosec} 2\beta - q\sec 2\beta\right\} = \frac{\sqrt{p^2+q^2}}{2}\left\{\frac{p}{\sqrt{p^2+q^2}}\operatorname{cosec} 2\beta - \frac{q}{\sqrt{p^2+q^2}}\sec 2\beta\right\}$$

$$= \frac{\sqrt{p^2+q^2}}{2}\left\{\sin\alpha\operatorname{cosec} 2\beta - \cos\alpha\sec 2\beta\right\} = \frac{\sqrt{p^2+q^2}}{2}\left\{\frac{\sin\alpha}{\sin 2\beta} - \frac{\cos\alpha}{\cos 2\beta}\right\}$$

$$= \frac{\sqrt{p^2+q^2}}{2}\left\{\frac{\sin\alpha\cos 2\beta - \cos\alpha\sin 2\beta}{\sin 2\beta\cos 2\beta}\right\} = \sqrt{p^2+q^2}\left\{\frac{\sin(\alpha-2\beta)}{2\sin 2\beta\cos 2\beta}\right\}$$

$$= \sqrt{p^2+q^2}\left\{\frac{\sin(6\beta-2\beta)}{\sin 4\beta}\right\} = \sqrt{p^2+q^2} = \text{RHS}. \qquad [\because \alpha = 6\beta]$$

EXAMPLE 32 *Prove that:* $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = \cot\alpha$

SOLUTION We find that

$$\cot x - \tan x = \frac{1}{\tan x} - \tan x = \frac{1-\tan^2 x}{\tan x} = 2\left\{\frac{1-\tan^2 x}{2\tan x}\right\} = \frac{2}{\tan 2x} = 2\cot 2x \quad ...(i)$$

$\therefore$

$$\begin{aligned}
\text{LHS} &= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha \\
&= \cot\alpha - \cot\alpha + \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha \\
&= \cot\alpha - \{\cot\alpha - \tan\alpha - 2\tan 2\alpha - 4\tan 4\alpha - 8\cot 8\alpha\} \\
&= \cot\alpha - \{(\cot\alpha - \tan\alpha) - 2\tan 2\alpha - 4\tan 4\alpha - 8\cot 8\alpha\} \\
&= \cot\alpha - \{2\cot 2\alpha - 2\tan 2\alpha - 4\tan 4\alpha - 8\cot 8\alpha\} \\
&= \cot\alpha - \{2(\cot 2\alpha - \tan 2\alpha) - 4\tan 4\alpha - 8\cot 8\alpha\} && \text{[Using (i)]} \\
&= \cot\alpha - \{2\times 2\cot 2(2\alpha) - 4\tan 4\alpha - 8\cot 8\alpha\} && \text{[Using (i)]} \\
&= \cot\alpha - \{4(\cot 4\alpha - \tan 4\alpha) - 8\cot 8\alpha\} && \text{[Using (i)]} \\
&= \cot\alpha - \{4\times 2\cot 2(4\alpha) - 8\cot 8\alpha\} = \cot\alpha - (8\cot 8\alpha - 8\cot 8\alpha) = \cot\alpha = \text{RHS}
\end{aligned}$$

EXAMPLE 33 *If* $\tan\beta = \frac{\tan\alpha + \tan\gamma}{1+\tan\alpha\tan\gamma}$, *prove that* $\sin 2\beta = \frac{\sin 2\alpha + \sin 2\gamma}{1+\sin 2\alpha\sin 2\gamma}$

SOLUTION It is given that $\tan\beta = \frac{\tan\alpha + \tan\gamma}{1+\tan\alpha\tan\gamma}$

$$\Rightarrow \quad \tan\beta = \frac{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\gamma}{\cos\gamma}}{1 + \frac{\sin\alpha}{\cos\alpha} \times \frac{\sin\gamma}{\cos\gamma}} = \frac{\sin\alpha\cos\gamma + \cos\alpha\sin\gamma}{\cos\alpha\cos\gamma + \sin\alpha\sin\gamma} = \frac{\sin(\alpha+\gamma)}{\cos(\alpha-\gamma)} \qquad \text{...(i)}$$

$$\therefore \quad \sin 2\beta = \frac{2\tan\beta}{1+\tan^2\beta} = \frac{\frac{2\sin(\alpha+\gamma)}{\cos(\alpha-\gamma)}}{1 + \frac{\sin^2(\alpha+\gamma)}{\cos^2(\alpha-\gamma)}} \qquad \text{[Using (i)]}$$

$$= \frac{2\sin(\alpha+\gamma)\cos(\alpha-\gamma)}{\cos^2(\alpha-\gamma)+\sin^2(\alpha+\gamma)} = \frac{2(\sin 2\alpha + \sin 2\gamma)}{2\cos^2(\alpha-\gamma) + 2\sin^2(\alpha+\gamma)}$$

$$= \frac{2(\sin 2\alpha + \sin 2\gamma)}{1+\cos(2\alpha-2\gamma)+1-\cos(2\alpha+2\gamma)} = \frac{2(\sin 2\alpha + \sin 2\gamma)}{2+\cos(2\alpha-2\gamma)-\cos(2\alpha+2\gamma)}$$

$$= \frac{2(\sin 2\alpha + \sin 2\gamma)}{2+2\sin 2\alpha\sin 2\gamma} = \frac{\sin 2\alpha + \sin 2\gamma}{1+\sin 2\alpha \sin 2\gamma}$$

EXAMPLE 34 *If* $\sin(\theta+\alpha) = a$ *and* $\sin(\theta+\beta) = b$, *prove that* $\cos 2(\alpha-\beta) - 4ab\cos(\alpha-\beta) = 1-2a^2-2b^2$.

[NCERT EXEMPLAR]

SOLUTION We have, $\sin(\theta+\alpha) = a$ and $\sin(\theta+\beta) = b$

$$\therefore \quad \cos(\theta+\alpha) = \sqrt{1-\sin^2(\theta+\alpha)} = \sqrt{1-a^2} \text{ and, } \cos(\theta+\beta) = \sqrt{1-\sin^2(\theta+\beta)} = \sqrt{1-b^2}$$

Now,

$$\cos(\alpha-\beta) = \cos\{(\theta+\alpha)-(\theta+\beta)\} = \cos(\theta+\alpha)\cos(\theta+\beta) + \sin(\theta+\alpha)\sin(\theta+\beta)$$

$$= \sqrt{1-a^2}\sqrt{1-b^2} + ab = ab + \sqrt{1-a^2-b^2+a^2b^2}$$

$$\therefore \quad \cos 2(\alpha-\beta) - 4ab\cos(\alpha-\beta) = 2\cos^2(\alpha-\beta) - 1 - 4ab\cos(\alpha-\beta)$$

$$= 2\left\{ab + \sqrt{1-a^2-b^2+a^2b^2}\right\}^2 - 1 - 4ab\left\{ab + \sqrt{1-a^2-b^2+a^2b^2}\right\}$$

$$= 2\left\{a^2b^2 + 2ab\sqrt{1-a^2-b^2+a^2b^2} + 1 - a^2 - b^2 + a^2b^2\right\} - 1 - 4a^2b^2 - 4ab\sqrt{1-a^2-b^2+a^2b^2}$$

$$= 2 - 2a^2 - 2b^2 - 1 = 1 - 2a^2 - 2b^2.$$

EXAMPLE 35 *If* $\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) = \tan^3\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$, *prove that* $\sin x = \frac{3\sin\alpha + \sin^3\alpha}{1+3\sin^2\alpha}$.

SOLUTION We have,

$$\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) = \tan^3\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

$$\Rightarrow \quad \frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}} = \left\{\frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}\right\}^3 \Rightarrow \frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}} = \left\{\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}\right\}^3$$

$$\Rightarrow \quad \left\{\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right\}^2 = \left[\left\{\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}\right\}^2\right]^3 \qquad \text{[On squaring both sides]}$$

$$\Rightarrow \quad \frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{1-2\sin\frac{x}{2}\cos\frac{x}{2}} = \left\{\frac{1+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{1-2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right\}^3 \Rightarrow \frac{1+\sin x}{1-\sin x} = \left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)^3$$

$$\Rightarrow \quad \frac{(1+\sin x)-(1-\sin x)}{(1+\sin x)+(1-\sin x)} = \frac{(1+\sin\alpha)^3-(1-\sin\alpha)^3}{(1+\sin\alpha)^3+(1-\sin\alpha)^3} \qquad \left[\begin{array}{l}\text{Applying componendo -}\\ \text{dividendo}\end{array}\right]$$

$$\Rightarrow \quad \frac{2\sin x}{2} = \frac{6\sin\alpha+2\sin^3\alpha}{2+6\sin^2\alpha} \Rightarrow \sin x = \frac{3\sin\alpha+\sin^3\alpha}{1+3\sin^2\alpha}$$

EXAMPLE 36 *If* $\tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\phi}{2}$, *prove that* $\cos\phi = \frac{\cos\theta-e}{1-e\cos\theta}$.

SOLUTION We have,

$$\cos\phi = \frac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}$$

$$\Rightarrow \quad \cos\phi = \frac{1-\frac{1+e}{1-e}\tan^2\frac{\theta}{2}}{1+\frac{1+e}{1-e}\tan^2\frac{\theta}{2}} \qquad \left[\because \tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\phi}{2} \Rightarrow \tan\frac{\phi}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{\theta}{2}\right]$$

$$\Rightarrow \quad \cos\phi = \frac{(1-e)-(1+e)\tan^2\frac{\theta}{2}}{(1-e)+(1+e)\tan^2\frac{\theta}{2}} = \frac{(1-\tan^2\theta/2)-e(1+\tan^2\theta/2)}{(1+\tan^2\theta/2)-e(1-\tan^2\theta/2)}$$

$$\Rightarrow \quad \cos\phi = \frac{\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}-e}{1-e\,\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}} \qquad \left[\text{Dividing numerator and denominator by } 1+\tan^2\frac{\theta}{2}\right]$$

$$\Rightarrow \quad \cos\phi = \frac{\cos\theta-e}{1-e\cos\theta} \qquad \left[\because \cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right]$$

EXAMPLE 37 *Prove that:*

$$\frac{2\cos 2^n x+1}{2\cos x+1} = (2\cos x-1)(2\cos 2x-1)(2\cos 2^2x-1)\ldots(2\cos 2^{n-1}x-1)$$

SOLUTION RHS $= (2\cos x-1)(2\cos 2x-1)(2\cos 2^2x-1)\ldots(2\cos 2^{n-1}x-1)$

$$= \frac{1}{(2\cos x+1)}\left\{(2\cos x+1)(2\cos x-1)(2\cos 2x-1)(2\cos 2^2 x-1)\ldots(2\cos 2^{n-1} x-1\right\}$$

$$= \frac{1}{(2\cos x+1)}\left\{(4\cos^2 x-1)(2\cos 2x-1)(2\cos 2^2 x-1)\ldots(2\cos 2^{n-1} x-1)\right\}$$

$$= \frac{1}{(2\cos x+1)}\left\{\{2(1+\cos 2x)-1\}(2\cos 2x-1)(2\cos 2^2 x-1)\ldots(2\cos 2^{n-1} x-1)\right\}$$

$$= \frac{1}{(2\cos x + 1)}\left\{(2\cos 2x + 1)(2\cos 2x - 1)(2\cos 2^2 x - 1)\dots(2\cos 2^{n-1} x - 1)\right\}$$

$$= \frac{1}{(2\cos x + 1)}\left\{(4\cos^2 2x - 1)(2\cos 2^2 x - 1)(2\cos 2^3 x - 1)\dots(2\cos 2^{n-1} x - 1)\right\}$$

$$= \frac{1}{(2\cos x + 1)}\left\{\left\{2(\cos 4x + 1) - 1\right\}(2\cos 2^2 x - 1)(2\cos 2^3 x - 1)\dots(2\cos 2^{n-1} x - 1)\right\}$$

$$= \frac{1}{(2\cos x + 1)}\left\{(2\cos 2^2 x + 1)(2\cos 2^2 x - 1)(2\cos 2^3 x - 1)\dots\dots(2\cos 2^{n-1} x - 1)\right\}$$

$$= \frac{1}{(2\cos x + 1)}\left\{(4\cos^2 2^2 x - 1)(2\cos 2^3 x - 1)\dots\dots(2\cos 2^{n-1} x - 1)\right\}$$

$$= \frac{1}{(2\cos x + 1)}\left\{\left\{2(1 + \cos 2^3 x) - 1\right\}(2\cos 2^3 x - 1)\dots(2\cos 2^{n-1} x - 1)\right\}$$

$$= \frac{1}{(2\cos x + 1)}\left\{(2\cos 2^3 x + 1)(2\cos 2^3 x - 1)\dots(2\cos 2^{n-1} x - 1)\right\}$$

....

$$= \frac{1}{(2\cos x + 1)}(2\cos 2^{n-1} x + 1)(2\cos 2^{n-1} x - 1)$$

$$= \frac{1}{(2\cos x + 1)}(4\cos^2 2^{n-1} x - 1) = \frac{1}{(2\cos x + 1)}\left\{2(\cos 2 . 2^{n-1} x + 1) - 1\right\}$$

$$= \frac{1}{(2\cos x + 1)}(2\cos 2^n x + 2 - 1) = \frac{2\cos 2^n x + 1}{2\cos x + 1} = \text{LHS}$$

Type VII ON CONDITIONAL IDENTITIES

EXAMPLE 38 *If* $\tan\frac{\theta}{2} = \sqrt{\frac{a-b}{a+b}}\tan\frac{\phi}{2}$, *prove that* $\cos\theta = \frac{a\cos\phi + b}{a + b\cos\phi}$.

