

CHAPTER 10

TRIGONOMETRIC RATIOS

10.1 INTRODUCTION

In this chapter, we intend to study an important branch of mathematics called "Trigonometry". The word 'Trigonometry' is derived from the Greek words: (i) trigonon and, (ii) metron. The word trigonon means a triangle and the word metron means a measure. Hence, trigonometry means the science of measuring triangles. In broader sense it is that branch of mathematics which deals with the measurement of the sides and the angles of a triangle and the problems allied with angles.

10.2 ANGLE

Consider a ray OA . If this ray rotates about its end point O and takes the position OB , then we say that the angle $\angle AOB$ has been generated.

Thus, an angle is considered as the figure obtained by rotating a given ray about its end-point.

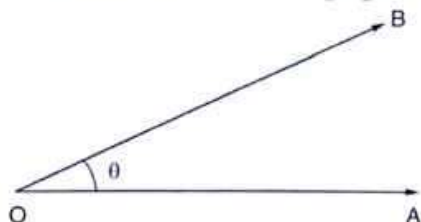


Fig. 10.1

The revolving ray is called the *generating line* of the angle. The initial position OA is called the *initial side* and the final position OB is called the *terminal side* of the angle. The end point O about which the ray rotates is called the *vertex* of the angle.

MEASURE OF AN ANGLE The measure of an angle is the amount of rotation from the initial side to the terminal side.

10.3 TRIGONOMETRIC RATIOS

The most important task of trigonometry is to find the remaining sides and angles of a triangle when some of its sides and angles are given. This problem is solved by using some ratios of the sides of a triangle with respect to its acute angles. These ratios of acute

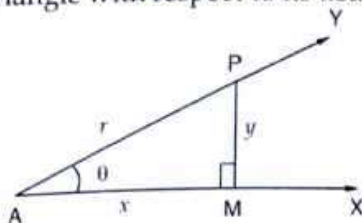


Fig. 10.2

angles are called *trigonometric ratios* of angles. Let us now define various trigonometric ratios.

Consider an acute angle $\angle YAX = \theta$ with initial side AX and terminal side AY . Take a point P on the terminal side AY . Draw PM perpendicular from P on AX to get the right angled triangle AMP in which $\angle PAM = \theta$.

In the right angled triangle AMP , Base = $AM = x$, Perpendicular = $PM = y$, Hypotenuse = $AP = r$.

We define the following six trigonometric ratios:

$$(i) \sin \theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{y}{r}, \text{ and is written as } \sin \theta$$

$$(ii) \cos \theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{x}{r}, \text{ and is written as } \cos \theta$$

$$(iii) \tan \theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{y}{x}, \text{ and is written as } \tan \theta$$

$$(iv) \operatorname{cosec} \theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{r}{y}, \text{ and is written as } \operatorname{cosec} \theta$$

$$(v) \sec \theta = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{r}{x}, \text{ and is written as } \sec \theta$$

$$(vi) \cot \theta = \frac{\text{Base}}{\text{Perpendicular}} = \frac{x}{y}, \text{ and is written as } \cot \theta$$

NOTE It should be noted that $\sin \theta$ is an abbreviation for "sine of angle θ ", it is not the \sin and θ . Similar is the case for other trigonometric ratios.

REMARK 1 The above trigonometric ratios are defined for an acute angle θ .

REMARK 2 The above trigonometric ratios depend only on the value of angle θ and are independent of the position of the point P on the terminal side AY of the acute angle $\angle XAY$. This means, if we choose P somewhere else on AY , then the lengths PM , AM and AP will change but the trigonometric ratios will remain same as proved in the following theorem.

THEOREM The trigonometric ratios are same for the same angle.

PROOF Let $\angle XAY = \theta$ be an acute angle with initial side AX and terminal side AY . Take two points P and Q on the terminal side AY , both different from A . Draw perpendiculars PM and QN from P and Q respectively on AX . We have to prove that the trigonometric ratios of angle θ are same in both the triangles viz. $\triangle AMP$ and $\triangle ANQ$.

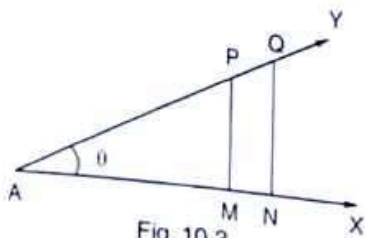


Fig. 10.3

In $\triangle AMP$ and $\triangle ANQ$, we have

$$\angle MAP = \angle XAY = \angle NAQ$$

and, $\angle AMP = \angle ANQ = \text{One right angle.}$

Thus, the corresponding angles of triangles AMP and ANQ are equal and, therefore by AAA similarity criterion, we have

$$\frac{AP}{AQ} = \frac{PM}{QN} = \frac{AM}{AN}$$

$$\Rightarrow \frac{PM}{AP} = \frac{QN}{AQ} \quad \dots(i)$$

In ΔAMP , we have

$$\sin \theta = \frac{PM}{AP}$$

$$\Rightarrow \sin \theta = \frac{QN}{AQ} \quad \text{[Using (i)]}$$

This shows that the value of $\sin \theta$ is independent of the position of point P .

Similarly, it can be proved that the other trigonometric ratios are independent of the position of point P .

Q.E.D.

10.4 RELATIONS BETWEEN TRIGONOMETRIC RATIOS

The trigonometric ratios $\sin \theta$, $\cos \theta$ and $\tan \theta$ of an angle θ are very closely connected by a relation. If any one of them is known, the other two can be easily calculated.

From Fig. 10.2, we have

$$\sin \theta = \frac{PM}{AP}, \cos \theta = \frac{AM}{AP} \text{ and } \tan \theta = \frac{PM}{AM}$$

$$\text{Now, } \tan \theta = \frac{PM}{AM}$$

$$\Rightarrow \tan \theta = \frac{\frac{PM}{AP}}{\frac{AM}{AP}}$$

$$\Rightarrow \tan \theta = \frac{\sin \theta}{\cos \theta}$$

REMARK It is clear from the definitions of the trigonometric ratios that for any acute angle θ , we have

$$(i) \operatorname{cosec} \theta = \frac{1}{\sin \theta} \text{ or, } \sin \theta = \frac{1}{\operatorname{cosec} \theta}$$

$$(ii) \sec \theta = \frac{1}{\cos \theta} \text{ or, } \cos \theta = \frac{1}{\sec \theta}$$

$$(iii) \cot \theta = \frac{1}{\tan \theta} \text{ or, } \tan \theta = \frac{1}{\cot \theta}$$

$$(iv) \cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$(v) \tan \theta \cdot \cot \theta = 1$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 In a $\triangle ABC$, right angled at A, if $AB = 12$, $AC = 5$ and $BC = 13$, find all the six trigonometric ratios of angle B.

SOLUTION With reference to $\angle B$, we have

Base = $AB = 12$, Perpendicular = $AC = 5$ and, Hypotenuse = $BC = 13$

Using the definitions of trigonometric ratios, we have

$$\sin B = \frac{AC}{BC} = \frac{5}{13}$$

$$\cos B = \frac{AB}{BC} = \frac{12}{13}$$

$$\tan B = \frac{AC}{AB} = \frac{5}{12}$$

$$\operatorname{cosec} B = \frac{BC}{AC} = \frac{13}{5}$$

$$\sec B = \frac{BC}{AB} = \frac{13}{12}$$

and, $\cot B = \frac{AB}{AC} = \frac{12}{5}$

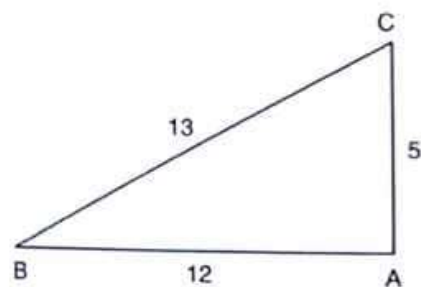


Fig. 10.4

EXAMPLE 2 In a $\triangle ABC$, right angled at A, if $AB = 5$, $AC = 12$ and $BC = 13$, find $\sin B$, $\cos C$ and $\tan B$.

SOLUTION With reference to $\angle B$, we have

Base = $AB = 5$, Perpendicular = $AC = 12$ and, Hypotenuse = $BC = 13$

$$\therefore \sin B = \frac{AC}{BC} = \frac{12}{13}$$

and, $\tan B = \frac{AC}{AB} = \frac{12}{5}$

With reference to $\angle C$, we have

Base = $AC = 12$, Perpendicular = $AB = 5$ and,
Hypotenuse = $BC = 13$

$$\therefore \cos C = \frac{AC}{BC} = \frac{12}{13}$$

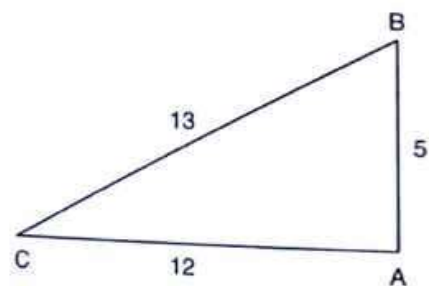


Fig. 10.5

EXAMPLE 3 In a $\triangle ABC$, right angled at B, if $AB = 4$ and $BC = 3$, find all the six trigonometric ratios of $\angle A$.

SOLUTION We have, $AB = 4$ and $BC = 3$.

By Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow AC = \sqrt{AB^2 + BC^2}$$

$$\Rightarrow AC = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

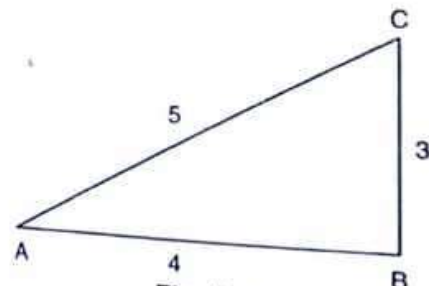


Fig. 10.6

When we consider the t -ratios of $\angle A$, we have

Base = $AB = 4$, Perpendicular = $BC = 3$ and, Hypotenuse = $AC = 5$.

$$\therefore \sin A = \frac{BC}{AC} = \frac{3}{5}, \cos A = \frac{AB}{AC} = \frac{4}{5}, \tan A = \frac{BC}{AB} = \frac{3}{4}$$

$$\operatorname{cosec} A = \frac{AC}{BC} = \frac{5}{3}, \sec A = \frac{AC}{AB} = \frac{5}{4} \text{ and, } \cot A = \frac{AB}{BC} = \frac{4}{3}$$

EXAMPLE 4 In a $\triangle ABC$, right angled at B , if $AB = 12$ and $BC = 5$, find:

(i) $\sin A$ and $\tan A$

(ii) $\cos C$ and $\cot C$

SOLUTION By Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow AC^2 = 12^2 + 5^2$$

$$\Rightarrow AC^2 = 169$$

$$\Rightarrow AC = 13$$

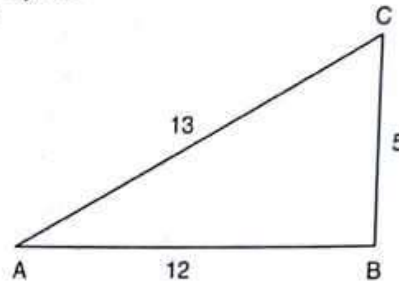


Fig. 10.7

(i) When we consider t -ratios of $\angle A$, we have

Base = $AB = 12$, Perpendicular = $BC = 5$ and, Hypotenuse = $AC = 13$

$$\therefore \sin A = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{5}{13} \text{ and, } \tan A = \frac{\text{Perpendicular}}{\text{Base}} = \frac{5}{12}$$

(ii) When we consider t -ratios of $\angle C$, we have

Base = $BC = 5$, Perpendicular = $AB = 12$ and, Hypotenuse = $AC = 13$

$$\therefore \cos C = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{5}{13} \text{ and, } \cot C = \frac{\text{Base}}{\text{Perpendicular}} = \frac{5}{12}$$

EXAMPLE 5 If $\sin A = \frac{3}{5}$, find $\cos A$ and $\tan A$.

