CHAPTER 1

REAL NUMBERS

1.1 INTRODUCTION

Since our childhood we have been using four fundamental operations of addition, subtraction, multiplication and division. We have applied these operations on natural numbers, integers, rational and irrational numbers. In this chapter, we will begin with a brief recall of divisibility on integers and will state some important properties of integers, namely, Euclid's division Lemma, Euclid's division algorithm and the Fundamental Theorem of Arithmetic which will be used in the remaining part of this chapter to learn more about integers and real numbers.

Euclid's division lemma tells us about divisibility of integers. It is quite easy to state and understand. It states that any positive integer a can be divided by any other positive integer b in such a way that it leaves a remainder r that is smaller than b. This is nothing but the usual long division process. Euclid's division lemma provides us a step-wise procedure to compute the HCF of two positive integers. This step-wise procedure is known as Euclid's algorithm. We will use the same for finding the HCF of positive integers.

The Fundamental Theorem of Arithmetic tells us about expressing positive integers as the product of prime integers. It states that every positive integer is either prime or it can be factorized (expressed) as a product of powers of prime integers. This theorem has many significant applications in mathematics and in other fields. We have learnt how to find the HCF and LCM of positive integers by using the Fundamental Theorem of Arithmetic in earlier classes. In this chapter, we will apply this theorem to prove the irrationality of many numbers such as $\sqrt{2}, \sqrt{3}, \sqrt{5}$ etc. We know that the decimal representation of a rational number is either terminating or if it is non-terminating then it is repeating. The prime factorisation of the denominator of a rational number completely reveals the nature of its decimal representation. In fact, by looking at the prime factorisation of the denominator of a rational number one can easily tell about its decimal representation whether it is terminating or non-terminating repeating. We will also use the Fundamental Theorem of Arithmetic to determine the nature of the decimal expansion of rational numbers.

Let us begin with the divisibility of integers.

1.2 DIVISIBILITY

We have been studying division of numbers for the last many years. Let us recall the same in a formal manner.

DEFINITION *A non-zero integer 'a' is said to divide an integer 'b' if there exists an integer c such that* $b = ac$.

The integer 'b' is called the *dividend*, integer 'a' is known as the divisor and integer 'c' is known as the quotient.

For example, 3 divides 36 because there is an integer 12 such that $36 = 3 \times 12$. However, 3 does not divide 35 because there do not exist an integer c such that $35 = 3 \times c$. In other words, $35 = 3 \times c$ is not true for any integer c.

If a non-zero integer 'a' divides an integer b, then we write $a \mid b$. This is read as "a divides b". When $a \mid b$, we say that 'b is divisible by a' or 'a is a factor of b' or 'b is a multiple of a' or 'a is a divisor of b'.

We write $a \nmid b$ to indicate that b is not divisible by a.

We observe that:

(i) $-4 \mid 20$, because there exists an integer -5 such that $20 = -4 \times (-5)$.

(ii) $4 \mid -20$, because there exists an integer -5 such that $-20 = 4 \times (-5)$.

(iii) $-4 \mid -20$, because there exists an integer 5 such that $-20 = -4 \times 5$.

ILLUSTRATION *State whether the following are true or not:*

(i) $3 \mid 93$ (ii) $6 \mid 28$ (iii) $0 \mid 4$ (iv) $5 \mid 0$ (v) $-2 \mid 8$

(vi) $-7 \mid -35$ (vii) $6 \mid 6$ (viii) $8 \mid -8$ (ix) $13 \mid -25$ (x) $1 \mid -1$

SOLUTION We observe that:

(i) $3 \mid 93$ is true, because $93 = 3 \times 31$

(ii) $6 \mid 28$ is not true, because $28 = 6c$ is not valid for any integer c.

(iii) $0 \mid 4$ is not true by definition.

(iv) $5 \mid 0$ is true, because $0 = 5 \times 0$

(v) $-2 \mid 8$ is true, because $8 = (-2) \times (-4)$

(vi) $-7 \mid -35$ is true, because $-35 = (-7) \times 5$

(vii) $6 \mid 6$ is true, because $6 = 6 \times 1$

(viii) $8 \mid -8$ is true, because $-8 = 8 \times -1$

(ix) $13 \mid -25$ is not true, because $-25 = 13 \times c$ is not valid for any integer c.

(x) $1 \mid -1$ is true, because $-1 = 1 \times -1$

Following are some properties of divisibility:

(i) ± 1 divides every non-zero integer.
i.e, $\pm 1 \mid a$ for every non-zero integer a.

(ii) 0 is divisible by every non-zero integer a.
i.e., $a \mid 0$ for every non-zero integer a.

(iii) 0 does not divide any integer.

(iv) If a is a non-zero integer and b is any integer, then
$$a \mid b \Rightarrow a \mid -b,\ -a \mid b \text{ and } -a \mid -b$$

(v) If a and b are non-zero integers, then
$$a \mid b \text{ and } b \mid a \Rightarrow a = \pm b$$

(vi) If a is a non-zero integer and b, c are any two integers, then
$$a \mid b \text{ and } a \mid c \Rightarrow \begin{cases} a \mid b \pm c \\ a \mid bc \\ a \mid bx & \text{for any integer } x. \end{cases}$$

(vii) If a and c are non-zero integers and b, d are any two integers, then

(i) $a \mid b$ and $c \mid d \Rightarrow ac \mid bd$

(ii) $ac \mid bc \Rightarrow a \mid b$

REMARK *If a divides b, then by property (iv) a divides $-b$ and $-a$ divides b. It is therefore enough if we consider positive divisors of positive integers only.*

Thus, whenever we will speak of divisors in this chapter we will mean positive divisors of positive integers.

1.3 EUCLID'S DIVISION LEMMA

Euclid was the first Greek Mathematician who initiated a new way of thinking the study of geometry. He is famous for his *Elements of Geometry* but few people are aware that he also made important contributions to the number theory. Among these is the Euclid's Lemma. A lemma is a proven statement which is used to prove other statements. This lemma was perhaps known for a long time, but was first recorded in Book VII of Euclid's Elements. Euclid's division algorithm is based on this lemma. This lemma is nothing but a restatement of the long division process we have been doing for the last many years.

Consider the division of one positive integer by another, say 58 by 9. The division can be carried out as follows:

$$9\overline{)\,58\,}(\,6$$
$$\underline{54}$$
$$4$$

While carrying out this division, we had to think of about the largest multiple of 9 which does not exceed 58 so that after subtraction the remainder 4 is less than the divisor 9. The result of this division is that we get two integers, namely, 6 which is called the quotient and 4 which is called the remainder. We can write the result in the following form:

$$58 = 9 \times 6 + 4, \quad 0 \leq 4 < 9$$

Let us now apply the same procedure to other pairs of positive integers to see whether such a representation is always possible or not.

Pair of integers	*Representation*	
25,7	$25 = 7 \times 3 + 4, 0 \leq 4 < 7$	[$\because$ 7 goes into 25 thrice and leaves remainder 4]
20, 3	$20 = 3 \times 6 + 2, \ 0 \leq 2 < 3$	[$\because$ 3 goes into 20 six times and leaves remainder 2]
7, 15	$7 = 15 \times 0 + 7, \ 0 \leq 7 < 15$	[$\because$ 15 is larger than 7. So, this relation is always possible]
35, 5	$35 = 5 \times 7 + 0, \ 0 \leq 0 < 5$	[$\because$ 5 goes into 35 seven times and leaves no remainder]

It is evident from the above discussion that the above representation also holds for other pairs of integers. We also observe that for each pair of positive integers a and b, we can find unique integers q and r satisfying the relation $a = bq + r, 0 \leq r < b$. In fact, this holds for every pair of positive integers as proved in the following lemma.

REMARK 1 *The above Lemma is nothing but a restatement of the long division process we have been doing all these years, and that the integers q and r are called the quotient and remainder, respectively.*

REMARK 2 *The above Lemma has been stated for positive integers only. But, it can be extended to all integers as stated below:*

Let a and b be any two integers with $b \neq 0$. Then, there exist unique integers q and r such that

$$a = bq + r, \text{ where } 0 \leq r < |b|$$

REMARK 3 (i) *When a positive integer is divided by 2, the remainder is either 0 or 1. So, any positive integer is of the form 2m, 2m + 1 for some interger m.*

(ii) *When any positive integer is divided by 3, the remainder is 0 or 1 or 2. So, any positive integer can be written in the form 3m, 3m + 1, 3m + 2 for some integer m.*

(iii) *When a positive integer is divided by 4, the remainder can be 0 or 1 or 2 or 3. So, any positive integer is of the form 4q or, 4q + 1 or, 4q + 2 or, 4q + 3.*

Let us now discuss some problems to illustrate the applications of Euclid's division Lemma.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Show that every positive even integer is of the form 2q, and that every positive odd integer is of the form $2q + 1$, where q is some integer.* [NCERT]

SOLUTION Let a be any positive integer and $b = 2$. Then, by Euclid's division Lemma there exist integers q and r such that

$$a = 2q + r, \text{ where } 0 \leq r < 2$$

Now, $0 \leq r < 2 \Rightarrow 0 \leq r \leq 1 \Rightarrow r = 0$ or, $r = 1$ [$\because$ r is an integer]

$\therefore$ $a = 2q$ or, $a = 2q + 1$

If $a = 2q$, then a is an even integer.

We know that an integer can be either even or odd. Therefore, any odd integer is of the form $2q + 1$.

EXAMPLE 2 *Show that any positive integer is of the form $3q$ or, $3q + 1$ or, $3q + 2$ for some integer q.*

SOLUTION Let a be any positive integer and $b = 3$. Applying division Lemma with a and $b = 3$, we have

$$a = 3q + r, \text{ where } 0 \leq r < 3 \text{ and } q \text{ is some integer}$$

$\Rightarrow$ $a = 3q + 0$ or, $a = 3q + 1$ or, $a = 3q + 2$

$\Rightarrow$ $a = 3q$ or, $a = 3q + 1$ or, $a = 3q + 2$ for some integer q.

EXAMPLE 3 *Show that any positive odd integer is of the form $4q + 1$ or $4q + 3$, where q is some integer.* [NCERT]

SOLUTION Let a be any odd positive integer and $b = 4$. By division Lemma there exists integers q and r such that

$$a = 4q + r, \text{ where } 0 \leq r < 4$$

EUCLID'S DIVISION LEMMA *Let a and b be any two positive integers. Then, there exist unique integers q and r such that*

$$a = bq + r, \; 0 \le r < b$$

If $b \mid a$, then $r = 0$. Otherwise, r satisfies the stronger inequality $0 < r < b$.

PROOF Consider the following arithmetic progression

$$\ldots\ldots, a - 3b, a - 2b, a - b, a, a + b, a + 2b, a + 3b, \ldots\ldots$$

Clearly, it is an arithmetic progression with common difference 'b' and it extends indefinitely in both directions.

Let r be the smallest non-negative term of this arithmetic progression. Then, there exists a non-negative integer q such that

$$a - bq = r \Rightarrow a = bq + r$$

As, r is the smallest non-negative integer satisfying the above result. Therefore, $0 \le r < b$

Thus, we have

$$a = bq + r, \text{ where } 0 \le r < b$$

We shall now prove the uniqueness of q and r.

Uniqueness To prove the uniqueness of q and r, let us assume that there is another pair q_1 and r_1 of non-negative integers satisfying the same relation i.e.,

$$a = bq_1 + r_1, \; 0 \le r_1 < b$$

We shall now prove that $r_1 = r$ and $q_1 = q$

We have,

$$a = bq + r \text{ and } a = bq_1 + r_1$$

$\Rightarrow \quad bq + r = bq_1 + r_1$

$\Rightarrow \quad r_1 - r = bq - bq_1$

$\Rightarrow \quad r_1 - r = b(q - q_1)$

$\Rightarrow \quad b \mid r_1 - r$

$\Rightarrow \quad r_1 - r = 0 \quad$ $[\because 0 \le r < b \text{ and } 0 \le r_1 < b \Rightarrow 0 \le r_1 - r < b]$

$\Rightarrow \quad r_1 = r$

Now, $\quad r_1 = r$

$\Rightarrow \quad -r_1 = -r \quad$ [Multiplying both sides by (–1)]

$\Rightarrow \quad a - r_1 = a - r \quad$ [Adding a on both sides]

$\Rightarrow \quad bq_1 = bq \quad$ $\left[\begin{array}{l} \because a = bq + r \text{ and } a = bq_1 + r_1 \\ \because bq = a - r \text{ and } bq_1 = a - r_1 \end{array}\right]$

$\Rightarrow \quad q_1 = q$

Hence, the representation $a = bq + r, \; 0 \le r < b$ is unique.

Q.E.D.

$\Rightarrow \quad a = 4q$ or, $a = 4q+1$ or, $a = 4q+2$ or, $a = 4q+3 \quad [\because 0 \le r < 4 \Rightarrow r = 0,1,2,3]$

$\Rightarrow \quad a = 4q+1$ or, $a = 4q+3 \quad [\because a$ is an odd integer $\therefore a \ne 4q, a \ne 4q+2]$

Hence, any odd integer is of the form $4q+1$ or, $4q+3$.

EXAMPLE 4 *Show that the square of an odd integer is of the form $4q+1$ for the some integer q.* [NCERT EXEMPLAR]

SOLUTION Let x be any odd integer. Then, $x = 2m+1$, for some integer m.

$\therefore \quad x^2 = (2m+1)^2 = 4m^2 + 4m + 1 = 4q + 1$, for some integer $q = m^2 + m$.

LEVEL-2

EXAMPLE 5 *Show that $n^2 - 1$ is divisible by 8, if n is an odd positive integer.*

[NCERT EXEMPLAR]

SOLUTION We know that any odd positive integer is of the form $4q+1$ or, $4q+3$ for some integer q.

So, we have the following cases:

CASE I *When $n = 4q+1$*: In this case, we have

$$n^2 - 1 = (4q+1)^2 - 1 = 16q^2 + 8q + 1 - 1 = 16q^2 + 8q = 8q(2q+1)$$

$\Rightarrow \quad n^2 - 1$ is divisible by 8 $\quad [\because 8q(2q+1)$ is divisible by $8]$

CASE II *When $n = 4q+3$*: In this case, we have

$$n^2 - 1 = (4q+3)^2 - 1 = 16q^2 + 24q + 9 - 1 = 16q^2 + 24q + 8$$

$\Rightarrow \quad n^2 - 1 = 8(2q^2 + 3q + 1) = 8(2q+1)(q+1)$

$\Rightarrow \quad n^2 - 1$ is divisible by 8 $\quad [\because 8(2q+1)(q+1)$ is divisible by $8]$

Hence, $n^2 - 1$ is divisible by 8.

EXAMPLE 6 *Prove that if x and y are odd positive integers, then $x^2 + y^2$ is even but not divisible by 4.*

SOLUTION We know that any odd positive integer is of the form $2q+1$ for some integer q.

So, let $x = 2m+1$ and $y = 2n+1$ for some integers m and n.

$\therefore \quad x^2 + y^2 = (2m+1)^2 + (2n+1)^2$

$\Rightarrow \quad x^2 + y^2 = 4(m^2 + n^2) + 4(m+n) + 2$

$\Rightarrow \quad x^2 + y^2 = 4\{(m^2 + n^2) + (m+n)\} + 2$

$\Rightarrow \quad x^2 + y^2 = 4q + 2$, where $q = (m^2 + n^2) + (m+n)$

$\Rightarrow \quad x^2 + y^2$ is even and leaves remainder 2 when divided by 4

$\Rightarrow \quad x^2 + y^2$ is even but not divisible by 4

EXAMPLE 7 *Prove that $n^2 - n$ is divisible by 2 for every positive integer n.*

SOLUTION We know that any positive integer is of the form $2q$ or $2q+1$, for some integer q.

So, following cases arise:

CASE I *When* $n = 2q$: In this case, we have

$$n^2 - n = (2q)^2 - 2q = 4q^2 - 2q = 2q(2q-1)$$

$\Rightarrow$ $n^2 - n = 2r$, where $r = q(2q-1)$

$\Rightarrow$ $n^2 - n$ is divisible by 2

CASE II *When* $n = 2q+1$: In this case, we have

$$n^2 - n = (2q+1)^2 - (2q+1) = (2q+1)(2q+1-1) = 2q(2q+1)$$

$\Rightarrow$ $n^2 - n = 2r$, where $r = q(2q+1)$

$\Rightarrow$ $n^2 - n$ is divisible by 2.

Hence, $n^2 - n$ is divisible by 2 for every positive integer n.

EXAMPLE 8 *Show that the square of any positive integer is of the form 3m or, 3m + 1 for some integer m.* **[NCERT, CBSE 2008]**

SOLUTION Let a be any positive integer. Then, it is of the form $3q$ or, $3q+1$ or, $3q+2$. So, we have the following cases:

CASE I *When* $a = 3q$: In this case, we have

$$a^2 = (3q)^2 = 9q^2 = 3q(3q) = 3m, \text{ where } m = 3q$$

CASE II *When* $a = 3q+1$: In this case, we have

$$a^2 = (3q+1)^2 = 9q^2 + 6q + 1 = 3q(3q+2) + 1 = 3m+1, \text{ where } m = q(3q+2)$$

CASE III *When* $a = 3q+2$: In this case, we have

$$a^2 = (3q+2)^2 = 9q^2 + 12q + 4 = 9q^2 + 12q + 3 + 1$$

$$= 3(3q^2 + 4q + 1) + 1 = 3m + 1, \text{ where } m = 3q^2 + 4q + 1$$

Hence, a is of the form $3m$ or $3m+1$.

EXAMPLE 9 *Use Euclid's division Lemma to show that the cube of any positive integer is either of the form 9m, 9m + 1 or, 9m + 8 for some integer m.* **[NCERT]**

SOLUTION Let x be any positive integer. Then, it is of the form $3q$ or, $3q+1$ or, $3q+2$. So, we have the following cases:

CASE I *When* $x = 3q$: In this case, we have

$$x^3 = (3q)^3 = 27q^3 = 9(3q^3) = 9m, \text{ where } m = 3q^3$$

CASE II *When* $x = 3q+1$: In this case, we have

$$x^3 = (3q+1)^3$$

$\Rightarrow$ $x^3 = 27q^3 + 27q^2 + 9q + 1$

$\Rightarrow$ $x^3 = 9q(3q^2 + 3q + 1) + 1$

$\Rightarrow$ $x^3 = 9m + 1$, where $m = q(3q^2 + 3q + 1)$

CASE III *When* $x = 3q+2$: In this case, we have

$$x^3 = (3q+2)^3$$

$\Rightarrow$ $x^3 = 27q^3 + 54q^2 + 36q + 8$

$\Rightarrow \quad x^3 = 9q\left(3q^2 + 6q + 4\right) + 8$

$\Rightarrow \quad x^3 = 9m + 8$, where $m = q\left(3q^2 + 6q + 4\right)$

Hence, x^3 is either of the form $9m$ or, $9m + 1$ or, $9m + 8$.