SOLUTION Putting $\tan\frac{\theta}{2} = \sqrt{\frac{a-b}{a+b}}\tan\frac{\phi}{2}$ in $\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$, we obtain

$$\Rightarrow \quad \cos\theta = \frac{1 - \frac{a-b}{a+b}\tan^2\frac{\phi}{2}}{1 + \frac{a-b}{a+b}\tan^2\frac{\phi}{2}} = \frac{(a+b) - (a-b)\tan^2\frac{\phi}{2}}{(a+b) + (a-b)\tan^2\frac{\phi}{2}}$$

$$\Rightarrow \quad \cos\theta = \frac{a\left(1 - \tan^2\frac{\phi}{2}\right) + b\left(1 + \tan^2\frac{\phi}{2}\right)}{a\left(1 + \tan^2\frac{\phi}{2}\right) + b\left(1 - \tan^2\frac{\phi}{2}\right)}$$

$$\Rightarrow \quad \cos\theta = \frac{a\left(\dfrac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}\right)+b}{a+b\left(\dfrac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}\right)} \qquad \left[\text{Dividing numerator and denominator } by\ 1+\tan^2\frac{\phi}{2}\right]$$

$$\Rightarrow \quad \cos\theta = \frac{a\cos\phi+b}{a+b\cos\phi}$$

EXAMPLE 39 *If* $\cos\theta = \cos\alpha\cos\beta$, *prove that* $\tan\dfrac{\theta+\alpha}{2}\tan\dfrac{\theta-\alpha}{2} = \tan^2\dfrac{\beta}{2}$.

SOLUTION We have,

$$\cos\theta = \cos\alpha\cos\beta$$

$$\Rightarrow \quad \cos\beta = \frac{\cos\theta}{\cos\alpha}$$

$$\Rightarrow \quad \frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}} = \frac{\cos\theta}{\cos\alpha}$$

$$\Rightarrow \quad \frac{\left(1-\tan^2\frac{\beta}{2}\right)+\left(1+\tan^2\frac{\beta}{2}\right)}{\left(1-\tan^2\frac{\beta}{2}\right)-\left(1+\tan^2\frac{\beta}{2}\right)} = \frac{\cos\theta+\cos\alpha}{\cos\theta-\cos\alpha} \qquad \text{[Applying componendo - dividendo]}$$

$$\Rightarrow \quad \frac{2}{-2\tan^2\frac{\beta}{2}} = \frac{2\cos\frac{\theta+\alpha}{2}\cos\frac{\theta-\alpha}{2}}{-2\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}}$$

$$\Rightarrow \quad \frac{1}{\tan^2\frac{\beta}{2}} = \frac{1}{\tan\frac{\theta+\alpha}{2}\tan\frac{\theta-\alpha}{2}}$$

$$\Rightarrow \quad \tan^2\frac{\beta}{2} = \tan\frac{\theta+\alpha}{2}\tan\frac{\theta-\alpha}{2}$$

EXAMPLE 40 *If* $\cos\theta = \dfrac{\cos\alpha-\cos\beta}{1-\cos\alpha\cos\beta}$, *prove that* $\tan\dfrac{\theta}{2} = \pm\tan\dfrac{\alpha}{2}\cot\dfrac{\beta}{2}$.

SOLUTION We have,

$$\cos\theta = \frac{\cos\alpha-\cos\beta}{1-\cos\alpha\cos\beta}$$

$$\Rightarrow \quad \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{\cos\alpha-\cos\beta}{1-\cos\alpha\cos\beta}$$

$$\Rightarrow \quad \frac{\left(1-\tan^2\frac{\theta}{2}\right)+\left(1+\tan^2\frac{\theta}{2}\right)}{\left(1-\tan^2\frac{\theta}{2}\right)-\left(1+\tan^2\frac{\theta}{2}\right)} = \frac{(\cos\alpha-\cos\beta)+(1-\cos\alpha\cos\beta)}{(\cos\alpha-\cos\beta)-(1-\cos\alpha\cos\beta)}$$

$$\Rightarrow \quad \frac{2}{-2\tan^2\frac{\theta}{2}} = \frac{1+\cos\alpha-\cos\beta-\cos\alpha\cos\beta}{-\{1-\cos\alpha+\cos\beta-\cos\alpha\cos\beta\}}$$

$$\Rightarrow \quad \frac{1}{\tan^2 \frac{\theta}{2}} = \frac{(1+\cos\alpha)(1-\cos\beta)}{(1-\cos\alpha)(1+\cos\beta)}$$

$$\Rightarrow \quad \frac{1}{\tan^2 \frac{\theta}{2}} = \frac{2\cos^2\frac{\alpha}{2} \times 2\sin^2\frac{\beta}{2}}{2\sin^2\frac{\alpha}{2} \times 2\cos^2\frac{\beta}{2}}$$

$$\Rightarrow \quad \tan^2\frac{\theta}{2} = \tan^2\frac{\alpha}{2}\cot^2\frac{\beta}{2} \Rightarrow \tan\frac{\theta}{2} = \pm\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$$

EXAMPLE 41 *If* $\cos\theta = \frac{\cos\alpha\cos\beta}{1-\sin\alpha\sin\beta}$, *prove that one value of* $\tan\frac{\theta}{2} = \frac{\tan\frac{\alpha}{2} - \tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}$.

SOLUTION We have, $\cos\theta = \frac{\cos\alpha\cos\beta}{1-\sin\alpha\sin\beta}$

Now, $\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta}$

$$\Rightarrow \quad \tan^2\frac{\theta}{2} = \frac{1-\frac{\cos\alpha\cos\beta}{1-\sin\alpha\sin\beta}}{1+\frac{\cos\alpha\cos\beta}{1-\sin\alpha\sin\beta}} = \frac{1-\sin\alpha\sin\beta-\cos\alpha\cos\beta}{1-\sin\alpha\sin\beta+\cos\alpha\cos\beta}$$

$$\Rightarrow \quad \tan^2\frac{\theta}{2} = \frac{1-(\cos\alpha\cos\beta+\sin\alpha\sin\beta)}{1+(\cos\alpha\cos\beta-\sin\alpha\sin\beta)} = \frac{1-\cos(\alpha-\beta)}{1+\cos(\alpha+\beta)} = \frac{2\sin^2\left(\frac{\alpha-\beta}{2}\right)}{2\cos^2\left(\frac{\alpha+\beta}{2}\right)}$$

$$\Rightarrow \quad \tan\frac{\theta}{2} = \pm\frac{\sin\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)} = \pm\frac{\sin\frac{\alpha}{2}\cos\frac{\beta}{2} - \cos\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}}$$

$$\Rightarrow \quad \tan\frac{\theta}{2} = \pm\frac{\tan\frac{\alpha}{2} - \tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}} \quad \left[\text{Dividing numerator and denominator by } \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\right]$$

EXAMPLE 42 *If* $\sin\alpha + \sin\beta = a$ *and* $\cos\alpha + \cos\beta = b$, *prove that*

(i) $\cos(\alpha-\beta) = \frac{a^2+b^2-2}{2}$ (ii) $\tan\frac{\alpha-\beta}{2} = \pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$

SOLUTION (i) We have,

$\sin\alpha + \sin\beta = a$ and $\cos\alpha + \cos\beta = b$

$$\Rightarrow \quad (\sin\alpha + \sin\beta)^2 + (\cos\alpha + \cos\beta)^2 = a^2 + b^2$$

$$\Rightarrow \quad (\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) + 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = a^2 + b^2$$

$$\Rightarrow \quad 2 + 2\cos(\alpha-\beta) = a^2 + b^2 \Rightarrow \cos(\alpha-\beta) = \frac{a^2+b^2-2}{2}$$

(ii) Now, $\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta}$

$$\Rightarrow \quad \tan^2\left(\frac{\alpha-\beta}{2}\right) = \frac{1-\cos(\alpha-\beta)}{1+\cos(\alpha-\beta)} \qquad [\text{Replacing } \theta \text{ by } (\alpha-\beta)]$$

$$\Rightarrow \quad \tan^2\left(\frac{\alpha-\beta}{2}\right) = \frac{1-\dfrac{a^2+b^2-2}{2}}{1+\dfrac{a^2+b^2-2}{2}} \qquad \left[\because \cos(\alpha-\beta) = \frac{a^2+b^2-2}{2}\right]$$

$$\Rightarrow \quad \tan^2\left(\frac{\alpha-\beta}{2}\right) = \frac{4-a^2-b^2}{a^2+b^2} \Rightarrow \tan\left(\frac{\alpha-\beta}{2}\right) = \pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$$

EXAMPLE 43 *If α and β are distinct roots of $a\cos x + b\sin x = c$, prove that* $\sin(\alpha+\beta) = \dfrac{2ab}{a^2+b^2}$.