SOLUTION We have,

$$\sin A = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{3}{5}$$

So, we draw a triangle ABC , right angled at B such that Perpendicular = $BC = 3$ units and, Hypotenuse = $AC = 5$ units.

By Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow 5^2 = AB^2 + 3^2$$

$$\Rightarrow AB^2 = 5^2 - 3^2$$

$$\Rightarrow AB^2 = 16$$

$$\Rightarrow AB = 4$$

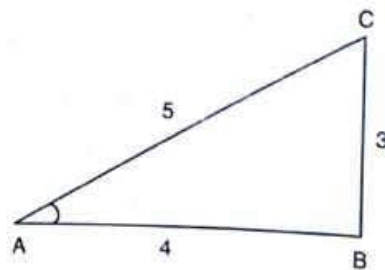


Fig. 10.8

When we consider the t -ratios of $\angle A$, we have

Base = $AB = 4$, Perpendicular = $BC = 3$, Hypotenuse = $AC = 5$

$$\therefore \cos A = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{4}{5} \text{ and, } \tan A = \frac{\text{Perpendicular}}{\text{Base}} = \frac{3}{4}$$

EXAMPLE 6 If $\cos B = \frac{1}{3}$, find the other five trigonometric ratios.

SOLUTION We have,

$$\cos B = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{1}{3}$$

So, we draw a triangle ABC , right angled at C such that

Base = $BC = 1$ unit and, Hypotenuse = $AB = 3$ units.

By Pythagoras theorem, we have

$$AB^2 = BC^2 + AC^2$$

$$\Rightarrow 3^2 = 1^2 + AC^2$$

$$\Rightarrow AC^2 = 9 - 1 = 8$$

$$\Rightarrow AC = \sqrt{8} = 2\sqrt{2}$$

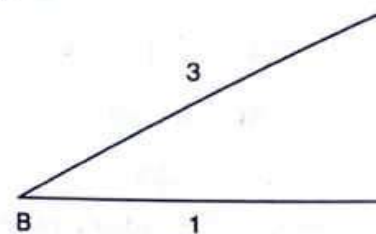


Fig. 10.9

When we consider the t -ratios of $\angle B$, we have

Base = $BC = 1$, Perpendicular = $AC = 2\sqrt{2}$ and, Hypotenuse = $AB = 3$

$$\therefore \sin B = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{2\sqrt{2}}{3}, \quad \tan B = \frac{\text{Perpendicular}}{\text{Base}} = \frac{2\sqrt{2}}{1} = 2\sqrt{2}$$

$$\operatorname{cosec} B = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{3}{2\sqrt{2}}, \quad \sec B = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{3}{1} = 3$$

$$\cot B = \frac{\text{Base}}{\text{Perpendicular}} = \frac{1}{2\sqrt{2}}$$

EXAMPLE 7 If $\cos \theta = \frac{8}{17}$, find the other five trigonometric ratios.

SOLUTION We have,

$$\cos \theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{8}{17}$$

So, we draw a triangle ABC right-angled at A such that

Base = $AB = 8$ units, Hypotenuse = $AC = 17$ units and, $\angle BAC = \theta$.

By Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow 17^2 = 8^2 + BC^2$$

$$\Rightarrow BC^2 = 17^2 - 8^2$$

$$\Rightarrow BC^2 = 289 - 64 = 225$$

$$\Rightarrow BC = \sqrt{225} = 15$$

When we consider the t -ratios of $\angle BAC = \theta$, we have

Base = $AB = 8$, Perpendicular = $BC = 15$, and Hypotenuse = $AC = 17$

$$\sin \theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{15}{17}$$

$$\tan \theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{15}{8}$$

$$\operatorname{cosec} \theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{17}{15}$$

$$\sec \theta = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{17}{8}$$

$$\text{and, } \cot \theta = \frac{\text{Base}}{\text{Perpendicular}} = \frac{8}{15}$$

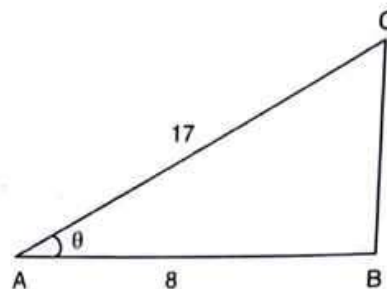


Fig. 10.10

EXAMPLE 8 If $\operatorname{cosec} A = \sqrt{10}$, find other five trigonometric ratios.

SOLUTION We have,

$$\operatorname{cosec} A = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{\sqrt{10}}{1}$$

So, we draw a right triangle ABC , right-angled at B such that

Perpendicular = $BC = 1$ unit and, Hypotenuse = $AC = \sqrt{10}$ units.

By Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow (\sqrt{10})^2 = AB^2 + 1^2$$

$$\Rightarrow AB^2 = 10 - 1 = 9$$

$$\Rightarrow AB = \sqrt{9} = 3$$

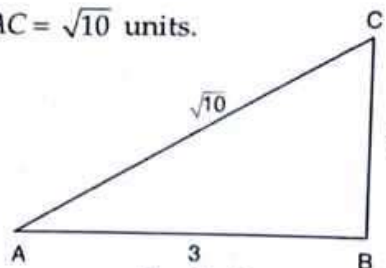


Fig. 10.11

When we consider the trigonometric ratios of $\angle A$, we have

Base = $AB = 3$, Perpendicular = $BC = 1$ and, Hypotenuse = $AC = \sqrt{10}$

$$\sin A = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{1}{\sqrt{10}}, \quad \cos A = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{3}{\sqrt{10}}$$

$$\tan A = \frac{\text{Perpendicular}}{\text{Base}} = \frac{1}{3}, \quad \sec A = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{\sqrt{10}}{3}$$

$$\text{and, } \cot A = \frac{\text{Base}}{\text{Perpendicular}} = \frac{3}{1} = 3$$

EXAMPLE 9 In a right triangle ABC right angled at B if $\sin A = \frac{3}{5}$, find all the six trigonometric ratios of $\angle C$.

SOLUTION We have,

$$\sin A = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{3}{5}$$

So, we draw a right triangle right angled at B such that Perpendicular = $BC = 3$ and, Hypotenuse = $AC = 5$

By Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow 5^2 = AB^2 + 3^2$$

$$\Rightarrow AB^2 = 25 - 9 = 16$$

$$\Rightarrow AB = \sqrt{16} = 4$$

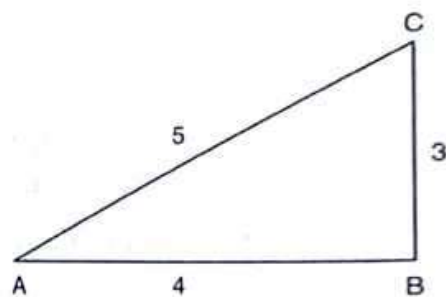


Fig. 10.12

When we consider the trigonometric ratios of $\angle C$, we have

Base = $BC = 3$, Perpendicular = $AB = 4$ and, Hypotenuse = $AC = 5$.

$$\therefore \sin C = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{4}{5}, \quad \cos C = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{3}{5}$$

$$\tan C = \frac{\text{Perpendicular}}{\text{Base}} = \frac{4}{3}, \quad \text{cosec } C = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{5}{4}$$

$$\sec C = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{5}{3} \quad \text{and,} \quad \cot C = \frac{\text{Base}}{\text{Perpendicular}} = \frac{3}{4}$$

EXAMPLE 10 If $\sin A = \frac{1}{3}$, evaluate $\cos A \operatorname{cosec} A + \tan A \sec A$.

SOLUTION We have,

$$\sin A = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{1}{3}$$

So, we draw a right triangle, right angled at B such that Perpendicular = $BC = 1$ unit, Hypotenuse = $AC = 3$ units.

By Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow 3^2 = AB^2 + 1^2$$

$$\Rightarrow 9 - 1 = AB^2$$

$$\Rightarrow AB = \sqrt{8} = 2\sqrt{2}$$

$$\text{Now,} \quad \cos A = \frac{AB}{AC} = \frac{2\sqrt{2}}{3}, \quad \operatorname{cosec} A = \frac{AC}{BC} = \frac{3}{1} = 3,$$

$$\tan A = \frac{BC}{AB} = \frac{1}{2\sqrt{2}} \quad \text{and} \quad \sec A = \frac{AC}{AB} = \frac{3}{2\sqrt{2}}$$

$$\therefore \cos A \operatorname{cosec} A + \tan A \sec A = \frac{2\sqrt{2}}{3} \times 3 + \frac{1}{2\sqrt{2}} \times \frac{3}{2\sqrt{2}}$$

$$\Rightarrow \cos A \operatorname{cosec} A + \tan A \sec A = 2\sqrt{2} + \frac{3}{8} = \frac{16\sqrt{2} + 3}{8}$$

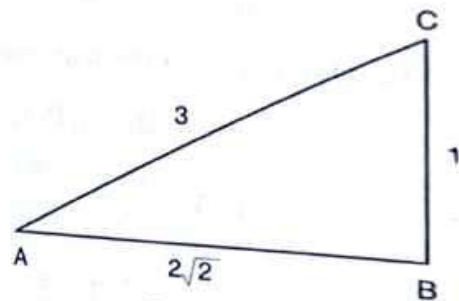


Fig. 10.13

EXAMPLE 11 If $\operatorname{cosec} A = 2$, find the value of $\frac{1}{\tan A} + \frac{\sin A}{1 + \cos A}$.

SOLUTION We have,

$$\operatorname{cosec} A = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{2}{1}$$

So, we draw a right triangle, right angled at B such that Perpendicular = $BC = 1$, Hypotenuse = $AC = 2$.

By Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow 2^2 = 1^2 + AB^2$$

$$\Rightarrow 4 - 1 = AB^2$$

$$\Rightarrow AB = \sqrt{3}$$

Now,

$$\tan A = \frac{BC}{AB} = \frac{1}{\sqrt{3}}, \sin A = \frac{BC}{AC} = \frac{1}{2} \text{ and, } \cos A = \frac{AB}{AC} = \frac{\sqrt{3}}{2}$$

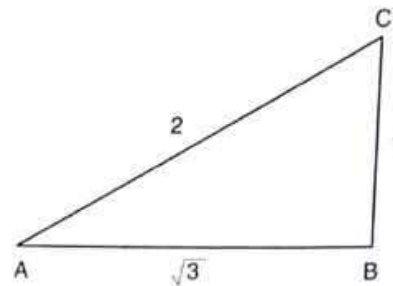


Fig. 10.14

$$\therefore \frac{1}{\tan A} + \frac{\sin A}{1 + \cos A} = \frac{1}{\frac{1}{\sqrt{3}}} + \frac{1/2}{1 + \frac{\sqrt{3}}{2}}$$

$$\Rightarrow \frac{1}{\tan A} + \frac{\sin A}{1 + \cos A} = \frac{\sqrt{3}}{1} + \frac{1/2}{\frac{2 + \sqrt{3}}{2}} = \frac{\sqrt{3}}{1} + \frac{1}{2 + \sqrt{3}}$$

$$\Rightarrow \frac{1}{\tan A} + \frac{\sin A}{1 + \cos A} = \sqrt{3} + \frac{1}{2 + \sqrt{3}} \times \frac{2 - \sqrt{3}}{2 - \sqrt{3}}$$

$$\Rightarrow \frac{1}{\tan A} + \frac{\sin A}{1 + \cos A} = \sqrt{3} + \frac{2 - \sqrt{3}}{2^2 - (\sqrt{3})^2} = \sqrt{3} + \frac{2 - \sqrt{3}}{4 - 3} = \sqrt{3} + (2 - \sqrt{3}) = 2$$

EXAMPLE 12 If $\tan A = \sqrt{2} - 1$ show that $\sin A \cos A = \frac{\sqrt{2}}{4}$

SOLUTION We have,

$$\tan A = \frac{\text{Perpendicular}}{\text{Base}} = \frac{\sqrt{2} - 1}{1}$$

So, we draw a right triangle ABC , right angled at B such that

Base = $AB = 1$ and, Perpendicular = $BC = \sqrt{2} - 1$

By Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow AC^2 = 1^2 + (\sqrt{2} - 1)^2$$

$$\Rightarrow AC^2 = 1 + 2 + 1 - 2\sqrt{2}$$

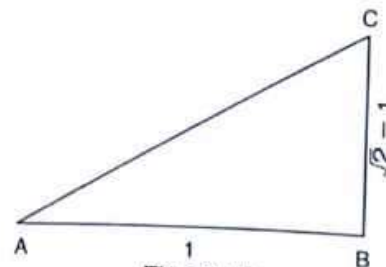


Fig. 10.15

$$\Rightarrow AC^2 = 4 - 2\sqrt{2}$$

$$\Rightarrow AC = \sqrt{4 - 2\sqrt{2}}$$

$$\text{Now, } \sin A = \frac{BC}{AC} = \frac{\sqrt{2} - 1}{\sqrt{4 - 2\sqrt{2}}} \text{ and, } \cos A = \frac{AB}{AC} = \frac{1}{\sqrt{4 - 2\sqrt{2}}}$$

$$\therefore \sin A \cos A = \frac{\sqrt{2} - 1}{\sqrt{4 - 2\sqrt{2}}} \times \frac{1}{\sqrt{4 - 2\sqrt{2}}} = \frac{\sqrt{2} - 1}{4 - 2\sqrt{2}} = \frac{\sqrt{2} - 1}{2\sqrt{2}(\sqrt{2} - 1)} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

EXAMPLE 13 In a right triangle ABC, right angled at C, if $\tan A = 1$, then verify that $2 \sin A \cos A = 1$. [NCERT]

SOLUTION In $\triangle ABC$, we have

$$\tan A = 1$$

$$\Rightarrow \frac{BC}{AC} = 1$$

$$\Rightarrow BC = x \text{ and } AC = x$$

By Pythagoras theorem, we have

$$AB^2 = AC^2 + BC^2$$

$$\Rightarrow AB^2 = x^2 + x^2$$

$$\Rightarrow AB = \sqrt{2}x$$

$$\therefore \sin A = \frac{BC}{AB} = \frac{x}{\sqrt{2}x} = \frac{1}{\sqrt{2}} \text{ and } \cos A = \frac{AC}{AB} = \frac{x}{\sqrt{2}x} = \frac{1}{\sqrt{2}} = 2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = 1$$

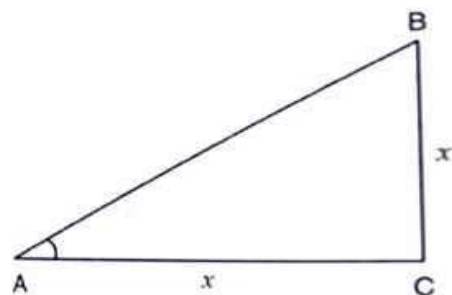


Fig. 10.16

EXAMPLE 14 If $\tan A = 1$ and $\tan B = \sqrt{3}$, evaluate $\cos A \cos B - \sin A \sin B$.

SOLUTION We have,

$$\tan A = \frac{\text{Perpendicular}}{\text{Base}} = \frac{1}{1}$$

So, we draw a right angled triangle ABC, right angled at C such that Base = AC = 1 and, Perpendicular = BC = 1

By Pythagoras theorem, we have

$$AB^2 = AC^2 + BC^2$$

$$\Rightarrow AB^2 = 1^2 + 1^2$$

$$\Rightarrow AB = \sqrt{2}$$

$$\therefore \cos A = \frac{AC}{AB} = \frac{1}{\sqrt{2}} \text{ and, } \sin A = \frac{BC}{AB} = \frac{1}{\sqrt{2}}$$

We have,

$$\tan B = \frac{\text{Perpendicular}}{\text{Base}} = \frac{\sqrt{3}}{1}$$

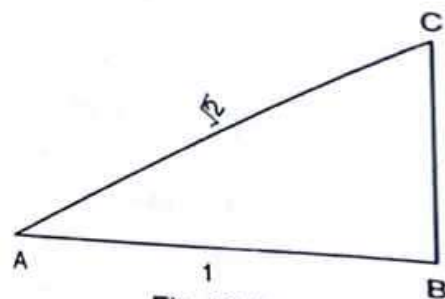


Fig. 10.17

So, we draw a right triangle ABC, right angled at A such that Base = AB = 1 and Perpendicular = AC = $\sqrt{3}$

By Pythagoras theorem, we have

$$BC^2 = AB^2 + AC^2$$

$$\Rightarrow BC^2 = 1^2 + (\sqrt{3})^2 = 4$$

$$\Rightarrow BC = 2$$

$$\therefore \sin B = \frac{AC}{BC} = \frac{\sqrt{3}}{2}, \text{ and } \cos B = \frac{AB}{BC} = \frac{1}{2}$$

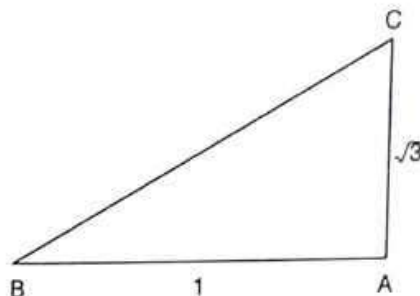


Fig. 10.18

Hence, $\cos A \cos B - \sin A \sin B = \frac{1}{\sqrt{2}} \times \frac{1}{2} - \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} = \frac{1 - \sqrt{3}}{2\sqrt{2}}$

EXAMPLE 15 In a ΔABC right angled at C , if $\tan A = \frac{1}{\sqrt{3}}$, find the value of $\sin A \cos B + \cos A \sin B$. [CBSE 2008]

SOLUTION Let us draw a ΔABC , right angled at C in which $\tan A = \frac{1}{\sqrt{3}}$.

Now, $\tan A = \frac{1}{\sqrt{3}}$

$$\Rightarrow \frac{BC}{AC} = \frac{1}{\sqrt{3}} \quad \left[\because \tan A = \frac{BC}{AC} \right]$$

$$\Rightarrow BC = x \text{ and } AC = \sqrt{3}x \quad \dots(i)$$

By Pythagoras theorem, we have

$$AB^2 = AC^2 + BC^2$$

$$\Rightarrow AB^2 = (\sqrt{3}x)^2 + x^2$$

$$\Rightarrow AB^2 = 3x^2 + x^2$$

$$\Rightarrow AB^2 = 4x^2$$

$$\Rightarrow AB = 2x$$

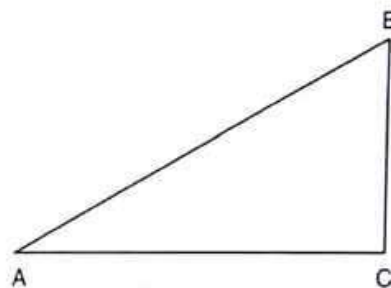


Fig. 10.19

To find the t -ratios of $\angle A$, we have

Base = $AC = \sqrt{3}x$, Perpendicular = $BC = x$ and Hypotenuse = $AB = 2x$

$$\therefore \sin A = \frac{BC}{AB} = \frac{x}{2x} = \frac{1}{2} \text{ and, } \cos A = \frac{AC}{AB} = \frac{\sqrt{3}x}{2x} = \frac{\sqrt{3}}{2}$$

When we consider the t -ratios of $\angle B$, we have

Base = $BC = x$, Perpendicular = $AC = \sqrt{3}x$ and, Hypotenuse = $AB = 2x$

$$\therefore \cos B = \frac{BC}{AB} = \frac{x}{2x} = \frac{1}{2} \text{ and, } \sin B = \frac{AC}{AB} = \frac{\sqrt{3}x}{2x} = \frac{\sqrt{3}}{2}$$

$$\therefore \sin A \cos B + \cos A \sin B = \frac{1}{2} \times \frac{1}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} = \frac{1}{4} + \frac{3}{4} = 1$$

EXAMPLE 16 If $\sec \alpha = \frac{5}{4}$, evaluate $\frac{1 - \tan \alpha}{1 + \tan \alpha}$.

SOLUTION We have,

$$\sec \alpha = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{5}{4}$$

So, we draw a right triangle ABC , right angled at B such that

Hypotenuse = $AC = 5$ units, Base = $AB = 4$ units, and $\angle BAC = \alpha$.

Applying Pythagoras theorem in ΔABC , we obtain

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow 5^2 = 4^2 + BC^2$$

$$\Rightarrow BC^2 = 5^2 - 4^2 = 9$$

$$\Rightarrow BC = \sqrt{9} = 3$$

$$\therefore \tan \alpha = \frac{BC}{AB} = \frac{3}{4}$$

$$\text{Now, } \frac{1 - \tan \alpha}{1 + \tan \alpha} = \frac{1 - \frac{3}{4}}{1 + \frac{3}{4}} = \frac{\frac{1}{4}}{\frac{7}{4}} = \frac{1}{7}$$

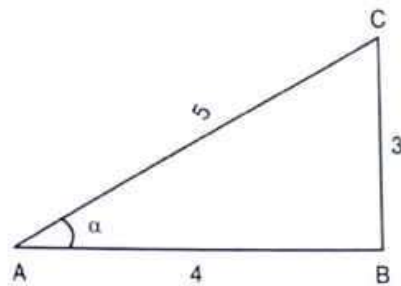


Fig. 10.20

NOTE It should be noted that: $\sin^2 \theta = (\sin \theta)^2$, $\sin^3 \theta = (\sin \theta)^3$, $\cos^3 \theta = (\cos \theta)^3$ etc.

EXAMPLE 17 If $\sin B = \frac{1}{2}$, show that $3\cos B - 4\cos^3 B = 0$.

SOLUTION We have,

$$\sin B = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{1}{2}$$

So, we draw a right triangle ABC , right angled at C such that

Perpendicular = $AC = 1$ unit, Hypotenuse = $AB = 2$ units.

Applying Pythagoras theorem in BCA , we obtain

$$AB^2 = BC^2 + AC^2$$

$$\Rightarrow 2^2 = BC^2 + 1^2$$

$$\Rightarrow BC^2 = 3$$

$$\Rightarrow BC = \sqrt{3}$$

$$\therefore \cos B = \frac{BC}{AB} = \frac{\sqrt{3}}{2}$$

Now,

$$3 \cos B - 4 \cos^3 B = 3 \times \frac{\sqrt{3}}{2} - 4 \left(\frac{\sqrt{3}}{2} \right)^3 = \frac{3\sqrt{3}}{2} - 4 \times \frac{3\sqrt{3}}{8} = \frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{2} = 0$$

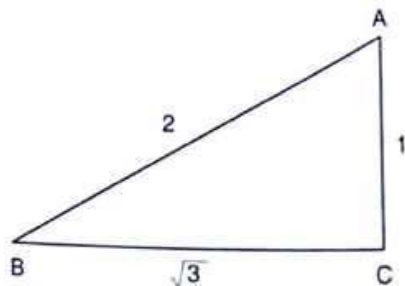


Fig. 10.21

EXAMPLE 18 If $\tan A = 2$, evaluate $\sec A \sin A + \tan^2 A - \operatorname{cosec} A$.