EXAMPLE 10 *Show that the cube of any positive integer is of the form $4m$, $4m + 1$, or $4m + 3$ for some integer m.*

SOLUTION Let n be any positive integer. Then, it is of the form $4q, 4q + 1, 4q + 2$, and $4q + 3$. So, we have the following cases:

CASE I *When* $n = 4q$: In this case, we have

$$n^3 = (4q)^3 = 64q^3 = 4(16q)^3 = 4m, \text{ where } m = 16q^3$$

CASE II *When* $n = 4q + 1$: In this case, we have

$$n^3 = (4q + 1)^3 = 64q^3 + 48q^2 + 12q + 1$$
$$= 4(16q^2 + 12q + 3q) + 1 = 4m + 1, \text{ where } m = 16q^2 + 12q + 3q$$

CASE III *When* $n = 4q + 2$: In this case, we have

$$n^3 = (4q + 2)^3 = 64q^3 + 96q^2 + 48q + 8$$
$$= 4(16q^3 + 24q^2 + 12q + 2) = 4m, \text{ where } m = 16q^3 + 24q^2 + 12q + 2$$

CASE IV *When* $n = 4q + 3$: In this case, we have

$$n^3 = (4q + 3)^3 = 64q^3 + 144q^2 + 108q + 27$$
$$= 64q^3 + 144q^2 + 108q + 24 + 3$$
$$= 4(16q^3 + 36q^2 + 27q + 6) + 3 = 4m + 3, \text{ where } m = 16q^3 + 36q^2 + 27q + 6$$

Hence, n is of the form $4m$, $4m + 1$ or $4m + 3$.

EXAMPLE 11 *Show that the square of any positive integer cannot be of the form $5q + 2$ or $5q + 3$ for any integer q.* [NCERT EXEMPLAR]

SOLUTION Let x be any positive integer. When we divide x by 5, the remainder is either 0 or 1 or 2 or 3 or 4. So, x can be written as $x = 5m$, or $x = 5m + 1$ or $x = 5m + 2$ or $x = 5m + 3$ or $x = 5m + 4$. Thus, we have the following cases:

CASE I *When* $x = 5m$: In this case,

$$x^2 = 25m^2 = 5(5m)^2 = 5q, \text{ where } q = 5m^2$$

CASE II *When* $x = 5m + 1$: In this case,

$$x^2 = (5m + 1)^2 = 25m^2 + 10m + 1 = 5(5m^2 + 2m) + 1 = 5q + 1, \text{ where } q = 5m^2 + 2m$$

CASE III *When* $x = 5m + 2$: In this case,

$$x^2 = (5m + 2)^2 = 25m^2 + 20m + 4 = 5(5m^2 + 4m) + 4 = 5q + 4, \text{ where } q = 5m^2 + 4m$$

CASE IV *When* $x = 5m + 3$: In this case,

$$x^2 = (5m + 3)^2 = 25m^2 + 30m + 9 = (5m^2 + 30m + 5) + 4$$
$$= 5(5m^2 + 6m + 1) + 4 = 5q + 4, \text{ where } q = 5m^2 + 6m + 1$$

CASE V *When* $x = 5m + 4$: In this case,

$$x^2 = (5m + 4)^2 = 25m^2 + 40m + 16$$

$$= 5(5m^2 + 8m + 3) + 1 = 5q + 1 \text{ where } q = 5m^2 + 8m + 3$$

Hence, x is of the form $5q$ or $5q + 1$, $5q + 4$. So, it cannot be of the form $5q + 2$ or $5q + 3$.

EXAMPLE 12 *Show that one and only one out of n, $n + 2$ or, $n + 4$ is divisible by 3, where n is any positive integer.*

SOLUTION We know that any positive integer is of the form $3q$ or, $3q + 1$ or, $3q + 2$ for some integer q and one and only one of these possibilities can occur.

So, we have following cases:

CASE I *When* $n = 3q$: In this case, we have

$n = 3q$, which is divisible by 3

Now, $n = 3q$

$\Rightarrow$ $n + 2 = 3q + 2$,

$\Rightarrow$ $n + 2$ leaves remainder 2 when divided by 3

$\Rightarrow$ $n + 2$ is not divisible by 3

Again, $n = 3q$

$\Rightarrow$ $n + 4 = 3q + 4 = 3(q + 1) + 1$

$\Rightarrow$ $n + 4$ leaves remainder 1 when divided by 3

$\Rightarrow$ $n + 4$ is not divisible by 3

Thus, n is divisible by 3 but $n + 2$ and $n + 4$ are not divisible by 3.

CASE II *When* $n = 3q + 1$: In this case, we have

$n = 3q + 1$

$\Rightarrow$ n leaves remainder 1 when divided by 3

$\Rightarrow$ n is not divisible by 3

Now, $n = 3q + 1$

$\Rightarrow$ $n + 2 = (3q + 1) + 2 = 3(q + 1)$

$\Rightarrow$ $n + 2$ is divisible by 3

Again, $n = 3q + 1$

$\Rightarrow$ $n + 4 = 3q + 1 + 4 = 3q + 5 = 3(q + 1) + 2$

$\Rightarrow$ $n + 4$ leaves remainder 2 when divided by 3

$\Rightarrow$ $n + 4$ is not divisible by 3

Thus, $n + 2$ is divisible by 3 but n and $n + 4$ are not divisible by 3

CASE III *When* $n = 3q + 2$: In this case, we have

$n = 3q + 2$

$\Rightarrow$ n leaves remainder 2 when divided by 3

$\Rightarrow$ n is not divisible by 3

Now, $n = 3q + 2$

$\Rightarrow \quad n + 2 = 3q + 2 + 2 = 3(q + 1) + 1$

$\Rightarrow \quad n + 2$ leaves remainder 1 when divided by 3

$\Rightarrow \quad n + 2$ is not divisible by 3

Again, $\quad n = 3q + 2$

$n + 4 = 3q + 2 + 4 = 3(q + 2)$

$\Rightarrow \quad n + 4$ is divisible by 3

Thus, $n + 4$ is divisible by 3 but n and $n + 2$ are not divisible by 3.

EXAMPLE 13 *Prove that one of every three consecutive positive integers is divisible by 3.*

SOLUTION Let n, $n + 1$, $n + 2$ be three consecutive positive integers. We know that n is of the form $3q$, $3q + 1$ or, $3q + 2$. So, we have the following cases:

CASE I *When* $n = 3q$: In this case,

n is divisible by 3 but $n + 1$ and $n + 2$ are not divisible by 3.

CASE II *When* $n = 3q + 1$: In this case,

$n + 2 = 3q + 1 + 2 = 3(q + 1)$ is divisible by 3 but n and $n + 1$ are not divisible by 3.

CASE III *When* $n = 3q + 2$: In this case,

$n + 1 = 3q + 1 + 2 = 3(q + 1)$ is divisible by 3 but n and $n + 2$ are not divisible by 3.

Hence, one of n, $n + 1$ and $n + 2$ is divisible by 3.

EXERCISE 1.1

LEVEL-1

1. If a and b are two odd positive integers such that $a > b$, then prove that one of the two numbers $\frac{a+b}{2}$ and $\frac{a-b}{2}$ is odd and the other is even.
2. Prove that the product of two consecutive positive integers is divisible by 2.
3. Prove that the product of three consecutive positive integer is divisible by 6.
4. For any positive integer n, prove that $n^3 - n$ divisible by 6. [NCERT EXEMPLAR]
5. Prove that if a positive integer is of the form $6q + 5$, then it is of the form $3q + 2$ for some integer q, but not conversely.
6. Prove that the square of any positive integer of the form $5q + 1$ is of the same form.
7. Prove that the square of any positive integer is of the form $3m$ or, $3m + 1$ but not of the form $3m + 2$.
8. Prove that the square of any positive integer is of the form $4q$ or $4q + 1$ for some integer q. [NCERT EXEMPLAR]
9. Prove that the square of any positive integer is of the form $5q$, $5q + 1$, $5q + 4$ for some integer q.
10. Show that the square of an odd positive integer is of the form $8q + 1$, for some integer q.

11. Show that any positive odd integer is of the form $6q+1$ or, $6q+3$ or, $6q+5$, where q is some integer. [NCERT]

LEVEL-2

12. Show that the square of any positive integer cannot be of the form $6m+2$ or $6m+5$ for any integer m. [NCERT EXEMPLAR]
13. Show that the cube of a positive integer is of the form $6q+r$, where q is an integer and $r = 0, 1, 2, 3, 4, 5$. [NCERT EXEMPLAR]
14. Show that one and only one out of $n, n+4, n+8, n+12$ and $n+16$ is divisible by 5, where n is any positive integer. [NCERT EXEMPLAR]
15. Show that the square of an odd positive integer can be of the form $6q+1$ or $6q+3$ for some integer q. [NCERT EXEMPLAR]
16. A positive integer is of the form $3q+1$, q being a natural number. Can you write its square in any form other than $3m+1$, $3m$ or $3m+2$ for some integer m? Justify your answer.
17. Show that the square of any positive integer cannot be of the form $3m+2$, where m is a natural number.

HINTS TO SELECTED PROBLEMS

2. Let $n-1$ and n be two consecutive positive integers. Then, their product is $(n-1)n = n^2 - n$. Now, proceed as in example 6.
3. Let n be any positive integer. Since any positive integer is of the form $6q$ or, $6q+1$ or, $6q+2$ or, $6q+3$ or, $6q+4$ or, $6q+5$.

   If $n = 6q$, then

   $n(n+1)(n+2) = 6q(6q+1)(6q+2)$, which is divisible by 6

   If $n = 6q+1$, then

   $n(n+1)(n+2) = (6q+1)(6q+2)(6q+3) = 6(6q+1)(3q+1)(2q+1)$, which is divisible by 6.

   If $n = 6q+2$, then

   $n(n+1)(n+2) = (6q+2)(6q+3)(6q+4) = 12(3q+1)(2q+1)(2q+3)$, which is divisible by 6.

   Similarly, $n(n+1)(n+2)$ is divisible by 6 if $n = 6q+3$ or, $6q+4$ or, $6q+5$.
4. We have,

   $n^3 - n = (n-1)(n)(n+1)$, which is product of three consecutive positive integers. So, proceed as in Q. No. 3.
5. Let $n = 6q+5$, where q is a positive integer. We know that any positive integer is of the form $3k$ or, $3k+1$ or, $3k+2$.

   $\therefore \quad q = 3k$ or, $3k+1$ or, $3k+2$

   If $q = 3k$, then

   $n = 6q+5 = 18k+5 = 3(6k+1)+2 = 3m+2$, where $m = 6k+1$

   If $q = 3k+1$, then

   $n = 6q+5 = 6(3k+1)+5 = 3(6k+3)+2 = 3m+2$, where $m = 6k+3$

If $q = 3k + 2$, then

$$n = 6q + 5 = 6(3k + 2) + 5 = 3(6k + 5) + 2 = 3m + 2, \text{ where } m = 6k + 5.$$

6. Let $n = 5q + 1$. Then,

$$n^2 = 25q^2 + 10q + 1 = 5(5q^2 + 2q) + 1 = 5m + 1, \text{ where } m = 5q^2 + 2q$$

$\Rightarrow$ n^2 is of the form $5m + 1$.

7. Any positive ingeger n is of the form $3q$, $3q + 1$ or, $3q + 2$.

If $n = 3q$, then

$$n^2 = 9q^2 = 3(3q^2) = 3m, \text{ where } m = 3q^2$$

If $n = 3q + 1$, then

$$n^2 = 9q^2 + 6q + 1 = 3q(3q + 2) + 1 = 3m + 1, \text{ where } m = q(3q + 2)$$

If $n = 3q + 2$, then

$$n^2 = (3q + 2)^2 = 9q^2 + 12q + 4 = 3(3q^2 + 4q + 1) + 1 = 3m + 1, \text{ where } m = 3q^2 + 4q + 1.$$

Hence, n^2 is of the form $3m$ or, $3m + 1$ but not of the form $3m + 2$.

8. Any positive integer n is of the form $2m$ or, $2m + 1$

If $n = 2m$, then

$$n^2 = 4m^2 = 4q, \text{ where } q = m^2$$

If $n = 2m + 1$, then

$$n^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 4m(m + 1) + 1 = 4q + 1, \text{ where } q = m(m + 1)$$

9. Any positive integer n is of the form $5m$ or $5m + 1$, or $5m + 2$ or $5m + 3$ or $5m + 4$.

If $n = 5m$, then

$$n^2 = 25m^2 = 5(5m^2) = 5q, \text{ where } q = 5m^2$$

If $n = 5m + 1$, then

$$n^2 = (5m + 1)^2 = 5m(5m + 2) + 1 = 5q + 1, \text{ where } q = m(5m + 2)$$

If $n = 5m + 2$, then

$$n^2 = (5m + 2)^2 = 5m(5m + 4) + 4 = 5q + 4, \text{ where } q = m(5m + 4)$$

If $n = 5m + 3$, then

$$n^2 = (5m + 3)^2 = 5(m^2 + 6m + 1) + 4 = 5q + 4, \text{ where } q = 5m^2 + 6m + 1$$

If $n = 5m + 4$, then

$$n^2 = 5(5m^2 + 8m + 3) + 1 = 5q + 1, \text{ where } q = 5m^2 + 8m + 3$$

Hence, n^2 is of the form $5q$ or, $5q + 1$ or, $5q + 4$.

10. Since, any odd positive integer n is of the form $4m + 1$ or $4m + 3$.

If $n = 4m + 1$, then

$$n^2 = (4m + 1)^2 = 16m^2 + 8m + 1 = 8m(m + 1) + 1 = 8q + 1 \text{ where } q = m(m + 1)$$

If $n = 4m + 3$, then

$$n^2 = (4m + 3)^2 = 16m^2 + 24m + 9 = 8(2m^2 + 3m + 1) + 1 = 8q + 1, \text{ where } q = 2m^2 + 3m + 1$$

Hence, n^2 is of the form $8q + 1$.

11. Let a be any odd positive integer and $b = 6$. Then, there exist integers q and r such that

$a = 6q + r, 0 \le r < 6$ [By division algorithm]

$\Rightarrow \quad a = 6q$ or, $6q + 1$ or, $6q + 2$ or, $6q + 3$ or, $6q + 4$ or, $6q + 5$

But, $6q, 6q + 2$ and $6q + 4$ are even positive integers.

$\therefore \quad a = 6q + 1$ or, $6q + 3$ or, $6q + 5$

12. We know that any positive integer x can be of the form $6m$, $6m + 1$, $6m + 2$, $6m + 3$, $6m + 4$ or $6m + 5$ *i.e.*, $x = 6m + r$, $r = 0, 1, 2, 3, 4, 5$.

Now, show that x^2 is of the form $6m$, $6m + 1$ or $6m + 4$.
Hence, x^2 cannot be of the form $6m + 2$ or $6m + 5$.

13. We know that any positive integer x can be of the form $6m$, $6m + 1$, $6m + 2$, $6m + 3$, $6m + 4$ or $6m + 5$.

CASE I *When* $x = 6q$: In this case,

$$x^3 = (6q)^3 = 6(36q^3) = 6m, \text{ where } m = 36q^3$$

CASE II *When* $x = 6q + 1$: In this case,

$$x^3 = (6q + 1)^3 = 216q^3 + 108q^2 + 18q + 1 = 6(36q^3 + 18q^2 + 3q) + 1$$

$$= 6m + 1, \text{ where } m = 36q^3 + 18q^2 + 3q \text{ and so on.}$$

14. We know that any positive integer can be of the form $5q$, $5q + 1$, $5q + 2$, $5q + 3$ or $5q + 4$.

15. We know that any positive integer can be of the form $6m$, $6m + 1$, $6m + 2$, $6m + 3$, $6m + 4$ or $6m + 5$ for some integer m. So, an odd positive integer x is of the form $6m + 1$ or $6m + 3$.

CASE I *When* $x = 6m + 1$: In this case,

$$x^2 = (6m + 1)^2 = 36m^2 + 12m + 1 = 6(6m^2 + 2m) + 1 = 6q + 1, \text{ where q} = 6m^2 + m$$

CASE II *When* $x = 6m + 3$: In this case,

$$x^2 = (6m + 3)^2 = 36m^2 + 36m + 9 = (36m^2 + 36m + 6) + 3$$

$$= 6(6m^2 + 6m + 1) + 3 = 6q + 3, \text{ where } q = 6m^2 + 6m + 3.$$

16. No. $(3q + 1)^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1 = 3m + 1$

17. Any positive integer x can be written as $3q$, $3q + 1$, $3q + 2$.

CASE I *When* $x = 3q$: In this case,

$$x^2 = (3q)^2 = 9q^2 = 3(3q^2) = 3m, \text{ where } m = 3q^2$$

CASE II *When* $x = 3q + 1$: In this case,

$$x^2 = (3q + 1)^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1 = 3m + 1, \text{ where } m = 3q^2 + 2q$$

CASE III *When* $x = 3q + 2$: In this case,

$$x^2 = (3q + 2)^2 = 9q^2 + 12q + 4 = (9q^2 + 12q + 3) + 1 = 3(3q^2 + 4q + 1) + 1 = 3m + 1,$$

$$\text{where } m = 3q^2 + 4q + 1$$

1.4 EUCLID'S DIVISION ALGORITHM

In the previous section, we have learnt about Euclid's division lemma and its applications. We have seen that the said lemma is nothing but a restatement of the long division process which we have been using all these years. In this section, we will learn one more application of Euclid's division lemma known as Euclid's division algorithm. The word algorithm comes from the name of 9th century Persian mathematician al-Khwarizmi. An algorithm means a series of well defined steps which provide a procedure of calculation repeated successively on the results of earlier steps till the desired result is obtained. Euclid's division algorithm is also an algorithm to compute the highest common factor (HCF) of two given positive integers. So, let us first have a brief review of HCF of positive integers.

In section 1.2, we have recalled that if an integer c divides each one of several integers $x_1, x_2, \ldots x_n$, then it is called a common divisor of these integers. For example, 7 is a common divisor of 42 and 63 as it divides both the integers. Throughout this chapter we will discuss positive divisors of positive integers only and the word integer will mean positive integer. Note that 1 is a common divisor of all positive integers. Two or more integers may have many common divisors. For example, common divisors of 24 and 42 are 1, 2, 3, 4 and 6. The largest among these common divisors is 6. This is called the Greatest Common Divisor (GCD) or Highest Common Factor (HCF) of integers 24 and 42. Thus, the largest or greatest among the common divisors of two or more integers is called the Greatest Common Divisor (GCD) or Highest Common Factor (HCF) of the given integers. The HCF of two or more positive integers always exists and it is unique. The proof of the same is beyond the scope of this book.

Let a and b be two positive integers such that $a > b$. If b is not a divisor of a, then by Euclid's division lemma there exist positive integers q and r such that $a = bq + r$, where $0 < r < b$. Common divisors of a and b are closely associated with the common divisors of b and r. In fact, every common divisor of b and r is a common divisor of a and b and vice-versa as stated and proved in the following theorem.

THEOREM *If a and b are positive integers such that $a = bq + r$, then every common divisor of a and b is a common divisor of b and r, and vice-versa.*

PROOF Let c be a common divisor of a and b. Then,

$$c \mid a \Rightarrow a = cq_1 \text{ for some integer } q_1 \qquad c \mid b \Rightarrow b = cq_2 \text{ for some integer } q_2$$

Now,

$$a = bq + r$$

$\Rightarrow \quad r = a - bq$

$\Rightarrow \quad r = cq_1 - cq_2q$

$\Rightarrow \quad r = c(q_1 - q_2q)$

$\Rightarrow \quad c \mid r$

$\Rightarrow \quad c \mid r$ and $c \mid b$ $\qquad [\because c \mid b \text{ (given)}]$

$\Rightarrow \quad c$ is a common divisor of b and r.

Hence, a common divisor of a and b is a common divisor of b and r.

Conversely, Let d be a common divisor of b and r. Then,

$$d \mid b \Rightarrow b = r_1 d \text{ for some integer } r_1$$

$$d \mid r \Rightarrow r = r_2 d \text{ for some integer } r_2$$

We will now show that d is a common divisor of a and b.
We have,

$$a = bq + r$$

$\Rightarrow$ $a = r_1 dq + r_2 d$

$\Rightarrow$ $a = (r_1 q + r_2)d$

$\Rightarrow$ $d \mid a$

$\Rightarrow$ $d \mid a$ and $d \mid b$ $[\because d \mid b \text{ (given)}]$

$\Rightarrow$ d is a common divisor of a and b.

