MATHEMATICS
WORKSHEET_090623
CHAPTER 12 - LIMITS AND DERIVATIVES
(ANSWERS)

SUBJECT: MATHEMATICS **MAX. MARKS : 40**
CLASS : XI **DURATION : 1½ hrs**

General Instructions:
(i). **All** questions are compulsory.
(ii). This question paper contains 2**0** questions divided into five Sections A, B, C, D and E.
(iii). **Section A** comprises of **10 MCQs** of **1 mark** each. **Section B** comprises of 4 questions of **2 marks** each. **Section C** comprises of 3 questions of **3 marks** each. **Section D** comprises of 1 question of **5 marks** each and **Section E** comprises of 2 Case Study Based Questions of **4 marks** each.
(iv). There is no overall choice.
(v). Use of Calculators is not permitted

SECTION – A
Questions 1 to 10 carry 1 mark each.

1. If $\lim\limits_{x \to 2} \dfrac{x^n - 2^n}{x-2} = 80$ then n is:
(a) 1 (b) 3 (c) 5 (d) 7
Ans. (c) 5
$n.2^{n-1} = 80$
$\Rightarrow n.2^{n-1} = 5 \times 16$
$\Rightarrow n.2^{n-1} = 5 \times 2^{5-1} \Rightarrow n = 5$

2. If $f(x) = \begin{cases} x^2+1, & x \ge 1 \\ 3x-1, & x < 1 \end{cases}$ then the value of $\lim\limits_{x \to 1} f(x)$ is:
(a) 2 (b) –2 (c) 1 (d) –1
Ans. (a) 2

3. The value of $\lim\limits_{x \to 0} \left(\dfrac{\tan^2 3x}{x^2}\right)$ is:
(a) 0 (b) 3 (c) ∞ (d) 9
Ans. (d) 9

4. If $f(x) = \begin{cases} x^2-1, & 0 < x < 2 \\ 2x+3, & 2 \le x \le 3 \end{cases}$ then the quadratic equations, whose roots are $\lim\limits_{x \to 2^-} f(x)$ and $\lim\limits_{x \to 2^+} f(x)$ is:
(a) $x^2 - 6x + 9 = 0$ (b) $x^2 - 7x + 8 = 0$
(c) $x^2 - 14x + 49 = 0$ (d) $x^2 - 10x + 21 = 0$
Ans. (d) $x^2 - 10x + 21 = 0$

5. $\lim\limits_{x \to 0} \dfrac{\tan 2x - x}{3x - \sin x}$ is:
(a) 2 (b) 1/2 (c) -1/2 (d) 1/4
Ans. (b) 1/2

6. Derivation of $\sin^3 x$ is:

(a) $\cos^3 x$ (b) $3 \sin^3 x$ (c) $3 \sin^2 x \cos x$ (d) 0

Ans. (c) $3 \sin^2 x \cos x$

7. If $f(x) = x^2 - 5x + 7$, then $f'(3) =$ .... :

(a) 11 (b) 1 (c) 4 (d) 0

Ans. (b) 1

Given, $f(x) = x^2 - 5x + 7$

$\Rightarrow f'(x) = 2x - 5$

$\Rightarrow f'(3) = 2 \times 3 - 5 = 6 - 5 = 1$

8. If $\sin(x + y) = \log(x + y)$ then $\frac{dy}{dx}$ equals

(a) 0 (b) 1 (c) –1 (d) none of these

Ans. (c) –1

Differentiating both sides with respect to x, we get

$$\cos(x+y)\left(1+\frac{dy}{dx}\right)=\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \left(1+\frac{dy}{dx}\right)\left[\cos(x+y)-\frac{1}{x+y}\right]=0$$

$$\Rightarrow 1+\frac{dy}{dx}=0 \quad \left[\because \cos(x+y)\neq\frac{1}{x+y}\right] \quad \Rightarrow \frac{dy}{dx}=-1$$

For Q9 and Q10, a statement of assertion (A) is followed by a statement of reason (R). Choose the correct answer out of the following choices.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true.