SOLUTION It is given that α and β are distinct roots of $a\cos x + b\sin x = c$

$$\therefore \quad a\cos\alpha + b\sin\alpha = c \text{ and } a\cos\beta + b\sin\beta = c$$

$$\Rightarrow \quad (a\cos\alpha + b\sin\alpha) - (a\cos\beta + b\sin\beta) = c - c$$

$$\Rightarrow \quad a(\cos\alpha - \cos\beta) + (b\sin\alpha - \sin\beta) = 0$$

$$\Rightarrow \quad -2a\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} + 2b\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2} = 0$$

$$\Rightarrow \quad 2a\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} = 2b\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\Rightarrow \quad \tan\frac{\alpha+\beta}{2} = \frac{b}{a} \qquad \left[\because \alpha \neq \beta \therefore \sin\frac{\alpha-\beta}{2} \neq 0\right]$$

$$\therefore \quad \sin(\alpha+\beta) = \frac{2\tan\dfrac{\alpha+\beta}{2}}{1+\tan^2\dfrac{\alpha+\beta}{2}} \Rightarrow \sin(\alpha+\beta) = \frac{\dfrac{2b}{a}}{1+\dfrac{b^2}{a^2}} = \frac{2ab}{a^2+b^2}.$$

ALITER We have,

$$a\cos x + b\sin x = c \qquad \ldots(i)$$

$$\Rightarrow \quad a\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) + b\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) = c \Rightarrow a\left(1-\tan^2\frac{x}{2}\right) + 2b\tan\frac{x}{2} = c\left(1+\tan^2\frac{x}{2}\right)$$

$$\Rightarrow \quad (c+a)\tan^2\frac{x}{2} - 2b\tan\frac{x}{2} + (c-a) = 0 \qquad \ldots(ii)$$

It is given that α and β are roots of the equation (i). Therefore, $\tan\dfrac{\alpha}{2}$ and $\tan\dfrac{\beta}{2}$ are roots of equation (ii).

$$\therefore \quad \tan\frac{\alpha}{2} + \tan\frac{\beta}{2} = \frac{2b}{c+a} \text{ and, } \tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{c-a}{c+a} \qquad \ldots(iii)$$

$$\text{Now,} \quad \tan\left(\frac{\alpha}{2}+\frac{\beta}{2}\right) = \frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}} \Rightarrow \tan\left(\frac{\alpha+\beta}{2}\right) = \frac{2b/c+a}{1-\dfrac{c-a}{c+a}} = \frac{b}{a} \qquad [\text{Using (iii)}]$$

$$\therefore \qquad \sin(\alpha+\beta) = \frac{2\tan\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)} = \frac{2\frac{b}{a}}{1+\frac{b^2}{a^2}} = \frac{2ab}{a^2+b^2}$$

EXERCISE 9.1

BASIC

Prove the following identities: (1–25)

1. $\sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}} = \tan x$

2. $\dfrac{\sin 2x}{1-\cos 2x} = \cot x$

3. $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$

4. $\sqrt{2+\sqrt{2+2\cos 4x}} = 2\cos x, 0 < x < \dfrac{\pi}{4}$

5. $\dfrac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x} = \tan x$

6. $\dfrac{\sin x+\sin 2x}{1+\cos x+\cos 2x} = \tan x$

7. $\dfrac{\cos 2x}{1+\sin 2x} = \tan\left(\dfrac{\pi}{4}-x\right)$

8. $\dfrac{\cos x}{1-\sin x} = \tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$

9. $\cos^2\dfrac{\pi}{8}+\cos^2\dfrac{3\pi}{8}+\cos^2\dfrac{5\pi}{8}+\cos^2\dfrac{7\pi}{8}=2$

10. $\sin^2\dfrac{\pi}{8}+\sin^2\dfrac{3\pi}{8}+\sin^2\dfrac{5\pi}{8}+\sin^2\dfrac{7\pi}{8}=2$

11. $(\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2 = 4\cos^2\left(\dfrac{\alpha-\beta}{2}\right)$

12. $\sin^2\left(\dfrac{\pi}{8}+\dfrac{x}{2}\right)-\sin^2\left(\dfrac{\pi}{8}-\dfrac{x}{2}\right)=\dfrac{1}{\sqrt{2}}\sin x$

13. $1+\cos^2 2x = 2(\cos^4 x+\sin^4 x)$

14. $\cos^3 2x+3\cos 2x = 4(\cos^6 x-\sin^6 x)$

15. $(\sin 3x+\sin x)\sin x+(\cos 3x-\cos x)\cos x=0$

16. $\cos^2\left(\dfrac{\pi}{4}-x\right)-\sin^2\left(\dfrac{\pi}{4}-x\right) = \sin 2x$

17. $\cos 4x = 1-8\cos^2 x+8\cos^4 x$

18. $\sin 4x = 4\sin x\cos^3 x-4\cos x\sin^3 x$ [NCERT EXEMPLAR]

BASED ON LOTS

19. $3(\sin x-\cos x)^4+6(\sin x+\cos x)^2+4(\sin^6 x+\cos^6 x) = 13$

20. $2(\sin^6 x+\cos^6 x)-3(\sin^4 x+\cos^4 x)+1 = 0$

21. $\cos^6 x-\sin^6 x = \cos 2x\left(1-\dfrac{1}{4}\sin^2 2x\right)$

22. $\tan\left(\dfrac{\pi}{4}+x\right)+\tan\left(\dfrac{\pi}{4}-x\right) = 2\sec 2x$

23. $\cot^2 x-\tan^2 x = 4\cot 2x\,\mathrm{cosec}\, 2x$

24. $\cos 4x-\cos 4\alpha = 8(\cos x-\cos\alpha)(\cos x+\cos\alpha)(\cos x-\sin\alpha)(\cos x+\sin\alpha)$

25. $\sin 3x + \sin 2x - \sin x = 4 \sin x \cos \frac{x}{2} \cos \frac{3x}{2}$ [NCERT]

26. Prove that : $\tan 82\frac{1}{2}^\circ = (\sqrt{3}+\sqrt{2})(\sqrt{2}+1) = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$

27. Prove that : $\cot \frac{\pi}{8} = \sqrt{2}+1$

28. (i) If $\cos x = -\frac{3}{5}$ and x lies in the IIIrd quadrant, find the values of $\cos \frac{x}{2}$, $\sin \frac{x}{2}$ and, $\sin 2x$.

(ii) If $\cos x = -\frac{3}{5}$ and x lies in IInd quadrant, find the values of $\sin 2x$ and $\sin \frac{x}{2}$.

29. If $\sin x = \frac{\sqrt{5}}{3}$ and x lies in IInd quadrant, find the values of $\cos \frac{x}{2}$, $\sin \frac{x}{2}$ and $\tan \frac{x}{2}$.

30. (i) If $0 \le x \le \pi$ and x lies in the IInd quadrant such that $\sin x = \frac{1}{4}$. Find the values of $\cos \frac{x}{2}$, $\sin \frac{x}{2}$ and $\tan \frac{x}{2}$.

(ii) If $\cos x = \frac{4}{5}$ and x is acute, find $\tan 2x$

(iii) If $\sin x = \frac{4}{5}$ and $0 < x < \frac{\pi}{2}$, find the value of $\sin 4x$.

31. If $\tan x = \frac{b}{a}$, then find the value of $\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}}$. [NCERT]

32. If $\tan A = \frac{1}{7}$ and $\tan B = \frac{1}{3}$, show that $\cos 2A = \sin 4B$.

33. Prove that: $\cos 7^\circ \cos 14^\circ \cos 28^\circ \cos 56^\circ = \frac{\sin 68^\circ}{16 \cos 83^\circ}$

BASED ON HOTS

34. Prove that: $\cos \frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15} \cos \frac{16\pi}{15} = \frac{1}{16}$

35. Prove that: $\cos \frac{\pi}{5} \cos \frac{2\pi}{5} \cos \frac{4\pi}{5} \cos \frac{8\pi}{5} = \frac{-1}{16}$

36. Prove that: $\cos \frac{\pi}{65} \cos \frac{2\pi}{65} \cos \frac{4\pi}{65} \cos \frac{8\pi}{65} \cos \frac{16\pi}{65} \cos \frac{32\pi}{65} = \frac{1}{64}$

37. If $2 \tan \alpha = 3 \tan \beta$, prove that $\tan (\alpha - \beta) = \frac{\sin 2\beta}{5 - \cos 2\beta}$.

38. If $\sin \alpha + \sin \beta = a$ and $\cos \alpha + \cos \beta = b$, prove that

(i) $\sin (\alpha + \beta) = \frac{2ab}{a^2 + b^2}$ (ii) $\cos (\alpha - \beta) = \frac{a^2 + b^2 - 2}{2}$

39. If $2 \tan \frac{\alpha}{2} = \tan \frac{\beta}{2}$, prove that $\cos \alpha = \frac{3 + 5 \cos \beta}{5 + 3 \cos \beta}$.

40. If $\cos x = \frac{\cos \alpha + \cos \beta}{1 + \cos \alpha \cos \beta}$, prove that $\tan \frac{x}{2} = \pm \tan \frac{\alpha}{2} \tan \frac{\beta}{2}$

41. If $\sec(x+\alpha)+\sec(x-\alpha)=2\sec x$, prove that $\cos x = \pm\sqrt{2}\cos\dfrac{\alpha}{2}$

42. If $\cos\alpha+\cos\beta=\dfrac{1}{3}$ and $\sin\alpha+\sin\beta=\dfrac{1}{4}$, prove that $\cos\dfrac{\alpha-\beta}{2}=\pm\dfrac{5}{24}$.

43. If $\sin\alpha=\dfrac{4}{5}$ and $\cos\beta=\dfrac{5}{13}$, prove that $\cos\dfrac{\alpha-\beta}{2}=\dfrac{8}{\sqrt{65}}$.

44. If $a\cos 2x+b\sin 2x=c$ has α and β as its roots, then prove that

(i) $\tan\alpha+\tan\beta=\dfrac{2b}{a+c}$ **[NCERT EXEMPLAR]** (ii) $\tan\alpha\tan\beta=\dfrac{c-a}{c+a}$

(iii) $\tan(\alpha+\beta)=\dfrac{b}{a}$ **[NCERT EXEMPLAR]**

45. If $\cos\alpha+\cos\beta=0=\sin\alpha+\sin\beta$, then prove that $\cos 2\alpha+\cos 2\beta=-2\cos(\alpha+\beta)$.

[NCERT EXEMPLAR]

ANSWERS

28. (i) $-\dfrac{1}{\sqrt{5}},\dfrac{2}{\sqrt{5}},\dfrac{24}{25}$ (ii) $-\dfrac{24}{25},\dfrac{2}{\sqrt{5}}$ 29. $\dfrac{1}{\sqrt{6}},\sqrt{\dfrac{5}{6}},\sqrt{5}$

30. (i) $\sqrt{\dfrac{4-\sqrt{15}}{8}},\sqrt{\dfrac{4+\sqrt{15}}{8}},4+\sqrt{15}$ (ii) $\dfrac{24}{7}$ (iii) $-\dfrac{336}{625}$ 31. $\dfrac{2\cos x}{\sqrt{\cos 2x}}$

HINTS TO SELECTED PROBLEMS

25. $$\begin{aligned}\text{LHS} &= \sin 3x+\sin 2x-\sin x = (\sin 3x-\sin x)+\sin 2x\\ &= 2\sin x\cos 2x+\sin 2x = 2\sin x\cos 2x+2\sin x\cos x\\ &= 2\sin x(\cos 2x+\cos x) = 2\sin x\left(2\cos\frac{3x}{2}\cos\frac{x}{2}\right) = 4\sin x\cos\frac{x}{2}\cos\frac{3x}{2} = \text{RHS}\end{aligned}$$

32. Use: $\cos 2A=\dfrac{1-\tan^2 A}{1+\tan^2 A}$, $\sin 4B=\dfrac{2\tan 2B}{1+\tan^2 2B}$, where $\tan 2B=\dfrac{2\tan B}{1-\tan^2 B}$

33. Let $A=7°$. Then, $\cos 7°\cos 14°\cos 28°\cos 56°$

$$=\cos A\cos 2A\cos 2^2A\cos 2^3A=\frac{\sin 2^4 A}{2^4\sin A}=\frac{\sin 16A}{16\sin A}=\frac{\sin 112°}{2\sin 7°}=\frac{\sin 68°}{2\cos 83°}$$

34. Let $A=\dfrac{2\pi}{15}$. Then,

$$\text{LHS}=\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15}=\cos A\cos 2A\cos 2^2A\cos 2^3A$$

$$=\frac{\sin 2^4A}{2^4\sin A}=\frac{\sin 16A}{16\sin A}=\frac{\sin\frac{32\pi}{15}}{16\sin\frac{2\pi}{15}}=\frac{\sin\left(2\pi+\frac{2\pi}{15}\right)}{16\sin\frac{2\pi}{15}}=\frac{\sin\frac{2\pi}{15}}{16\sin\frac{2\pi}{15}}=\frac{1}{16}=\text{RHS}$$

44. We have,

$$a\cos 2x+b\sin 2x=c \quad \ldots\text{(i)}$$

$$\Rightarrow a\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)+\frac{2b\tan x}{1+\tan^2 x}=c\Rightarrow (c+a)\tan^2 x-2b\tan x+(c-a)=0 \quad \ldots\text{(ii)}$$

It is given that α,β are roots of equation (i). Therefore, $\tan\alpha$, $\tan\beta$ are roots of equation (ii).