Q.E.D

Let us now discuss an application of this theorem and Euclid's division lemma.
Consider integers 117 and 45.

Let $a = 117$ and $b = 45$. By Euclid's division lemma, we obtain

$$117 = 45 \times 2 + 27 \qquad \ldots(\text{i}) \qquad \left[\because 45 \overline{)117} (2 \quad \frac{90}{27}\right]$$

or, $a = bq_1 + r_1$, where $q_1 = 2$ and $r_1 = 27$

By using the above theorem, we observe that the common divisors of $a = 117$ and $b = 45$ are also the common divisors of $b = 45$ and $r_1 = 27$ and vice-versa.

Applying Euclid's division lemma on divisor $b = 45$ and remainder $r_1 = 27$, we get

$$45 = 27 \times 1 + 18 \qquad \ldots(\text{ii}) \qquad \left[\because 27 \overline{)45} (1 \quad \frac{27}{18}\right]$$

or, $b = q_2 r_1 + r_2$, where $q_2 = 1$ and $r_2 = 18$

Using the above theorem, we find that the common divisors of $r_1 = 27$ and $r_2 = 18$ are the common divisors of $b = 45$ and $r_1 = 27$ and vice-versa. But, common divisors of $b = 45$ and $r_1 = 27$ are the common divisors of $a = 117$ and $b = 45$ and vice-versa. Therefore, common divisors of $r_1 = 27$ and $r_2 = 18$ are the common divisors of $a = 117$ and $b = 45$ and vice-versa.

Applying Euclid's division lemma on $r_1 = 27$ and $r_2 = 18$, we get

$$27 = 18 \times 1 + 9 \qquad \ldots(\text{iii}) \qquad \left[\because 18 \overline{)27} (1 \quad \frac{18}{9}\right]$$

or, $r_1 = q_3 r_2 + r_3$, where $q_3 = 1$ and $r_3 = 9$

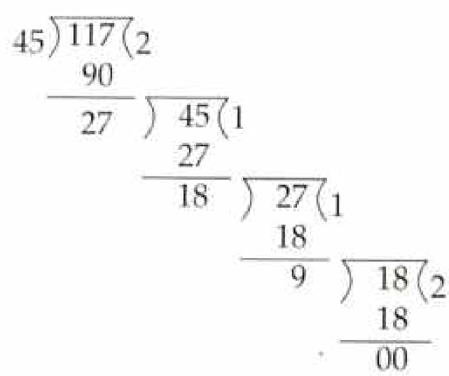
Again by using the above theorem, we find that common divisors of $r_2 = 18$ and $r_3 = 9$ are the common divisors of $a = 117$ and $b = 45$ and vice-versa.

Using Euclid's division lemma on $r_2 = 18$ and $r_3 = 9$, we get

$$18 = 9 \times 2 + 0 \qquad \text{...(iv)}$$

Therefore, $r_3 = 9$ is a divisor of $r_2 = 18$ and $r_3 = 9$. Also, it is the greatest common divisor (or HCF) of r_2 and r_3. Hence, $r_3 = 9$ is the greatest common divisor (or HCF) of $a = 117$ and $b = 45$. We also observe that $r_3 = 9$ is the last non-zero remainder in the above process of repeated application of Euclid's division lemma on the divisor and the remainder in the next step.

The set of equation (i) to (iv) is called Euclid's division algorithm for 117 and 45. The last divisor, or the last but one non-zero remained which is 9 is the HCF (or GCD) of 117 and 45. The above process of finding HCF can also be carried out by successive divisons as follows:



OR,

1	45 27	117 90	2
2	18 18	27 18	1
	00	(9)	

In the general form Euclid's division algorithm may be described as follows:

Let a, b be positive integers such that $a > b$.

Applying Euclid's division lemma in succession as discussed above, we obtain.

$a = bq_1 + r_1 \qquad 0 < r_1 < b \qquad$ ... (i)

$b = r_1q_2 + r_2 \qquad 0 < r_2 < r_1 \qquad$ ... (ii)

$r_1 = r_2q_3 + r_3 \qquad 0 < r_3 < r_2 \qquad$ ... (iii)

$r_2 = r_3q_4 + r_4 \qquad 0 < r_4 < r_3 \qquad$ ... (iv)

$\vdots$

$$r_{n-3} = r_{n-2}\, q_{n-1} + r_{n-1} \quad 0 < r_{n-1} < r_{n-2} \qquad \ldots (n-1)$$

$$r_{n-2} = r_{n-1}\, q_n + r_n \qquad 0 = r_n \qquad \ldots (n)$$

Clearly, $a > b > r_1 > r_2 > r_3 > \ldots > r_{n-2} > r_{n-1} > 0 = r_n$

Thus, the remainders are positive and decreasing. Therefore, for some natural number n, r_n must be zero and the process of applying division lemma ends there.

The set of equations (i) to (n) is called Euclid's division algorithm for numbers a and b.

This algorithm is not only useful for computing the HCF of very large positive integers, but also because it is one of the earliest examples of an algorithm that a computer had been programmed to carry out. Although we have stated Euclid's division algorithm for only positive integers, but it can be extended for all integers except zero, i.e. $b \neq 0$. However, we shall not discuss this aspect in this chapter.

REMARK *The HCF of numbers is a common divisor of the numbers which are divisors of their LCM. Consequently, HCF is a divisor of LCM.*

In order to compute the HCF of two positive integers, say a and b, with $a > b$ we may follow the following steps:

STEP I *Apply Euclid's division lemma to a and b and obtain whole numbers q_1 and r_1 such that $a = bq_1 + r_1, 0 \leq r_1 < b$.*

STEP II *If $r_1 = 0$, b is the HCF of a and b*

STEP III *If $r_1 \neq 0$, apply Euclid's division lemma to b and r_1 and obtain two whole numbers q_1 and r_2 such that $b = q_1 r_1 + r_2$.*

STEP IV *If $r_2 = 0$, then r_1 is the HCF of a and b.*

STEP V *If $r_2 \neq 0$, then apply Euclid's division lemma to r_1 and r_2 and continue the above process till the remainder r_n is zero. The divisor at this stage i.e. r_{n-1}, or the non-zero remainder at the previous stage, is the HCF of a and b.*

Following examples will illustrate the same.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE HCF OF TWO POSITIVE INTEGERS**

EXAMPLE 1 *Use Euclid's division algorithm to find the HCF of 210 and 55.*

SOLUTION Given integers are 210 and 55. Clearly, 210 > 55. Applying Euclid's division lemma to 210 and 55, we get

$$210 = 55 \times 3 + 45 \qquad \ldots \text{(i)} \qquad \left[\because\ 55 \overline{)\,210\,}\ (3 \quad \begin{array}{r} 165 \\ \hline 45 \end{array}\right]$$

Since the remainder $45 \neq 0$. So, we apply the division lemma to the divisor 55 and remainder 45 to get

$$55 = 45 \times 1 + 10 \quad \ldots(ii) \qquad \left[\because 45\,\overline{)\,55\,}(\,1 \quad \frac{45}{10}\right]$$

Now, we apply division lemma to the new divisor 45 and new remainder 10 to get

$$45 = 10 \times 4 + 5 \quad \ldots(iii) \qquad \left[\because 10\,\overline{)\,45\,}(\,4 \quad \frac{40}{5}\right]$$

We now consider the new divisor 10 and the new remainder 5, and apply division lemma to get

$$10 = 5 \times 2 + 0 \quad \ldots(iv)$$

The remainder at this stage is zero. So, the divisor at this stage or the remainder at the previous stage i.e. 5 is the HCF of 210 and 55.

EXAMPLE 2 *Use Euclid's division algorithm to find the HCF of 4052 and 12576.*

SOLUTION Given integers are 4052 and 12576 such that 12576 > 4052. Applying Euclid's division lemma to 12576 and 4052, we get

$$12576 = 4052 \times 3 + 420 \quad \ldots(i) \qquad \left[\because 4052\,\overline{)\,12576\,}(\,3 \quad \frac{12156}{420}\right]$$

Since the remainder $420 \neq 0$. So, we apply the division lemma to 4052 and 420, to get

$$4052 = 420 \times 9 + 272 \quad \ldots(ii) \qquad \left[\because 420\,\overline{)\,4052\,}(\,9 \quad \frac{3780}{272}\right]$$

We consider the new divisor 420 and the new remainder 272 and apply division lemma to get

$$420 = 272 \times 1 + 148 \quad \ldots(iii) \qquad \left[\because 272\,\overline{)\,420\,}(\,1 \quad \frac{272}{148}\right]$$

Let us now consider the new divisor 272 and the new remainder 148 and apply division lemma to get

$$272 = 148 \times 1 + 124 \quad \ldots(iv) \qquad \left[\because 148\,\overline{)\,272\,}(\,1 \quad \frac{148}{124}\right]$$

We consider now the new divisor 148 and the new remainder 124 and apply division lemma to get

$$148 = 124 \times 1 + 24 \quad \ldots(v) \qquad \left[\because 124\,\overline{)\,148\,}(\,1 \quad \frac{124}{24}\right]$$

We consider now the new divisor 124 and the new remainder 24 and apply division lemma to get

$$124 = 24 \times 5 + 4 \quad \ldots(vi) \qquad \left[\because 24 \overline{)124}(5 \atop {120 \atop \overline{4}}\right]$$

We consider the new divisor 24 and the new remainder 4 and apply division lemma to get

$$24 = 4 \times 6 + 0 \quad \ldots(vii) \qquad \left[\because 4 \overline{)24}(6 \atop {24 \atop \overline{0}}\right]$$

We observe that the remainder at this stage is zero. Therefore, the divisor at this stage i.e. 4 (or the remainder at the earlier stage) is the HCF of 4052 and 12576.

Type II ON FINDING THE HCF OF THREE NUMBERS

To find the hcf of three numbers, we use the following steps:

STEP I *Find the HCF of any two of the given numbers.*

STEP II *Find the HCF of the third given number and the HCF obtained in step I.*

STEP III *The HCF obtained in step II is the HCF of three given numbers.*

EXAMPLE 3 *Use Euclid's division algorithm to find the HCF of 441, 567 and 693.*

[NCERT EXEMPLAR]

SOLUTION Let us first find the HCF of 441 and 567 by using Euclid's lemma. Applying Euclid's division lemma to 441 and 567, we obtain

$$567 = 441 \times 1 + 126$$

We find that the remainder is 126 which is *a* non-zero number. So, we apply Euclid's division lemma to 441 (divisor) and 126 (remainder) to get

$$441 = 126 \times 3 + 63$$

Now, we apply Euclid's division lemma to the divisor 126 and the remainder 63, to get

$$126 = 63 \times 2 + 0$$

The remainder at this stage is 0. So, the divisor at the previous stage *i.e.*, 63 is the HCF of 441 and 567.

Now, we use Euclid's division lemma to find the HCF of 63 and 693. We observe that

$$693 = 63 \times 11 + 0$$

So, the HCF of the third number 693 and 63 (the HCF of first two numbers 441 and 567) is 63.

Hence, the HCF of 441, 567 and 693 is 63.

Type III ON SOME APPLICATIONS OF HCF

EXAMPLE 4 *A sweet seller has 420 Kaju burfis and 130 Badam burfis she wants to stack them in such a way that each stack has the same number, and they take up the least area of the tray. What is the number of burfis that can be placed in each stack for this purpose?* **[NCERT]**

SOLUTION The area of the tray that is used up in stacking the burfis will be least if the sweet seller stacks maximum number of burfis in each stack. Since each stack must have the same number of burfis. Therefore, the number of stacks will be least if the number of burfis in each stack is equal to the HCF of 420 and 130.

In order to find the HCF of 420 and 130, let us apply Euclid's division lemma to 420 and 130 to get

$$420 = 130 \times 3 + 30 \quad \ldots(i)$$

Let us now consider the divisor 130 and the remainder 30 and apply division lemma to get

$$130 = 30 \times 4 + 10 \quad \ldots(ii)$$

Considering now divisor 30 and the remainder 10 and apply division lemma, we get

$$30 = 3 \times 10 + 0 \quad \ldots(iii)$$

Since, the remainder at this stage is zero. Therefore, last divisor 10 is the HCF of 420 and 130.

Hence, the sweet seller can make stacks of 10 burfis of each kind to cover the least area of the tray.

EXAMPLE 5 *Any contingent of 616 members is to march behind an army band of 32 members in a parade. The two groups are to march in the same number of columns. What is the maximum number of columns in which they can march?* [NCERT]

SOLUTION The maximum number of columns is the HCF of 616 and 32. In order to find the HCF of 616 and 32, let us apply Euclid's division lemma to 616 and 32 to get

$$616 = 32 \times 19 + 8$$

Let us now take the divisor 32 as dividend and remainder 8 as divisor and apply Euclid's division lemma to get

$$32 = 8 \times 4 + 0$$

Since, the remainder at this stage is 0. Therefore, the last divisor i.e. 8 is the HCF of 616 and 32.

Hence, the maximum number of columns in which they can march is 8.

EXAMPLE 6 *Two tankers contain 850 litres and 680 litres of petrol respectively. Find the maximum capacity of a container which can measure the petrol of either tanker in exact number of times.*

SOLUTION Clearly, the maximum capacity of the container is the HCF of 850 and 680 in litres. So, Let us find the HCF of 850 and 680 by Euclid's algorithm.

4	680 680	850 680	1
	0 (Remainder)	(170) (HCF)	

Clearly, HCF of 850 and 680 is 170.

Hence, capacity of the container must be 170 litres.

EXAMPLE 7 *Find the largest number which divides 245 and 1029 leaving remainder 5 in each case.*

SOLUTION It is given that the required number when divides 245 and 1029, the remainder is 5 in each case. This means that $245 - 5 = 240$ and $1029 - 5 = 1024$ are completely divisible by the required number.

It follows from this that the required number is a common factor of 240 and 1024. It is also given that the required number is the largest number satisfying the given property. Therefore, it is the HCF of 240 and 1024.

Let us now find the HCF of 240 and 1024 by Euclid's algorithm.

3	240 192	1024 960	4
3	48 48	64 48	1
	0 (Remainder)	(16) (HCF)	

Clearly, HCF of 240 and 1024 is the last divisor i.e. 16. Hence, required number = 16.

EXAMPLE 8 *Find the largest number that divides 2053 and 967 and leaves a remainder of 5 and 7 respectively.*

SOLUTION It is given that on dividing 2053 by the required number, there is a remainder of 5. This means that 2053 − 5 = 2048 is exactly divisible by the required number.

Similarly, 967 − 7 = 960 is also exactly divisible by the required number.

Also, the required number is the largest number satisfying the above property. Therefore, it is the HCF of 2048 and 960.

Let us now find the HCF of 2048 and 960 by Euclid's algorithm.

7	960 896	2048 1920	2
	(64) (HCF)	128 128	2
		0 (Remainder)	

Clearly, HCF of 960 and 2048 is the last divisor i.e. 64. Hence, required number = 64.

EXAMPLE 9 *Find the largest number that will divide 398, 436 and 542 leaving remainders 7, 11 and 15 respectively.*

SOLUTION Clearly, the required number is the HCF of the numbers

398 − 7 = 391, 436 − 11 = 425, and, 542 − 15 = 527.

First we find the HCF of 391 and 425 by Euclid's algorithm as given below.

11	391 374	425 391	1
	17 (HCF)	34 34	2
		0 (Remainder)	

Clearly, H.C.F. of 391 and 425 is 17.

Let us now the HCF of 17 and the third number 527 by Euclid's algorithm:

17 (HCF)	527 51	3
	17 17	1
	0 (Remainder)	

The HCF of 17 and 527 is 17. Hence, HCF of 391, 4250 and 527 is 17.

Hence, the required number is 17.

EXAMPLE 10 *Can two numbers have 18 as their HCF and 380 as their LCM? Justify your answer.*

SOLUTION We know that HCF of two numbers is a divisor of their LCM. Here, 18 is not a divisor of 380. So, 18 and 380 cannot be respectively HCF and LCM of two numbers.

EXAMPLE 11 *The numbers 525 and 3000 are both divisible only by 3, 5, 15, 25 and 75. What is HCF (525, 3000)?*

SOLUTION It is given that 3, 5, 15, 25 and 75 are the only common factors of 525 and 3000. The highest of these common factors is 75. Hence, HCF (525, 3000) = 75.

LEVEL-2

Type IV ON EXPRESSING THE HCF OF TWO NUMBERS a AND b IN THE FORM $xa + xy$

EXAMPLE 12 *If the HCF of 210 and 55 is expressible in the form* $210 \times 5 + 55y$, *find* y.

SOLUTION Let us first find the HCF of 210 and 55.

Applying Euclid's division lemma on 210 and 55, we get

$$210 = 55 \times 3 + 45 \qquad \ldots \text{(i)} \qquad \left[\because 55\overline{)210}(3,\ 165,\ 45\right]$$

Since, the remainder $45 \neq 0$. So, we now apply division lemma on the divisor 55 and the remainder 45 to get

$$55 = 45 \times 1 + 10 \qquad \ldots \text{(ii)} \qquad \left[\because 45\overline{)55}(1,\ 45,\ 10\right]$$

We consider the divisor 45 and the remainder 10 and apply division lemma to get

$$45 = 4 \times 10 + 5 \qquad \ldots \text{(iii)} \qquad \left[\because 10\overline{)45}(4,\ 40,\ 5\right]$$

We consider the divisor 10 and the remainder 5 and apply division lemma to get

$$10 = 5 \times 2 + 0 \qquad \ldots \text{(iv)}$$

We observe that the remainder at this stage is zero. So, the last divisor i.e. 5 is the HCF of 210 and 55.

$$5 = 210 \times 5 + 55y$$

$\Rightarrow \qquad 55y = 5 - 210 \times 5 = 5 - 1050$

$\Rightarrow \qquad 55y = -1045$

$\Rightarrow \qquad y = \dfrac{-1045}{55} = -19$

EXAMPLE 13 *In a seminar, the number of participants in Hindi, English and Mathematics are 60, 84 and 108, respectively. Find the minimum number of rooms required if in each room the same number of participants are to be seated and all of them being in the same subject.*

SOLUTION The number of participants in each room must be the HCF of 60, 84 and 108. In order to find the HCF of 60, 84 and 108, we first find the HCF of 60 and 84 by Euclid's division algorithm:

2	60 48	84 60	1
	12 (HCF)	24 24	2
		0 (Remainder)	

Clearly, HCF of 60 and 84 is 12

Now, we find the HCF of 12 and 108

12 (HCF)	108 108	9
	0 (Remainder)	

Clearly, HCF of 12 and 108 is 12. Hence, the HCF of 60, 84 and 108 is 12.

Therefore, in each room maximum 12 participants can be seated.

We have,

Total number of participants $= 60 + 84 + 108 = 252$

$\therefore \qquad$ Number of rooms required $= \dfrac{252}{12} = 21.$

EXAMPLE 14 *Three sets of English, Hindi and Mathematics books have to be stacked in such a way that all the books are stored topic-wise and the height of each stack is the same. The number of English books is 96, the number of Hindi books is 240 and the number of Mathematics books is 336. Assuming that the books are of the same thickness, determine the number of stacks of English, Hindi and Mathematics books.*

SOLUTION In order to arrange the books as required, we have to find the largest number that divides 96, 240 and 336 exactly. Clearly, such a number is their HCF Computation of HCF of 96 and 240:

2	96 96	240 192	2
	0 (Remainder)	48 (HCF)	

Clearly, HCF of 96 and 240 is 48.
Computation of HCF of 48 and 336:

48 (HCF)	336 336	7
	0 (Remainder)	

Clearly, HCF of 48 and 336 is 48. Thus, HCF of 96, 240 and 336 is 48.
Hence, there must be 48 books in each stack.