9. **Assertion (A):** The derivation of $f(x) = x^3$ is x^2.

Reason (R): The derivation of $f(x) = x^n$ is nx^{n-1}.

Ans. (d) A is false but R is true.

10. **Assertion (A):** $\lim_{x \to 0} \frac{\sin ax}{bx}$ is equal to $\frac{a}{b}$.

Reason (R): $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Ans. (a) Both (A) and (R) are true and (R) is the correct explanation of (A).

SECTION – B

Questions 11 to 14 carry 2 marks each.

11. Evaluate: $\lim_{z \to 1}\left(\frac{z^{1/3}-1}{z^{1/6}-1}\right)$

Ans.

$$\text{Here } \lim_{z\to 1}\frac{z^{1/3}-1}{z^{1/6}-1}\left[\frac{0}{0}\text{ form}\right]$$

$$=\lim_{z\to 1}\frac{(z^{1/6})^2-(1)^2}{z^{1/6}-1}=\lim_{z\to 1}\frac{(z^{1/6}+1)(z^{1/6}-1)}{(z^{1/6}-1)}=\lim_{z\to 1}(z^{1/6}+1)=(1)^{1/6}+1=1+1=2$$

12. Evaluate: $\lim\limits_{x \to 0} \left(\dfrac{\sin ax + bx}{ax + \sin bx} \right), \; a, b, a+b \neq 0$

Ans.

Here $\lim\limits_{x \to 0} \dfrac{\sin ax + bx}{ax + \sin bx} \left[\dfrac{0}{0} \text{ form} \right]$

Dividing numerator and denominator by ax

$$= \lim_{x \to 0} \frac{\dfrac{\sin ax}{ax} + \dfrac{bx}{ax}}{\dfrac{ax}{ax} + \dfrac{\sin bx}{ax}} = \lim_{x \to 0} \frac{\dfrac{\sin ax}{ax} + \dfrac{bx}{ax}}{\dfrac{ax}{ax} + \dfrac{\sin bx}{bx} \times \dfrac{bx}{ax}} = \frac{1 + \dfrac{b}{a}}{1 + \dfrac{b}{a}} = 1$$

13. For some constants a and b find the derivative of $(x - a)(x - b)$.

Ans.

Here $f(x) = (x - a)(x - b)$

$$\therefore f'(x) = \frac{d}{dx}(x - a)(x - b)$$

$$= (x - a)\frac{d}{dx}(x - b) + (x - b)\frac{d}{dx}(x - a)$$

$$= (x - a) \times 1 + (x - b) \times 1 = x - a + x - b = 2x - a - b$$

14. Find the derivative of $(5x^3 + 3x - 1)(x - 1)$

Ans.

Here $f(x) = (5x^3 + 3x - 1)(x - 1)$

$$\therefore f'(x) = \frac{d}{dx}[(5x^3 + 3x - 1)(x - 1)]$$

$$= (5x^3 + 3x - 1)\frac{d}{dx}(x - 1) + (x - 1)\frac{d}{dx}(5x^3 + 3x - 1)$$

$$= (5x^3 + 3x - 1) \times 1 + (x - 1)(15x^2 + 3)$$

$$= 5x^3 + 3x - 1 + 15x^3 + 3x - 15x^2 - 3 = 20x^3 - 15x^2 + 6x - 4$$

OR

Find the derivative of $\sin x \cos x$

Ans.

Here $f(x) = \sin x \cos x$

$$\therefore f'(x) = \frac{d}{dx}(\sin x \cos x) = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x)$$

$$= \sin x \times (-\sin x) + \cos x \times \cos x$$

$$= -\sin^2 x + \cos^2 x = \cos^2 x - \sin^2 x = \cos 2x$$

SECTION – C

Questions 15 to 17 carry 3 marks each.

15. Prove that the derivative of $\sin x$ with respect to x is $\cos x$ using first principle of derivative.

Ans.