$\therefore \quad \tan\alpha+\tan\beta=\dfrac{2b}{c+a}$ and, $\tan\alpha\tan\beta=\dfrac{c-a}{c+a}$

Hence, $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\dfrac{2b/c+a}{1-\dfrac{c-a}{c+a}}=\dfrac{b}{a}.$

45. We have, $\cos\alpha+\cos\beta=0$ and $\sin\alpha+\sin\beta=0$

$\therefore \quad (\cos\alpha+\cos\beta)^2-(\sin\alpha+\sin\beta)^2=0^2-0^2$

$\Rightarrow \quad (\cos^2\alpha-\sin^2\alpha)+(\cos^2\beta-\sin^2\beta)+2(\cos\alpha\cos\beta-\sin\alpha\sin\beta)=0$

$\Rightarrow \quad \cos 2\alpha+\cos 2\beta+2\cos(\alpha+\beta)=0 \Rightarrow \cos 2\alpha+\cos 2\beta=-2\cos(\alpha+\beta)$

9.3 VALUES OF TRIGONOMETRIC FUNCTIONS AT $3x$ IN TERMS VALUES AT x

THEOREM *For the values of angle x, for which the two sides are meaningful prove that:*

(i) $\sin 3x = 3\sin x-4\sin^3 x$ (ii) $\cos 3x = 4\cos^3 x-3\cos x$

(iii) $\tan 3x = \dfrac{3\tan x-\tan^3 x}{1-3\tan^2 x}$

PROOF (i) Replacing y by $2x$ in $\sin(x+y) = \sin x\cos y+\cos x\sin y$, we obtain

$\sin(x+2x) = \sin x\cos 2x+\cos x\sin 2x$

$\Rightarrow \quad \sin 3x = \sin x(1-2\sin^2 x)+\cos x(2\sin x\cos x)$

$[\because \cos 2x=1-2\sin^2 x \;\&\; \sin 2x=2\sin x\cos x]$

$\Rightarrow \quad \sin 3x = \sin x-2\sin^3 x+2\sin x(1-\sin^2 x)$

$\Rightarrow \quad \sin 3x = 3\sin x-4\sin^3 x$

Hence, $\sin 3x = 3\sin x-4\sin^3 x$

(ii) Replacing y by $2x$ in $\cos(x+y) = \cos x\cos y-\sin x\sin y$, we obtain

$\cos(x+2x) = \cos x\cos 2x-\sin x\sin 2x$ [Replacing y by $2x$]

$\Rightarrow \quad \cos 3x = \cos x\cos 2x-\sin x(2\sin x\cos x)$ $[\because \sin 2x=2\sin x\cos x]$

$\Rightarrow \quad \cos 3x = \cos x(2\cos^2 x-1)-2\cos x(1-\cos^2 x)$ $[\because \cos 2x=2\cos^2 x-1]$

$\Rightarrow \quad \cos 3x = 4\cos^3 x-3\cos x$

Hence, $\cos 3x = 4\cos^3 x-3\cos x$

(iii) Replacing y by $2x$ in $\tan(x+y) = \dfrac{\tan x+\tan y}{1-\tan x\tan y}$, we obtain

$$\tan(x+2x) = \frac{\tan x+\tan 2x}{1-\tan x\tan 2x} = \frac{\tan x+\dfrac{2\tan x}{1-\tan^2 x}}{1-\tan x\times\dfrac{2\tan x}{1-\tan^2 x}} \quad \left[\because \tan 2x = \frac{2\tan x}{1-\tan^2 x}\right]$$

$$\Rightarrow \quad \tan 3x = \frac{\tan x(1-\tan^2 x)+2\tan x}{1-\tan^2 x-2\tan^2 x} = \frac{3\tan x-\tan^3 x}{1-3\tan^2 x}$$

Hence, $\tan 3x = \dfrac{3\tan x-\tan^3 x}{1-3\tan^2 x}$

Q.E.D.

NOTE *It should be noted the angle on the RHS of these formulae is one third of the angle on LHS.*

$\therefore\ \sin\frac{\pi}{3}=3\sin\frac{\pi}{9}-4\sin^3\frac{\pi}{9},\ \sin\frac{\pi}{6}=3\sin\frac{\pi}{18}-4\sin^3\frac{\pi}{18},\ \cos\frac{2\pi}{3}=4\cos^3\frac{2\pi}{9}-3\cos\frac{2\pi}{9}$ etc.

9.4 VALUES OF TRIGONOMETRIC FUNCTIONS AT x IN TERMS OF VALUES AT $\frac{x}{3}$

Replacing x by $x/3$ in the formulas in the previous section, we obtain the following formulae:

(i) $\sin x=3\sin\frac{x}{3}-4\sin^3\frac{x}{3}$ (ii) $\cos x=4\cos^3\frac{x}{3}-3\cos\frac{x}{3}$ (iii) $\tan x=\dfrac{3\tan\frac{x}{3}-\tan^3\frac{x}{3}}{1-3\tan^2\frac{x}{3}}$

ILLUSTRATIVE EXAMPLES

BASIC

EXAMPLE 1 *Prove that :* $8\cos^3\frac{\pi}{9}-6\cos\frac{\pi}{9}=1$

SOLUTION $\text{LHS}=2\left(4\cos^3\frac{\pi}{9}-3\cos\frac{\pi}{9}\right)=2\cos\left(3\times\frac{\pi}{9}\right)=2\cos\frac{\pi}{3}=1=\text{RHS}$

EXAMPLE 2 *Prove that:* $108\sin\frac{\pi}{18}-144\sin^3\frac{\pi}{18}=18$

SOLUTION $\text{LHS}=108\sin\frac{\pi}{18}-144\sin^3\frac{\pi}{18}$

$$=36\left(3\sin\frac{\pi}{18}-4\sin^3\frac{\pi}{18}\right)=36\sin\left(3\times\frac{\pi}{18}\right)=36\sin\frac{\pi}{6}=36\times\frac{1}{2}=18=\text{RHS}$$

EXAMPLE 3 *Prove that:* $15\sin\frac{5\pi}{12}+15\cos\frac{5\pi}{12}-20\sin^3\frac{5\pi}{12}-20\cos^3\frac{5\pi}{12}=0$

SOLUTION $\text{LHS}=15\sin\frac{5\pi}{12}+15\cos\frac{5\pi}{12}-20\sin^3\frac{5\pi}{12}-20\cos^3\frac{5\pi}{12}$

$$=\left(15\sin\frac{5\pi}{12}-20\sin^3\frac{5\pi}{12}\right)-\left(20\cos^3\frac{5\pi}{12}-15\cos\frac{5\pi}{12}\right)$$

$$=5\left(3\sin\frac{5\pi}{12}-4\sin^3\frac{5\pi}{12}\right)-5\left(4\cos^3\frac{5\pi}{12}-3\cos\frac{5\pi}{12}\right)$$

$$=5\sin\left(3\times\frac{5\pi}{12}\right)-5\cos\left(3\times\frac{5\pi}{12}\right)=5\sin\frac{5\pi}{4}-5\cos\frac{5\pi}{4}=-5\sin\frac{\pi}{4}+5\cos\frac{\pi}{4}=0$$

BASED ON LOTS

EXAMPLE 4 *Prove that:* $\cos 6x=32\cos^6 x-48\cos^4 x+18\cos^2 x-1$

SOLUTION $\text{LHS}=\cos 6x=2\cos^2 3x-1$ $[\because \cos 2x=2\cos^2 x-1]$

$$=2(4\cos^3 x-3\cos x)^2-1=2(16\cos^6 x+9\cos^2 x-24\cos^4 x)-1$$

$$=32\cos^6 x-48\cos^4 x+18\cos^2 x-1=\text{RHS}$$

EXAMPLE 5 *Prove that:* $\cos x\cos\left(\frac{\pi}{3}-x\right)\cos\left(\frac{\pi}{3}+x\right)=\frac{1}{4}\cos 3x$

SOLUTION $\text{LHS}=\cos x\cos\left(\frac{\pi}{3}-x\right)\cos\left(\frac{\pi}{3}+x\right)$

$$= \cos x\left(\cos^2\frac{\pi}{3}-\sin^2 x\right) \qquad [\because \cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B]$$

$$= \cos x\left(\frac{1}{4}-\sin^2 x\right)=\cos x\left\{\frac{1}{4}-(1-\cos^2 x)\right\}=\cos x\left(-\frac{3}{4}+\cos^2 x\right)$$

$$= \frac{1}{4}\cos x(-3+4\cos^2 x) = \frac{1}{4}(4\cos^3 x-3\cos x) = \frac{1}{4}\cos 3x = \text{RHS}$$

EXAMPLE 6 *Prove that:* $\sin x \sin\left(\frac{\pi}{3}-x\right)\sin\left(\frac{\pi}{3}+x\right)=\frac{1}{4}\sin 3x$

SOLUTION LHS $= \sin x\sin\left(\frac{\pi}{3}-x\right)\sin\left(\frac{\pi}{3}+x\right)$

$$= \sin x\left(\sin^2\frac{\pi}{3}-\sin^2 x\right) \qquad [\because \sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B]$$

$$= \sin x\left(\frac{3}{4}-\sin^2 x\right)=\frac{1}{4}\sin x(3-4\sin^2 x)=\frac{1}{4}(3\sin x-4\sin^3 x)=\frac{1}{4}\sin 3x=\text{RHS}$$

NOTE *Reader is advised to learn the results derived in the above two examples as standard results. The following example is an application of the above results.*

EXAMPLE 7 *Prove that:* $\sin 20°\sin 40°\sin 60°\sin 80°=\frac{3}{16}$

SOLUTION LHS $= \frac{\sqrt{3}}{2}\left\{\sin 20°\sin(60°-20°)\sin(60+20°)\right\}$

$$= \frac{\sqrt{3}}{2}\left\{(\sin x\sin\left(\frac{\pi}{3}-x\right)\sin\left(\frac{\pi}{3}+x\right)\right\}, \text{ where } x=20°$$

$$= \frac{\sqrt{3}}{2}\times\frac{1}{4}\sin 3x=\frac{\sqrt{3}}{8}\times\sin\frac{\pi}{3}=\frac{\sqrt{3}}{8}\times\frac{\sqrt{3}}{2}=\frac{3}{16}=\text{RHS}$$

EXAMPLE 8 *Prove that:*

(i) $\tan x+\tan\left(\frac{\pi}{3}+x\right)-\tan\left(\frac{\pi}{3}-x\right)=3\tan 3x$

(ii) $\cot x+\cot\left(\frac{\pi}{3}+x\right)-\cot\left(\frac{\pi}{3}-x\right)=3\cot 3x$

SOLUTION (i) LHS $=\tan x+\tan\left(\frac{\pi}{3}+x\right)-\tan\left(\frac{\pi}{3}-x\right)=\tan x+\frac{\sqrt{3}+\tan x}{1-\sqrt{3}\tan x}-\frac{\sqrt{3}-\tan x}{1+\sqrt{3}\tan x}$

$$= \tan x+\frac{(\sqrt{3}+\tan x)(1+\sqrt{3}\tan x)-(\sqrt{3}-\tan x)(1-\sqrt{3}\tan x)}{(1-\sqrt{3}\tan x)(1+\sqrt{3}\tan x)}$$

$$= \tan x+\frac{8\tan x}{1-3\tan^2 x}=\frac{9\tan x-3\tan^3 x}{1-3\tan^2 x}=3\left(\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}\right)=3\tan 3x=\text{RHS}$$