Now, Number of stacks of English books $= \dfrac{\text{Number of English books}}{\text{Number of books in each stack}} = \dfrac{96}{48} = 2$

Number of stacks of Hindi books $= \dfrac{\text{Number of Hindi books}}{\text{Number of books in each stack}} = \dfrac{240}{48} = 5$

and, Number of stacks of Mathematics books $= \dfrac{\text{No. of Mathematics books}}{\text{No. of books in each stack}} = \dfrac{336}{48} = 7$

LEVEL-3

Type V **ON EXPRESSING THE HCF OF TWO NUMBERS a AND b IN THE FORM $xa + yb$**

EXAMPLE 15 *Find the HCF of 81 and 237 and express it as a linear combination of 81 and 237.*
SOLUTION Given integers are 81 and 237 such that $81 < 237$.
Applying division lemma to 81 and 237, we get

$$237 = 81 \times 2 + 75 \quad \ldots(i)$$

[∵ 81) 237 (2
162
75]

Since the remainder $75 \neq 0$. So, consider the divisor 81 and the remainder 75 and apply division lemma to get

$$81 = 75 \times 1 + 6 \quad \ldots(ii)$$

[∵ 75) 81 (1
75
6]

We consider the new divisor 75 and the new remainder 6 and apply division lemma to get

$75 = 6 \times 12 + 3$ ...(iii) $\left[\because 6\overline{)75}(12 \quad \frac{72}{3}\right]$

We consider the new divisor 6 and the new remainder 3 and apply division lemma to get

$6 = 3 \times 2 + 0$ ...(iv) $\left[\because 3\overline{)6}(2 \quad \frac{6}{0}\right]$

The remainder at this stage is zero. So, the divisor at this stage or the remainder at the earlier stage i.e. 3 is the HCF of 81 and 237.

To represent the HCF as a linear combination of the given two numbers, we start from the last but one step and successively eliminate the previous remainders as follows:

From (iii), we have

$3 = 75 - 6 \times 12$

$\Rightarrow \quad 3 = 75 - (81 - 75 \times 1) \times 12$ $\left[\begin{array}{l}\text{Substituting } 6 = 81 - 75 \times 1 \\ \text{obtained from (ii)}\end{array}\right]$

$\Rightarrow \quad 3 = 75 - 12 \times 81 + 12 \times 75$

$\Rightarrow \quad 3 = 13 \times 75 - 12 \times 81$ $\left[\begin{array}{l}\text{Substituting } 75 = 237 - 81 \times 2 \\ \text{obtained from (i)}\end{array}\right]$

$\Rightarrow \quad 3 = 13 \times (237 - 81 \times 2) - 12 \times 81$

$\Rightarrow \quad 3 = 13 \times 237 - 26 \times 81 - 12 \times 81$

$\Rightarrow \quad 3 = 13 \times 237 - 38 \times 81$

$\Rightarrow \quad 3 = 237x + 81y$, where $x = 13$ and $y = -38$.

REMARK *It follows from the above example that the HCF (say d) of two positive integers a and b can be expressed as a linear combination of a and b i.e., $d = xa + yb$ for some integers x and y. Also, this representation is not unique. Because,*

$d = xa + yb$

$\Rightarrow \quad d = xa + yb + ab - ab$

$\Rightarrow \quad d = (x + b)\,a + (y - a)\,b$

In the above example, we had

$3 = 13 \times 237 - 38 \times 81$

$\Rightarrow \quad 3 = 13 \times 237 - 38 \times 81 + 237 \times 81 - 237 \times 81$

$\Rightarrow \quad 3 = (13 \times 237 + 237 \times 81) + (-38 \times 81 - 237 \times 81)$

$\Rightarrow \quad 3 = (13 + 81) \times 237 + (-38 - 237) \times 81$

$\Rightarrow \quad 3 = 94 \times 237 - 275 \times 81$

$\Rightarrow \quad 3 = 94 \times 237 + (-275) \times 81$

EXAMPLE 16 *Find the HCF of 65 and 117 and express it in the form 65 m + 117 n.*

SOLUTION Given integers are 65 and 117 such that $117 > 65$.

Applying division lemma to 65 and 117, we get

$$117 = 65 \times 1 + 52 \quad \ldots(i) \qquad \left[\because 65\overline{)117}(1,\ \ 65,\ \ 52\right]$$

Since the remainder $52 \neq 0$. So, we apply the division lemma to the divisor 65 and the remainder 52 to get

$$65 = 52 \times 1 + 13 \quad \ldots(ii) \qquad \left[\because 52\overline{)65}(1,\ \ 52,\ \ 13\right]$$

We consider the new divisor 52 and the new remainder 13 and apply division lemma, to get

$$52 = 13 \times 4 + 0 \quad \ldots(iii)$$

At this stage the remainder is zero. So, the last divisor or the non-zero remainder at the earlier stage i.e. 13 is the HCF of 65 and 117.

From (ii), we have

$$13 = 65 - 52 \times 1$$

$\Rightarrow$ $13 = 65 - (117 - 65 \times 1)$ [Substituting $52 = 117 - 65 \times 1$ obtain from (i)]

$\Rightarrow$ $13 = 65 - 117 + 65 \times 1$

$\Rightarrow$ $13 = 65 \times 2 + 117 \times (-1)$

$\Rightarrow$ $13 = 65 - 117 + 65 \times 1$

$\Rightarrow$ $13 = 65m + 117n$, where $m = 2$ and $n = -1$.

EXAMPLE 17 *If d is the HCF of 56 and 72, find x, y satisfying $d = 56x + 72y$. Also, show that x and y are not unique.*

SOLUTION Applying Euclid's division lemma to 56 and 72, we get

$$72 = 56 \times 1 + 16 \quad \ldots(i) \qquad \left[\because 56\overline{)72}(1,\ \ 56,\ \ 16\right]$$

Since, the remainder $16 \neq 0$. So, we consider the divisor 56 and the remainder 16 and apply division lemma to get

$$56 = 16 \times 3 + 8 \quad \ldots(ii) \qquad \left[\because 16\overline{)56}(3,\ \ 48,\ \ 8\right]$$

We consider the divisor 16 and the remainder 8 and apply division algorithm to get

$$16 = 8 \times 2 + 0 \quad \ldots(iii) \qquad \left[\because 8\overline{)16}(2,\ \ 16,\ \ 0\right]$$

We observe that the remainder at this stage is zero. Therefore, last divisor 8 (or the remainder at the earlier stage) is the HCF of 56 and 72.
From (ii), we get

$8 = 56 - 16 \times 3$

$\Rightarrow \quad 8 = 56 - (72 - 56 \times 1) \times 3 \qquad [\because 16 = 72 - 56 \times 1 \text{ (from (i))}]$

$\Rightarrow \quad 8 = 56 - 3 \times 72 + 56 \times 3$

$\Rightarrow \quad 8 = 56 \times 4 + (-3) \times 72$

$\therefore \quad x = 4$ and $y = -3$.

Now, $\quad 8 = 56 \times 4 + (-3) \times 72$

$8 = 56 \times 4 + (-3) \times 72 - 56 \times 72 + 56 \times 72$

$\Rightarrow \quad 8 = 56 \times 4 - 56 \times 72 + (-3) \times 72 + 56 \times 72$

$\Rightarrow \quad 8 = 56 \times (4 - 72) + \{(-3) + 56\} \times 72$

$\Rightarrow \quad 8 = 56 \times (-68) + (53) \times 72$

$\therefore \quad x = -68$ and $y = 53$.

Hence, x and y are not unique.

EXERCISE 1.2

LEVEL-1

1. Define HCF of two positive integers and find the HCF of the following pairs of numbers:
   (i) 32 and 54 (ii) 18 and 24 (iii) 70 and 30
   (iv) 56 and 88 (v) 475 and 495 (vi) 75 and 243.
   (vii) 240 and 6552 (viii) 155 and 1385 (ix) 100 and 190 [CBSE 2009]
   (x) 105 and 120 [CBSE 2009]
2. Use Euclid's division algorithm to find the HCF of
   (i) 135 and 225 (ii) 196 and 38220 (iii) 867 and 255. [NCERT]
   (iv) 184, 230 and 276 (v) 136, 170 and 255
3. Find the HCF of the following pairs of integers and express it as a linear combination of them.
   (i) 963 and 657 (ii) 592 and 252 (iii) 506 and 1155
   (iv) 1288 and 575
4. Find the largest number which divides 615 and 963 leaving remainder 6 in each case.
5. If the HCF of 408 and 1032 is expressible in the form $1032\,m - 408 \times 5$, find m.
6. If the HCF of 657 and 963 is expressible in the form $657\,x + 963 \times -15$, find x.
7. An army contingent of 616 members is to march behind an army band of 32 members in a parade. The two groups are to march in the same number of columns. What is the maximum number of columns in which they can march?
8. A merchant has 120 litres of oil of one kind, 180 litres of another kind and 240 litres of third kind. He wants to sell the oil by filling the three kinds of oil in tins of equal capacity. What should be the greatest capacity of such a tin?
9. During a sale, colour pencils were being sold in packs of 24 each and crayons in packs of 32 each. If you want full packs of both and the same number of pencils and crayons, how many of each would you need to buy?

10. 144 cartons of Coke Cans and 90 cartons of Pepsi Cans are to be stacked in a Canteen. If each stack is of the same height and is to contain cartons of the same drink, what would be the greatest number of cartons each stack would have?
11. Find the greatest number which divides 285 and 1249 leaving remainders 9 and 7 respectively.
12. Find the largest number which exactly divides 280 and 1245 leaving remainders 4 and 3, respectively.
13. What is the largest number that divides 626, 3127 and 15628 and leaves remainders of 1, 2 and 3 respectively?
14. Find the greatest number that will divide 445, 572 and 699 leaving remainders 4, 5 and 6 respectively.
15. Find the greatest number which divides 2011 and 2623 leaving remainders 9 and 5 respectively.
16. Using Euclid's division algorithm, find the largest number that divides 1251, 9377 and 15628 leaving remainders 1, 2 and 3 respectively. [NCERT EXEMPLAR]

LEVEL-2

17. Two brands of chocolates are available in packs of 24 and 15 respectively. If I need to buy an equal number of chocolates of both kinds, what is the least number of boxes of each kind I would need to buy?
18. A mason has to fit a bathroom with square marble tiles of the largest possible size. The size of the bathroom is 10 ft. by 8 ft. What would be the size in inches of the tile required that has to be cut and how many such tiles are required?
19. 15 pastries and 12 biscuit packets have been donated for a school fete. These are to be packed in several smaller identical boxes with the same number of pastries and biscuit packets in each. How many biscuit packets and how many pastries will each box contain?
20. 105 goats, 140 donkeys and 175 cows have to be taken across a river. There is only one boat which will have to make many trips in order to do so. The lazy boatman has his own conditions for transporting them. He insists that he will take the same number of animals in every trip and they have to be of the same kind. He will naturally like to take the largest possible number each time. Can you tell how many animals went in each trip?
21. The length, breadth and height of a room are 8 m 25 cm, 6 m 75 cm and 4 m 50 cm, respectively. Determine the longest rod which can measure the three dimensions of the room exactly.
22. Express the HCF of 468 and 222 as $468x + 222y$ where x, y are integers in two different ways.

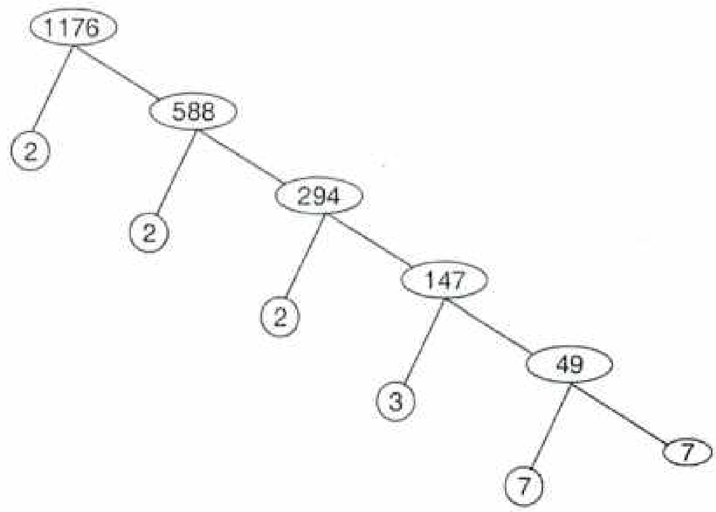
ANSWERS

1. (i) 2 (ii) 6 (iii) 10 (iv) 8 (v) 5 (vi) 3 (vii) 24 (viii) 5 (ix) 10 (x) 15
2. (i) 45, $45 = (-1)\,225 + 2 \times 135$ (ii) 196, $196 = 38220 \times 1 + (-194) \times 196$
   (iii) $51 = 51 = (-2)\,867 + 7 \times 255$ (iv) 46 (v) 17
3. (i) $9 = (-15) \times 963 + 22 \times 657$ (ii) $4 = 77 \times 252 + (-20)\,592$
   (iii) $11 = 16 \times 506 + (-7) \times 1155$ (iv) $23 = (-4) \times 1288 + 9 \times 575$
4. 87 5. 2 6. 22 7. 8 columns 8. 60 litres

9. 4 packets of colour pencils, 3 packets of crayons 10. 18 11. 138 12. 138
13. 625 14. 63 15. 154 16. 625 17. 5 of first kind, 8 of second kind
18. 24 inches, 20 tiles 19. 4 biscuit packets, 5 pastries
20. 35 21. 75 cm 22. $6 = 468 \times -9 + 222 \times 19, 6 = 468 \times 213 + 222 \times (-449)$

1.5 THE FUNDAMENTAL THEOREM ARITHMETIC

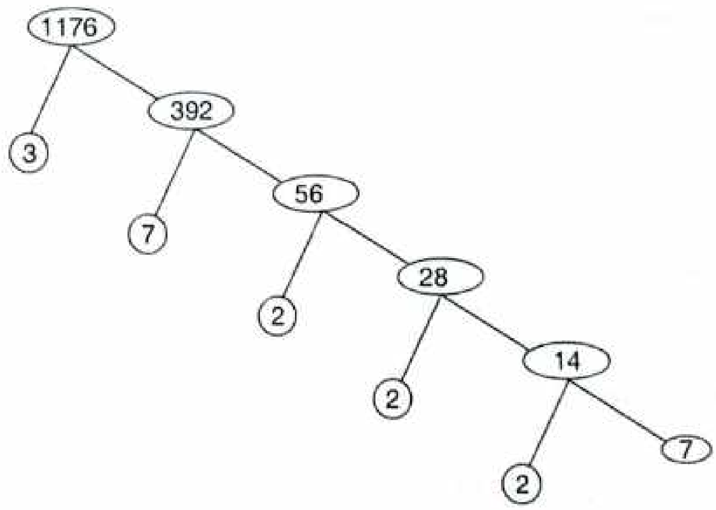
In earlier classes, we have learnt about prime and composite numbers. Let us recall that a positive integer p is prime if $p \neq 1$ and the only positive divisors of p are 1 and p. For example, 2, 3, 5, 7, 11, 13, 19, 23, 29, 31, 37, ... are the first few primes. We have learnt that every positive integer, other than 1, is either prime or composite. If a given positive integer is a composite number, it can be written as the product of two of its factors. These factors in turn are also either prime or composite. If composite, the factors can be split up further. If we keep on doing this factorization, ultimately we will arrive at a stage when all the factors are prime numbers as shown below for the positive integer 1176.



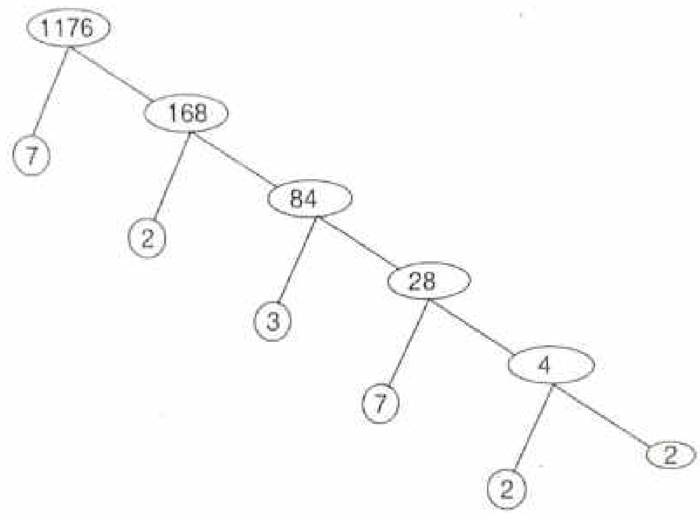
Thus, we have

$$1176 = 2 \times 2 \times 2 \times 3 \times 7 \times 7$$

Also, we have



and,



$\therefore \quad 1176 = 3 \times 7 \times 2 \times 2 \times 2 \times 7$

and, $\quad 1176 = 7 \times 2 \times 3 \times 7 \times 2 \times 2$ etc.

We observe that in all these prime factorizations of 1176, the prime numbers appearing are same, although the order in which they appear are different. Thus, the prime factorization of 1176 is unique except for the order in which the primes occur.

Let us now try another positive integer, say, 32760. This can be written as

$2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 13$

i.e., $\quad 32760 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 13$

$= 3 \times 2 \times 2 \times 3 \times 5 \times 7 \times 13 \times 2$ etc.

We observe that the above observation is also true for the positive integer 32760. This leads us to a conjecture that every positive integer is either prime or it can be expressed as the product of primes. In fact, this statement is true, and is called the *Fundamental Theorem of Arithmetic* because of its basic importance in the development of number theory. Let us now formally state this theorem.

THEOREM 1: (FUNDAMENTAL THEOREM OF ARITHMETIC) *Every composite number can be expressed (factorised) as a product of primes, and this factorization is unique except for the order in which the prime factors occur.*

While writing a positive integer as the product of primes, if we decide to write the prime factors in ascending order and we combine the same primes, then the integer is expressed as the product of powers of primes and the representation is unique. So, we can say that every composite number can be expressed as the products of powers distinct primes in ascending or descending order in a unique way.

Following theorem is a direct consequence of the Fundamental Theorem of Arithmetic.

THEOREM 2 *Let p be a prime number and a be a positive integer. If p divides a^2, then p divides a.* **[NCERT]**

PROOF From the Fundamental Theorem of Arithmetic integer a can be factorised as the product of primes. Let $a = p_1 p_2 p_3 \cdots p_n$ be the prime factorisation of a, where $p_1, p_2, \ldots, p_n$ are primes, not necessarily distinct.