Let $f(x) = \sin x$. Then, $f(x + h) = \sin(x + h)$

$$\therefore \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{2\sin\left(\dfrac{h}{2}\right)\cos\left(\dfrac{2x + h}{2}\right)}{h} \qquad \left[\because \sin C - \sin D = 2\sin\frac{C - D}{2}\cos\frac{C + D}{2}\right]$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(2 \sin h/2) \cos (x + h/2)}{2(h/2)} \Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \cos\left(x + \frac{h}{2}\right) \lim_{h \to 0} \frac{\sin (h/2)}{(h/2)}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = (\cos x) \times 1 = \cos x \qquad \left[\because \lim_{h \to 0} \frac{\sin (h/2)}{(h/2)} = 1\right]$$

16. Find $\lim\limits_{x \to 0} f(x)$ and $\lim\limits_{x \to 1} f(x)$ where $f(x) = \begin{cases} 2x+3, & x \le 0 \\ 3(x+1), & x > 0 \end{cases}$.

Ans.

$$\text{Here } f(x) = \begin{cases} 2x+3 & x \le 0 \\ 3(x+1) & x > 0 \end{cases}$$

$$\text{Now, LHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x+3) = 2 \times 0 + 3 = 3$$

$$\text{RHL} = \lim_{x \to 0^+} 3(x+1) = \lim_{x \to 0^+} (3x+3) = 3$$

$$f(0) = 2 \times 0 + 3 = 3$$

$$\text{Here LHL} = \text{RHL} = f(0) \quad \Rightarrow \quad \lim_{x \to 0} f(x) = 3$$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} 3(x+1) = 3 \times 2 = 6$$

OR

Evaluate: $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\tan 2x}{x - \frac{\pi}{2}}$

Ans.

$$\text{Here } \lim_{x \to \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}} \left[\frac{0}{0} \text{ form}\right]$$

Put $x = \dfrac{\pi}{2} + y$ as $x \to \dfrac{\pi}{2}, y \to 0$

$$\therefore \lim_{y \to 0} \frac{\tan 2\left(\frac{\pi}{2} + y\right)}{\frac{\pi}{2} + y - \frac{\pi}{2}} = \lim_{y \to 0} \frac{\tan (\pi + 2y)}{y} = \lim_{y \to 0} \frac{\tan 2y}{y} = \lim_{y \to 0} \frac{\tan 2y}{2y} \times 2 = 1 \times 2 = 2$$

17. Find the derivative of $\dfrac{2}{x+1} - \dfrac{x^2}{3x-1}$

Ans.

$$\text{Here } f(x) = \frac{2}{x+1} - \frac{x^2}{3x-1}$$

$$\therefore f'(x) = \frac{d}{dx}\left[\frac{2}{x+1} - \frac{x^2}{3x-1}\right] = \frac{d}{dx}\left(\frac{2}{x+1}\right) - \frac{d}{dx}\left(\frac{x^2}{3x-1}\right)$$

$$= \frac{(x+1)\frac{d}{dx}(2) - 2\frac{d}{dx}(x+1)}{(x+1)^2} - \frac{(3x-1)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(3x-1)}{(3x-1)^2} \text{ [Quotient rule]}$$

$$= \frac{(x+1) \times 0 - 2 \times 1}{(x+1)^2} - \frac{(3x-1)(2x) - x^2 \times 3}{(3x-1)^2} = \frac{-2}{(x+1)^2} - \frac{6x^2 - 2x - 3x^2}{(3x-1)^2} = \frac{-2}{(x+1)^2} - \frac{3x^2 - 2x}{(3x-1)^2}$$

OR

If $y=\sqrt{x}+\frac{1}{\sqrt{x}}$ then show that $2x\frac{dy}{dx}+y=2\sqrt{x}$.

Ans.

Given $y = \sqrt{x}+\frac{1}{\sqrt{x}} = x^{1/2}+x^{-1/2}$ ...(*i*)

$$\therefore \quad \frac{dy}{dx}=\frac{1}{2}x^{-1/2}+\left(-\frac{1}{2}\right)x^{-3/2}=\frac{1}{2\sqrt{x}}-\frac{1}{2x^{3/2}}$$

Multiplying both sides by $2x$, we get

$$2x\frac{dy}{dx}=\sqrt{x}-\frac{1}{\sqrt{x}} \qquad ...(ii)$$

Adding (*i*) and (*ii*), we get $2x\frac{dy}{dx}+y=2\sqrt{x}$

SECTION – D

Questions 18 carry 5 marks.