(ii) LHS $=\cot x+\cot\left(\frac{\pi}{3}+x\right)-\cot\left(\frac{\pi}{3}-x\right)$

$$= \frac{1}{\tan x}+\frac{1}{\tan\left(\frac{\pi}{3}+x\right)}-\frac{1}{\tan\left(\frac{\pi}{3}-x\right)}=\frac{1}{\tan x}+\frac{1-\sqrt{3}\tan x}{\sqrt{3}+\tan x}-\frac{1+\sqrt{3}\tan x}{\sqrt{3}-\tan x}$$

$$= \frac{1}{\tan x}+\frac{(1-\sqrt{3}\tan x)(\sqrt{3}-\tan x)-(1+\sqrt{3}\tan x)(\sqrt{3}+\tan x)}{(\sqrt{3}+\tan x)(\sqrt{3}-\tan x)}$$

$$= \frac{1}{\tan x} - \frac{8 \tan x}{3 - \tan^2 x} = \frac{3 - 9 \tan^2 x}{3 \tan x - \tan^3 x} = 3\left(\frac{1 - 3 \tan^2 x}{3 \tan x - \tan^3 x}\right) = \frac{3}{\tan 3x} = 3 \cot 3x = \text{RHS}$$

BASED ON HOTS

EXAMPLE 9 *If* $\cos \alpha + \cos \beta + \cos \gamma = 0$, *then prove that*

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 12 \cos \alpha \cos \beta \cos \gamma.$$

SOLUTION $\cos 3\alpha + \cos 3\beta + \cos 3\gamma$

$$= (4 \cos^3 \alpha - 3 \cos \alpha) + (4 \cos^3 \beta - 3 \cos \beta) + (4 \cos^3 \gamma - 3 \cos \gamma)$$

$$= 4 (\cos^3 \alpha + \cos^3 \beta + \cos^3 \gamma) - 3 (\cos \alpha + \cos \beta + \cos \gamma)$$

$$= 4 (\cos^3 \alpha + \cos^3 \beta + \cos^3 \gamma) - 3 \times 0 \qquad [\because \cos\alpha + \cos\beta + \cos\gamma = 0]$$

$$= 4 \times 3 \cos \alpha \cos \beta \cos \gamma = 12 \cos \alpha \cos \beta \cos \gamma \quad [\because a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc]$$

EXAMPLE 10 *Prove that:* $\sin 3x \sin^3 x + \cos 3x \cos^3 x = \cos^3 2x$

SOLUTION We know that

$$\sin 3x = 3 \sin x - 4 \sin^3 x \Rightarrow \sin^3 x = \frac{3 \sin x - \sin 3x}{4}$$

Similarly, $\cos 3x = 4 \cos^3 x - 3 \cos x \Rightarrow \cos^3 x = \dfrac{\cos 3x + 3 \cos x}{4}$

$$\therefore \quad \text{LHS} = \sin 3x \sin^3 x + \cos 3x \cos^3 x$$

$$\Rightarrow \quad \text{LHS} = \sin 3x \left\{\frac{3 \sin x - \sin 3x}{4}\right\} + \cos 3x \left\{\frac{\cos 3x + 3 \cos x}{4}\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4}\left\{3 (\cos x \cos 3x + \sin x \sin 3x) + (\cos^2 3x - \sin^2 3x)\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{4}\left\{3 \cos (3x - x) + \cos 2 (3x)\right\} = \frac{1}{4}\left\{3 \cos 2x + \cos 3 (2x)\right\}$$

$$= \frac{1}{4}\left\{3 \cos 2x + (4 \cos^3 2x - 3 \cos 2x)\right\} = \cos^3 2x = \text{RHS}$$

EXAMPLE 11 *Prove that:* $\cos^3 x + \cos^3\left(\dfrac{2\pi}{3} + x\right) + \cos^3\left(\dfrac{4\pi}{3} + x\right) = \dfrac{3}{4} \cos 3x$

SOLUTION We know that $\cos 3x = 4 \cos^3 x - 3 \cos x$. Therefore, $\cos^3 x = \dfrac{1}{4}(\cos 3x + 3 \cos x)$.

Using this, we obtain

$$\text{LHS} = \frac{1}{4}\left\{\cos 3x + 3 \cos x\right\} + \frac{1}{4}\left\{\cos (2\pi + 3x) + 3 \cos\left(\frac{2\pi}{3} + x\right)\right\}$$

$$+ \frac{1}{4}\left\{\cos (4\pi + 3x) + 3 \cos\left(\frac{4\pi}{3} + x\right)\right\}$$

$$= \frac{1}{4}\left\{\cos 3x + 3 \cos x\right\} + \frac{1}{4}\left\{\cos 3x + 3 \cos\left(\frac{2\pi}{3} + x\right)\right\} + \frac{1}{4}\left\{\cos 3x + 3 \cos\left(\frac{4\pi}{3} + x\right)\right\}$$

$$= \frac{3}{4} \cos 3x + \frac{3}{4}\left\{\cos x + \cos\left(\frac{2\pi}{3} + x\right) + \cos\left(\frac{4\pi}{3} + x\right)\right\}$$

$$= \frac{3}{4} \cos 3x + \frac{3}{4}\left\{\cos x + 2 \cos (\pi + x) \cos \frac{\pi}{3}\right\}$$

$$= \frac{3}{4} \cos 3x + \frac{3}{4}\left\{\cos x - 2 \cos x \times \frac{1}{2}\right\} = \frac{3}{4} \cos 3x = \text{RHS}$$

ALITER $\cos x + \cos\left(\frac{2\pi}{3}+x\right) + \cos\left(\frac{4\pi}{3}+x\right)$

$$= \cos x + 2\cos\left(\frac{\frac{4\pi}{3}+x+\frac{2\pi}{3}+x}{2}\right)\cos\left(\frac{\frac{4\pi}{3}+x-\frac{2\pi}{3}-x}{2}\right)$$

$$= \cos x + 2\cos(\pi+x)\cos\frac{\pi}{3} = \cos x - 2(\cos x)\times\frac{1}{2} = \cos x - \cos x = 0$$

$\therefore$ $\cos^3 x + \cos^3\left(\frac{2\pi}{3}+x\right) + \cos^3\left(\frac{4\pi}{3}+x\right)$

$$= 3\cos x\cos\left(\frac{2\pi}{3}+x\right)\cos\left(\frac{4\pi}{3}+x\right) \qquad [\because a+b+c=0 \Rightarrow a^3+b^3+c^3=3abc]$$

$$= 3\cos x\cos\left(\pi-\frac{\pi}{3}+x\right)\cos\left(\pi+\frac{\pi}{3}+x\right)$$

$$= 3\cos x\cos\left\{\pi-\left(\frac{\pi}{3}-x\right)\right\}\cos\left\{\pi+\left(\frac{\pi}{3}+x\right)\right\}$$

$$= (3\cos x)\left\{-\cos\left(\frac{\pi}{3}-x\right)\right\}\left\{-\cos\left(\frac{\pi}{3}+x\right)\right\}$$

$$= 3\cos x\cos\left(\frac{\pi}{3}-x\right)\cos\left(\frac{\pi}{3}+x\right) = 3\times\frac{1}{4}\cos 3x = \frac{3}{4}\cos 3x$$

EXAMPLE 12 *Prove that* $\frac{\tan 3x}{\tan x}$ *never lies between* $\frac{1}{3}$ *and 3.*

SOLUTION Let $y = \frac{\tan 3x}{\tan x}$. Then,

$$y = \frac{3\tan x - \tan^3 x}{\tan x(1-3\tan^2 x)} \Rightarrow y = \frac{3-\tan^2 x}{1-3\tan^2 x} \Rightarrow (3y-1)\tan^2 x = y-3 \Rightarrow \tan^2 x = \frac{y-3}{3y-1}$$

But, $\tan^2 x \geq 0$ for all x

Fig. 9.1 Signs of $\frac{y-3}{3y-1}$ for different values of y

$\therefore$ $\frac{y-3}{3y-1} \geq 0$

$\Rightarrow$ $y < \frac{1}{3}$ or, $y \geq 3$

$\Rightarrow$ y does not lie between 1/3 and 3.

Hence, $\frac{\tan 3x}{\tan x}$ never lies between $\frac{1}{3}$ and 3.

EXAMPLE 13 *Prove that:* $\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$

SOLUTION $\cos 5x = \cos(3x+2x) = \cos 3x\cos 2x - \sin 3x\sin 2x$

$\Rightarrow$ $\cos 5x = (4\cos^3 x - 3\cos x)(2\cos^2 x - 1) - (3\sin x - 4\sin^3 x)(2\sin x\cos x)$

$\Rightarrow$ $\cos 5x = (4\cos^3 x - 3\cos x)(2\cos^2 x - 1) - (3 - 4\sin^2 x)(2\sin^2 x\cos x)$

$\Rightarrow$ $\cos 5x = (4\cos^3 x - 3\cos x)(2\cos^2 x - 1) - \{3 - 4(1-\cos^2 x)\}2(1-\cos^2 x)\cos x$

$\Rightarrow$ $\cos 5x = (8\cos^5 x - 10\cos^3 x + 3\cos x) - 2\cos x(1-\cos^2 x)(4\cos^2 x - 1)$

$\Rightarrow$ $\cos 5x = (8\cos^5 x - 10\cos^3 x + 3\cos x) - 2\cos x(5\cos^2 x - 4\cos^4 x - 1)$

$\Rightarrow$ $\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$

EXERCISE 9.2

BASIC

Prove that:

1. $\sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x$
2. $4(\cos^3 10° + \sin^3 20°) = 3(\cos 10° + \sin 20°)$
3. $\cos^3 x \sin 3x + \sin^3 x \cos 3x = \frac{3}{4}\sin 4x$

BASED ON LOTS

4. $\tan x \tan\left(x+\frac{\pi}{3}\right) + \tan x \tan\left(x-\frac{\pi}{3}\right) + \tan\left(x+\frac{\pi}{3}\right)\tan\left(x-\frac{\pi}{3}\right) = -3$
5. $\tan x + \tan\left(\frac{\pi}{3}+x\right) - \tan\left(\frac{\pi}{3}-x\right) = 3\tan 3x$
6. $\cot x + \cot\left(\frac{\pi}{3}+x\right) - \cot\left(\frac{\pi}{3}-x\right) = 3\cot 3x$
7. $\cot x + \cot\left(\frac{\pi}{3}+x\right) + \cot\left(\frac{2\pi}{3}+x\right) = 3\cot 3x$

BASED ON HOTS

8. $\sin 5x = 5\cos^4 x \sin x - 10\cos^2 x \sin^3 x + \sin^5 x$
9. $\sin^3 x + \sin^3\left(\frac{2\pi}{3}+x\right) + \sin^3\left(\frac{4\pi}{3}+x\right) = -\frac{3}{4}\sin 3x.$
10. $\left|\sin x \sin\left(\frac{\pi}{3}-x\right)\sin\left(\frac{\pi}{3}+x\right)\right| \le \frac{1}{4}$ for all values of x.
11. $\left|\cos x \cos\left(\frac{\pi}{3}-x\right)\cos\left(\frac{\pi}{3}+x\right)\right| \le \frac{1}{4}$ for all values of x.

9.5 VALUES OF TRIGONOMETRICAL FUNCTIONS AT SOME IMPORTANT POINTS

By using the formulae introduced in the previous sections we can now find the values of trigonometrical functions at some important points like $\frac{\pi}{10}, \frac{\pi}{5}, \frac{3\pi}{10}$ etc.

THEOREM 1 *Prove that:* $\sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$.

PROOF Let $x = \frac{\pi}{10}$. Then,

$$5x = \frac{\pi}{2} \Rightarrow 2x + 3x = \frac{\pi}{2} \Rightarrow 2x = \frac{\pi}{2} - 3x \Rightarrow \sin 2x = \sin\left(\frac{\pi}{2} - 3x\right) \Rightarrow \sin 2x = \cos 3x$$

$$\Rightarrow \quad 2\sin x\cos x = 4\cos^3 x - 3\cos x \Rightarrow \cos x(2\sin x - 4\cos^2 x + 3) = 0$$

$$\Rightarrow \quad 2\sin x - 4\cos^2 x + 3 = 0 \qquad \left[\because \cos x = \cos\frac{\pi}{10} \ne 0\right]$$

$$\Rightarrow \quad 2\sin x - 4(1-\sin^2 x) + 3 = 0 \Rightarrow 4\sin^2 x + 2\sin x - 1 = 0$$

$$\Rightarrow \quad \sin x = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

$$\Rightarrow \quad \sin x = \frac{-1+\sqrt{5}}{4} = \frac{\sqrt{5}-1}{4} \qquad [\because x \text{ lies in Ist quadrant } \therefore \sin x > 0]$$

Hence, $\sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$

Q.E.D.