Now,

$$a = p_1 p_2 p_3 \cdots p_n$$

$$\Rightarrow \quad a^2 = (p_1 p_2 p_3 \ldots p_n)(p_1 p_2 p_3 \ldots p_n)$$

$$\Rightarrow \quad a^2 = p_1^2 p_2^2 p_3^2 \ldots p_n^2$$

It is given that p is prime and it divides a^2. Therefore, p is a prime factor of a^2. From the uniqueness part of the Fundamental Theorem of Arithmetic it follows that the only prime factors of a^2 are $p_1, p_2, p_3, \ldots, p_n$. Therefore, p is one of $p_1, p_2, p_3, \ldots, p_n$. This implies that

$$p \mid p_1 p_2 p_3 \cdots p_n \Rightarrow p \mid a.$$ **Q.E.D.**

ILLUSTRATIVE EXAMPLES

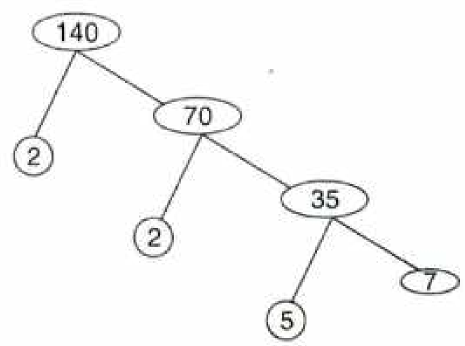
LEVEL-1

Type I **ON EXPRESSING A POSITIVE INTEGER AS THE PRODUCT OF ITS PRIME FACTORS**

EXAMPLE 1 *Express each of the following positive integers as the product of its prime factors:*

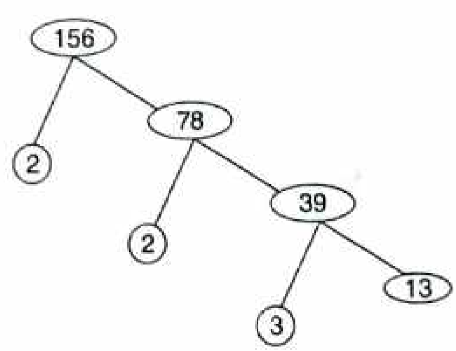
(i) 140 *(ii)* 156 *(iii)* 234

SOLUTION (i) Using the factor tree for prime factorization, we have



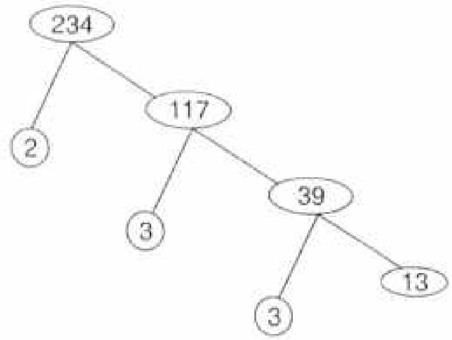
$$\therefore \quad 140 = 2 \times 2 \times 5 \times 7 = 2^2 \times 5 \times 7$$

(ii) Using the factor tree for prime factorisation, we have



$$\therefore \quad 156 = 2 \times 2 \times 3 \times 13$$

(iii) Using the factor tree for prime factorization, we have

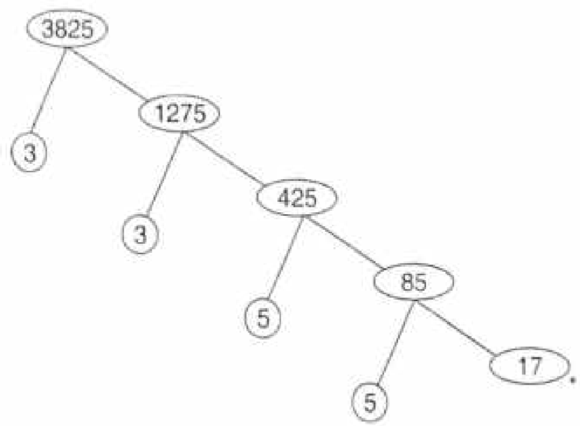


$\therefore \quad 234 = 2 \times 3 \times 3 \times 13 = 2 \times 3^2 \times 13$

EXAMPLE 2 *Express each of the following positive integers as the product of its prime factors:*

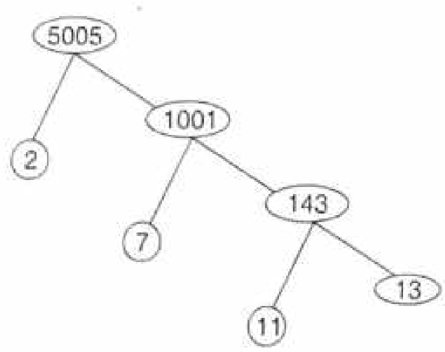
(i) 3825 *(ii) 5005* *(iii) 7429*

SOLUTION (i) Using the factor tree, we have



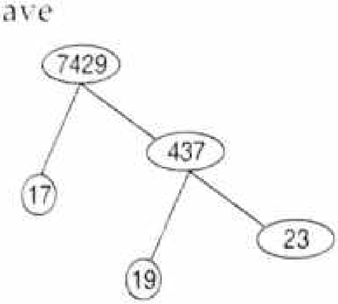
$\therefore \quad 3825 = 3 \times 3 \times 5 \times 5 \times 17 = 3^2 \times 5^2 \times 17$

(ii) Using the factor tree, we have



$\therefore \quad 5005 = 5 \times 7 \times 11 \times 13$

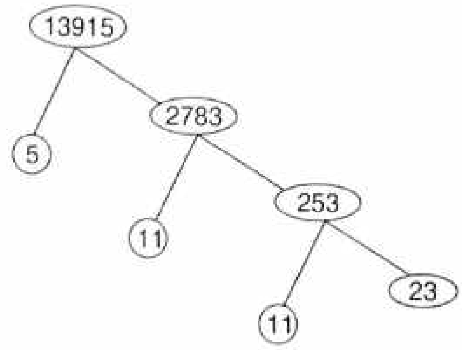
(iii) Using the factor tree, we have



$\therefore \quad 7429 = 17 \times 19 \times 23$

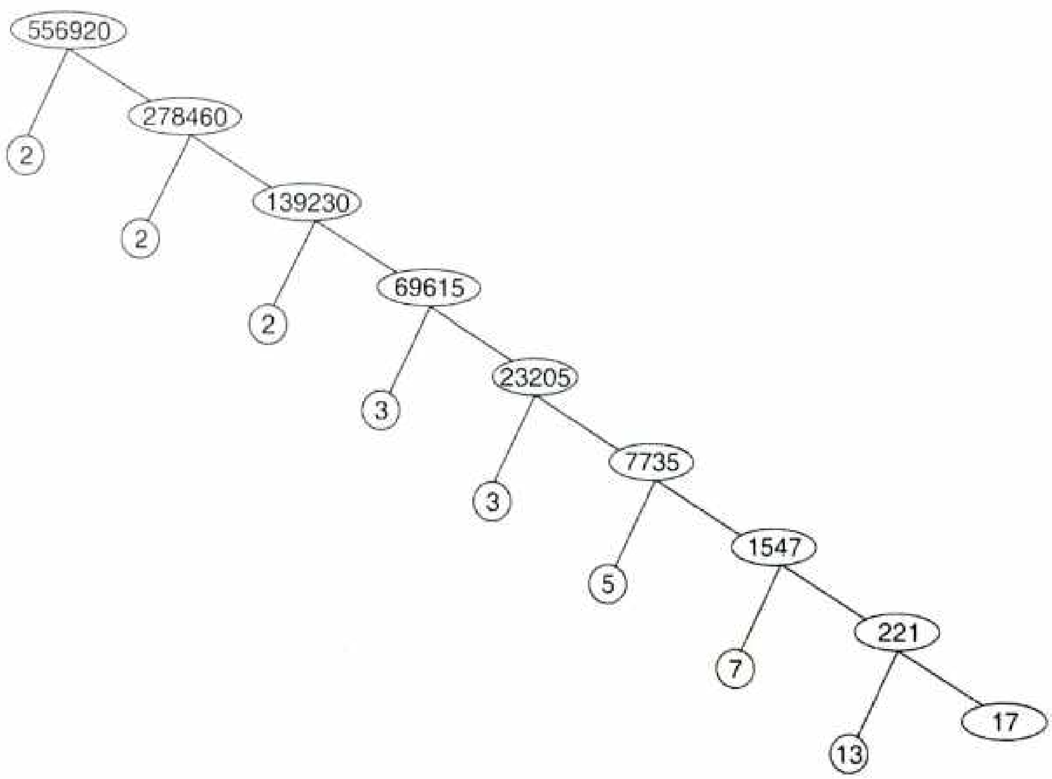
EXAMPLE 3 *Determine the prime factorization of each of the following numbers:*
(i) 13915 *(ii) 556920*

SOLUTION (i) Using the prime factorization tree, we have



$\therefore \quad 13915 = 5 \times 11 \times 11 \times 23 = 5 \times 11^2 \times 23$

(ii) Using the prime factorisation tree, we have



$\therefore \quad 556920 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 13 \times 17 = 2^3 \times 3^2 \times 5 \times 7 \times 13 \times 17$

Type II **ON MORE APPLICATIONS OF THE FUNDAMENTAL THEOREM OF ARITHMETIC**

EXAMPLE 4 *Prove that there is no natural number for which 4^n ends with the digit zero.* [NCERT]

SOLUTION We know that any positive integer ending with the digit zero is divisible by 5 and so its prime factorization must contain the prime 5.

We have,

$4^n = (2^2)^n = 2^{2n}$

$\Rightarrow$ The only prime in the factorization of 4^n is 2.

$\Rightarrow$ There is no other primes in the factorization of $4^n = 2^{2n}$

[By uniqueness of the Fundamental Theorem of Arithmetic]

$\Rightarrow$ 5 does not occur in the prime factorization of 4^n for any n.

$\Rightarrow$ 4^n does not end with the digit zero for any natural number n.

EXAMPLE 5 *Show that 12^n cannot end with digit 0 or 5 for any natural number n.* [NCERT EXEMPLAR]

SOLUTION Expressing 12 as the product of primes, we obtain

$$12 = 2^2 \times 3$$

$$\Rightarrow \quad 12^n = (2^2 \times 3)^n = (2^2)^n \times 3^n = (2)^{2n} \times 3^n$$

So, only primes in the factorisation of 12^n are 2 and 3 and, not 5.

Hence, 12^n cannot end with digit 0 or 5.

LEVEL-2

EXAMPLE 6 *Show that there are infinitely many positive primes.*

SOLUTION If possible, let there be finite number of positive primes $p_1, p_2, \dots, p_n$. Such that $p_1 < p_2 < p_3 < \dots < p_n$.

Let $x = 1 + p_1 p_2 p_3 \dots p_n$. Clearly, $p_1 p_2 \dots p_n$ is divisible by each of $p_1, p_2, p_3, \dots, p_n$.

$\therefore$ $x = 1 + p_1 p_2 p_3 \dots p_n$ is not divisible by any one of $p_1, p_2, \dots, p_n$

$\Rightarrow$ x is a prime or it has prime divisors other than $p_1, p_2, \dots, p_n$

There exists a positive prime different from $p_1, p_2, \dots, p_n$

This contradicts that there are finite number of positive primes.

Hence, the number of positive primes is infinite.

EXAMPLE 7 *Prove that every positive integer different from 1 can be expressed as a product of a non-negative power of 2 and an odd number.*

SOLUTION Let n be a positive integer other than 1. By the fundamental theorem of Arithmetic n can be uniquely expressed as powers of primes in ascending order. So, let $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ be the unique factorisation of n into primes with $p_1 < p_2 < p_3 < \dots < p_k$. Clearly, either $p_1 = 2$ and $p_2, p_3, \dots, p_k$ are odd positive integers or each of $p_1, p_2 \dots, p_k$ is an odd positive integer.

Therefore, we have the following cases:

CASE I *When $p_1 = 2$ and $p_2, p_3, \dots, p_k$ are odd positive integers:*

In this case, we have

$$n = 2^{a_1} p_2^{a_2} p_3^{a_3} \dots p_k^{a_k}$$

$$\Rightarrow \quad n = 2^{a_1} \times (p_2^{a_2} \, p_3^{a_3} \dots p_k^{a_k})$$

$$\Rightarrow \quad n = 2^{a_1} \times \text{An odd positive integer.}$$

$$\Rightarrow \quad n = (\text{A non-negative power of 2}) \times (\text{An odd positive integer})$$

CASE II *When each of $p_1, p_2, p_3, \dots, p_k$ is an odd positive integer:*

In this case, we have

$$n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \ldots p_k^{a_k}$$

$\Rightarrow \quad n = 2^0 \times (p_1^{a_1} p_2^{a_2} p_3^{a_3} \ldots p_k^{a_k})$

$\Rightarrow \quad n =$ (A non-negative power of 2) $\times$ (An odd positive integer)

Hence, in either case n is expressible as the product of a non-negative power of 2 and an odd positive integer.

EXAMPLE 8 *Prove that a positive integer n is prime number, if no prime p less than or equal to $\sqrt{n}$ divides n.*

SOLUTION Let n be a positive integer such that no prime less than or equal to $\sqrt{n}$ divides n. Then, we have to prove that n is prime. Suppose n is not a prime integer. Then, we may write

$$n = ab \text{ where } 1 < a \leq b$$

$\Rightarrow \quad a \leq \sqrt{n}$ and $b \geq \sqrt{n}$

Let p be a prime factor of a. Then, $p \leq a \leq \sqrt{n}$ and $p | a$

$\Rightarrow \quad p | ab$

$\Rightarrow \quad p | n$

$\Rightarrow \quad$ a prime less than $\sqrt{n}$ divides n.

This contradicts our assumption that no prime less than $\sqrt{n}$ divides n. So, our assumption is is wrong. Hence, n is a prime.

EXERCISE 1.3

LEVEL-1

1. Express each of the following integers as a product of its prime factors:
   (i) 420 (ii) 468 (iii) 945 (iv) 7325
2. Determine the prime factorisation of each of the following positive integer:
   (i) 20570 (ii) 58500 (iii) 45470971
3. Explain why $7 \times 11 \times 13 + 13$ and $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 + 5$ are composite numbers.
4. Check whether 6^n can end with the digit 0 for any natural number n. [NCERT]
5. Explain why $3 \times 5 \times 7 + 7$ is a composite number. [NCERT EXEMPLAR]

ANSWERS

1. (i) $2^2 \times 3 \times 5 \times 7$ (ii) $2^2 \times 3^2 \times 13$ (iii) $3^3 \times 5 \times 7$ (iv) $5^2 \times 293$
2. (i) $2 \times 5 \times 11^2 \times 17$ (ii) $2^2 \times 3^2 \times 5^3 \times 13$ (iii) $7^2 \times 13^2 \times 17^2 \times 19$
3. Since $7 \times 11 \times 13 + 13 = (7 \times 11 + 1) \times 13$

   and, $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 + 5 = (7 \times 6 \times 4 \times 3 \times 2 \times 1 + 1) \times 5$
4. No

HINTS TO SELECTED PROBLEMS

4. We have, $6^n = (2 \times 3)^n = 2^n \times 3^n$. Therefore, prime factorisation of 6^n does not contain 5 as a factor. Hence, 6^n can never end with the digit 0 for any natural number.
5. Since $3 \times 5 \times 7 + 7 = (3 \times 5 + 1) \times 7 = (15 + 1) \times 7 = 16 \times 7$. Hence, it is a composite number.

1.6 SOME APPLICATIONS OF THE FUNDAMENTAL THEOREM OF ARITHMETIC

In this section, we will learn about various applications of the Fundamental Theorem of Arithmetic. In fact, we have studied about some of these applications in earlier classes even without realising their dependence on the Fundamental Theorem of Arithmetic. For example, we have have used prime factorisation method to find the HCF and LCM of positive integers. In this method, we use the Fundamental Theorem of Arithmetic in expressing the given integers as the product of primes. We will also discuss some other applications of the Fundamental Theorem of Arithmetic.

1.6.1 FINDING HCF AND LCM OF POSITIVE INTEGERS

In order to find the HCF and LCM of two or more positive integers, we may use the following algorithm.

ALGORITHM

STEP I *Factorize each of the given positive integers and express them as a product of powers of primes in ascending order of magnitudes of primes.*

STEP II *To find the HCF, identify common prime factors and find the smallest (least) exponent of these common factors. Now raise these common prime factors to their smallest exponents and multiply them to get the HCF.*

To find the LCM, list all prime factors (once only) occuring in the prime factorisation of the given positive integers.

For each of these factors, find the greatest exponent and raise each prime factor to the greatest exponent and multiply them to get the LCM.

REMARK *To find the LCM of two positive integers a and b, we can also use the following result, if we have already found the HCF.*

$$HCF \times LCM = a \times b.$$

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE HCF AND LCM BY PRIME FACTORISATION**

EXAMPLE 1 *Find the HCF and LCM of 90 and 144 by the prime factorisation method.*

SOLUTION Using the factor tree for the prime factorisation of 90 and 144, we have

$$90 = 2 \times 3^2 \times 5 \text{ and } 144 = 2^4 \times 3^2$$

To find the HCF, we list the common prime factors and their smallest exponents in 90 and 144 as under:

Common prime factors	*Least exponents*
2	1
3	2

$\therefore \quad HCF = 2^1 \times 3^2 = 2 \times 9 = 18$

To find the LCM, we list all prime factors of 90 and 144 and their greatest exponents as follows:

Prime factors of 90 and 144	*Greatest exponents*
2	4
3	2
5	1

$\therefore \quad LCM = 2^4 \times 3^2 \times 5^1 = 16 \times 9 \times 5 = 720$

EXAMPLE 2 *Find the HCF and LCM of 144, 180 and 192 by prime factorisation method.*

SOLUTION Using the factor tree for the prime factorisation of 144, 180 and 192, we have

$$144 = 2^4 \times 3^2,\ 180 = 2^2 \times 3^2 \times 5 \text{ and } 192 = 2^6 \times 3$$

To find the HCF, we list the common prime factors and their smallest exponents in 144, 180 and 192 as follows:

Common prime factors	*Least exponents*
2	2
3	1

$\therefore \quad HCF = 2^2 \times 3^1 = 12$

To find the LCM, we list all prime factors of 144, 180, 192 and their greatest exponents as follows:

Prime factors of 144, 180 and 192	*Greatest exponents*
2	6
3	2
5	1

$\therefore \quad LCM = 2^6 \times 3^2 \times 5^1 = 64 \times 9 \times 5 = 2880$

EXAMPLE 3 *Find the HCF of 96 and 404 by prime factorisation method. Hence, find their LCM.* [NCERT]

SOLUTION We have,

$$96 = 2^5 \times 3 \text{ and } 404 = 2^2 \times 101$$

$\therefore \quad HCF = 2^2 = 4$

Now, $\quad HCF \times LCM = 96 \times 404$

$$\Rightarrow \quad LCM = \frac{96 \times 404}{HCF} = \frac{96 \times 404}{4} = 96 \times 101 = 9696$$

REMARK *The product of two positive integers is equal to the product of their HCF and LCM, but the same is not true for three or more positive integers.*

Type II ON APPLICATIONS OF HCF AND LCM

EXAMPLE 4 *Find the largest positive integer that will divide 398, 436 and 542 leaving remainders 7, 11 and 15 respectively.*

SOLUTION It is given that on dividing 398 by the required number, there is a remainder of 7. This means that $398 - 7 = 391$ is exactly divisible by the required number. In other words required number is a factor of 391.

Similarly, required positive integer is a factor of $436 - 11 = 425$ and $542 - 15 = 527$.

Clearly, required number is the HCF of 391, 425 and 527.

Using the factor tree the prime factorisations of 391, 425 and 527 are as follows:

$$391 = 17 \times 23,\ 425 = 5^2 \times 17 \text{ and } 527 = 17 \times 31$$

$\therefore$ HCF of 391, 425 and 527 is 17

Hence, required number = 17

EXAMPLE 5 *There is a circular path around a sports field. Priya takes 18 minutes to drive one round of the field, while Ravish takes 12 minutes for the same. Suppose they both start at the same point and at the same time, and go in the same direction. After how many minutes will they meet again at the starting point?* [NCERT]

SOLUTION Required number of minutes is the LCM of 18 and 12.

We have,

$$18 = 2 \times 3^2 \text{ and } 12 = 2^2 \times 3$$

$\therefore$ LCM of 18 and 12 is $2^2 \times 3^2 = 36$

Hence, Ravish and Priya will meet again at the starting point after 36 minutes.

EXAMPLE 6 *In a school there are two sections – section A and section B of classX. There are 32 students in section A and 36 students in section B. Determine the minimum number of books required for their class library so that they can be distributed equally among students of section A or section B.*

SOLUTION Since the books are to be distributed equally among the students of section *A* or section *B*. Therefore, number of books must be a multiple of 32 as well as 36. Hence, required number of books is the LCM of 32 and 36.