18. Evaluate: $\lim\limits_{x\to 0}\frac{\tan x+4\tan 2x-3\tan 3x}{x^2\tan x}$

Ans.

$$\lim_{x\to 0}\frac{\tan x+4\tan 2x-3\tan 3x}{x^2\tan x}$$

$$=\lim_{x\to 0}\frac{\tan x+4\frac{2\tan x}{1-\tan^2 x}-3\left(\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}\right)}{x^2\tan x}=\lim_{x\to 0}\frac{1+\frac{8}{1-\tan^2 x}-3\left(\frac{3-\tan^2 x}{1-3\tan^2 x}\right)}{x^2}$$

$$=\lim_{x\to 0}\frac{(1-\tan^2 x)(1-3\tan^2 x)+8(1-3\tan^2 x)-3(3-\tan^2 x)(1-\tan^2 x)}{x^2(1-\tan^2 x)(1-3\tan^2 x)}$$

$$=\lim_{x\to 0}\frac{1-4\tan^2 x+3\tan^4 x+8-24\tan^2 x-9+12\tan^2 x-3\tan^4 x}{x^2(1-\tan^2 x)(1-3\tan^2 x)}$$

$$=\lim_{x\to 0}\frac{-16\tan^2 x}{x^2(1-\tan^2 x)(1-3\tan^2 x)}=-16\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^2\times\frac{1}{1-\tan^2 x}\times\frac{1}{1-3\tan^2 x}=-16$$

OR

Evaluate: $\lim\limits_{y\to 0}\frac{(x+y)\sec(x+y)-x\sec x}{y}$

Ans.

$$\lim_{y\to 0}\frac{(x+y)\sec(x+y)-x\sec x}{y}$$

$$=\lim_{y\to 0}\frac{x\sec(x+y)-x\sec x+y\sec(x+y)}{y}=\lim_{y\to 0}\left[\frac{x\sec(x+y)-x\sec x}{y}+\frac{y\sec(x+y)}{y}\right]$$

$$=\lim_{y\to 0}\frac{x}{y}\left(\frac{1}{\cos(x+y)}-\frac{1}{\cos x}\right)+\lim_{y\to 0}\sec(x+y) = \lim_{y\to 0}\frac{x(\cos x-\cos(x+y))}{(y\cos x\cos(x+y))}+\sec x$$

$$= \lim_{y \to 0} \frac{x \cdot 2 \sin \frac{x+x+y}{2} \sin \frac{x+y-x}{2}}{y \cos x \cos (x+y)} + \sec x = \lim_{y \to 0} \frac{x \sin \left(x + \frac{y}{2}\right)}{\cos x \cos (x+y)} \cdot \lim_{\frac{y}{2} \to 0} \frac{\sin \frac{y}{2}}{\frac{y}{2}} + \sec x$$

$$= \frac{x \sin x}{\cos x \cos x} \cdot 1 + \sec x = x \tan x \sec x + \sec x = \sec x (x \tan x + 1)$$

SECTION – E (Case Study Based Questions)

Questions 19 to 20 carry 4 marks each.

19. A function f is said to be a rational function, if $f(x) = \frac{g(x)}{h(x)}$, where g (x) and h (x) are polynomial functions such that h (x) $\neq$ 0.

Then, $\lim_{x \to a} f(x) = \lim_{x \to a} \frac{g(x)}{h(x)} = \frac{\lim_{x \to a} g(x)}{\lim_{x \to a} h(x)} \frac{g(a)}{h(a)}$

However, if h (a) = 0, then there are two cases arise,
(i) g (a) $\neq$ 0 (ii) g (a) = 0.
In the first case, we say that the limit does not exist.
In the second case, we can find limit.
Based on above information, answer the following questions.