SOLUTION We have,

$$\tan A = \frac{\text{Perpendicular}}{\text{Base}} = \frac{2}{1}$$

So, we draw a right triangle ABC , right angled at C such that Perpendicular = $BC = 2$ unit and Base = $AB = 1$ unit.

By Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow AC^2 = 2^2 + 1^2 = 5$$

$$\Rightarrow AC = \sqrt{5}$$

$$\therefore \sec A = \frac{AC}{AB} = \sqrt{5}, \tan A = \frac{BC}{AB} = \frac{2}{1} = 2, \sin A = \frac{BC}{AC} = \frac{2}{\sqrt{5}}$$

$$\text{and, } \operatorname{cosec} A = \frac{AC}{BC} = \frac{\sqrt{5}}{2}$$

$$\text{Now, } \sec A \sin A + \tan^2 A - \operatorname{cosec} A = \sqrt{5} \times \frac{2}{\sqrt{5}} + (2)^2 - \left(\frac{\sqrt{5}}{2}\right) = 2 + 4 - \frac{\sqrt{5}}{2} = 6 - \frac{\sqrt{5}}{2} = \frac{12 - \sqrt{5}}{2}$$

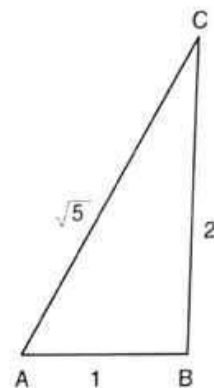


Fig. 10.22

EXAMPLE 19 Given $\triangle ACB$ right angled at C in which $AB = 29$ units, $BC = 21$ units and $\angle ABC = \theta$. Determine the values of

(i) $\cos^2 \theta + \sin^2 \theta$

(ii) $\cos^2 \theta - \sin^2 \theta$

[NCERT]

SOLUTION In $\triangle ACB$, we have

$$AB^2 = AC^2 + BC^2$$

$$\Rightarrow AC = \sqrt{AB^2 - BC^2} = \sqrt{29^2 - 21^2} = \sqrt{(29+21)(29-21)} = \sqrt{400} = 20 \text{ units}$$

$$\therefore \sin \theta = \frac{AC}{AB} = \frac{20}{29} \text{ and } \cos \theta = \frac{BC}{AB} = \frac{21}{29}$$

(i) Using the values of $\sin \theta$ and $\cos \theta$, we obtain

$$\begin{aligned} \cos^2 \theta + \sin^2 \theta &= \left(\frac{21}{29}\right)^2 + \left(\frac{20}{29}\right)^2 \\ &= \frac{441 + 400}{841} = 1 \end{aligned}$$

(ii) Using the values of $\sin \theta$ and $\cos \theta$, we obtain

$$\cos^2 \theta - \sin^2 \theta = \left(\frac{21}{29}\right)^2 - \left(\frac{20}{29}\right)^2 = \frac{21^2 - 20^2}{29^2} = \frac{(21+20)(21-20)}{841} = \frac{41}{841}$$

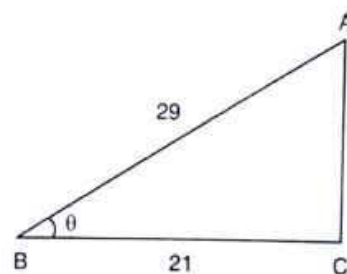


Fig. 10.23

EXAMPLE 20 If $\cot B = \frac{12}{5}$, prove that $\tan^2 B - \sin^2 B = \sin^4 B \sec^2 B$.

SOLUTION We have,

$$\cot B = \frac{\text{Base}}{\text{Perpendicular}} = \frac{12}{5}$$

So, we draw a right triangle ABC , right angled at C such that Base = $BC = 12$ units and, Perpendicular = $AC = 5$ units.

Applying Pythagoras Theorem in $\triangle ABC$, we obtain

$$AB^2 = BC^2 + AC^2$$

$$\Rightarrow AB^2 = 12^2 + 5^2 = 169$$

$$\Rightarrow AB = \sqrt{169} = 13$$

$$\therefore \sin B = \frac{AC}{AB} = \frac{5}{13}, \tan B = \frac{AC}{BC} = \frac{5}{12} \text{ and, } \sec B = \frac{AB}{BC} = \frac{13}{12}$$

Now, $LHS = \tan^2 B - \sin^2 B$

$$\Rightarrow LHS = (\tan B)^2 - (\sin B)^2$$

$$\Rightarrow LHS = \left(\frac{5}{12}\right)^2 - \left(\frac{5}{13}\right)^2 = \frac{25}{144} - \frac{25}{169}$$

$$\Rightarrow LHS = 25 \left(\frac{1}{144} - \frac{1}{169} \right) = 25 \left(\frac{169 - 144}{144 \times 169} \right)$$

$$\Rightarrow LHS = 25 \times \frac{25}{144 \times 169} = \frac{25 \times 25}{144 \times 169} = \frac{5^2 \times 5^2}{12^2 \times 13^2} \quad \dots(i)$$

and, $RHS = \sin^4 B \sec^2 B$

$$\Rightarrow RHS = (\sin B)^4 (\sec B)^2 = \left(\frac{5}{13}\right)^4 \times \left(\frac{13}{12}\right)^2 = \frac{5^4 \times 13^2}{13^4 \times 12^2} = \frac{5^4}{13^2 \times 12^2} = \frac{5^2 \times 5^2}{13^2 \times 12^2}$$

From (i) and (ii), we have

$$\tan^2 B - \sin^2 B = \sin^4 B \sec^2 B.$$

EXAMPLE 21 If $\sec \alpha = \frac{5}{4}$, verify that $\frac{\tan \alpha}{1 + \tan^2 \alpha} = \frac{\sin \alpha}{\sec \alpha}$

SOLUTION We have,

$$\sec \alpha = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{5}{4}$$

So, we draw a right triangle ABC , right angled at B such that

$\angle BAC = \alpha$, Base = $AB = 4$ and Hypotenuse = $AC = 5$.

By Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow 5^2 = 4^2 + BC^2$$

$$\Rightarrow BC^2 = 25 - 16 = 9$$

$$\Rightarrow BC = 3$$

$$\therefore \tan \alpha = \frac{BC}{AB} = \frac{3}{4} \text{ and } \sin \alpha = \frac{BC}{AC} = \frac{3}{5}$$

Now,
$$\frac{\tan \alpha}{1 + \tan^2 \alpha} = \frac{\frac{3}{4}}{1 + \left(\frac{3}{4}\right)^2} = \frac{\frac{3}{4}}{1 + \frac{9}{16}} = \frac{\frac{3}{4}}{\frac{16+9}{16}} = \frac{\frac{3}{4}}{\frac{25}{16}} = \frac{3}{4} \times \frac{16}{25} = \frac{12}{25} \quad \dots(i)$$

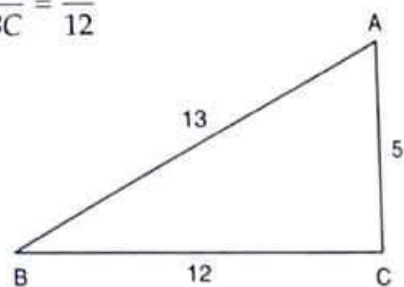


Fig. 10.24

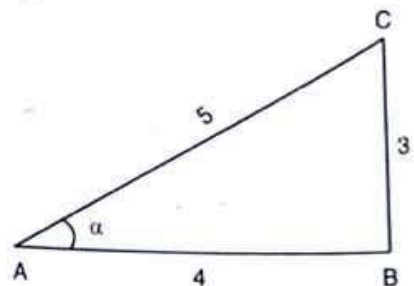


Fig. 10.25

$$\text{and, } \frac{\sin \alpha}{\sec \alpha} = \frac{3/5}{5/4} = \frac{3}{5} \times \frac{4}{5} = \frac{12}{25} \quad \dots(\text{ii})$$

From (i) and (ii), we obtain

$$\frac{\tan \alpha}{1 + \tan^2 \alpha} = \frac{\sin \alpha}{\sec \alpha}$$

EXAMPLE 22 If $\sin \theta = \frac{4}{5}$, find the value of $\frac{4 \tan \theta - 5 \cos \theta}{\sec \theta + 4 \cot \theta}$.

SOLUTION We have,

$$\sin \theta = \frac{4}{5} \Rightarrow \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{4}{5}$$

So, we construct a right triangle ABC , right angled at B such that $\angle BAC = \theta$, Perpendicular = $BC = 4$ and, Hypotenuse = $AC = 5$.

Applying Pythagoras theorem in $\triangle ABC$, we obtain

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow 5^2 = AB^2 + 4^2$$

$$\Rightarrow AB^2 = 25 - 16 = 9$$

$$\Rightarrow AB = \sqrt{9} \Rightarrow AB = 3$$

$$\therefore \cos \theta = \frac{AB}{AC} = \frac{3}{5}, \tan \theta = \frac{BC}{AB} = \frac{4}{3}, \cot \theta = \frac{AB}{BC} = \frac{3}{4} \text{ and, } \sec \theta = \frac{AC}{AB} = \frac{5}{3}$$

$$\therefore \frac{4 \tan \theta - 5 \cos \theta}{\sec \theta + 4 \cot \theta} = \frac{4 \times \frac{4}{3} - 5 \times \frac{3}{5}}{\frac{5}{3} + 4 \times \frac{3}{4}} = \frac{\frac{16}{3} - 3}{\frac{5}{3} + 3} = \frac{\frac{16-9}{3}}{\frac{5+9}{3}} = \frac{7}{14} = \frac{1}{2}$$

EXAMPLE 23 In a right triangle ABC , right angled at B , the ratio of AB to AC is $1 : \sqrt{2}$. Find the values of

$$(i) \frac{2 \tan A}{1 + \tan^2 A} \quad \text{and,}$$

$$(ii) \frac{2 \tan A}{1 - \tan^2 A}$$

SOLUTION We have,

$$AB : AC = 1 : \sqrt{2} \Rightarrow \frac{AB}{AC} = \frac{1}{\sqrt{2}}$$

$$\therefore AB = x \text{ and } AC = \sqrt{2}x, \text{ for some } x.$$

Applying Pythagoras theorem in $\triangle ABC$, we obtain

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow (\sqrt{2}x)^2 = x^2 + BC^2$$

$$\Rightarrow BC^2 = 2x^2 - x^2 = x^2$$

$$\Rightarrow BC = x$$

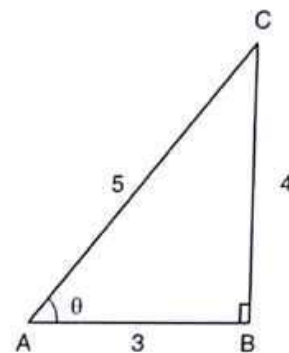


Fig. 10.26

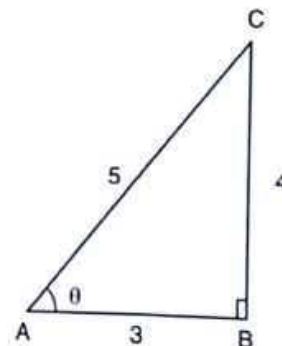


Fig. 10.27

$$\therefore \tan A = \frac{BC}{AB} = \frac{x}{x} = 1$$

$$(i) \frac{2 \tan A}{1 + \tan^2 A} = \frac{2 \times 1}{1 + 1^2} = \frac{2}{2} = 1$$

$$(ii) \frac{2 \tan A}{1 - \tan^2 A} = \frac{2 \times 1}{1 - 1} = \frac{2}{0}, \text{ which is undefined.}$$

EXAMPLE 24 In a right triangle ABC right angled at B , $\angle ACB = \theta$, $AB = 2$ cm and $BC = 1$ cm. Find the value of $\sin^2 \theta + \tan^2 \theta$.