We have,

$$32 = 2^5 \text{ and } 36 = 2^2 \times 3^2$$

$\therefore$ LCM of 32 and 36 is $2^5 \times 3^2 = 288$

Hence, required number of books is 288.

EXAMPLE 7 *On a morning walk, three persons step off together and their steps measure 40 cm, 42 cm and 45 cm respectively. What is the minimum distance each should walk so that each can cover the same distance and complete steps?* [NCERT EXEMPLAR]

SOLUTION Each person will cover the same distance in complete steps if the distance covered in cm is the LCM of 40, 42 and 45.

Now,

$$40 = 2^3 \times 5,\ 42 = 2 \times 3 \times \times 7 \text{ and } 45 = 3^2 \times 5$$

$\therefore$ LCM of 40, 42 and 45 is $2^3 \times 3^2 \times 5 \times 7 = 8 \times 9 \times 5 \times 7 = 2520$

Hence, minimum distance each should walk = 2520 cm.

LEVEL-2

EXAMPLE 8 *In a seminar, the number of participants in Hindi, English and Mathematics are 60, 84 and 108 respectively. Find the minimum number of rooms required if in each room the same number of participants are to be seated and all of them being in the same subject.*

SOLUTION The Number of room will be minimum if each room accomodates maximum number of participants. Since in each room the same number of participants are to be seated and all of them must be of the same subject. Therefore, the number of participants in each room must be the HCF of 60, 84 and 108. The prime factorisations of 60, 84 and 108 are as under:

$$60 = 2^2 \times 3 \times 5, 84 = 2^2 \times 3 \times 7 \text{ and } 108 = 2^2 \times 3^3$$

$\therefore$ HCF of 60, 84 and 108 is $2^2 \times 3 = 12$

Therefore, in each room 12 participants can be seated.

$$\therefore \quad \text{Number of rooms required} = \frac{\text{Total number of participants}}{12}$$

$$= \frac{60 + 84 + 108}{12} = \frac{252}{12} = 21$$

EXAMPLE 9 *Three sets of English, Hindi and Mathematics books have to be stacked in such a way that all the books are stored topic wise and the height of each stack is the same. The number of English books is 96, the number of Hindi books is 240 and the number of Mathematics books is 336. Assuming that the books are of the same thickness, determine the number of stacks of English, Hindi and Mathematics books.*

SOLUTION In order to arrange the books as required, we have to find the largest number that divides 96, 240 and 336 exactly. Clearly, such a number is their HCF.

We have,

$$96 = 2^5 \times 3, 240 = 2^4 \times 3 \times 5 \text{ and } 336 = 2^4 \times 3 \times 7$$

$\therefore$ HCF of 96, 240 and 336 is $2^4 \times 3 = 48$

So, there must be 48 books in each stack.

$$\therefore \quad \text{Number of stacks of English books} = \frac{96}{48} = 2$$

$$\text{Number of stacks of Hindi books} = \frac{240}{48} = 5$$

$$\text{Number of stacks of Mathematics books} = \frac{336}{48} = 7$$

EXERCISE 1.4

LEVEL-1

1. Find the LCM and HCF of the following pairs of integers and verify that LCM × HCF = Product of the integers:
   (i) 26 and 91 (ii) 510 and 92 (iii) 336 and 54 (iv) 404 and 96 [CBSE 2018]
2. Find the LCM and HCF of the following integers by applying the prime factorisation method:
   (i) 12, 15 and 21 [NCERT] (ii) 17, 23 and 29 [NCERT] (iii) 8, 9 and 25 [NCERT]
   (iv) 40, 36 and 126 (v) 84, 90 and 120 (vi) 24, 15 and 36
3. Given that HCF (306, 657) = 9, find LCM (306, 657). [NCERT]

4. Can two numbers have 16 as their HCF and 380 as their LCM? Give reason.
5. The HCF of two numbers is 145 and their LCM is 2175. If one number is 725, find the other.
6. The HCF of two numbers is 16 and their product is 3072. Find their LCM.
7. The LCM and HCF of two numbers are 180 and 6 respectively. If one of the numbers is 30, find the other number.

LEVEL-2

8. Find the smallest number which when increased by 17 is exactly divisible by both 520 and 468.
9. Find the smallest number which leaves remainders 8 and 12 when divided by 28 and 32 respectively.
10. What is the smallest number that, when divided by 35, 56 and 91 leaves remainders of 7 in each case?
11. A rectangular courtyard is 18 m 72 cm long and 13 m 20 cm broad. It is to be paved with square tiles of the same size. Find the least possible number of such tiles.
12. Find the greatest number of 6 digits exactly divisible by 24, 15 and 36.
13. Determine the number nearest to 110000 but greater than 100000 which is exactly divisible by each of 8, 15 and 21.
14. Find the least number that is divisible by all the numbers between 1 and 10 (both inclusive).
15. A circular field has a circumference of 360 km. Three cyclists start together and can cycle 48, 60 and 72 km a day, round the field. When will they meet again?
16. In a morning walk three persons step off together, their steps measure 80 cm, 85 cm and 90 cm respectively. What is the minimum distance each should walk so that he can cover the distance in complete steps?

ANSWERS

1. (i) LCM = 182, HCF = 13, (ii) LCM = 23 460, HCF = 2, (iii) LCM = 3024, HCF = 6

(iv) LCM = 9696, HCF = 4

2.	LCM	HCF
(i)	420	3
(ii)	1139	1
(iii)	1800	1
(iv)	2520	2
(v)	2520	6
(vi)	360	3

3. 22338
4. No
5. 435
6. 192
7. 36
8. 4663
9. 204
10. 3647
11. 4290
12. 999720
13. 109200
14. 2520
15. 30 days
16. 122 m 40 cm

HINT TO SELECTED PROBLEM

12. Greatest number of 6 digits is 999999. Required number must be divisible by the LCM of 24, 15 and 36 i.e., by 360.

Hence, required number = 999999 – Remainder when 999999 is divided by 360

1.6.2 PROVING IRRATIONALITY OF NUMBERS

In class IX, we have learnt about irrational numbers and their properties. We have also learnt about the existence of irrational numbers and their representation on the number line. Recall that a number is an irrational number if it cannot be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. For example, $\sqrt{2}, \sqrt{3}, \sqrt{7}, \frac{\sqrt{2}}{\sqrt{5}}, \pi$ etc. are irrational numbers. In this section, we will prove that $\sqrt{2}, \sqrt{3}, \sqrt{5}$ etc. are irrational numbers by using the Fundamental Theorem of Arithmetic. In fact, for any prime number p, $\sqrt{p}$ is an irrational number. In proving the irrationality of these numbers, we will use the result that if a prime p divides a^2, then it divides a also (see Theorem 2 on page 1.30). We will prove the irrationality of numbers by using the method of contradiction. In class IX, we have also learnt that the sum or difference of a rational and an irrational number is an irrational number. Also, the product and quotient of a non-zero rational number and an irrational number is an irrational number. We will also prove these results in the following examples.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Prove that $\sqrt{2}$ is an irrational number.* [NCERT, CBSE 2010]

SOLUTION Let us assume on the contrary that $\sqrt{2}$ is a rational number. Then, there exist positive integers a and b such that

$\sqrt{2} = \frac{a}{b}$ where, a and b, are co-prime i.e. their HCF is 1

$\Rightarrow \quad (\sqrt{2})^2 = \left(\frac{a}{b}\right)^2$

$\Rightarrow \quad 2 = \frac{a^2}{b^2}$

$\Rightarrow \quad 2b^2 = a^2$

$\Rightarrow \quad 2 \mid a^2$ $\quad [\because 2 \mid 2b^2 \text{ and } 2b^2 = a^2]$

$\Rightarrow \quad 2 \mid a$ $\quad$ [Using Theorem 2 on page 1.29] ...(i)

$\Rightarrow \quad a = 2c$ for some integer c

$\Rightarrow \quad a^2 = 4c^2$

$\Rightarrow \quad 2b^2 = 4c^2$ $\quad [\because 2b^2 = a^2]$

$\Rightarrow \quad b^2 = 2c^2$

$\Rightarrow \quad 2 \mid b^2$ $\quad [\because 2 \mid 2c^2]$

$\Rightarrow \quad 2 \mid b$ $\quad$ ...(ii)

From (i) and (ii), we obtain that 2 is a common factor of a and b. But, this contradicts the fact that a and b have no common factor other than 1. This means that our supposition is wrong.

Hence, $\sqrt{2}$ is an irrational number.

EXAMPLE 2 *Prove that $\sqrt{3}$ is an irrational number.* [NCERT, CBSE 2009, 2010]

SOLUTION Let us assume on the contrary that $\sqrt{3}$ is a rational number. Then, there exist positive integers a and b such that

$$\sqrt{3} = \frac{a}{b}, \text{ where } a \text{ and } b \text{ are co-prime i.e. their HCF is 1.}$$

Now,

$$\sqrt{3} = \frac{a}{b}$$

$$\Rightarrow \quad 3 = \frac{a^2}{b^2}$$

$$\Rightarrow \quad 3b^2 = a^2$$

$$\Rightarrow \quad 3 \mid a^2 \qquad [\because 3 \mid 3b^2]$$

$$\Rightarrow \quad 3 \mid a \qquad \text{[By Theorem 2 on page 1.29]} \quad \ldots \text{(i)}$$

$$\Rightarrow \quad a = 3c \text{ for some integer } c$$

$$\Rightarrow \quad a^2 = 9c^2$$

$$\Rightarrow \quad 3b^2 = 9c^2 \qquad [\because a^2 = 3b^2]$$

$$\Rightarrow \quad b^2 = 3c^2$$

$$\Rightarrow \quad 3 \mid b^2 \qquad [\because 3 \mid 3c^2]$$

$$\Rightarrow \quad 3 \mid b \qquad \text{[By Theorem 2 on page 1.29]} \quad \ldots \text{(ii)}$$

From (i) and (ii), we observe that a and b have at least 3 as a common factor. But this contradicts the fact that a and b are co-prime. This means that our assumption is not correct.

Hence, $\sqrt{3}$ is an irrational number.

EXAMPLE 3 *Prove that $3\sqrt{2}$ is irrational.* [NCERT]

SOLUTION Let us assume, to the contrary, that $3\sqrt{2}$ is rational. Then, there exist co-prime positive integers a and b such that

$$3\sqrt{2} = \frac{a}{b}$$

$$\Rightarrow \quad \sqrt{2} = \frac{a}{3b}$$

$\Rightarrow \quad \sqrt{2}$ is rational $\quad \left[\because\ 3, a \text{ and } b \text{ are integers} \therefore \frac{a}{3b} \text{ is a rational number}\right]$

This contradicts the fact that $\sqrt{2}$ is irrational. So, our assumption is not correct.

Hence, $3\sqrt{2}$ is an irrational number.

EXAMPLE 4 *Prove that $\sqrt{5}$ is an irrational number.* [NCERT, CBSE 2009, 2010]

SOLUTION Let us assume on the contrary that $\sqrt{5}$ is a rational number. Then, there exist co-prime positive integers a and b such that

$$\sqrt{5} = \frac{a}{b}$$

$\Rightarrow \quad 5b^2 = a^2$

$\Rightarrow \quad 5 \mid a^2 \quad [\because\ 5 \mid 5b^2]$

$\Rightarrow \quad 5 \mid a \quad$ [See Theorem 2 on page 1.29] ...(i)

$\Rightarrow \quad a = 5c$ for some positive integer c

$\Rightarrow \quad a^2 = 25c^2$

$\Rightarrow \quad 5b^2 = 25c^2 \quad [\because\ a^2 = 5b^2]$

$\Rightarrow \quad b^2 = 5c^2$

$\Rightarrow \quad 5 \mid b^2 \quad [\because\ 5 \mid 5c^2]$

$\Rightarrow \quad 5 \mid b \quad$ [See Theorem 2 on page 1.29] ...(ii)

From (i) and (ii), we find that a and b have at least 5 as a common factor. This contradicts the fact that a and b are co-prime.

Hence, $\sqrt{5}$ is an irrational number.

EXAMPLE 5 *Prove that $5 - \sqrt{3}$ is an irrational number.* [NCERT]

SOLUTION Let us assume on the contrary that $5 - \sqrt{3}$ is rational. Then, there exist co-prime positive integers a and b such that

$$5 - \sqrt{3} = \frac{a}{b}$$

$\Rightarrow \quad 5 - \frac{a}{b} = \sqrt{3}$

$\Rightarrow \quad \frac{5b - a}{b} = \sqrt{3}$

$\Rightarrow \quad \sqrt{3}$ is rational $\quad \left[\because\ a, b \text{ are integers} \therefore \frac{5b-a}{b} \text{ is a rational number}\right]$

This contradicts the fact that $\sqrt{3}$ is irrational. So, our assumption is incorrect. Hence, $5 - \sqrt{3}$ is an irrational number.

EXAMPLE 6 *Prove that* $3+2\sqrt{5}$ *is irrational.* [NCERT]

SOLUTION Let us assume on the contrary that $3+2\sqrt{5}$ is rational. Then there exist co-prime positive integers a and b such that

$$3+2\sqrt{5}=\frac{a}{b}$$

$$\Rightarrow \quad 2\sqrt{5}=\frac{a}{b}-3$$

$$\Rightarrow \quad \sqrt{5}=\frac{a-3b}{2b}$$

$\Rightarrow$ $\sqrt{5}$ is rational $\left[\because a, b \text{ are integers } \therefore \frac{a-3b}{2b} \text{ is a rational}\right]$

This contradicts the fact that $\sqrt{5}$ is irrational. So, our supposition is incorrect. Hence, $3+2\sqrt{5}$ is an irrational number.

EXAMPLE 7 *Prove that* $\sqrt{2}+\sqrt{5}$ *is irrational.*

SOLUTION Let us assume on the contrary that $\sqrt{2}+\sqrt{5}$ is a rational number. Then, there exist co-prime positive integers a and b such that

$$\sqrt{2}+\sqrt{5}=\frac{a}{b}$$

$$\Rightarrow \quad \frac{a}{b}-\sqrt{2}=\sqrt{5}$$

$$\Rightarrow \quad \left(\frac{a}{b}-\sqrt{2}\right)^2=(\sqrt{5})^2 \quad \text{[Squaring both sides]}$$

$$\Rightarrow \quad \frac{a^2}{b^2}-\frac{2a}{b}\sqrt{2}+2=5$$

$$\Rightarrow \quad \frac{a^2}{b^2}-3=\frac{2a}{b}\sqrt{2}$$

$$\Rightarrow \quad \frac{a^2-3b^2}{2ab}=\sqrt{2}$$

$\Rightarrow$ $\sqrt{2}$ is a rational number $\left[\because a, b \text{ are integers } \because \frac{a^2-3b^2}{2ab} \text{ is rational}\right]$

This contradicts the fact that $\sqrt{2}$ is irrational. So, our assumption is wrong.

Hence, $\sqrt{2}+\sqrt{5}$ is irrational.

LEVEL-2

EXAMPLE 8 *Show that there is no positive integer n for which* $\sqrt{n-1}+\sqrt{n+1}$ *is rational.*

SOLUTION If possible, let there be a positive integer n for which $\sqrt{n-1}+\sqrt{n+1}$ is rational equal to $\frac{a}{b}$ (say), where a, b are positive integers. Then,

$$\frac{a}{b} = \sqrt{n-1} + \sqrt{n+1} \qquad \dots \text{(i)}$$

$$\Rightarrow \quad \frac{b}{a} = \frac{1}{\sqrt{n-1} + \sqrt{n+1}}$$

$$\Rightarrow \quad \frac{b}{a} = \frac{\sqrt{n+1} - \sqrt{n-1}}{\{\sqrt{n+1} + \sqrt{n-1}\}\{\sqrt{n+1} - \sqrt{n-1}\}} = \frac{\sqrt{n+1} - \sqrt{n-1}}{(n+1)-(n-1)} = \frac{\sqrt{n+1} - \sqrt{n-1}}{2}$$

$$\Rightarrow \quad \frac{2b}{a} = \sqrt{n+1} - \sqrt{n-1} \qquad \dots \text{(ii)}$$

Adding (i) and (ii) and subtracting (ii) from (i), we get

$$2\sqrt{n+1} = \frac{a}{b} + \frac{2b}{a} \text{ and } 2\sqrt{n-1} = \frac{a}{b} - \frac{2b}{a}$$

$$\Rightarrow \quad \sqrt{n+1} = \frac{a^2 + 2b^2}{2ab} \text{ and } \sqrt{n-1} = \frac{a^2 - 2b^2}{2ab}$$

$\Rightarrow$ $\sqrt{n+1}$ and $\sqrt{n-1}$ are rationals $\left[\because a, b \text{ are integers } \therefore \frac{a^2 + 2b^2}{2ab} \text{ and } \frac{a^2 - 2b^2}{2ab} \text{ are rationals.}\right]$

$\Rightarrow$ $(n+1)$ and $(n-1)$ are perfect squares of positive integers.

This is not possible as any two perfect squares differ at least by 3.

Hence, there is no positive integer n for which $(\sqrt{n-1} + \sqrt{n+1})$ is rational.

EXAMPLE 9 *Let a and b be positive integers. Show that* $\sqrt{2}$ *always lies between* $\frac{a}{b}$ and $\frac{a+2b}{a+b}$.

SOLUTION We do not know whether $\frac{a}{b} < \frac{a+2b}{a+b}$ or, $\frac{a}{b} > \frac{a+2b}{a+b}$.

Therefore, to compare these two numbers, let us compute $\frac{a}{b} - \frac{a+2b}{a+b}$

We have,

$$\frac{a}{b} - \frac{a+2b}{a+b} = \frac{a(a+b) - b(a+2b)}{b(a+b)} = \frac{a^2 + ab - ab - 2b^2}{b(a+b)} = \frac{a^2 - 2b^2}{b(a+b)}$$

$$\therefore \quad \frac{a}{b} - \frac{a+2b}{a+b} > 0$$

$$\Rightarrow \quad \frac{a^2 - 2b^2}{b(a+b)} > 0$$

$$\Rightarrow \quad a^2 - 2b^2 > 0$$

$$\Rightarrow \quad a^2 > 2b^2$$

$$\Rightarrow \quad a > \sqrt{2}\, b$$

and, $\frac{a}{b} - \frac{a+2b}{a+b} < 0$

$\Rightarrow \quad \frac{a^2 - 2b^2}{b(a+b)} < 0$

$\Rightarrow \quad a^2 - 2b^2 < 0$

$\Rightarrow \quad a^2 < 2b^2$

$\Rightarrow \quad a < \sqrt{2}\,b$

Thus, $\frac{a}{b} > \frac{a+2b}{a+b}$, if $a > \sqrt{2}\,b$ and $\frac{a}{b} < \frac{a+2b}{a+b}$, if $a < \sqrt{2}\,b$.