THEOREM 2 *Prove that:* $\cos\frac{\pi}{10} = \frac{\sqrt{10+2\sqrt{5}}}{4}$.

PROOF Putting $x = \frac{\pi}{10}$ in $\cos x = \sqrt{1-\sin^2 x}$, we get

$$\cos\frac{\pi}{10} = \sqrt{1-\sin^2\frac{\pi}{10}} = \sqrt{1-\left(\frac{\sqrt{5}-1}{4}\right)^2} = \sqrt{\frac{16-(5+1-2\sqrt{5})}{16}} = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

Q.E.D.

REMARK *The complement of* $\frac{\pi}{10}$ is $\frac{2\pi}{5}$.

$$\therefore \sin\frac{2\pi}{5} = \sin\left(\frac{\pi}{2}-\frac{\pi}{10}\right) = \cos\frac{\pi}{10} = \frac{\sqrt{10+2\sqrt{5}}}{4} \text{ and, } \cos\frac{2\pi}{5} = \cos\left(\frac{\pi}{2}-\frac{\pi}{10}\right) = \sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$$

The values of the remaining trigonometrical functions at $\frac{\pi}{10}$ *may be obtained from the above values.*

THEOREM 3 *Prove that:* $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$.

PROOF We have, $\cos 2x = 1-2\sin^2 x$. Putting $x = \frac{\pi}{10}$, we obtain

$$\cos\frac{\pi}{5} = 1-2\sin^2\frac{\pi}{10} = 1-2\left(\frac{\sqrt{5}-1}{4}\right)^2 = 1-2\left(\frac{6-2\sqrt{5}}{16}\right) = 1-\left(\frac{3-\sqrt{5}}{4}\right) = \frac{\sqrt{5}+1}{4}$$

Q.E.D.

THEOREM 4 *Prove that:* $\sin\frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4}$.

PROOF Putting $x = \frac{\pi}{5}$ in $\sin x = \sqrt{1-\cos^2 x}$, we obtain

$$\sin\frac{\pi}{5} = \sqrt{1-\cos^2\frac{\pi}{5}} = \sqrt{1-\left(\frac{\sqrt{5}+1}{4}\right)^2} = \sqrt{\frac{16-(6+2\sqrt{5})}{16}} = \frac{\sqrt{10-2\sqrt{5}}}{4}$$

Q.E.D.

REMARK *The complement of* $\frac{\pi}{5}$ *is* $\frac{3\pi}{10}$.

$$\therefore \quad \sin\frac{3\pi}{10} = \sin\left(\frac{\pi}{2}-\frac{\pi}{5}\right) = \cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4} \text{ and, } \cos\frac{3\pi}{10} = \cos\left(\frac{\pi}{2}-\frac{\pi}{5}\right) = \sin\frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4}$$

The other trigonometrical ratios of $\frac{\pi}{5}$ *may be obtained from the above values.*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Prove that:*

(i) $\sin^2\frac{2\pi}{5} - \sin^2\frac{\pi}{3} = \frac{\sqrt{5}-1}{8}$ (ii) $\cos^2\frac{4\pi}{15} - \sin^2\frac{\pi}{15} = \frac{\sqrt{5}+1}{8}$

(iii) $\sin\frac{\pi}{10} + \sin\frac{13\pi}{10} = -\frac{1}{2}$ (iv) $\sin\frac{\pi}{10}\sin\frac{13\pi}{10} = -\frac{1}{4}$

(v) $\sin^2 24° - \sin^2 6° = \frac{\sqrt{5}-1}{8}$

SOLUTION (i) LHS $= \sin^2\frac{2\pi}{5} - \sin^2\frac{\pi}{3} = \cos^2\frac{\pi}{10} - \sin^2\frac{\pi}{3}$ $\left[\because \sin\frac{\pi}{5} = \cos\frac{\pi}{10}\right]$

$= \left\{\frac{\sqrt{10+2\sqrt{5}}}{4}\right\}^2 - \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{10+2\sqrt{5}}{16} - \frac{3}{4} = \frac{2\sqrt{5}-2}{16} = \frac{\sqrt{5}-1}{8} = \text{RHS}$

(ii) LHS $= \cos^2\frac{4\pi}{15} - \sin^2\frac{\pi}{15}$

$= \cos\left(\frac{4\pi}{15}+\frac{\pi}{15}\right)\cos\left(\frac{4\pi}{15}-\frac{\pi}{15}\right)$ $[\because \cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B)]$

$= \cos\frac{\pi}{3}\cos\frac{\pi}{5} = \frac{1}{2}\times\frac{\sqrt{5}+1}{4} = \frac{\sqrt{5}+1}{8} = \text{RHS}$

(iii) LHS $= \sin\frac{\pi}{10} + \sin\frac{13\pi}{10}$

$= \sin\frac{\pi}{10} + \sin\left(\frac{3\pi}{2}-\frac{\pi}{5}\right) = \sin\frac{\pi}{10} - \cos\frac{\pi}{5} = \frac{\sqrt{5}-1}{4} - \frac{\sqrt{5}+1}{4} = -\frac{1}{2} = \text{RHS}$

(iv) LHS $= \sin\frac{\pi}{10}\sin\frac{13\pi}{10} = \sin\frac{\pi}{10}\sin\left(\frac{3\pi}{2}-\frac{\pi}{5}\right)$

$= -\sin\frac{\pi}{10}\cos\frac{\pi}{5} = -\frac{\sqrt{5}-1}{4}\times\frac{\sqrt{5}+1}{4} = -\left(\frac{5-1}{16}\right) = -\frac{1}{4} = \text{RHS}$

(v) LHS $= \sin^2 24° - \sin^2 6°$

$= \sin(24°+6°)\sin(24°-6°)$ $[\because \sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B]$

$= \sin 30° \sin 18° = \sin\frac{\pi}{6}\sin\frac{\pi}{10} = \frac{1}{2}\times\frac{\sqrt{5}-1}{4} = \frac{\sqrt{5}-1}{8} = \text{RHS}$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 2 *Prove that:* $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$

SOLUTION If $x + y = \pi$, then $x = \pi - y \Rightarrow \sin x = \sin(\pi - y) \Rightarrow \sin x = \sin y$

$\therefore \frac{\pi}{5} + \frac{4\pi}{5} = \pi \Rightarrow \sin\frac{\pi}{5} = \sin\frac{4\pi}{5}$ and $\frac{2\pi}{5} + \frac{3\pi}{5} = \pi \Rightarrow \sin\frac{2\pi}{5} = \sin\frac{3\pi}{5}$

Using these values, we obtain

LHS $= \sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5}$

$= \sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{2\pi}{5}\sin\frac{\pi}{5}$ $\left[\because \sin\frac{3\pi}{5} = \sin\frac{2\pi}{5}, \sin\frac{4\pi}{5} = \sin\frac{\pi}{5}\right]$

$= \left(\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\right)^2 = \left\{\sin\frac{\pi}{5}\sin\left(\frac{\pi}{2}-\frac{\pi}{10}\right)\right\}^2 = \left\{\sin\frac{\pi}{5}\cos\frac{\pi}{10}\right\}^2$

$$= \left\{ \frac{\sqrt{10-2\sqrt{5}}}{4} \times \frac{\sqrt{10+2\sqrt{5}}}{4} \right\}^2 = \frac{10-2\sqrt{5}}{16} \times \frac{10+2\sqrt{5}}{16} = \frac{100-20}{256} = \frac{80}{256} = \frac{5}{16} = \text{RHS}$$

EXAMPLE 3 *Prove that:* $16 \cos \frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15} \cos \frac{14\pi}{15} = 1$

SOLUTION LHS $= 16 \cos \frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15} \cos \frac{14\pi}{15} = 4\left(2 \cos \frac{2\pi}{15} \cos \frac{4\pi}{15}\right)\left(2 \cos \frac{4\pi}{15} \cos \frac{14\pi}{15}\right)$

$$= 4\left(\cos \frac{2\pi}{3} + \cos \frac{2\pi}{5}\right)\left(\cos \frac{6\pi}{5} + \cos \frac{2\pi}{3}\right) = 4\left(-\sin \frac{\pi}{6} + \sin \frac{\pi}{10}\right)\left(-\cos \frac{\pi}{5} - \sin \frac{\pi}{6}\right)$$

$$= 4\left(-\frac{1}{2} + \frac{\sqrt{5}-1}{4}\right)\left(-\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right) = 4\left(\frac{\sqrt{5}-3}{4}\right)\left(\frac{-\sqrt{5}-3}{4}\right) = 4\left(\frac{3-\sqrt{5}}{4}\right)\left(\frac{3+\sqrt{5}}{4}\right)$$

$$= \left(\frac{9-5}{4}\right) = 1 = \text{RHS}$$

EXAMPLE 4 *Prove that:* $\sin \frac{\pi}{15} \sin \frac{4\pi}{15} \sin \frac{3\pi}{10} = \frac{1}{8}$.

SOLUTION LHS $= \frac{1}{2}\left(2 \sin \frac{4\pi}{15} \sin \frac{\pi}{15}\right) \sin \frac{3\pi}{10}$

$$= \frac{1}{2}\left(\cos \frac{\pi}{5} - \cos \frac{\pi}{3}\right) \cos \frac{\pi}{5} \qquad [\because\ 2 \sin A \sin B = \cos (A-B) - \cos (A+B)]$$

$$= \frac{1}{2}\left(\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right)\left(\frac{\sqrt{5}+1}{4}\right) = \frac{1}{2}\left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4}\right) = \frac{1}{8} = \text{RHS}$$

EXAMPLE 5 *Prove that:* $\left(1 + \cos \frac{\pi}{10}\right)\left(1 + \cos \frac{3\pi}{10}\right)\left(1 + \cos \frac{7\pi}{10}\right)\left(1 + \cos \frac{9\pi}{10}\right) = \frac{1}{16}$.

SOLUTION If $A + B = \pi$, then $\cos A = \cos (\pi - B) = -\cos B$

$\therefore \quad \frac{\pi}{10} + \frac{9\pi}{10} = \pi \Rightarrow \cos \frac{9\pi}{10} = -\cos \frac{\pi}{10}$ and, $\frac{3\pi}{10} + \frac{7\pi}{10} = \pi \Rightarrow \cos \frac{7\pi}{10} = -\cos \frac{3\pi}{10}$

Using these values, we obtain

$$\text{LHS} = \left(1 + \cos \frac{\pi}{10}\right)\left(1 + \cos \frac{3\pi}{10}\right)\left(1 - \cos \frac{3\pi}{10}\right)\left(1 - \cos \frac{\pi}{10}\right)$$

$$= \left(1 - \cos^2 \frac{\pi}{10}\right)\left(1 - \cos^2 \frac{3\pi}{10}\right) = \sin^2 \frac{\pi}{10} \sin^2 \frac{3\pi}{10} = \sin^2 \frac{\pi}{10} \sin^2 \left(\frac{\pi}{2} - \frac{2\pi}{10}\right)$$

$$= \sin^2 \frac{\pi}{10} \cos^2 \frac{2\pi}{10} = \left(\sin \frac{\pi}{10} \cos \frac{\pi}{5}\right)^2 = \left(\frac{\sqrt{5}-1}{4} \times \frac{\sqrt{5}+1}{4}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16} = \text{RHS}$$

EXAMPLE 6 *Prove that:* $\tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ = 1$.

