

Class 8 Mathematics: The Baudhāyana-Pythagoras Theorem

Comprehensive Enriched Study Guide • NCERT Ganita Prakash (Part-II, Chapter 2)

MASTER
NOTES

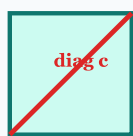
1. Doubling and Halving a Square: Geometric Foundations

Around 800 BCE, the ancient Indian mathematician **Baudhāyana** recorded in the *Śulba-Sūtra* (manuals for altar geometry) a fundamental question: *How can one construct a square having double the area of a given square?*

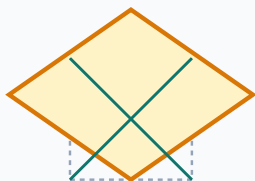
Common Fallacy: If a square has side length s , doubling each side to $2s$ does **not** double the area. Instead, Area = $(2s)^2 = 4s^2$ (the area quadruples). To double the area, we must find a square of area $2s^2$.

Baudhāyana's Sulba-Sūtra (Verse 1.9) — The Diagonal Rule:

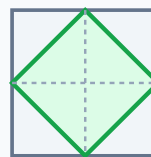
"The diagonal of a square produces a square of double the area of the original square."



Base Square (Area = 1)



Doubled Square (4 Triangles)



Halved Square PQRS (Area = 1/2)

Geometric Proof by Dissection

A square of side a divides along its diagonal into **2 congruent isosceles right triangles** (each area = $a^2/2$). The square constructed on this diagonal incorporates **4 identical congruent triangles**. Thus:
Area = $4 \times (a^2/2) = 2a^2$.

Halving a Square

Connecting the midpoints of the 4 edges of a square of side s produces an inscribed tilted square. The outer 4 corner triangles account for half the area, leaving the inscribed square with exactly **half the original area** ($s^2/2$).

SOLVED EXAMPLE 1: HYPOTENUSE OF AN ISOSCELES RIGHT TRIANGLE

Problem: In an isosceles right triangle with equal sides of length 1 unit, find the exact length of the hypotenuse c .
Solution:

- Two such triangles form a unit square of area $1 \times 1 = 1$ sq. unit.
- The square erected on hypotenuse c has Area = $2 \times 1 = 2$ sq. units.
- Since Area of square = c^2 , we have $c^2 = 2 \Rightarrow c = \sqrt{2}$ units.
- For an isosceles right triangle with equal legs a : $c^2 = 2a^2 \Rightarrow c = a\sqrt{2}$.

2. Decimal Approximations and The Nature of $\sqrt{2}$

Because $(1)^2 = 1 < 2$ and $(2)^2 = 4 > 2$, the exact length $\sqrt{2}$ lies strictly between 1 and 2. We obtain closer rational bounds by systematically squaring decimals:

Decimal Step	Lower Bound ($x^2 < 2$)	Upper Bound ($x^2 > 2$)	Sandwich Inequality
Tenths	$1.4^2 = 1.96$	$1.5^2 = 2.25$	$1.4 < \sqrt{2} < 1.5$
Hundredths	$1.41^2 = 1.9881$	$1.42^2 = 2.0164$	$1.41 < \sqrt{2} < 1.42$
Thousandths	$1.414^2 = 1.999396$	$1.415^2 = 2.002225$	$1.414 < \sqrt{2} < 1.415$
Sulba-Sūtra Value	$1 + 1/3 + 1/(3 \times 4) - 1/(3 \times 4 \times 34) = 577/408 \approx 1.41421568...$ (accurate to 5 decimal places!)		

Why $\sqrt{2}$ Cannot Terminate

If $\sqrt{2}$ were a terminating decimal with last digit $d \neq 0$, its square would end in the last digit of d^2 . For instance, any number ending in 4 squares to end in 6 ($...4^2 = ...6$). A finite decimal squared can never end in an infinite string of zeros (.000...). Thus, $\sqrt{2}$ has an **infinite, non-terminating decimal representation**.

Proof that $\sqrt{2}$ is Not a Fraction (Irrational)

Assume $\sqrt{2} = m/n$ where m, n are positive integers.

Squaring both sides: $2n^2 = m^2$.

In any integer square (m^2 and n^2), the prime factor 2 must occur an **even number of times**.

Therefore, the left side ($2 \times n^2$) has an **odd** power of 2, while m^2 has an **even** power of 2.