SOLUTION We have,

$$AB = 2 \text{ cm and } BC = 1 \text{ cm.}$$

Applying Pythagoras theorem in $\triangle ABC$, we obtain

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow AC^2 = 2^2 + 1 = 5$$

$$\Rightarrow AC = \sqrt{5}$$

Thus, Base = $BC = 1$ cm, Perpendicular = $AB = 2$ cm and, Hypotenuse = $AC = \sqrt{5}$ cm

$$\therefore \sin \theta = \frac{AB}{AC} = \frac{2}{\sqrt{5}} \text{ and, } \tan \theta = \frac{AB}{BC} = \frac{2}{1} = 2$$

Now,

$$\therefore \sin^2 \theta + \tan^2 \theta = (\sin \theta)^2 + (\tan \theta)^2 = \left(\frac{2}{\sqrt{5}}\right)^2 + (2)^2 = \frac{4}{5} + 4 = \frac{24}{5}$$

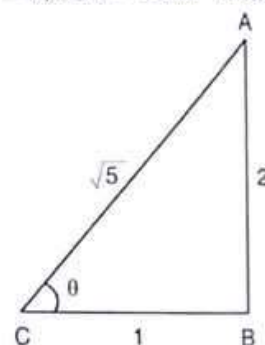


Fig. 10.28

EXAMPLE 25 In a $\triangle ABC$, right angled at C and $\angle A = \angle B$,

(i) Is $\cos A = \cos B$?

(ii) Is $\tan A = \tan B$?

What about the other trigonometric ratios for $\angle A$ and $\angle B$? Will they be equal?

SOLUTION We have,

$$\angle A = \angle B$$

$$\Rightarrow BC = AC$$

$$\Rightarrow BC = AC = x \text{ (say)}$$

[\because Sides opposite to equal angles are equal]

Using Pythagoras theorem in $\triangle ACB$, we obtain

$$AB^2 = AC^2 + BC^2$$

$$\Rightarrow AB^2 = x^2 + x^2$$

$$\Rightarrow AB = \sqrt{2}x$$

(i) We have,

$$\cos A = \frac{AC}{AB} = \frac{x}{\sqrt{2}x} = \frac{1}{\sqrt{2}} \text{ and, } \cos B = \frac{BC}{AB} = \frac{x}{\sqrt{2}x} = \frac{1}{\sqrt{2}}$$

$$\therefore \cos A = \cos B$$

(ii) We have,

$$\tan A = \frac{BC}{AC} = \frac{x}{x} = 1 \text{ and, } \tan B = \frac{AC}{BC} = \frac{x}{x} = 1$$

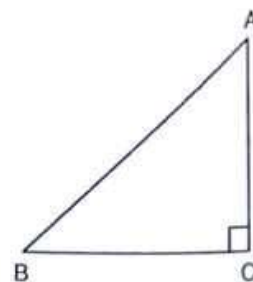


Fig. 10.29

$$\therefore \tan A = \tan B$$

$$\text{Now, } \sin A = \frac{BC}{AB} = \frac{x}{\sqrt{2x}} = \frac{1}{\sqrt{2}} \text{ and, } \sin B = \frac{AC}{AB} = \frac{x}{\sqrt{2x}} = \frac{1}{\sqrt{2}}$$

$$\therefore \sin A = \sin B$$

$$\cot A = \frac{AC}{BC} = \frac{x}{x} = 1 \text{ and } \cot B = \frac{BC}{AC} = \frac{x}{x} = 1$$

$$\therefore \cot A = \cot B$$

$$\sec A = \frac{AB}{AC} = \frac{\sqrt{2x}}{x} = \sqrt{2} \text{ and } \sec B = \frac{AB}{BC} = \frac{\sqrt{2x}}{x} = \sqrt{2}$$

$$\therefore \sec A = \sec B$$

$$\operatorname{cosec} A = \frac{AB}{BC} = \frac{\sqrt{2x}}{x} = \sqrt{2} \text{ and } \operatorname{cosec} B = \frac{AB}{AC} = \frac{\sqrt{2x}}{x} = \sqrt{2}$$

$$\therefore \operatorname{cosec} A = \operatorname{cosec} B$$

EXAMPLE 26 In $\triangle ABC$, right angled at B, if $\tan A = \frac{1}{\sqrt{3}}$, find the value of

(i) $\sin A \cos C + \cos A \sin C$

(ii) $\cos A \cos C - \sin A \sin C$

[NCERT]

SOLUTION We have,

$$\tan A = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \frac{BC}{AB} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow BC = x \text{ and } AB = \sqrt{3}x$$

Using Pythagoras theorem in $\triangle ABC$, we have

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow AC^2 = (\sqrt{3}x)^2 + x^2$$

$$\Rightarrow AC^2 = 4x^2$$

$$\Rightarrow AC = 2x$$

$$\text{Now, } \sin A = \frac{BC}{AC} = \frac{x}{2x} = \frac{1}{2}, \cos A = \frac{AB}{AC} = \frac{\sqrt{3}x}{2x} = \frac{\sqrt{3}}{2}$$

$$\sin C = \frac{AB}{AC} = \frac{\sqrt{3}x}{2x} = \frac{\sqrt{3}}{2} \text{ and, } \cos C = \frac{BC}{AC} = \frac{x}{2x} = \frac{1}{2}$$

$$(i) \sin A \cos C + \cos A \sin C = \frac{1}{2} \times \frac{1}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} = \frac{1}{4} + \frac{3}{4} = 1$$

$$(ii) \cos A \cos C - \sin A \sin C = \frac{\sqrt{3}}{2} \times \frac{1}{2} - \frac{1}{2} \times \frac{\sqrt{3}}{2} = 0$$

EXAMPLE 27 Given that $16 \cot A = 12$; find the value of $\frac{\sin A + \cos A}{\sin A - \cos A}$.

SOLUTION We have, $16 \cot A = 12 \Rightarrow \cot A = \frac{12}{16} \Rightarrow \cot A = \frac{3}{4}$.

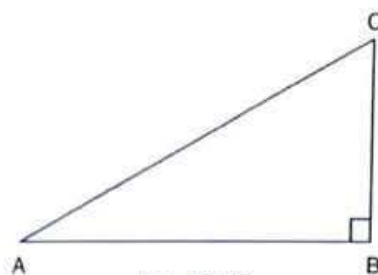


Fig. 10.30

$$\text{Now, } \frac{\sin A + \cos A}{\sin A - \cos A} = \frac{\frac{\sin A + \cos A}{\sin A}}{\frac{\sin A - \cos A}{\sin A}} \quad [\text{Dividing Numerator and Denominator by } \sin A]$$

$$= \frac{\frac{\sin A}{\sin A} + \frac{\cos A}{\sin A}}{\frac{\sin A}{\sin A} - \frac{\cos A}{\sin A}}$$

$$= \frac{1 + \cot A}{1 - \cot A} = \frac{1 + \frac{12}{16}}{1 - \frac{12}{16}}$$

$$\left[\because 16 \cot A = 12 \Rightarrow \cot A = \frac{12}{16} \right]$$

$$= \frac{1 + \frac{3}{4}}{1 - \frac{3}{4}} = \frac{\frac{4}{4} + \frac{3}{4}}{\frac{4}{4} - \frac{3}{4}} = 7$$

EXAMPLE 28 If $5 \tan \alpha = 4$, show that $\frac{5 \sin \alpha - 3 \cos \alpha}{5 \sin \alpha + 2 \cos \alpha} = \frac{1}{6}$

SOLUTION We have,

$$5 \tan \alpha = 4 \Rightarrow \tan \alpha = \frac{4}{5}$$

$$\text{Now, } \frac{5 \sin \alpha - 3 \cos \alpha}{5 \sin \alpha + 2 \cos \alpha} = \frac{\frac{5 \sin \alpha - 3 \cos \alpha}{\cos \alpha}}{\frac{5 \sin \alpha + 2 \cos \alpha}{\cos \alpha}} \quad [\text{Dividing Numerator and Denominator by } \cos \alpha]$$

$$= \frac{\frac{5 \sin \alpha}{\cos \alpha} - \frac{3 \cos \alpha}{\cos \alpha}}{\frac{5 \sin \alpha}{\cos \alpha} + \frac{2 \cos \alpha}{\cos \alpha}} = \frac{5 \tan \alpha - 3}{5 \tan \alpha + 2} = \frac{5 \times \frac{4}{5} - 3}{5 \times \frac{4}{5} + 2}$$

$$\left[\because \tan \alpha = \frac{4}{5} \right]$$

$$= \frac{4 - 3}{4 + 2} = \frac{1}{6}$$

EXAMPLE 29 If $\tan \theta = \frac{12}{13}$, evaluate $\frac{2 \sin \theta \cos \theta}{\cos^2 \theta - \sin^2 \theta}$.

SOLUTION We have, $\tan \theta = \frac{12}{13}$

$$\text{Now, } \frac{2 \sin \theta \cos \theta}{\cos^2 \theta - \sin^2 \theta} = \frac{\frac{2 \sin \theta \cos \theta}{\cos^2 \theta}}{\frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta}}$$

[Dividing N^r and D^r by $\cos^2 \theta$]

$$= \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2 \times \frac{12}{13}}{1 - \left(\frac{12}{13}\right)^2} = \frac{\frac{24}{13}}{1 - \frac{144}{169}} = \frac{\frac{24}{13}}{\frac{25}{169}} = \frac{24}{13} \times \frac{169}{25} = \frac{312}{25}$$

LEVEL-2

EXAMPLE 30 If $\sin \theta = \frac{a^2 - b^2}{a^2 + b^2}$, find the values of other five trigonometric ratios.