(a) Evaluate: $\lim_{x \to -1} \left(\frac{x^{10} + x^5 + 1}{x - 1} \right)$ (1)

(b) Evaluate: $\lim_{x \to -1} \frac{(x-1)^2 + 3x^2}{(x^4+1)^2}$ (1)

(c) Find the value of $\lim_{x \to 2} \left(\frac{x^2 - 4}{x^3 - 4x^2 + 4x} \right)$ (2)

Ans. (a)

$$\lim_{x \to -1} \frac{x^{10} + x^5 + 1}{x - 1} = \frac{(-1)^{10} + (-1)^5 + 1}{-1 - 1} = \frac{1 - 1 + 1}{-2} = \frac{-1}{2}$$

(b)

$$\lim_{x \to -1} \frac{(x-1)^2 + 3x^2}{(x^4+1)^2} = \frac{(-1-1)^2 + 3(-1)^2}{((-1)^4+1)^2} = \frac{(-2)^2 + 3(1)}{(1+1)^2} = \frac{4+3}{2^2} = \frac{7}{4}$$

(c)

$$\lim_{x \to 2} \frac{x^2 - 4}{x^3 - 4x^2 + 4x} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x(x-2)^2} = \lim_{x \to 2} \frac{(x+2)}{x(x-2)} = \frac{2+2}{2(2-2)} = \frac{4}{0}$$

which is not defined.
Therefore, limit does not exist.

20. Let *f* and *g* be two functions such that their derivatives are defined in a common domain. We define the derivative of product of two functions is given by the product rule *i.e.*,

$$\frac{d}{dx}[f(x).g(x)] = g(x).\frac{d}{dx} f(x) + f(x).\frac{d}{dx} g(x)$$

The derivative of quotient of two functions is given by quotient rule i.e.,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x).\frac{d}{dx} f(x) - f(x).\frac{d}{dx} g(x)}{[g(x)]^2}; g(x) \neq 0$$

Based on above information, answer the following questions.

(a) Find the derivative of $x(x + 2)$. (1)
(b) Find the value of $f'(x)$, if $f(x) = \sin x \cdot \cos x$ (2)
(c) Find $\frac{d}{dx}\left(\frac{x+1}{x-1}\right)$ (2)

OR

(c) Find the value of derivative of $\frac{x^2 \cos\left(\frac{\pi}{4}\right)}{\sin x}$ at $x = \frac{\pi}{2}$. (2)

Ans. (a)
$$\frac{d}{dx}(x(x+2)) = \frac{d}{dx}(x)\cdot(x+2) + x\frac{d}{dx}(x+2) = 1\cdot(x + 2) + x\cdot 1 = 2x + 2 = 2(x + 1)$$
(b)
$$f'(x) = \cos x \cdot \frac{d}{dx}(\sin x) + \sin x \cdot \frac{d}{dx}(\cos x) = \cos x \cdot \cos x + \sin x\,(-\sin x)$$
$$= \cos^2 x - \sin^2 x = \cos 2x$$
(c)
$$\frac{d}{dx}\left(\frac{x+1}{x-1}\right) = \frac{\frac{d}{dx}(x+1).(x-1) - (x+1).\frac{d}{dx}(x-1)}{(x-1)^2} = \frac{(x-1)-(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

OR

(c)
$$f'(x) = \frac{\frac{d}{dx}(x^2)\cdot(\sqrt{2}\sin x) - x^2\frac{d}{dx}(\sqrt{2}\sin x)}{(\sqrt{2}\sin x)^2} = \frac{2x\cdot\sqrt{2}\sin x - \sqrt{2}x^2\cos x}{2\sin^2 x}$$
$$= \frac{2x\sin x - x^2\cos x}{\sqrt{2}\sin^2 x} \Rightarrow f'\left(\frac{\pi}{2}\right) = \frac{2\times\frac{\pi}{2}.1 - 0}{\sqrt{2}\cdot(1)^2} = \frac{\pi}{\sqrt{2}}$$