By the Fundamental Theorem of Arithmetic, this is impossible! Hence, $\sqrt{2}$ cannot be written as a fraction m/n .

3. The General Baudhāyana Formula for Isosceles Right Triangles

Relationship: $c^2 = 2a^2$ (or $c = a\sqrt{2}$)

Given equal perpendicular legs of length a and hypotenuse c :

- Finding Hypotenuse: $c = \sqrt{2a^2} = a\sqrt{2}$.
- Finding Legs from Hypotenuse: $a^2 = c^2 / 2 \Rightarrow a = c / \sqrt{2}$.

SOLVED EXAMPLE 2: REVERSING THE RELATIONSHIP

Problem: The hypotenuse of an isosceles right triangle is 10 units. Find the length of each of its two equal sides.

Solution:

- Using $c^2 = 2a^2$ with $c = 10$: $10^2 = 2a^2 \Rightarrow 100 = 2a^2$.
- $a^2 = 50 \Rightarrow a = \sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}$ units.
- Numerical bound: Since $7^2 = 49$ and $7.1^2 = 50.41$, $7 < a < 7.1$ (approx 7.07 units).

4. Combining Two Different Squares: The General Theorem

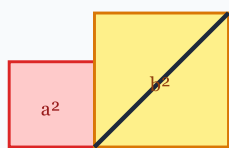
What happens when we wish to combine two squares of *different* sizes, say with side lengths a and b , into a single larger square of area equal to their sum?

Baudhāyana's Śulba-Sūtra (Verse 1.12 & Verse 2.1) — General Theorem:

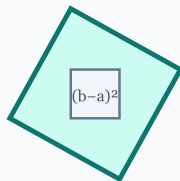
"The area of the square produced by the diagonal of a rectangle is the sum of the areas of the squares produced by its two sides."

In modern notation: For any right-angled triangle with perpendicular sides a and b , and hypotenuse c :

$$a^2 + b^2 = c^2$$



Step 1: Joined (a , b)



Step 2: Square c^2 Assembled

Algebraic Conservation:

$$\begin{aligned}\text{Area of Big Square} &= c^2 \\ &= 4 \times (\text{Right } \triangle) + (b-a)^2 \\ &= 4(\tfrac{1}{2}ab) + (b^2 - 2ab + a^2) = a^2 + b^2\end{aligned}$$

The Baudhayana Dissection (Verse 2.1)

Place square a^2 alongside square b^2 . Cut a right triangle of perpendicular sides a and b from the larger square and rotate it. Repeating this cut creates **4 congruent right-angled triangles** of base a and height b plus a central square of side $(b-a)$. Assembled together, they form a perfect square on hypotenuse c !

SOLVED EXAMPLE 3: FINDING THE UNKNOWN SIDELENGTH

Problem: A right-angled triangle has a perpendicular side of 8 cm and a hypotenuse of 17 cm. Find the length of the third side.

Solution:

1. Let perpendicular legs be $a = 8$ and b , and hypotenuse $c = 17$.
2. By the Baudhāyana-Pythagoras theorem: $a^2 + b^2 = c^2$.
3. $8^2 + b^2 = 17^2 \Rightarrow 64 + b^2 = 289 \Rightarrow b^2 = 289 - 64 = 225$.
4. $b = \sqrt{225} = 15$ cm. The third side is exactly 15 cm.

5. Baudhāyana-Pythagorean Triples & Parametric Generation

A set of three positive integers (a, b, c) that satisfy $a^2 + b^2 = c^2$ is called a **Baudhāyana triple**. If the greatest common divisor $\gcd(a, b, c) = 1$, it is called a **primitive triple**.

Scaling Property

If (a, b, c) is a Baudhāyana triple, then for any positive integer k , **(ka, kb, kc)** is also a triple:

$(ka)^2 + (kb)^2 = k^2a^2 + k^2b^2 = k^2(a^2 + b^2) = k^2c^2 = (kc)^2$.

Example: $(3, 4, 5)$ scales to $(6, 8, 10)$, $(9, 12, 15)$, $(30, 40, 50)$...

Generation via Odd Squares

Recall: Sum of first n odd numbers is n^2 .

Thus: **$(n - 1)^2 + (2n - 1) = n^2$** .