SOLUTION We have,

$$\sin \theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{a^2 - b^2}{a^2 + b^2}$$

So, draw a right triangle right angled at B such that

Perpendicular = $a^2 - b^2$, Hypotenuse = $a^2 + b^2$ and, $\angle BAC = \theta$

By Pythagoras theorem, we have

$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ \Rightarrow (a^2 + b^2)^2 &= AB^2 + (a^2 - b^2)^2 \\ \Rightarrow AB^2 &= (a^2 + b^2)^2 - (a^2 - b^2)^2 \\ \Rightarrow AB^2 &= (a^4 + b^4 + 2a^2b^2) - (a^4 + b^4 - 2a^2b^2) \\ \Rightarrow AB^2 &= 4a^2b^2 = (2ab)^2 \\ \Rightarrow AB &= 2ab \end{aligned}$$

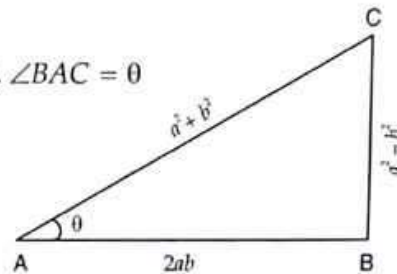


Fig. 10.31

When we consider the trigonometric ratios of $\angle BAC = \theta$, we have

Base = $AB = 2ab$, Perpendicular = $BC = a^2 - b^2$ and, Hypotenuse = $AC = a^2 + b^2$

$$\therefore \cos \theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{2ab}{a^2 + b^2}, \quad \tan \theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{a^2 - b^2}{2ab}$$

$$\Rightarrow \operatorname{cosec} \theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{a^2 + b^2}{a^2 - b^2}, \quad \sec \theta = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{a^2 + b^2}{2ab}$$

$$\text{and, } \cot \theta = \frac{\text{Base}}{\text{Perpendicular}} = \frac{2ab}{a^2 - b^2}$$

EXAMPLE 31 From Fig. 10.32, write the values of:

- (i) $\sin A$ (ii) $\cot A$ (iii) $\tan B$ (iv) $\sin^2 B + \cos^2 B$

SOLUTION In $\triangle ACD$, we have

$$\begin{aligned} AC^2 &= AD^2 + CD^2 \\ \Rightarrow AC^2 &= 3^2 + 4^2 \\ \Rightarrow AC^2 &= 25 \\ \Rightarrow AC &= \sqrt{25} = 5 \end{aligned}$$

In $\triangle BCD$, we have

$$\begin{aligned} BC^2 &= BD^2 + CD^2 \\ \Rightarrow BC^2 &= 6^2 + 4^2 \end{aligned}$$

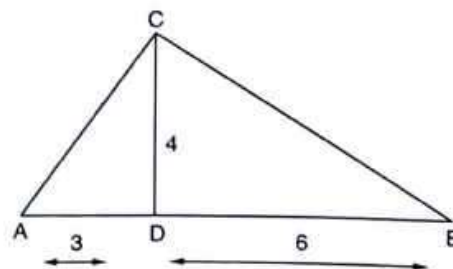


Fig. 10.32

$$\Rightarrow BC^2 = 36 + 16$$

$$\Rightarrow BC^2 = 52$$

$$\Rightarrow BC = \sqrt{52} = \sqrt{13 \times 4} = 2\sqrt{13}$$

$$\text{Now, } \sin A = \frac{CD}{AC} = \frac{4}{5}, \cot A = \frac{AD}{CD} = \frac{3}{4}$$

$$\tan B = \frac{CD}{DB} = \frac{4}{6} = \frac{2}{3}, \sin B = \frac{CD}{BC} = \frac{4}{2\sqrt{13}} = \frac{2}{\sqrt{13}}$$

$$\text{and, } \cos B = \frac{BD}{BC} = \frac{6}{2\sqrt{13}} = \frac{3}{\sqrt{13}}$$

$$\therefore \sin^2 B + \cos^2 B = \left(\frac{2}{\sqrt{13}}\right)^2 + \left(\frac{3}{\sqrt{13}}\right)^2 = \frac{4}{13} + \frac{9}{13} = \frac{13}{13} = 1$$

EXAMPLE 32 In Fig. 10.33, $AD = DB$ and $\angle B$ is a right angle. Determine:

(i) $\sin \theta$

(ii) $\cos \theta$

(iii) $\tan \theta$

(iv) $\sin^2 \theta + \cos^2 \theta$

SOLUTION We have,

$$AB = a$$

$$\Rightarrow AD + DB = a$$

$$\Rightarrow AD + AD = a$$

$$\Rightarrow 2AD = a$$

$$\Rightarrow AD = \frac{a}{2}$$

$$\text{Thus, } AD = DB = \frac{a}{2}$$

By Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow b^2 = a^2 + BC^2$$

$$\Rightarrow BC^2 = b^2 - a^2$$

$$\Rightarrow BC = \sqrt{b^2 - a^2}$$

Thus, in $\triangle BCD$, we have

$$\text{Base} = BC = \sqrt{b^2 - a^2} \text{ and Perpendicular} = BD = \frac{a}{2}$$

Applying Pythagoras theorem in $\triangle BCD$, we have

$$\Rightarrow BC^2 + BD^2 = CD^2$$

$$\Rightarrow (\sqrt{b^2 - a^2})^2 + \left(\frac{a}{2}\right)^2 = CD^2$$

$$\Rightarrow CD^2 = b^2 - a^2 + \frac{a^2}{4}$$

$$\Rightarrow CD^2 = \frac{4b^2 - 4a^2 + a^2}{4}$$

$[\because AD = DB]$

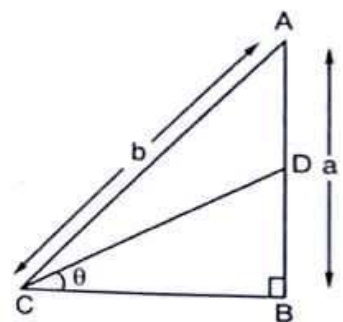


Fig. 10.33

$$\Rightarrow CD^2 = \frac{4b^2 - 3a^2}{4}$$

$$\Rightarrow CD = \frac{\sqrt{4b^2 - 3a^2}}{2}$$

Now,

$$(i) \quad \sin \theta = \frac{BD}{CD} \Rightarrow \sin \theta = \frac{a/2}{\frac{\sqrt{4b^2 - 3a^2}}{2}} = \frac{a}{\sqrt{4b^2 - 3a^2}}$$

$$(ii) \quad \cos \theta = \frac{BC}{CD} \Rightarrow \cos \theta = \frac{\sqrt{b^2 - a^2}}{\frac{\sqrt{4b^2 - 3a^2}}{2}} = \frac{2\sqrt{b^2 - a^2}}{\sqrt{4b^2 - 3a^2}}$$

$$(iii) \quad \tan \theta = \frac{BD}{BC} \Rightarrow \tan \theta = \frac{a/2}{\sqrt{b^2 - a^2}} = \frac{a}{2\sqrt{b^2 - a^2}}$$

$$(iv) \quad \sin^2 \theta + \cos^2 \theta = \left(\frac{a}{\sqrt{4b^2 - 3a^2}} \right)^2 + \left(\frac{2\sqrt{b^2 - a^2}}{\sqrt{4b^2 - 3a^2}} \right)^2$$

$$= \frac{a^2}{4b^2 - 3a^2} + \frac{4(b^2 - a^2)}{4b^2 - 3a^2} = \frac{4b^2 - 3a^2}{4b^2 - 3a^2} = 1$$

EXAMPLE 33 In $\triangle OPQ$ right angled at P , $OP = 7$ cm, $OQ - PQ = 1$ cm. Determine the values of $\sin Q$ and $\cos Q$. [NCERT]

SOLUTION In $\triangle OPQ$, we have

$$OQ^2 = OP^2 + PQ^2$$

$$\Rightarrow (PQ + 1)^2 = OP^2 + PQ^2 \quad [\because OQ - PQ = 1 \Rightarrow OQ = 1 + PQ]$$

$$\Rightarrow PQ^2 + 2PQ + 1 = OP^2 + PQ^2$$

$$\Rightarrow 2PQ + 1 = 49$$

$$\Rightarrow 2PQ = 48$$

$$\Rightarrow PQ = 24 \text{ cm}$$

$$\therefore OQ - PQ = 1 \text{ cm}$$

$$\Rightarrow OQ = (PQ + 1) \text{ cm} = 25 \text{ cm}$$

Now $\sin Q = \frac{OP}{OQ} = \frac{7}{25}$

and, $\cos Q = \frac{PQ}{OQ} = \frac{24}{25}$

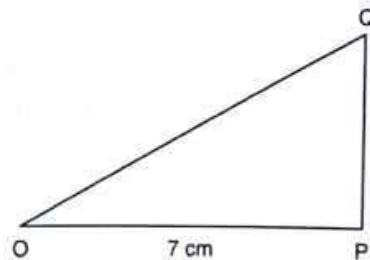


Fig. 10.34

EXAMPLE 34 In $\triangle PQR$, right angled at Q , $PR + QR = 25$ cm and $PQ = 5$ cm. Determine the values of $\sin P$, $\cos P$ and $\tan P$. [NCERT]

SOLUTION In $\triangle PQR$, we have

$$PR^2 = PQ^2 + QR^2$$

$$\Rightarrow (25 - QR)^2 = 5^2 + QR^2 \quad [\because PR + QR = 25 \text{ cm} \Rightarrow PR = 25 - QR]$$

$$\begin{aligned} \Rightarrow 625 - 50QR + QR^2 &= 25 + QR^2 \\ \Rightarrow 600 - 50QR &= 0 \\ \Rightarrow QR &= \frac{600}{50} = 12 \text{ cm} \\ \text{Now, } PR + QR &= 25 \\ \Rightarrow PR &= 25 - QR = (25 - 12) = 13 \text{ cm} \\ \therefore \sin P &= \frac{QR}{PR} = \frac{12}{13}, \cos P = \frac{PQ}{PR} = \frac{5}{13} \text{ and, } \tan P = \frac{QR}{PQ} = \frac{12}{5} \end{aligned}$$

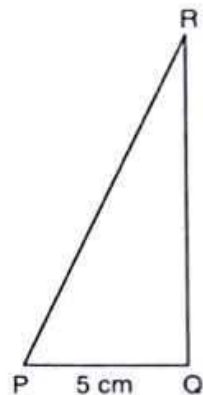


Fig. 10.35

EXAMPLE 35 If $\tan \theta + \frac{1}{\tan \theta} = 2$, find the value of $\tan^2 \theta + \frac{1}{\tan^2 \theta}$.

SOLUTION We have,

$$\begin{aligned} \tan \theta + \frac{1}{\tan \theta} &= 2 \\ \Rightarrow \left(\tan \theta + \frac{1}{\tan \theta} \right)^2 &= 2^2 \\ \Rightarrow \tan^2 \theta + \frac{1}{\tan^2 \theta} + 2 \times \tan \theta \times \frac{1}{\tan \theta} &= 4 \\ \Rightarrow \tan^2 \theta + \frac{1}{\tan^2 \theta} + 2 &= 4 \\ \Rightarrow \tan^2 \theta + \frac{1}{\tan^2 \theta} &= 2 \end{aligned}$$

ALITER We have,

$$\begin{aligned} \tan \theta + \frac{1}{\tan \theta} &= 2 \\ \Rightarrow \tan^2 \theta + 1 &= 2 \tan \theta \\ \Rightarrow \tan^2 \theta - 2 \tan \theta + 1 &= 0 \\ \Rightarrow (\tan \theta - 1)^2 &= 0 \\ \Rightarrow \tan \theta &= 1 \\ \therefore \tan^2 \theta + \frac{1}{\tan^2 \theta} &= 1 + 1 = 2 \end{aligned}$$