SOLUTION LHS $= \frac{\sin 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ}{\cos 6^\circ \cos 42^\circ \cos 66^\circ \cos 78^\circ} = \frac{(2 \sin 66^\circ \sin 6^\circ)(2 \sin 78^\circ \sin 42^\circ)}{(2 \cos 66^\circ \cos 6^\circ)(2 \cos 78^\circ \cos 42^\circ)}$

$$= \frac{(\cos 60^\circ - \cos 72^\circ)(\cos 36^\circ - \cos 120^\circ)}{(\cos 60^\circ + \cos 72^\circ)(\cos 36^\circ + \cos 120^\circ)}$$

$$= \frac{(\cos 60^\circ - \sin 18^\circ)(\cos 36^\circ + \sin 30^\circ)}{(\cos 60^\circ + \sin 18^\circ)(\cos 36^\circ - \sin 30^\circ)}$$

$$= \frac{\left(\frac{1}{2} - \frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4} + \frac{1}{2}\right)}{\left(\frac{1}{2} + \frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right)} = \frac{(3-\sqrt{5})(3+\sqrt{5})}{(\sqrt{5}+1)(\sqrt{5}-1)} = \frac{9-5}{5-1} = 1 = \text{RHS}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 7 *Prove that* : $4 \sin 27° = \sqrt{(5+\sqrt{5})} - \sqrt{(3-\sqrt{5})}$.

SOLUTION $16 \sin^2 27° = 8(2\sin^2 27°) = 8\left(1 - \cos\frac{3\pi}{10}\right) = 8\left(1 - \sin\frac{\pi}{5}\right)$

$$= 8\left\{1 - \frac{\sqrt{10-2\sqrt{5}}}{4}\right\} = 2\left\{4 - \sqrt{10-2\sqrt{5}}\right\} = 8 - 2\sqrt{10-2\sqrt{5}}$$

$$= (5+\sqrt{5}) + (3-\sqrt{5}) - 2\sqrt{(5+\sqrt{5})(3-\sqrt{5})}$$

$$= \left\{\sqrt{5+\sqrt{5}}\right\}^2 + \left\{\sqrt{3-\sqrt{5}}\right\}^2 - 2\sqrt{(5+\sqrt{5})(3-\sqrt{5})}$$

$$= \left\{\sqrt{5+\sqrt{5}} - \sqrt{3-\sqrt{5}}\right\}^2$$

Taking square roots of both sides, we obtain

$\therefore \quad 4\sin 27° = \sqrt{5+\sqrt{5}} - \sqrt{3-\sqrt{5}}$ [$\because$ $\sin 27°$ is positive]

EXAMPLE 8 *Find the value of* $\tan 9° - \tan 27° - \tan 63° + \tan 81°$. [NCERT EXEMPLAR]

SOLUTION $\tan 9° - \tan 27° - \tan 63° + \tan 81° = (\tan 9° + \tan 81°) - (\tan 27° + \tan 63°)$

$= (\tan 9° + \cot 9°) - (\tan 27° + \cot 27°)$ $\left[\because \begin{array}{l} \tan 81° = \tan(90°-9°) = \cot 9°, \\ \tan 63° = \tan(90°-27°) = \cot 27° \end{array}\right]$

$= \frac{1}{\sin 9° \cos 9°} - \frac{1}{\sin 27° \cos 27°}$ $\left[\because \tan x + \cot x = \frac{1}{\sin x \cos x}\right]$

$$= \frac{2}{\sin 18°} - \frac{2}{\sin 54°} = \frac{2}{\sin 18°} - \frac{2}{\cos 36°} = \frac{8}{\sqrt{5}-1} - \frac{8}{\sqrt{5}+1} = \frac{8\times 2}{5-1} = 4$$

EXERCISE 9.3

BASIC

Prove that:

1. $\sin^2\frac{2\pi}{5} - \sin^2\frac{\pi}{3} = \frac{\sqrt{5}-1}{8}$

2. $\sin^2 24° - \sin^2 6° = \frac{\sqrt{5}-1}{8}$

3. $\sin^2 42° - \cos^2 78° = \frac{\sqrt{5}+1}{8}$

BASED ON LOTS

4. $\cos 78° \cos 42° \cos 36° = \frac{1}{8}$

5. $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{16}$

6. $\cos 6° \cos 42° \cos 66° \cos 78° = \frac{1}{16}$

7. $\sin 6° \sin 42° \sin 66° \sin 78° = \frac{1}{16}$

8. $\cos 36° \cos 42° \cos 60° \cos 78° = \frac{1}{16}$

BASED ON HOTS

9. $\sin \frac{\pi}{5} \sin \frac{2\pi}{5} \sin \frac{3\pi}{5} \sin \frac{4\pi}{5} = \frac{5}{16}$

10. $\cos \frac{\pi}{15} \cos \frac{2\pi}{15} \cos \frac{3\pi}{15} \cos \frac{4\pi}{15} \cos \frac{5\pi}{15} \cos \frac{6\pi}{15} \cos \frac{7\pi}{15} = \frac{1}{128}$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. If $\dfrac{1-\tan^2\left(\frac{\pi}{4}-x\right)}{1+\tan^2\left(\frac{\pi}{4}-x\right)} = \sin kx$, then $k =$ ....

2. If $\cos x \cos 2x \cos 2^2 x \ldots\ldots \cos 2^{n-1} x = \lambda \dfrac{\sin 2^n \lambda}{\sin x}$, then $\lambda =$ ....

3. The value of $\cos\frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15} \cos\frac{16\pi}{15}$ is ....

4. If $\tan x = \dfrac{1-\cos y}{\sin y}$, then $\tan 2x =$ ....

5. If $k = \sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}$, then the numerical value of k is ....

6. In a triangle ABC with $\angle C = \frac{\pi}{2}$ the equation whose roots are $\tan A$ and $\tan B$ is ....

7. The value of $\cos^2 48° - \sin^2 12°$ is ....

8. The least value of $2\sin^2\theta + 3\cos^2\theta$ is ....

9. If $\cos^6 x + \sin^6 x + k\sin^2 2x = 1$, then $k =$.............................. .

10. The value of $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$ is ....

11. The value of $\dfrac{\cot x - \tan x}{\cot 2x}$ is ....

12. If $\tan\theta = t$, then $\tan 2\theta + \sec 2\theta =$ ....

13. If $\tan\theta = \frac{a}{b}$, then $a\sin 2\theta + b\cos 2\theta$ is equal to ....

14. If $\tan x = \frac{1}{7}$, $\tan y = \frac{1}{3}$ and $\cos 2x = \sin ky$, then $k =$ ....

15. The value of $\cos\frac{\pi}{5}\cos\frac{2\pi}{5}\cos\frac{4\pi}{5}\cos\frac{8\pi}{5}$ is ....

16. The value of $\sin\frac{3\pi}{10}$ is ....

17. The value of $\cos^2 6° - \cos^2 24°$ is ....

18. If $\frac{\pi}{2} < x < \pi$, then $\sqrt{\frac{1-\cos 2x}{1+\cos 2x}} =$ ....

19. If $\frac{\pi}{4} < x < \frac{\pi}{2}$, then $\sqrt{2+\sqrt{2+2\cos 4x}} =$ ....

20. The value of $108 \sin\frac{\pi}{9} - 144 \sin^3\frac{\pi}{9}$ is ....

ANSWERS

1. 2 2. $\frac{1}{2^n}$ 3. $\frac{1}{16}$ 4. $\tan y$ 5. $\frac{1}{8}$ 6. $x^2 - \frac{2}{\sin 2A}x + 1 = 0$

7. $\frac{\sqrt{5}+1}{8}$ 8. 2 9. $\frac{3}{4}$ 10. $\frac{1}{8}$ 11. 2 12. $\frac{1+t}{1-t}$ 13. b 14. 4

15. $-\frac{1}{16}$ 16. $\frac{\sqrt{5}+1}{4}$ 17. $\frac{\sqrt{5}-1}{8}$ 18. $-\tan x$ 19. $2\sin x$ 20. $18\sqrt{3}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If $\cos 4x = 1 + k \sin^2 x \cos^2 x$, then write the value of k.

2. If $\tan \frac{x}{2} = \frac{m}{n}$, then write the value of $m \sin x + n \cos x$.

3. If $\frac{\pi}{2} < x < \frac{3\pi}{2}$, then write the value of $\sqrt{\frac{1 + \cos 2x}{2}}$.

4. If $\frac{\pi}{2} < x < \pi$, then write the value of $\sqrt{2 + \sqrt{2 + 2\cos 2x}}$ in the simplest form.

5. If $\frac{\pi}{2} < x < \pi$, then write the value of $\sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}}$.

6. If $\pi < x < \frac{3\pi}{2}$, then write the value of $\sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}}$.

7. In a right angled triangle ABC, write the value of $\sin^2 A + \sin^2 B + \sin^2 C$.

8. Write the value of $\cos^2 76° + \cos^2 16° - \cos 76° \cos 16°$.

9. If $\frac{\pi}{4} < x < \frac{\pi}{2}$, then write the value of $\sqrt{1 - \sin 2x}$.

10. Write the value of $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$.

11. If $\tan A = \frac{1 - \cos B}{\sin B}$, then find the value of $\tan 2A$.

12. If $\sin x + \cos x = a$, find the value of $\sin^6 x + \cos^6 x$.

13. If $\sin x + \cos x = a$, find the value of $|\sin x - \cos x|$.

ANSWERS

1. -8 2. n 3. $-\cos x$ 4. $2\sin\frac{x}{2}$ 5. $-\tan x$ 6. $\tan x$ 7. 2 8. $\frac{3}{4}$
9. $\sin x - \cos x$ 10. $-\frac{1}{8}$ 11. $\tan B$ 12. $\frac{1}{4}\{4-3(a^2-1)^2\}$ 13. $\sqrt{2-a^2}$.

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. $8\sin\frac{x}{8}\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}$ is equal to
   (a) $8\cos x$ (b) $\cos x$ (c) $8\sin x$ (d) $\sin x$
2. $\frac{\sec 8A-1}{\sec 4A-1}$ is equal to
   (a) $\frac{\tan 2A}{\tan 8A}$ (b) $\frac{\tan 8A}{\tan 2A}$ (c) $\frac{\cot 8A}{\cot 2A}$ (d) none of these
3. The value of $\cos\frac{\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}$ is
   (a) $\frac{1}{8}$ (b) $\frac{1}{16}$ (c) $\frac{1}{32}$ (d) none of these
4. If $\cos 2x + 2\cos x = 1$ then, $(2-\cos^2 x)\sin^2 x$ is equal to
   (a) 1 (b) -1 (c) $-\sqrt{5}$ (d) $\sqrt{5}$
5. For all real values of x, $\cot x - 2\cot 2x$ is equal to
   (a) $\tan 2x$ (b) $\tan x$ (c) $-\cot 3x$ (d) none of these
6. The value of $2\tan\frac{\pi}{10} + 3\sec\frac{\pi}{10} - 4\cos\frac{\pi}{10}$ is
   (a) 0 (b) $\sqrt{5}$ (c) 1 (d) none of these
7. If in a ΔABC, $\tan A + \tan B + \tan C = 0$, then $\cot A\cot B\cot C =$
   (a) 6 (b) 1 (c) $\frac{1}{6}$ (d) none of these
8. If $\cos x = \frac{1}{2}\left(a+\frac{1}{a}\right)$, and $\cos 3x = \lambda\left(a^3+\frac{1}{a^3}\right)$, then $\lambda =$
   (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 1 (d) none of these
9. If $2\tan\alpha = 3\tan\beta$, then $\tan(\alpha-\beta) =$
   (a) $\frac{\sin 2\beta}{5-\cos 2\beta}$ (b) $\frac{\cos 2\beta}{5-\cos 2\beta}$ (c) $\frac{\sin 2\beta}{5+\cos 2\beta}$ (d) none of these
10. If $\tan\alpha = \frac{1-\cos\beta}{\sin\beta}$, then
    (a) $\tan 3\alpha = \tan 2\beta$ (b) $\tan 2\alpha = \tan\beta$ (c) $\tan 2\beta = \tan\alpha$ (d) none of these
11. If $\sin\alpha + \sin\beta = a$ and $\cos\alpha - \cos\beta = b$, then $\tan\frac{\alpha-\beta}{2} =$
    (a) $-\frac{a}{b}$ (b) $-\frac{b}{a}$ (c) $\sqrt{a^2+b^2}$ (d) none of these