If the n -th odd number **$(2n - 1)$** is itself a square number **k^2** , we obtain a primitive triple where **$b = c - 1$** !

Odd Base (k)	Odd Square: $2n - 1 = k^2$	Index ($n = (k^2+1)/2$)	Leg a = k	Leg b = n - 1	Hypotenuse c = n	Resulting Triple
k = 3	$2n - 1 = 9$	n = 5	3	4	5	(3, 4, 5)
k = 5	$2n - 1 = 25$	n = 13	5	12	13	(5, 12, 13)
k = 7	$2n - 1 = 49$	n = 25	7	24	25	(7, 24, 25)
k = 9	$2n - 1 = 81$	n = 41	9	40	41	(9, 40, 41)
k = 11	$2n - 1 = 121$	n = 61	11	60	61	(11, 60, 61)

6. Fermat’s Last Theorem & Historical Milestones

While an infinite number of integer solutions satisfy $x^2 + y^2 = z^2$, French mathematician **Pierre de Fermat** (1637) postulated that **no positive integer solutions exist for higher powers**:

$x^n + y^n = z^n$ has no natural number solutions for any integer $n > 2$.

Fermat famously wrote: *"I have discovered a truly marvelous proof of this, which this margin is too narrow to contain."* After 350+ years of unresolved efforts by the world’s greatest minds, British mathematician **Sir Andrew Wiles** definitively proved the theorem in 1994!

SOLVED EXAMPLE 4: CLASSIC LOTUS PROBLEM FROM BHĀSKARĀCHĀRYA’S LĪLĀVATĪ (C. 1150 CE)

Problem: A lotus flower peeks 1 unit above the surface of a calm lake. A gentle wind blows it, and the flower tip grazes the water surface at a distance of 3 units from where it emerged. Find the depth of the lake.

Solution:

- Let the depth of the lake be x . The stem is rooted at the bottom, so total stem length = $x + 1$.
- When blown, the stem tilts into the hypotenuse of a right triangle of legs 3 (horizontal) and x (vertical depth).
- By Baudhāyana-Pythagoras: **$3^2 + x^2 = (x + 1)^2 \Rightarrow 9 + x^2 = x^2 + 2x + 1$** .
- Subtracting x^2 : $9 = 2x + 1 \Rightarrow 2x = 8 \Rightarrow$ **$x = 4$ units**. Lake depth = 4 units.

7. Comprehensive Practice Questions (Textbook + Advanced Extras)

Attempt each problem rigorously. Complete step-by-step solutions are printed on Page 6.

Q1 (Textbook). Construct a square whose area is equal to the difference of the areas of squares of side lengths 7 units and 5 units. What is its exact side length?

Q2 (Textbook). Find the missing side length in each right triangle:

- (i) Legs $a = 7$, $b = 9$; hypotenuse $c = ?$
- (ii) Leg $a = 40$, hypotenuse $c = 41$; leg $b = ?$
- (iii) Isosceles right triangle with hypotenuse $c = \sqrt{200}$, equal legs $= ?$
- (iv) Leg $a = 10$, leg $b = \sqrt{150}$; hypotenuse $c = ?$

Q3 (Textbook). A rhombus has diagonals of lengths 24 units and 70 units. Find the side length and perimeter of the rhombus.

Q4 (Textbook). Find the area of an equilateral triangle with side length 6 units. (Hint: Show that the altitude bisects the base to create two 30° - 60° - 90° right triangles).

Q5 (Coordinate Grid Square Construction). Using the intersection dots of a standard square grid (integer lattice), is it possible to draw a square having an area of: (a) 2 sq. units, (b) 3 sq. units, (c) 4 sq. units, (d) 5 sq. units? State the mathematical condition for realizable integer areas.

Q6 (Triples Analysis). True or False:

- (i) Every Baudhāyana triple is either a primitive triple or a scaled version of a primitive triple.
- (ii) The hypotenuse is always strictly the longest side in any right-angled triangle.
- (iii) Every primitive triple can be generated by the odd square formula $(n - 1)^2 + (2n - 1) = n^2$.

Q7 (Extra: Broken Bamboo Problem). A bamboo 32 cubits tall is broken by the wind such that its tip meets the ground 16 cubits away from its base. At what height from the ground did the bamboo break?