EXAMPLE 36 If $\angle B$ and $\angle Q$ are acute angles such that $\sin B = \sin Q$, then prove that $\angle B = \angle Q$.
SOLUTION Consider two right triangles ABC and PQR such that $\sin B = \sin Q$.
 We have, **[NCERT]**

$$\sin B = \frac{AC}{AB} \text{ and, } \sin Q = \frac{PR}{PQ}$$

$$\therefore \sin B = \sin Q$$

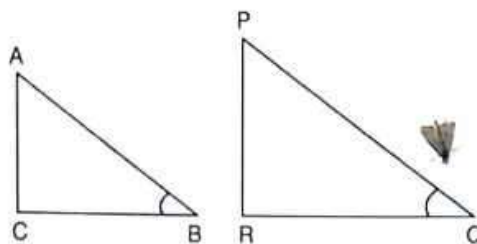


Fig. 10.36

$$\Rightarrow \frac{AC}{AB} = \frac{PR}{PQ}$$

$$\Rightarrow \frac{AC}{PR} = \frac{AB}{PQ} = k, \text{ (say)} \quad \dots(i)$$

$$\Rightarrow AC = k PR \text{ and } AB = k PQ \quad \dots(ii)$$

Using Pythagoras theorem in triangles ABC and PQR , we obtain

$$AB^2 = AC^2 + BC^2 \text{ and } PQ^2 = PR^2 + QR^2$$

$$\Rightarrow BC = \sqrt{AB^2 - AC^2} \text{ and } QR = \sqrt{PQ^2 - PR^2}$$

$$\Rightarrow \frac{BC}{QR} = \frac{\sqrt{AB^2 - AC^2}}{\sqrt{PQ^2 - PR^2}} = \frac{\sqrt{k^2 PQ^2 - k^2 PR^2}}{\sqrt{PQ^2 - PR^2}} \quad \text{[Using (ii)]}$$

$$\Rightarrow \frac{BC}{QR} = \frac{k\sqrt{PQ^2 - PR^2}}{\sqrt{PQ^2 - PR^2}} = k \quad \dots(iii)$$

From (i) and (iii), we have

$$\frac{AC}{PR} = \frac{AB}{PQ} = \frac{BC}{QR}$$

$$\Rightarrow \Delta ACB \sim \Delta PRQ$$

$$\Rightarrow \angle B = \angle Q$$

EXERCISE 10.1

LEVEL-1

1. In each of the following, one of the six trigonometric ratios is given. Find the values of the other trigonometric ratios.

(i) $\sin A = \frac{2}{3}$ (ii) $\cos A = \frac{4}{5}$ (iii) $\tan \theta = 11$ (iv) $\sin \theta = \frac{11}{15}$

(v) $\tan \alpha = \frac{5}{12}$ (vi) $\sin \theta = \frac{\sqrt{3}}{2}$ (vii) $\cos \theta = \frac{7}{25}$ (viii) $\tan \theta = \frac{8}{15}$

(ix) $\cot \theta = \frac{12}{5}$ (x) $\sec \theta = \frac{13}{5}$ (xi) $\operatorname{cosec} \theta = \sqrt{10}$ (xii) $\cos \theta = \frac{12}{15}$

2. In a ΔABC , right angled at B , $AB = 24$ cm, $BC = 7$ cm. Determine

(i) $\sin A, \cos A$

(ii) $\sin C, \cos C$

[NCERT]

3. In Fig. 10.37, find $\tan P$ and $\cot R$. Is $\tan P = \cot R$?

[NCERT]

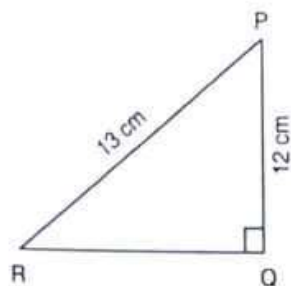


Fig. 10.37

4. If $\sin A = \frac{9}{41}$, compute $\cos A$ and $\tan A$.
5. Given $15 \cot A = 8$, find $\sin A$ and $\sec A$. [NCERT]
6. In ΔPQR , right angled at Q , $PQ = 4$ cm and $RQ = 3$ cm. Find the values of $\sin P$, $\sin R$, $\sec P$ and $\sec R$.
7. If $\cot \theta = \frac{7}{8}$, evaluate:
- (i) $\frac{(1 + \sin \theta)(1 - \sin \theta)}{(1 + \cos \theta)(1 - \cos \theta)}$ (ii) $\cot^2 \theta$ [NCERT]
8. If $3 \cot A = 4$, check whether $\frac{1 - \tan^2 A}{1 + \tan^2 A} = \cos^2 A - \sin^2 A$ or not. [NCERT]
9. If $\tan \theta = \frac{a}{b}$, find the value of $\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta}$
10. If $3 \tan \theta = 4$, find the value of $\frac{4 \cos \theta - \sin \theta}{2 \cos \theta + \sin \theta}$
11. If $3 \cot \theta = 2$, find the value of $\frac{4 \sin \theta - 3 \cos \theta}{2 \sin \theta + 6 \cos \theta}$
12. If $\tan \theta = \frac{a}{b}$, prove that $\frac{a \sin \theta - b \cos \theta}{a \sin \theta + b \cos \theta} = \frac{a^2 - b^2}{a^2 + b^2}$
13. If $\sec \theta = \frac{13}{5}$, show that $\frac{2 \sin \theta - 3 \cos \theta}{4 \sin \theta - 9 \cos \theta} = 3$
14. If $\cos \theta = \frac{12}{13}$, show that $\sin \theta (1 - \tan \theta) = \frac{35}{156}$
15. If $\cot \theta = \frac{1}{\sqrt{3}}$, show that $\frac{1 - \cos^2 \theta}{2 - \sin^2 \theta} = \frac{3}{5}$
16. If $\tan \theta = \frac{1}{\sqrt{7}}$, show that $\frac{\operatorname{cosec}^2 \theta - \sec^2 \theta}{\operatorname{cosec}^2 \theta + \sec^2 \theta} = \frac{3}{4}$
17. If $\sec \theta = \frac{5}{4}$, find the value of $\frac{\sin \theta - 2 \cos \theta}{\tan \theta - \cot \theta}$
18. If $\tan \theta = \frac{12}{13}$, find the value of $\frac{2 \sin \theta \cos \theta}{\cos^2 \theta - \sin^2 \theta}$

19. If $\cos \theta = \frac{3}{5}$, find the value of $\frac{\sin \theta - 1}{2 \tan \theta}$
20. If $\sin \theta = \frac{3}{5}$, evaluate $\frac{\cos \theta - 1}{2 \cot \theta}$
21. If $\tan \theta = \frac{24}{7}$, find that $\sin \theta + \cos \theta$
22. If $\sin \theta = \frac{a}{b}$, find $\sec \theta + \tan \theta$ in terms of a and b .
23. If $8 \tan A = 15$, find $\sin A - \cos A$
24. If $\tan \theta = \frac{20}{21}$, show that $\frac{1 - \sin \theta + \cos \theta}{1 + \sin \theta + \cos \theta} = \frac{3}{7}$
25. If $\operatorname{cosec} A = 2$, find the value of $\frac{1}{\tan A} + \frac{\sin A}{1 + \cos A}$
26. If $\angle A$ and $\angle B$ are acute angles such that $\cos A = \cos B$, then show that $\angle A = \angle B$.

[NCERT]

27. In a $\triangle ABC$, right angled at A , if $\tan C = \sqrt{3}$, find the value of $\sin B \cos C + \cos B \sin C$.

[CBSE 2008]

28. State whether the following are true or false. Justify your answer.

- (i) The value of $\tan A$ is always less than 1.
- (ii) $\sec A = \frac{12}{5}$ for some value of angle A .
- (iii) $\cos A$ is the abbreviation used for the cosecant of angle A .
- (iv) $\cot A$ is the product of \cot and A .
- (v) $\sin \theta = \frac{4}{3}$ for some angle θ .

LEVEL-2

29. If $\sin \theta = \frac{12}{13}$, find the value of $\frac{\sin^2 \theta - \cos^2 \theta}{2 \sin \theta \cos \theta} \times \frac{1}{\tan^2 \theta}$
30. If $\cos \theta = \frac{5}{13}$, find the value of $\frac{\sin^2 \theta - \cos^2 \theta}{2 \sin \theta \cos \theta} \times \frac{1}{\tan^2 \theta}$
31. If $\sec A = \frac{5}{4}$, verify that $\frac{3 \sin A - 4 \sin^3 A}{4 \cos^3 A - 3 \cos A} = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$
32. If $\sin \theta = \frac{3}{4}$, prove that $\sqrt{\frac{\operatorname{cosec}^2 \theta - \cot^2 \theta}{\sec^2 \theta - 1}} = \frac{\sqrt{7}}{3}$
33. If $\sec A = \frac{17}{8}$, verify that $\frac{3 - 4 \sin^2 A}{4 \cos^2 A - 3} = \frac{3 - \tan^2 A}{1 - 3 \tan^2 A}$

34. If $\cot \theta = \frac{3}{4}$, prove that $\sqrt{\frac{\sec \theta - \operatorname{cosec} \theta}{\sec \theta + \operatorname{cosec} \theta}} = \frac{1}{\sqrt{7}}$

35. If $3 \cos \theta - 4 \sin \theta = 2 \cos \theta + \sin \theta$, find $\tan \theta$.

36. If $\angle A$ and $\angle P$ are acute angles such that $\tan A = \tan P$, then show that $\angle A = \angle P$.

ANSWERS

1. (i) $\cos A = \frac{\sqrt{5}}{3}$, $\tan A = \frac{2}{\sqrt{5}}$, $\sec A = \frac{3}{\sqrt{5}}$, $\operatorname{cosec} A = \frac{3}{2}$, $\cot A = \frac{\sqrt{5}}{2}$

(ii) $\sin A = \frac{3}{5}$, $\tan A = \frac{3}{4}$, $\sec A = \frac{5}{4}$, $\operatorname{cosec} A = \frac{5}{3}$, $\cot A = \frac{4}{3}$

(iii) $\sin \theta = \frac{11}{\sqrt{122}}$, $\cos \theta = \frac{1}{\sqrt{122}}$, $\operatorname{cosec} \theta = \frac{\sqrt{122}}{11}$, $\sec \theta = \sqrt{122}$, $\cot \theta = \frac{1}{11}$

(iv) $\cos \theta = \frac{2\sqrt{26}}{15}$, $\tan \theta = \frac{11}{2\sqrt{26}}$, $\sec \theta = \frac{15}{2\sqrt{26}}$, $\cot \theta = \frac{2\sqrt{26}}{11}$, $\operatorname{cosec} \theta = \frac{15}{11}$

(v) $\sin \alpha = \frac{5}{13}$, $\cos \alpha = \frac{12}{13}$, $\cot \alpha = \frac{12}{5}$, $\operatorname{cosec} \alpha = \frac{13}{5}$, $\sec \alpha = \frac{13}{12}$

(vi) $\cos \theta = \frac{1}{2}$, $\tan \theta = \sqrt{3}$, $\sec \theta = 2$, $\operatorname{cosec} \theta = \frac{2}{\sqrt{3}}$, $\cot \theta = \frac{1}{\sqrt{3}}$

(vii) $\sin \theta = \frac{24}{25}$, $\tan \theta = \frac{24}{7}$, $\sec \theta = \frac{25}{7}$, $\operatorname{cosec} \theta = \frac{25}{24}$, $\cot \theta = \frac{7}{24}$

(viii) $\sin \theta = \frac{8}{17}$, $\cos \theta = \frac{15}{17}$, $\cot \theta = \frac{15}{8}$, $\operatorname{cosec} \theta = \frac{17}{8}$, $\sec \theta = \frac{17}{15}$

(ix) $\tan \theta = \frac{5}{12}$, $\sin \theta = \frac{5}{13}$, $\cos \theta = \frac{12}{13}$, $\operatorname{cosec} \theta = \frac{13}{5}$, $\sec \theta = \frac{13}{12}$

(x) $\sin \theta = \frac{12}{13}$, $\cos \theta = \frac{5}{13}$, $\tan \theta = \frac{12}{5}$, $\operatorname{cosec} \theta = \frac{13}{12}$, $\cot \theta = \frac{5}{12}$

(xi) $\sin \theta = \frac{1}{\sqrt{10}}$, $\cos \theta = \frac{3}{\sqrt{10}}$, $\tan \theta = \frac{1}{3}$, $\sec \theta = \frac{\sqrt{10}}{3}$, $\cot \theta = 3$

(xii) $\sin \theta = \frac{9}{15}$, $\tan \theta = \frac{9}{12}$, $\operatorname{cosec} \theta = \frac{15}{9}$, $\cot \theta = \frac{12}{9}$, $\sec \theta = \frac{15}{12}$

2. (i) $\frac{7}{25}$, $\frac{24}{25}$ (ii) $\frac{24}{25}$, $\frac{7}{25}$

4. $\cos A = \frac{40}{41}$, $\tan A = \frac{9}{40}$

6. $\sin P = \frac{3}{5}$, $\sec P = \frac{5}{4}$, $\sec R = \frac{5}{3}$, $\sin R = \frac{4}{5}$

9. $\frac{b+a}{b-a}$

10. $\frac{4}{5}$

11. $\frac{1}{3}$

17. $\frac{12}{7}$

18. $\frac{312}{25}$

19. $\frac{3}{160}$

20. $\frac{-1}{5}$

21. $\frac{31}{25}$

22. $\sqrt{\frac{b+a}{b-a}}$

23. $\frac{7}{17}$

25. 2

27. 1

28. (i) False

(ii) True

(iii) False

(iv) False (v) False

29. $\frac{595}{3456}$

30. $\frac{595}{3456}$

35. $\frac{1}{5}$

3. $\tan P = \frac{5}{12}$, $\cot R = \frac{5}{12}$. Yes.