So, we have the following cases:

CASE I *When* $a > \sqrt{2}\,b$

In this case, we have

$$\frac{a}{b} > \frac{a+2b}{a+b} \text{ i.e., } \frac{a+2b}{a+b} < \frac{a}{b}$$

We have to prove that

$$\frac{a+2b}{a+b} < \sqrt{2} < \frac{a}{b}$$

We have,

$a > \sqrt{2}b$

$\Rightarrow \quad a^2 > 2b^2$

$\Rightarrow \quad a^2 + a^2 > a^2 + 2b^2$ [Adding a^2 on both sides]

$\Rightarrow \quad 2a^2 + 2b^2 > (a^2 + 2b^2) + 2b^2$ [Adding $2b^2$ on both sides]

$\Rightarrow \quad 2(a^2 + b^2) + 4ab > a^2 + 4b^2 + 4ab$ [Adding $4ab$ on both sides]

$\Rightarrow \quad 2(a^2 + 2ab + b^2) > a^2 + 4ab + 4b^2$

$\Rightarrow \quad 2(a+b)^2 > (a+2b)^2$

$\Rightarrow \quad \sqrt{2}(a+b) > a + 2b$

$\Rightarrow \quad \sqrt{2} > \frac{a+2b}{a+b}$ ... (i)

Again,

$a > \sqrt{2}\,b \Rightarrow \frac{a}{b} > \sqrt{2}$ ... (ii)

From (i) and (ii), we get

$$\frac{a+2b}{a+b} < \sqrt{2} < \frac{a}{b}$$

CASE II *When* $a < \sqrt{2}\,b$

In this case, we have

$$\frac{a}{b} < \frac{a+2b}{a+b}$$

We have to show that $\frac{a}{b} < \sqrt{2} < \frac{a+2b}{a+b}$

We have,

$$a < \sqrt{2}\,b$$

$\Rightarrow \quad a^2 < 2b^2$

$\Rightarrow \quad a^2 + a^2 < a^2 + 2b^2$ [Adding a^2 on both sides]

$\Rightarrow \quad 2a^2 + 2b^2 < a^2 + 4b^2$ [Adding $2b^2$ on both sides]

$\Rightarrow \quad 2a^2 + 4ab + 2b^2 < a^2 + 4ab + 4b^2$

$\Rightarrow \quad 2(a+b)^2 < (a+2b)^2$

$\Rightarrow \quad \sqrt{2}\,(a+b) < a+2b$

$\Rightarrow \quad \sqrt{2} < \frac{a+2b}{a+b}$ ... (iii)

$\Rightarrow \quad a < \sqrt{2}\,b \Rightarrow \frac{a}{b} < \sqrt{2}$ ... (iv)

From (iii) and (iv), we get

$$\frac{a}{b} < \sqrt{2} < \frac{a+2b}{a+b}$$

Hence, $\sqrt{2}$ lies between $\frac{a}{b}$ and $\frac{a+2b}{a+b}$.

EXAMPLE 10 *Let a, b, c, d be positive rationals such that $a + \sqrt{b} = c + \sqrt{d}$, then either $a = c$ and $b = d$ or b and d are squares of rationals.*

SOLUTION If $a = c$, then

$$a + \sqrt{b} = c + \sqrt{d} \Rightarrow \sqrt{b} = \sqrt{d} \Rightarrow b = d.$$

So, let $a \neq c$. Then, there exists a positive rational number x such that $a = c + x$.

Now,

$$a + \sqrt{b} = c + \sqrt{d}$$

$\Rightarrow \quad c + x + \sqrt{b} = c + \sqrt{d}$ [$\because a = c + x$]

$\Rightarrow \quad x + \sqrt{b} = \sqrt{d}$ ...(i)

$\Rightarrow \quad (x + \sqrt{b})^2 = (\sqrt{d})^2$

$\Rightarrow \quad x^2 + 2\sqrt{b}\,x + b = d$

$\Rightarrow \quad d - x^2 - b = 2x\sqrt{b}$

$\Rightarrow \quad \sqrt{b} = \frac{d - x^2 - b}{2x}$

$\Rightarrow \quad \sqrt{b}$ is rational $\quad \left[\because d, x, b \text{ are rationals } \therefore \frac{d - x^2 - b^2}{2x} \text{ is rational}\right]$

$\Rightarrow \quad b$ is the square of a rational number.

From (i), we have

$$\sqrt{d} = x + \sqrt{b}$$

$\Rightarrow \quad \sqrt{d}$ is rational $\quad [\because \sqrt{b} \text{ is rational}]$

$\Rightarrow \quad d$ is the square of a rational number.

Hence, either $a = c$ and $b = d$ or b and d are the squares of rationals.

EXAMPLE 11 *Let a, b, c, p be rational numbers such that p is not a perfect cube. If $a + b\,p^{\frac{1}{3}} + c\,p^{\frac{2}{3}} = 0$, then prove that $a = b = c = 0$.*

SOLUTION We have,

$$a + b\,p^{\frac{1}{3}} + c\,p^{\frac{2}{3}} = 0 \quad \ldots \text{(i)}$$

Multiplying both sides by $p^{\frac{1}{3}}$, we get

$$a\,p^{\frac{1}{3}} + b\,p^{\frac{2}{3}} + cp = 0 \quad \ldots \text{(ii)}$$

Multiplying (i) by b and (ii) by c and subtracting, we get

$$(ab + b^2 p^{1/3} + bcp^{2/3}) - (acp^{1/3} + bcp^{2/3} + c^2 p) = 0$$

$$\Rightarrow \quad (b^2 - ac)\, p^{1/3} + ab - c^2 p = 0$$

$$\Rightarrow \quad b^2 - ac = 0 \text{ and } ab - c^2 p = 0 \quad [\because p^{1/3} \text{ is irrational}]$$

$$\Rightarrow \quad b^2 = ac \text{ and } ab = c^2 p$$

$$\Rightarrow \quad b^2 = ac \text{ and } a^2 b^2 = c^4 p^2$$

$$\Rightarrow \quad a^2(ac) = c^4 p^2 \quad \left[\text{Putting } b^2 = ac \text{ in } a^2 b^2 = c^4 p^2\right]$$

$$\Rightarrow \quad a^3 c - p^2 c^4 = 0$$

$$\Rightarrow \quad (a^3 - p^2 c^3)\, c = 0$$

$$\Rightarrow \quad a^3 - p^2 c^3 = 0, \text{ or } c = 0$$

Now, $\quad a^3 - p^2 c^3 = 0$

$$\Rightarrow \quad p^2 = \frac{a^3}{c^3} \Rightarrow (p^2)^{1/3} = \left(\frac{a^3}{c^3}\right)^{1/3} \Rightarrow (p^{1/3})^2 = \left\{\left(\frac{a}{c}\right)^3\right\}^{1/3} \Rightarrow (p^{1/3})^2 = \frac{a}{c}$$

This is not possible as $p^{1/3}$ is irrational and $\frac{a}{c}$ is rational.

$\therefore \qquad a^3 - p^2c^3 \neq 0$

Hence, $\quad c = 0$

Putting $c = 0$ in $b^2 - ac = 0$, we get $b = 0$

Putting $b = 0$ and $c = 0$ in $a + b\,p^{1/3} + cp^{2/3} = 0$, we get $a = 0$

Hence, $\quad a = b = c = 0.$

EXAMPLE 12 *For any positive real number x, prove that there exists an irrational number y such that* $0 < y < x$.

SOLUTION If x is irrational, then $y = \frac{x}{2}$ is also an irrational number such that $0 < y < x$.

If x is rational, then $\frac{x}{\sqrt{2}}$ is an irrational number such that $\frac{x}{\sqrt{2}} < x$ as $\sqrt{2} > 1$.

$\therefore \qquad y = \frac{x}{\sqrt{2}}$ is an irrational number such that $0 < y < x$.

EXERCISE 1.5

LEVEL-1

1. Show that the following numbers are irrational.

   (i) $\frac{1}{\sqrt{2}}$ (ii) $7\sqrt{5}$ (iii) $6 + \sqrt{2}$ (iv) $3 - \sqrt{5}$

2. Prove that following numbers are irrationals:

   (i) $\frac{2}{\sqrt{7}}$ (ii) $\frac{3}{2\sqrt{5}}$ (iii) $4 + \sqrt{2}$ (iv) $5\sqrt{2}$

3. Show that $2 - \sqrt{3}$ is an irrational number. [CBSE 2008]
4. Show that $3 + \sqrt{2}$ is an irrational number. [CBSE 2009]
5. Prove that $4 - 5\sqrt{2}$ is an irrational number. [CBSE 2010]
6. Show that $5 - 2\sqrt{3}$ is an irrational number. [CBSE 2009]
7. Prove that $2\sqrt{3} - 1$ is an irrational number. [CBSE 2010]
8. Prove that $2 - 3\sqrt{5}$ is an irrational number. [CBSE 2010]
9. Prove that $\sqrt{5} + \sqrt{3}$ is irrational. [NCERT EXEMPLAR]
10. Prove that $\sqrt{2} + \sqrt{3}$ is an irrational number. [NCERT EXEMPLAR]
11. Given that $\sqrt{2}$ is irrational, prove that $(5 + 3\sqrt{2})$ is an irrational number. [CBSE 2018]

LEVEL-2

12. Prove that for any prime positive integer p, $\sqrt{p}$ is an irrational number.
13. If p, q are prime positive integers, prove that $\sqrt{p} + \sqrt{q}$ is an irrational number. [NCERT EXEMPLAR]

HINTS TO SELECTED PROBLEMS

1. (i) If possible, let $\frac{1}{\sqrt{2}}$ be rational. Then, there exist positive co-primes a and b such that

$$\frac{1}{\sqrt{2}} = \frac{a}{b}$$

$\Rightarrow \quad 2a^2 = b^2$

$\Rightarrow \quad 2 \mid b^2$ $\quad [\because 2 \mid 2a^2]$

$\Rightarrow \quad 2 \mid b$

$\Rightarrow \quad b = 2c$ for some positive integer c

$\therefore \quad 2a^2 = b^2 \Rightarrow 2a^2 = 4c^2 \Rightarrow a^2 = 2c^2 \Rightarrow 2 \mid a^2$ $\quad [\because 2 \mid 2c^2]$

$\Rightarrow \quad 2 \mid a$

This is a contradiction to the fact that a, b are co-primes.

Hence, $\frac{1}{\sqrt{2}}$ is irrational.

(ii) Let $7\sqrt{5}$ be rational. Then,

$$7\sqrt{5} = \frac{a}{b} \Rightarrow \sqrt{5} = \frac{a}{7b} \Rightarrow \sqrt{5} \text{ is rational, a contradiction.}$$

$\therefore \quad 7\sqrt{5}$ is irrational.

(iii) Let $6 + \sqrt{2}$ be a rational number equal to $\frac{a}{b}$, where a, b are positive co-primes. Then,

$$6 + \sqrt{2} = \frac{a}{b}$$

$\Rightarrow \quad \sqrt{2} = \frac{a}{b} - 6$

$\Rightarrow \quad \sqrt{2} = \frac{a - 6b}{b}$

$\Rightarrow \quad \sqrt{2}$ is rational.

This is a contradiction.

Hence, $6 + \sqrt{2}$ is irrational

(iv) Let $3 - \sqrt{5}$ be a rational equal to $\frac{a}{b}$ Then,

$$3 - \sqrt{5} = \frac{a}{b}$$

$\Rightarrow \quad \sqrt{5} = 3 - \frac{a}{b}$

$\Rightarrow \quad \sqrt{5} = \dfrac{3b - a}{b}$

$\Rightarrow \quad \sqrt{5}$ is rational.

This is a contradiction.

Hence, $3 - \sqrt{5}$ is irrational.

9. Let $\sqrt{5} + \sqrt{3}$ be rational equal to $\frac{a}{b}$. Then,

$$\sqrt{5} + \sqrt{3} = \frac{a}{b}$$

$$\Rightarrow \quad \sqrt{5} = \frac{a}{b} - \sqrt{3}$$

$$\Rightarrow \quad (\sqrt{5})^2 = \left(\frac{a}{b} - \sqrt{3}\right)^2 \qquad \text{[Squaring both sides]}$$

$$\Rightarrow \quad 5 = \frac{a^2}{b^2} - \frac{2a\sqrt{3}}{b} + 3$$

$$\Rightarrow \quad 2 = \frac{a^2}{b^2} - 2\sqrt{3}\,\frac{a}{b}$$

$$\Rightarrow \quad 2\sqrt{3}\,\frac{a}{b} = \frac{a^2 - 2b^2}{b^2}$$

$\Rightarrow \quad \sqrt{3} = \dfrac{a^2 - 2b^2}{2ab} \Rightarrow \sqrt{3}$ is rational, a contradiction.

Hence, $\sqrt{5} + \sqrt{3}$ is irrational.

12. Let us assume on the contrary that $\sqrt{p}$ is rational. Then, there exist positive co-primes a and b such that

$$\sqrt{p} = \frac{a}{b}$$

$$\Rightarrow \quad p = \frac{a^2}{b^2}$$

$$\Rightarrow \quad b^2 p = a^2$$

$$\Rightarrow \quad p \mid a^2 \qquad [\because\ p \mid b^2 p]$$

$$\Rightarrow \quad p \mid a$$

$\Rightarrow \quad a = pc$ for some positive integer c.

Now, $b^2 p = a^2$

$$\Rightarrow \quad b^2 p = p^2 c^2 \qquad [\because\ a = pc]$$

$$\Rightarrow \quad b^2 = pc^2$$

$\Rightarrow \quad p \mid b^2$

$\Rightarrow \quad p \mid b$

$\therefore \quad p \mid a$ and $p \mid b$

This contradicts that a and b are co-primes.

Hence, $\sqrt{p}$ is irrational.

13. Let us assume that $\sqrt{p} + \sqrt{q}$ is a rational number equal to $\frac{a}{b}$, where a and b are integers having no common factor.

Now, $\sqrt{p} + \sqrt{q} = \frac{a}{b}$

$$\Rightarrow \quad \sqrt{p} = \frac{a}{b} - \sqrt{q}$$

$$\Rightarrow \quad \left(\sqrt{p}\right)^2 = \left(\frac{a}{b} - \sqrt{q}\right)^2$$

$$\Rightarrow \quad p = \frac{a^2}{b^2} - 2\left(\frac{a}{b}\right)\sqrt{q} + q$$

$$\Rightarrow \quad 2\left(\frac{a}{b}\right)\sqrt{q} = \frac{a^2}{b^2} + q - p$$

$$\Rightarrow \quad 2\frac{a}{b}\sqrt{q} = \frac{a^2 + b^2(q-p)}{b^2}$$

$$\Rightarrow \quad \sqrt{q} = \frac{a^2 + b^2(q-p)}{2ab}$$

$\Rightarrow \quad \sqrt{q}$ is a rational number.

This is a contradiction as $\sqrt{q}$ is an irrational number.

Hence, $\sqrt{p} + \sqrt{q}$ is an irrational number.

1.6.3 DETERMINING THE NATURE OF THE DECIMAL EXPANSIONS OF RATIONAL NUMBERS

In class IX, we have studied that the decimal expansion of a rational number is either terminating or non-terminating repeating (or recurring) without knowing when it is terminating and when it is non-terminating repeating. In this section, we will explore exactly when the decimal expansion of a rational number is terminating and when it is non-terminating repeating. In earlier classes, we have also learnt that any rational number having terminating decimal expansion can be written as a rational number whose denominator is some power of 10. For example,

(i) $0.875 = \frac{875}{1000} = \frac{875}{10^3}$ (ii) $1.512 = \frac{1512}{1000} = \frac{1512}{10^3}$

(iii) $0.01764 = \frac{1764}{100000} = \frac{1764}{10^5}$ (iv) $26.7624 = \frac{267624}{10000} = \frac{267624}{10^4}$ etc.

As we know that 2 and 5 are the only prime factors of 10. Therefore, any positive integral power of 10, say 10^n, is expressible in the form $(2 \times 5)^n = 2^n \times 5^n$. For example,

$$10 = 2 \times 5, \quad 100 = 10^2 = 2^2 \times 5^2, \quad 1000 = 10^3 = 2^3 \times 5^3, \quad 10000 = 10^4 = 2^4 \times 5^4 \text{ etc.}$$

Therefore, denominators of rational numbers having terminating decimal expansions are of the form $2^n \times 5^n$. If we express the numerators of such rational numbers as products of primes and cancel out the common factors between the numerators and the corresponding denominators, we find that the prime factorisations of their denominators are of the form $2^m \times 5^n$, where m and n are non-negative integers. For example,

(i) $0.875 = \dfrac{875}{10^3} = \dfrac{5^3 \times 7}{2^3 \times 5^3} = \dfrac{7}{2^3} = \dfrac{7}{2^3 \times 5^0}$

(ii) $1.512 = \dfrac{1512}{10^3} = \dfrac{2^3 \times 3^3 \times 7}{2^3 \times 5^3} = \dfrac{3^3 \times 7}{5^3} = \dfrac{189}{2^0 \times 5^3}$

(iii) $0.01764 = \dfrac{1764}{10^5} = \dfrac{2^2 \times 3^2 \times 7^2}{2^5 \times 5^5} = \dfrac{7^2}{2^3 \times 5^3} = \dfrac{49}{2^3 \times 5^3}$

(iv) $27.7624 = \dfrac{277624}{10^4} = \dfrac{2^3 \times 3^2 \times 7 \times 531}{2^4 \times 5^4} = \dfrac{3^2 \times 7 \times 531}{2 \times 5^4} = \dfrac{33453}{2^1 \times 5^4}$

It follows from the above discussion that the denominators of the rational numbers having terminating decimal expansions are expressible in the form $2^m \times 5^n$, where m, n are non-negative integers.

This result can be stated formally as a theorem as follows:

THEOREM 1 *Let x be a rational number whose decimal expansion terminates. Then, x can expressed in the form $\dfrac{p}{q}$, where p and q are co-primes, and the prime factorisation of q is of the form $2^m \times 5^n$, where m, n are non-negative integers.*

Let us now see whether the converse of this theorem is also true or not. That is, if we have a rational number of the form $\dfrac{p}{q}$, and the prime factorisation of q is of the form $2^m \times 5^n$, where m, n are non-negative integers, then does $\dfrac{p}{q}$ have a terminating decimal?

Let $\dfrac{a}{b}$ be a rational number in the lowest form such that the prime factorisation of b is of the form $2^m \times 5^n$, where m, n are non-negative integers.

We have the following cases:

CASE I *When $m = n$:*

In this case, we have

$$\frac{a}{b} = \frac{a}{2^m \times 5^n} = \frac{a}{2^m \times 5^m} = \frac{a}{(10)^m}$$

CASE II When $m > n$:

In this case, we have

$$m = n + p, \text{ where } p \text{ is a positive integer.}$$

$$\therefore \quad \frac{a}{b} = \frac{a}{2^m \times 5^n} = \frac{a \times 5^p}{2^m \times 5^{n+p}} = \frac{a \times 5^p}{2^m \times 5^m} = \frac{a \times 5^p}{(2 \times 5)^m} = \frac{c}{10^m}, \text{ where } c = a \times 5^p$$

CASE III When $m < n$:

In this case, we have

$$n = m + p, \text{ where } p \text{ is a positive integer.}$$

$$\therefore \quad \frac{a}{b} = \frac{a}{2^m \times 5^n} = \frac{a \times 2^p}{2^{m+p} \times 5^n} = \frac{a \times 2^p}{2^n \times 5^n} = \frac{a \times 2^p}{(2 \times 5)^n} = \frac{c}{10^n}, \text{ where } c = a \times 2^p$$

Thus, a rational number whose denominator is of the form $2^m \times 5^n$, where m, n are non-negative integers, can be converted to an equivalent rational number of the form $\frac{c}{d}$, where d is a power of 10.

For example,

(i) $\frac{7}{8} = \frac{7}{2^3} = \frac{7 \times 5^3}{2^3 \times 5^3} = \frac{7 \times 125}{(2 \times 5)^3} = \frac{875}{10^3}$

(ii) $\frac{189}{125} = \frac{189}{5^3} = \frac{2^3 \times 189}{2^3 \times 5^3} = \frac{8 \times 189}{(2 \times 5)^3} = \frac{1512}{10^3}$

(iii) $\frac{49}{500} = \frac{49}{2^2 \times 5^3} = \frac{49 \times 2}{2^3 \times 5^3} = \frac{98}{(2 \times 5)^3} = \frac{98}{10^3}$

(iv) $\frac{2139}{1250} = \frac{2139}{2^1 \times 5^4} = \frac{2139 \times 2^3}{2^4 \times 5^4} = \frac{2139 \times 8}{(2 \times 5)^4} = \frac{17112}{10^4}$

$$\therefore \quad \frac{7}{8} = \frac{875}{10^3} = 0.875$$

$$\frac{189}{125} = \frac{1512}{10^3} = 1.512$$

$$\frac{49}{500} = \frac{98}{10^3} = 0.098$$

and, $$\frac{2139}{1250} = \frac{17112}{10^4} = 1.7112$$

This shows that the decimal expansion of a rational number whose denominator is of the form $2^m \times 5^n$, where m, n are non-negative integers, is terminating. Also, it terminates after k places of decimals, where k is the larger of m and n.