Q8 (Extra: 3D Diagonal). A rectangular box has dimensions $3 \text{ cm} \times 4 \text{ cm} \times 12 \text{ cm}$. Find the length of the longest rigid rod that can fit entirely inside this box.

Q9 (Logic Puzzle: Box Labeling). Three boxes contain only Red balls, only Blue balls, and a Mixture (both colors). All 3 boxes are labeled incorrectly (RED, BLUE, MIXED). How can you determine the correct contents by drawing only one ball from one box?

8. Complete Step-by-Step Solutions & Answers

Sol 1: Difference of areas = $7^2 - 5^2 = 49 - 25 = 24$.

Geometric construction: Construct a right triangle with hypotenuse 7 and one leg 5. By Baudhāyana: other leg² = $7^2 - 5^2 = 24$. A square built on this leg has area **24 sq. units** with side $\sqrt{24} = 2\sqrt{6} \approx 4.899$ units.

Sol 2:

(i) $c^2 = 7^2 + 9^2 = 49 + 81 = 130 \Rightarrow c = \sqrt{130} \approx 11.40$ units.

(ii) $b^2 = 41^2 - 40^2 = (41 - 40)(41 + 40) = 1 \times 81 = 81 \Rightarrow b = 9$ units.

(iii) $2a^2 = (\sqrt{200})^2 = 200 \Rightarrow a^2 = 100 \Rightarrow a = 10$ units.

(iv) $c^2 = 10^2 + (\sqrt{150})^2 = 100 + 150 = 250 \Rightarrow c = \sqrt{250} = 5\sqrt{10} \approx 15.81$ units.

Sol 3: The diagonals of a rhombus are perpendicular bisectors of each other.

Half-diagonals: $d_1/2 = 24/2 = 12$, $d_2/2 = 70/2 = 35$.

Side length $s = \sqrt{(12^2 + 35^2)} = \sqrt{(144 + 1225)} = \sqrt{1369} = 37$ units.

Perimeter = $4 \times 37 = 148$ units.

Sol 4: In an equilateral triangle of side 6, the altitude splits the base into halves of 3 units.

Altitude $h = \sqrt{(6^2 - 3^2)} = \sqrt{(36 - 9)} = \sqrt{27} = 3\sqrt{3}$ units.

Area = $\frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 6 \times 3\sqrt{3} = 9\sqrt{3}$ sq. units ≈ 15.588 sq. units.

Sol 5: On a square lattice, the area of any square whose vertices lie on grid points is $a^2 + b^2$ for integers a, b (representing horizontal and vertical shifts between adjacent vertices):

- Area 2 = $1^2 + 1^2$: **Possible** (tilted square of leg 1, 1).
- Area 3: Cannot be written as sum of two integer squares (0+3, 1+2 impossible): **Impossible!**
- Area 4 = $2^2 + 0^2$: **Possible** (axis-aligned 2×2 square).
- Area 5 = $2^2 + 1^2$: **Possible** (tilted square using vector (2, 1)).

Sol 6:

(i) **True:** Dividing any non-primitive triple by its gcd produces a primitive triple.

(ii) **True:** In a right triangle, the right angle (90°) is the greatest angle; hence the opposite side (hypotenuse) is strictly the longest.

(iii) **False:** The odd square method only produces triples where $c - b = 1$. Primitive triples such as **(8, 15, 17)** and **(20, 21, 29)** have $c - b = 2$ and cannot be generated by this formula alone.

Sol 7: Let break height = h . Top segment = $32 - h$ (hypotenuse). Base = 16.

$h^2 + 16^2 = (32 - h)^2 \Rightarrow h^2 + 256 = 1024 - 64h + h^2$.

$64h = 1024 - 256 = 768 \Rightarrow h = 12$ cubits.

Sol 8: 3D space diagonal $D = \sqrt{(l^2 + w^2 + h^2)} = \sqrt{(3^2 + 4^2 + 12^2)} = \sqrt{(9 + 16 + 144)} = \sqrt{169} = 13$ cm.

Sol 9: Draw 1 ball from the box labeled **MIXED**. Since all labels are incorrect, this box MUST contain only one color! If the ball drawn is Red, then that box is **RED**. Then the box labeled **BLUE** must be **MIXED**, and the box labeled **RED** must be **BLUE**.

✓ Chapter 2 Complete Mastery & Rigorous Foundation Achieved

Ganita Prakash Grade 8 Part-II