12. The value of $\left(\cot\frac{x}{2}-\tan\frac{x}{2}\right)^2(1-2\tan x\cot 2x)$ is

(a) 1 (b) 2 (c) 3 (d) 4

13. The value of $\tan x\sin\left(\frac{\pi}{2}+x\right)\cos\left(\frac{\pi}{2}-x\right)$ is

(a) 1 (b) -1 (c) $\frac{1}{2}\sin 2x$ (d) none of these

14. The value of $\sin^2\left(\frac{\pi}{18}\right)+\sin^2\left(\frac{\pi}{9}\right)+\sin^2\left(\frac{7\pi}{18}\right)+\sin^2\left(\frac{4\pi}{9}\right)$ is

(a) 1 (b) 2 (c) 4 (d) none of these

15. If $5\sin\alpha=3\sin(\alpha+2\beta)\neq 0$, then $\tan(\alpha+\beta)$ is equal to

(a) $2\tan\beta$ (b) $3\tan\beta$ (c) $4\tan\beta$ (d) $6\tan\beta$

16. The value of $2\cos x-\cos 3x-\cos 5x-16\cos^3 x\sin^2 x$ is

(a) 2 (b) 1 (c) 0 (d) -1

17. If $A=2\sin^2 x-\cos 2x$, then A lies in the interval

(a) $[-1, 3]$ (b) $[1, 2]$ (c) $[-2, 4]$ (d) none of these

18. The value of $\frac{\cos 3x}{2\cos 2x-1}$ is equal to

(a) $\cos x$ (b) $\sin x$ (c) $\tan x$ (d) none of these

19. If $\tan(\pi/4+x)+\tan(\pi/4-x)=\lambda\sec 2x$, then

(a) 3 (b) 4 (c) 1 (d) 2

20. The value of $\cos^2\left(\frac{\pi}{6}+x\right)-\sin^2\left(\frac{\pi}{6}-x\right)$ is

(a) $\frac{1}{2}\cos 2x$ (b) 0 (c) $-\frac{1}{2}\cos 2x$ (d) $\frac{1}{2}$

21. $\frac{\sin 3x}{1+2\cos 2x}$ is equal to

(a) $\cos x$ (b) $\sin x$ (c) $-\cos x$ (d) $\sin x$

22. The value of $2\sin^2 B+4\cos(A+B)\sin A\sin B+\cos 2(A+B)$ is

(a) 0 (b) $\cos 3A$ (c) $\cos 2A$ (d) none of these

23. The value of $\frac{2(\sin 2x+2\cos^2 x-1)}{\cos x-\sin x-\cos 3x+\sin 3x}$ is

(a) $\cos x$ (b) $\sec x$ (c) $\operatorname{cosec} x$ (d) $\sin x$

24. $2(1-2\sin^2 7x)\sin 3x$ is equal to

(a) $\sin 17x-\sin 11x$ (b) $\sin 11x-\sin 17x$

(c) $\cos 17x-\cos 11x$ (d) $\cos 17x+\cos 11x$

25. If α and β are acute angles satisfying $\cos 2\alpha=\frac{3\cos 2\beta-1}{3-\cos 2\beta}$, then $\tan\alpha=$

(a) $\sqrt{2}\tan\beta$ (b) $\frac{1}{\sqrt{2}}\tan\beta$ (c) $\sqrt{2}\cot\beta$ (d) $\frac{1}{\sqrt{2}}\cot\beta$

26. If $\tan\frac{x}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\alpha}{2}$, then $\cos\alpha =$

(a) $1 - e\cos(\cos x + e)$ (b) $\frac{1+e\cos x}{\cos x - e}$ (c) $\frac{1-e\cos x}{\cos x - e}$ (d) $\frac{\cos x - e}{1 - e\cos x}$

27. If $(2^n+1)x = \pi$, then $2^n \cos x \cos 2x \cos 2^2 x \ldots \cos 2^{n-1} x =$

(a) -1 (b) 1 (c) $1/2$ (d) none of these

28. If $\tan x = t$ then $\tan 2x + \sec 2x$ is equal to

(a) $\frac{1+t}{1-t}$ (b) $\frac{1-t}{1+t}$ (c) $\frac{2t}{1-t}$ (d) $\frac{2t}{1+t}$

29. The value of $\cos^4 x + \sin^4 x - 6\cos^2 x \sin^2 x$ is

(a) $\cos 2x$ (b) $\sin 2x$ (c) $\cos 4x$ (d) none of these

30. The value of $\cos(36° - A)\cos(36° + A) + \cos(54° - A)\cos(54° + A)$ is

(a) $\cos 2A$ (b) $\sin 2A$ (c) $\cos A$ (d) 0

31. The value of $\tan x \tan\left(\frac{\pi}{3} - x\right)\tan\left(\frac{\pi}{3} + x\right)$ is

(a) $\cot 3x$ (b) $2\cot 3x$ (c) $\tan 3x$ (d) $3\tan 3x$

32. The value of $\tan x + \tan\left(\frac{\pi}{3} + x\right) + \tan\left(\frac{2\pi}{3} + x\right)$ is

(a) $3\tan 3x$ (b) $\tan 3x$ (c) $3\cot 3x$ (d) $\cot 3x$

33. The value of $\frac{\sin 5\alpha - \sin 3\alpha}{\cos 5\alpha + 2\cos 4\alpha + \cos 3\alpha}$ is

(a) $\cot\alpha/2$ (b) $\cot\alpha$ (c) $\tan\alpha/2$ (d) none of these

34. $\frac{\sin 5x}{\sin x}$ is equal to

(a) $16\cos^4 x - 12\cos^2 x + 1$ (b) $16\cos^4 x + 12\cos^2 x + 1$

(c) $16\cos^4 x - 12\cos^2 x - 1$ (d) $16\cos^4 x + 12\cos^2 x - 1$

35. If $n = 1, 2, 3, \ldots$, then $\cos\alpha\cos 2\alpha\cos 4\alpha \ldots \cos 2^{n-1}\alpha$ is equal to

(a) $\frac{\sin 2n\alpha}{2n\sin\alpha}$ (b) $\frac{\sin 2^n\alpha}{2^n\sin 2^{n-1}\alpha}$ (c) $\frac{\sin 4^{n-1}\alpha}{4^{n-1}\sin\alpha}$ (d) $\frac{\sin 2^n\alpha}{2^n\sin\alpha}$

36. If $\tan x = \frac{a}{b}$, then $b\cos 2x + a\sin 2x$ is equal to

(a) a (b) b (c) $\frac{a}{b}$ (d) $\frac{b}{a}$

[NCERT EXEMPLAR]

37. If $\tan\alpha = \frac{1}{7}$, $\tan\beta = \frac{1}{3}$, then $\cos 2\alpha$ is equal to

(a) $\sin 2\beta$ (b) $\sin 4\beta$ (c) $\sin 3\beta$ (d) $\cos 2\beta$

[NCERT EXEMPLAR]

38. The value of $\cos^2 48° - \sin^2 12°$ is

(a) $\frac{\sqrt{5}+1}{8}$ (b) $\frac{\sqrt{5}-1}{8}$ (c) $\frac{\sqrt{5}+1}{5}$ (d) $\frac{\sqrt{5}+1}{2\sqrt{2}}$

[NCERT EXEMPLAR]

39. The value of $\frac{1-\tan^2 15°}{1+\tan^2 15°}$ is

(a) 1 (b) $\sqrt{3}$ (c) $\frac{\sqrt{3}}{2}$ (d) 2

[NCERT EXEMPLAR]

40. The value of $\tan 75° - \cot 75°$ is

(a) $2\sqrt{3}$ (b) $2+\sqrt{3}$ (c) $2-\sqrt{3}$ (d) 1

[NCERT EXEMPLAR]

41. $\cos 2\theta \cos 2\phi + \sin^2(\theta-\phi) - \sin^2(\theta+\phi)$ is equal to

(a) $\sin 2(\theta+\phi)$ (b) $\cos 2(\theta+\phi)$ (c) $\sin 2(\theta-\phi)$ (d) $\cos 2(\theta-\phi)$

[NCERT EXEMPLAR]

42. If $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$, then $\tan(2A+B)$ is equal

(a) 1 (b) 2 (c) 3 (d) 4

[NCERT EXEMPLAR]

43. If $\sin\theta + \cos\theta = 1$, then the value of $\sin 2\theta$ is equal to

(a) 1 (b) $\frac{1}{2}$ (c) 0 (d) -1

[NCERT EXEMPLAR]

44. The value of $\sin\frac{\pi}{10}\sin\frac{13\pi}{10}$ is

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) $-\frac{1}{4}$ (d) 1

[NCERT EXEMPLAR]

45. If $\sin\theta = -\frac{4}{5}$ and θ lies in third quadrant, then the value of $\cos\frac{\theta}{2}$ is

(a) $\frac{1}{5}$ (b) $-\frac{1}{\sqrt{10}}$ (c) $-\frac{1}{\sqrt{5}}$ (d) $\frac{1}{\sqrt{10}}$

[NCERT EXEMPLAR]

46. The value of $\cos 12° + \cos 84° + \cos 156° + \cos 132°$ is

(a) $\frac{1}{2}$ (b) 1 (c) $-\frac{1}{2}$ (d) $-\frac{1}{8}$

ANSWERS

1. (d)	2. (b)	3. (d)	4. (a)	5. (b)	6. (a)	7. (d)	8. (b)
9. (a)	10. (b)	11. (b)	12. (d)	13. (d)	14. (b)	15. (c)	16. (c)
17. (a)	18. (a)	19. (d)	20. (a)	21. (b)	22. (c)	23. (c)	24. (a)
25. (a)	26. (d)	27. (b)	28. (a)	29. (c)	30. (a)	31. (c)	32. (a)
33. (c)	34. (a)	35. (d)	36. (b)	37. (b)	38. (a)	39. (c)	40. (a)
41. (b)	42. (c)	43. (c)	44. (c)	45. (c)	46. (c)		

SUMMARY

1. (i) $\sin 2x = 2 \sin x \cos x$ (ii) $\cos 2x = \cos^2 x - \sin^2 x$

 (iii) $\cos 2x = 2\cos^2 x - 1$ or, $1 + \cos 2x = 2\cos^2 x$

 (iv) $\cos 2x = 1 - 2\sin^2 x$ or, $1 - \cos 2x = 2\sin^2 x$

 (v) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$ (vi) $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$ (vii) $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

2. (i) $\sin 3x = 3\sin x - 4\sin^3 x$ (ii) $\cos 3x = 4\cos^3 x - 3\cos x$

 (iii) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

3. (i) $\sin \dfrac{x}{2} = \sqrt{\dfrac{1-\cos x}{2}}$ (ii) $\cos \dfrac{x}{2} = \sqrt{\dfrac{1+\cos x}{2}}$ (iii) $\tan \dfrac{x}{2} = \sqrt{\dfrac{1-\cos x}{1+\cos x}}$

4. (i) $\cos x \cos 2x \cos 2^2 x \cos 2^3 x \ldots \cos 2^{n-1} x = \dfrac{\sin 2^n x}{2^n \sin x}$

 (ii) $\sin x \sin\left(\dfrac{\pi}{3} - x\right) \sin\left(\dfrac{\pi}{3} + x\right) = \dfrac{1}{4} \sin 3x$

 (iii) $\cos x \cos\left(\dfrac{\pi}{3} - x\right) \cos\left(\dfrac{\pi}{3} + x\right) = \dfrac{1}{4} \cos 3x$

5. (i) $\sin \dfrac{\pi}{10} = \dfrac{\sqrt{5}-1}{4}$ (ii) $\cos \dfrac{\pi}{5} = \dfrac{\sqrt{5}+1}{4}$

 (iii) $\cos \dfrac{\pi}{10} = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$ (iv) $\sin \dfrac{\pi}{5} = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$