This result can be stated formally as a theorem as follows:

THEOREM 2 *Let $x = \frac{p}{q}$ be a rational number, such that the prime factorisation of q is of the form $2^m \times 5^n$, where m, n are non-negative integers. Then, x has a decimal expansion which terminates after k places of decimals, where k is the larger of m and n.*

Let us now consider rational numbers whose decimal expansions are non-terminating and repeating. For example,

(i) $\frac{5}{3} = 1.66666\cdots\cdots$ (ii) $\frac{17}{6} = 2.83333\cdots\cdots$ (iii) $\frac{1}{7} = 0.142857142857\cdots\cdots$

We observe that the prime factorisation of the denominators of these rational numbers are not of the form $2^m \times 5^n$, where m, n are non-negative integers.

So, we arrive at the following conclusion.

THEOREM 3 *Let $x = \frac{p}{q}$ be a rational number, such that the prime factorisation of q is not of the form $2^m \times 5^n$, where m, n are non-negative integers. Then, x has a decimal expansion which is non-terminating repeating.*

Let us now discuss some examples to determine the nature of the decimal expansions of rational numbers by using the above theorems.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Without actually performing the long division, state whether the following rational numbers will have terminating decimal expansion or a non-terminating repeating decimal expansion. Also, find the number of places of decimals after which the decimal expansion terminates.*

(i) $\frac{17}{8}$ [NCERT] (ii) $\frac{64}{455}$ [NCERT] (iii) $\frac{29}{343}$ [NCERT]

(iv) $\frac{15}{1600}$ [NCERT] (v) $\frac{13}{3125}$ [NCERT] (vi) $\frac{23}{2^3 5^2}$ [NCERT]

SOLUTION (i) We have, $\frac{17}{8} = \frac{17}{2^3 \times 5^0}$

So, the denominator 8 of $\frac{17}{8}$ is of the form $2^m \times 5^n$, where m, n are non-negative integers.

Hence, $\frac{17}{8}$ has terminating decimal expansion. The decimal expansion of $\frac{17}{8}$ terminates after three places of decimals.

(ii) We have,

$$\frac{64}{455} = \frac{64}{5 \times 7 \times 13}$$

Clearly, 455 is not of the form $2^m \times 5^n$. So, the decimal expansion of $\frac{64}{455}$ is non-terminating repeating.

(iii) We have,

$$\frac{29}{343} = \frac{29}{3^5}$$

Clearly, 343 is not of the form $2^m \times 5^n$.

Hence, the decimal expansion of $\frac{29}{343}$ is non-terminating repeating.

(iv) We have,

$$\frac{15}{1600} = \frac{3}{320} = \frac{3}{2^6 \times 5}$$

This means that the prime factorisation of the denominator of $\frac{15}{1600}$ is of the form $2^m \times 5^n$. Hence, it has terminating decimal expansion which terminates after 6 places of decimals.

(v) We have,

$$\frac{13}{3125} = \frac{13}{2^0 \times 5^5}$$

This shows that the prime factorisation of the denominator of $\frac{13}{3125}$ is of the form $2^m \times 5^n$. Hence, it has terminating decimal expansion which terminates after 5 places of decimals.

(vi) Clearly, the prime factorisation of the denominator of $\frac{23}{2^3 \times 5^2}$ is of the form $2^m \times 5^n$. So, it has terminating decimal expansion which terminates after 3 places of decimals.

LEVEL-2

EXAMPLE 2 *What can you say about the prime factorisations of the denominators of the following rationals:*

(i) 34.12345 (ii) $34.\overline{5678}$

SOLUTION (i) Since 34.12345 has terminating decimal expansion. So, its denominator is of the form $2^m \times 5^n$, where m, n are non-negative integers.

(ii) Since $34.\overline{5678}$ has non-terminating repeating decimal expansion. So, its denominator has factors other than 2 or 5.

EXERCISE 1.6

LEVEL-1

1. Without actually performing the long division, state whether the following rational numbers will have a terminating decimal expansion or a non-terminating repeating decimal expansion.

(i) $\frac{23}{8}$ (ii) $\frac{125}{441}$ (iii) $\frac{35}{50}$ [NCERT]

(iv) $\frac{77}{210}$ [NCERT] (v) $\frac{129}{2^2 \times 5^7 \times 7^{17}}$ (vi) $\frac{987}{10500}$ [NCERT EXEMPLAR]

2. Write down the decimal expansions of the following rational numbers by writing their denominators in the form $2^m \times 5^n$, where m, n are non-negative integers.

(i) $\frac{3}{8}$ (ii) $\frac{13}{125}$ (iii) $\frac{7}{80}$

(iv) $\frac{14588}{625}$ (v) $\frac{129}{2^2 \times 5^7}$ [NCERT]

3. Write the denominator of the rational number $\frac{257}{5000}$ in the form $2^m \times 5^n$, where m, n are non-negative integers. Hence, write the decimal expansion, without actual division.

LEVEL-2

4. What can you say about the prime factorisations of the denominators of the following rationals:

(i) 43.123456789 (ii) $43.\overline{123456789}$

(iii) $27.\overline{142857}$ [CBSE 2010] (iv) 0.120120012000120000.... [NCERT]

5. A rational number in its decimal expansion is 327.7081. What can you say about the prime factors of q, when this number is expressed in the form $\frac{p}{q}$? Give reasons. [NCERT EXEMPLAR]

ANSWERS

1. (i) Terminating (ii) Non-terminating repeating (iii) Terminating (iv) Non-terminating repeating (v) Non-terminating repeating. (vi) Terminating
2. (i) 0.375 (ii) 0.104 (iii) 0.0875 (iv) 23.3408 (v) 0.0004128
3. $2^3 \times 5^4$; 0.0514
4. (i) Prime factorisation of the denominator is of the form $2^m \times 5^n$, where m, n are non-negative integers.
   (ii) Prime factorisation of the denominator contains factors other than 2 or 5.
   (iii) Prime factorisation of the denominator contains factors other than 2 or 5.
   (iv) Prime factorisation of the denominator contains factors other than 2 or 5.
5. Since 327.7081 is a terminating decimal number, so, q must be of the form $2^m \times 5^n$; m, n are natural numbers.

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

Answer each of the following questions either in one word or one sentence or as per requirement of the questions:

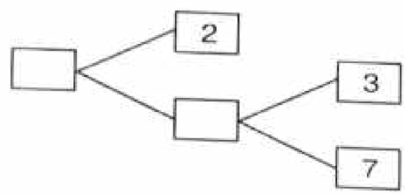
1. State Euclid's division lemma.
2. State Fundamental Theorem of Arithmetic.
3. Write 98 as product of its prime factors.
4. Write the exponent of 2 in the prime factorization of 144.
5. Write the sum of the exponents of prime factors in the prime factorization of 98.
6. If the prime factorization of a natural number n is $2^3 \times 3^2 \times 5^2 \times 7$, write the number of consecutive zeros in n.
7. If the product of two numbers is 1080 and their HCF is 30, find their LCM.

8. Write the condition to be satisfied by q so that a rational number $\frac{p}{q}$ has a terminating decimal expansion.

[CBSE 2008]

9. Write the condition to be satisfied by q so that a rational number $\frac{p}{q}$ has a non-terminating decimal expansion.

10. Complete the missing entries in the following factor tree.



[CBSE 2008]

11. The decimal expansion of the rational number $\frac{43}{2^4 \times 5^3}$ will terminate after how many places of decimals?

[CBSE 2009]

12. Has the rational number $\frac{441}{2^2 \times 5^7 \times 7^2}$ a terminating or a nonterminating decimal representation?

[CBSE 2010]

13. Write whether $\frac{2\sqrt{45} + 3\sqrt{20}}{2\sqrt{5}}$ on simplification gives a rational or an irrational number.

[CBSE 2010]

14. What is an algorithm?
15. What is a lemma?
16. If p and q are two prime numbers, then what is their HCF?
17. If p and q are two prime numbers, then what is their LCM?
18. What is the total number of factors of a prime number?
19. What is a composite number?
20. What is the HCF of the smallest composite number and the smallest prime number?

[CBSE 2018]

21. HCF of two numbers is always a factor of their LCM (True/False).
22. π is an irrational number (True/False).
23. The sum of two prime numbers is always a prime number (True/False).
24. The product of any three consecutive natural numbers is divisible by 6 (True/False).
25. Every even integer is of the form $2m$, where m is an integer (True/False).
26. Every odd integer is of the form $2m - 1$, where m is an integer (True/False).
27. The product of two irrational numbers is an irrational number (True/False).
28. The sum of two irrational numbers is an irrational number (True/False).
29. For what value of n, $2^n \times 5^n$ ends in 5.
30. If a and b are relatively prime numbers, then what is their HCF?
31. If a and b are relatively prime numbers, then what is their LCM?
32. Two numbers have 12 as their HCF and 350 as their LCM (True/False).

ANSWERS

1. See text 2. See text 3. 2×7^2 4. 4
5. 3 6. 2 7. 36
8. The prime factorization of q must be of the form $2^m \times 5^n$, where m, n are non-negative integers.
9. The prime factorization of q is not of the form $2^m \times 5^n$, where m, n are non-negative integers. 10. 42, 21 11. 4
12. Non-terminating 13. Rational Number 16. 1
17. $p \times q$ 18. 2 20. 2 21. True
22. True 23. False 24. True 25. True
26. True 27. False 28. False 29. No value of n
30. 1 31. ab 32. False

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following questions.

1. The exponent of 2 in the prime factorisation of 144, is
   (a) 4 (b) 5 (c) 6 (d) 3
2. The LCM of two numbers is 1200. Which of the following cannot be their HCF?
   (a) 600 (b) 500 (c) 400 (d) 200
3. If $n = 2^3 \times 3^4 \times 5^4 \times 7$, then the number of consecutive zeros in n, where n is a natural number, is
   (a) 2 (b) 3 (c) 4 (d) 7
4. The sum of the exponents of the prime factors in the prime factorisation of 196, is
   (a) 1 (b) 2 (c) 4 (d) 6
5. The number of decimal places after which the decimal expansion of the rational number $\frac{23}{2^2 \times 5}$ will terminate, is
   (a) 1 (b) 2 (c) 3 (d) 4
6. If p_1 and p_2 are two odd prime numbers such that $p_1 > p_2$, then $p_1^2 - p_2^2$ is
   (a) an even number (b) an odd number
   (c) an odd prime number (d) a prime number
7. If two positive ingeters a and b are expressible in the form $a = pq^2$ and $b = p^3q$; p, q being prime numbers, then LCM (a, b) is
   (a) pq (b) p^3q^3 (c) p^3q^2 (d) p^2q^2

8. In Q. No. 7, HCF (a, b) is

(a) pq (b) p^3q^3 (c) p^3q^2 (d) p^2q^2

9. If two positive integers m and n are expressible in the form $m = pq^3$ and $n = p^3q^2$, where p, q are prime numbers, then HCF $(m, n) =$

(a) pq (b) pq^2 (c) p^3q^3 (d) p^2q^3

10. If the LCM of a and 18 is 36 and the HCF of a and 18 is 2, then $a =$

(a) 2 (b) 3 (c) 4 (d) 1

11. The HCF of 95 and 152, is

(a) 57 (b) 1 (c) 19 (d) 38

12. If HCF (26, 169) = 13, then LCM (26, 169) =

(a) 26 (b) 52 (c) 338 (d) 13

13. If $a = 2^3 \times 3$, $b = 2 \times 3 \times 5$, $c = 3^n \times 5$ and LCM $(a, b, c) = 2^3 \times 3^2 \times 5$, then $n =$

(a) 1 (b) 2 (c) 3 (d) 4

14. The decimal expansion of the rational number $\frac{14587}{1250}$ will terminate after

(a) one decimal place (b) two decimal place

(c) three decimal place (d) four decimal place

15. If p and q are co-prime numbers, then p^2 and q^2 are

(a) coprime (b) not coprime (c) even (d) odd

16. Which of the following rational numbers have terminating decimal?

(i) $\frac{16}{225}$ (ii) $\frac{5}{18}$ (iii) $\frac{2}{21}$ (iv) $\frac{7}{250}$

(a) (i) and (ii) (b) (ii) and (iii) (c) (i) and (iii) (d) (i) and (iv)

17. If 3 is the least prime factor of number a and 7 is the least prime factor of number b, then the least prime factor of $a + b$, is

(a) 2 (b) 3 (c) 5 (d) 10

18. $3.\overline{27}$ is

(a) an integer (b) a rational number

(c) a natural number (d) an irrational number

19. The smallest number by which $\sqrt{27}$ should be multiplied so as to get a rational number is

(a) $\sqrt{27}$ (b) $3\sqrt{3}$ (c) $\sqrt{3}$ (d) 3

20. The smallest rational number by which $\frac{1}{3}$ should be multiplied so that its decimal expansion terminates after one place of decimal, is

(a) $\frac{3}{10}$ (b) $\frac{1}{10}$ (c) 3 (d) $\frac{3}{100}$

21. If n is a natural number, then $9^{2n} - 4^{2n}$ is always divisible by

(a) 5 (b) 13 (c) both 5 and 13 (d) None of these

[Hint: $9^{2n} - 4^{2n}$ is of the form $a^{2n} - b^{2n}$ which is divisible by both $a - b$ and $a + b$. So, $9^{2n} - 4^{2n}$ is divisible by both $9 - 4 = 5$ and $9 + 4 = 13$.]

22. If n is any natural number, then $6^n - 5^n$ always ends with

(a) 1 (b) 3 (c) 5 (d) 7

[Hint: For any $n \in N$, 6^n and 5^n end with 6 and 5 respectively. Therefore, $6^n - 5^n$ always ends with $6 - 5 = 1$.]

23. The LCM and HCF of two rational numbers are equal, then the numbers must be

(a) prime (b) co-prime (c) composite (d) equal

24. If the sum of LCM and HCF of two numbers is 1260 and their LCM is 900 more than their HCF, then the product of two numbers is

(a) 203400 (b) 194400 (c) 198400 (d) 205400

25. The remainder when the square of any prime number greater than 3 is divided by 6, is

(a) 1 (b) 3 (c) 2 (d) 4

[Hint: Any prime number greater than 3 is of the form $6k \pm 1$, where k is a natural number and $(6k \pm 1)^2 = 36k^2 \pm 12k + 1 = 6k(6k \pm 2) + 1$]

26. For some integer m, every even integer is of the form

(a) m (b) $m + 1$ (c) $2m$ (d) $2m + 1$

27. For some integer q, every odd integer is of the form

(a) q (b) $q + 1$ (c) $2q$ (d) $2q + 1$

28. $n^2 - 1$ is divisible by 8, if n is

(a) an integer (b) a natural number

(c) an odd integer (d) an even integer

29. The decimal expansion of the rational number $\frac{33}{2^2 \times 5}$ will terminate after

(a) one decimal place (b) two decimal places

(c) three decimal places(d)more than 3 decimal places

30. If two positive integers a and b are written as $a = x^3y^2$ and $b = xy^3$; x, y are prime numbers, then HCF (a, b) is

(a) xy (b) xy^2 (c) x^3y^3 (d) x^2y^2

31. The least number that is divisible by all the numbers from 1 to 10 (both inclusive) is

(a) 10 (b) 100 (c) 504 (d) 2520

32. The largest number which divides 70 and 125, leaving remainders 5 and 8, respectively, is

(a) 13 (b) 65 (c) 875 (d) 1750

33. If the HCF of 65 and 117 is expressible in the form $65m - 117$, then the value of m is

(a) 4 (b) 2 (c) 1 (d) 3

34. The decimal expansion of the rational number $\frac{14587}{1250}$ will terminate after:

(a) one decimal place (b) two decimal places

(c) three decimal places (d) four decimal places

35. Euclid's division lemma states that for two positive integers a and b, there exist unique integers q and r such that $a = bq + r$, where r must satisfy

(a) $1 < r < b$ (b) $0 < r \le b$ (c) $0 \le r < b$ (d) $0 < r < b$

ANSWERS

1. (a)	2. (b)	3. (b)	4. (c)	5. (b)
6. (a)	7. (c)	8. (a)	9. (b)	10. (c)
11. (c)	12. (c)	13. (b)	14. (d)	15. (a)
16. (d)	17. (a)	18. (b)	19. (c)	20. (a)
21. (c)	22. (a)	23. (d)	24. (b)	25. (a)
26. (c)	27. (d)	28. (c)	29. (b)	30. (b)
31. (d)	32. (a)	33. (b)	34. (d)	35. (c)

SUMMARY

1. *Euclid's division lemma*: Given positive integers a and b there exist whole numbers q and r satisfying $a = bq + r, 0 \leq r < b$.

2. *Euclid's division algorithm*: In order to compute the HCF of two positive integers, say a and b, with $a > b$ by using Euclid's algorithm we follow the following steps:

   STEP I *Apply Euclid's division lemma to a and b and obtain whole numbers q_1 and r_1 such that* $a = bq_1 + r_1, 0 \leq r_1 < b$.

   STEP II *If $r_1 = 0$, b is the HCF of a and b*

   STEP III *If $r_1 \neq 0$, apply Euclid's division lemma to b and r_1 and obtain two whole numbers q_1 and r_2 such that $b = q_1 r_1 + r_2$.*

   STEP IV *If $r_2 = 0$, then r_1 is the HCF of a and b.*

   STEP V *If $r_2 \neq 0$, then apply Euclid's division lemma to r_1 and r_2 and continue the above process till the remainder r_n is zero. The divisor at this stage i.e. r_{n-1}, or the non-zero remainder at the previous stage, is the HCF of a and b.*

3. *The Fundamental Theorem of Arithmetic*: Every composite number can be expressed (factorised) as a product of primes, and this factorisation is unique except for the order in which the prime factors occur.

4. Every composite number can be uniquely expressed as the product of powers of primes in ascending or descending order.

5. Let a be a positive integer and p be a prime number such that $p \mid a^2$, then $p \mid a$.

6. There are infinitely many positive primes.

7. Every positive integer different from 1 can be expressed as a product of non-negative power of 2 and an odd number.

8. A positive integer n is prime, if it is not divisible by any prime less than or equal to $\sqrt{n}$.

9. If p is a positive prime, then $\sqrt{p}$ is an irrational number. For example, $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}$ etc. are irrational numbers.

10. Let x be a rational number whose decimal expansion terminates. Then, x can be expressed in the form $\frac{p}{q}$, where p and q are co-prime, and the prime factorization of q is of the form $2^m \times 5^n$, where m, n are non-negative integers.

11. Let $x = \frac{p}{q}$ be a rational number, such that the prime factorization of q is of the form $2^m \times 5^n$ where m, n are non-negative integers. Then, x has a terminating decimal expansion which terminates after k places of decimals, where k is the larger of m and n.

12. Let $x = \frac{p}{q}$ be a rational number, such that the prime factorization of q is not of the form $2^m \times 5^n$, where m, n are non-negative integers. Then, x has non-terminating repeating decimal expansion.

NOTE: Formative assessment also includes lab activities, projects, assignments (Home work), oral and visual testing.