5. $\sin A = \frac{15}{17}$, $\sec A = \frac{17}{8}$

7. (i) $\frac{49}{64}$ (ii) $\frac{49}{64}$

8. Yes.

10.5 TRIGONOMETRIC RATIOS OF SOME SPECIFIC ANGLES

In this section we shall find the trigonometric ratios of some standard angles by using some elementary knowledge of geometry. For other angles, we can make use of the trigonometric tables.

10.5.1 TRIGONOMETRIC RATIOS OF 0° AND 90°

Let $\angle XAY = \theta$ be an acute angle and let P be a point on its terminal side AY . Draw perpendicular PM from P on AX .

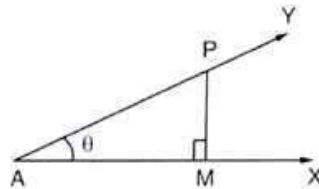


Fig. 10.38

In $\triangle AMP$, we have

$$\sin \theta = \frac{PM}{AP}, \quad \cos \theta = \frac{AM}{AP} \text{ and } \tan \theta = \frac{PM}{AM}$$

It is evident from $\triangle AMP$ that as θ becomes smaller and smaller, line segment PM also becomes smaller and smaller; and finally when θ becomes 0° ; the point P will coincide with M . Consequently, we have

$$PM = 0 \text{ and } AP = AM$$

$$\therefore \sin 0^\circ = \frac{PM}{AP} = \frac{0}{AP} = 0, \quad \cos 0^\circ = \frac{AM}{AP} = \frac{AP}{AP} = 1$$

$$\text{and, } \tan 0^\circ = \frac{PM}{AM} = \frac{0}{AP} = 0$$

Thus, we have

$$\sin 0^\circ = 0, \quad \cos 0^\circ = 1 \text{ and } \tan 0^\circ = 0$$

From $\triangle AMP$, it is evident that as θ increase, line segment AM becomes smaller and smaller and finally when θ becomes 90° the point M will coincide with A . Consequently, we have

$$AM = 0, \quad AP = PM$$

$$\therefore \sin 90^\circ = \frac{PM}{AP} = \frac{PM}{PM} = 1 \text{ and } \cos 90^\circ = \frac{AM}{AP} = \frac{0}{AP} = 0$$

Thus, we have

$$\sin 90^\circ = 1 \text{ and } \cos 90^\circ = 0$$

REMARK It is evident from the above discussion that $\tan 90^\circ = \frac{PM}{AM} = \frac{PM}{0}$ is not defined. Similarly, $\sec 90^\circ$, $\operatorname{cosec} 0^\circ$, $\cot 0^\circ$ are not defined.

10.5.2 TRIGONOMETRIC RATIOS OF 30° AND 60°

Consider an equilateral triangle ABC with each side of length $2a$. Since each angle of an equilateral triangle is of 60° . Therefore, each angle of $\triangle ABC$ is of 60° . Let AD be perpendicular from A on BC . Since the triangle is equilateral. Therefore, AD is the bisector of $\angle A$ and D is the mid-point of BC .

$$\therefore BD = DC = a \text{ and } \angle BAD = 30^\circ$$

Thus, in $\triangle ABD$, $\angle D$ is a right angle, hypotenuse $AB = 2a$ and $BD = a$.

So, by Pythagoras theorem, we have

$$\begin{aligned} AB^2 &= AD^2 + BD^2 \\ \Rightarrow (2a)^2 &= AD^2 + a^2 \\ \Rightarrow AD^2 &= 4a^2 - a^2 \\ \Rightarrow AD &= \sqrt{3}a \end{aligned}$$

Trigonometric ratios of 30° :

In right triangle ADB , we have

Base = $AD = \sqrt{3}a$, Perpendicular = $BD = a$, Hypotenuse = $AB = 2a$ and $\angle DAB = 30^\circ$

$$\begin{aligned} \therefore \sin 30^\circ &= \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2} \\ \cos 30^\circ &= \frac{AD}{AB} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2} \\ \tan 30^\circ &= \frac{BD}{AD} = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}} \\ \operatorname{cosec} 30^\circ &= \frac{1}{\sin 30^\circ} = 2 \end{aligned}$$

$$\sec 30^\circ = \frac{1}{\cos 30^\circ} = \frac{2}{\sqrt{3}} \text{ and, } \cot 30^\circ = \frac{1}{\tan 30^\circ} = \sqrt{3}$$

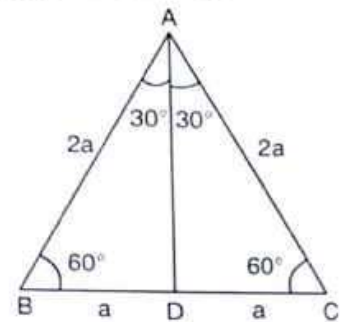


Fig. 10.39

[CBSE 2009, 2010]

Trigonometric ratios of 60° :

In right triangle ADB , we have

Base = $BD = a$, Perpendicular = $AD = \sqrt{3}a$, Hypotenuse = $AB = 2a$ and $\angle ABD = 60^\circ$

$$\begin{aligned} \therefore \sin 60^\circ &= \frac{AD}{AB} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}, \quad \cos 60^\circ = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2} \\ \tan 60^\circ &= \frac{AD}{BD} = \frac{\sqrt{3}a}{a} = \sqrt{3}, \quad \operatorname{cosec} 60^\circ = \frac{1}{\sin 60^\circ} = \frac{2}{\sqrt{3}} \\ \sec 60^\circ &= \frac{1}{\cos 60^\circ} = 2 \text{ and, } \cot 60^\circ = \frac{1}{\tan 60^\circ} = \frac{1}{\sqrt{3}} \end{aligned}$$

10.5.3 TRIGONOMETRIC RATIOS OF 45°

Consider a right triangle ABC with right angle at B such that $\angle A = 45^\circ$. Then,

$$\begin{aligned} \angle A + \angle B + \angle C &= 180^\circ \\ \Rightarrow 45^\circ + 90^\circ + \angle C &= 180^\circ \\ \Rightarrow \angle C &= 45^\circ \\ \therefore \angle A &= \angle C \\ \Rightarrow AB &= BC \end{aligned}$$

Let $AB = BC = a$. Then, by Pythagoras theorem, we have

$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ \Rightarrow AC^2 &= a^2 + a^2 \\ \Rightarrow AC^2 &= 2a^2 \\ \Rightarrow AC &= \sqrt{2}a \end{aligned}$$

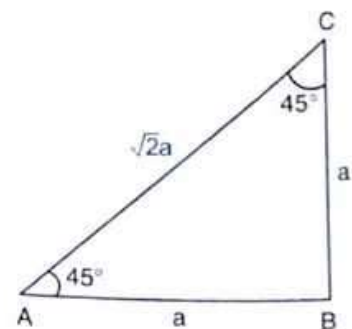


Fig. 10.40

Thus, in $\triangle ABC$, we have

$\angle A = 45^\circ$, Base = $AB = a$, Perpendicular = $BC = a$, and Hypotenuse = $AC = \sqrt{2}a$.

$$\therefore \sin 45^\circ = \frac{BC}{AC} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}, \quad \cos 45^\circ = \frac{AB}{AC} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{BC}{AB} = \frac{a}{a} = 1, \quad \operatorname{cosec} 45^\circ = \frac{1}{\sin 45^\circ} = \sqrt{2}$$

$$\sec 45^\circ = \frac{1}{\cos 45^\circ} = \sqrt{2} \quad \text{and} \quad \cot 45^\circ = \frac{1}{\tan 45^\circ} = \frac{1}{1} = 1$$

Following table gives the values of various trigonometric ratios of $0^\circ, 30^\circ, 45^\circ, 60^\circ$ and 90° for ready reference.

θ	0°	30°	45°	60°	90°
T. ratios					
$\sin \theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
$\operatorname{cosec} \theta$	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec \theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
$\cot \theta$	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Following examples will illustrate the use of the values of various trigonometric ratios given in the above table to find the values of trigonometric expressions.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 Evaluate each of the following in the simplest form:

- (i) $\sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ$ (ii) $\sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ$
 (iii) $\cos 60^\circ \cos 30^\circ + \sin 60^\circ \sin 30^\circ$ [NCERT] (iv) $\cos 60^\circ \cos 30^\circ - \sin 60^\circ \sin 30^\circ$

SOLUTION (i) We have,

$$\sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ = \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4} + \frac{1}{4} = 1$$

(ii) We have,

$$\sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(iii) We have,

$$\cos 60^\circ \cos 30^\circ + \sin 60^\circ \sin 30^\circ = \frac{1}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \times \frac{1}{2} = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = 2 \left(\frac{\sqrt{3}}{4} \right) = \frac{\sqrt{3}}{2}$$

(iv) We have,

$$\cos 60^\circ \cos 30^\circ - \sin 60^\circ \sin 30^\circ = \frac{1}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \times \frac{1}{2} = \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = 0$$

EXAMPLE 2 Evaluate each of the following in the simplest form:

(i) $\operatorname{cosec} 30^\circ + \cot 45^\circ$

(ii) $\cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ$

(iii) $\tan 30^\circ \sec 45^\circ + \tan 60^\circ \sec 30^\circ$

(iv) $\sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ$

SOLUTION (i) We have,

$$\operatorname{cosec} 30^\circ + \cot 45^\circ = 2 + 1 = 3$$

(ii) We have,

$$\cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

(iii) We have,

$$\tan 30^\circ \sec 45^\circ + \tan 60^\circ \sec 30^\circ = \frac{1}{\sqrt{3}} \times \sqrt{2} + \sqrt{3} \times \frac{2}{\sqrt{3}} = \frac{\sqrt{2} + 2\sqrt{3}}{\sqrt{3}}$$

(iv) We have,

$$\sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ = \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{3} + 1}{2\sqrt{2}}$$

EXAMPLE 3 Evaluate the following expressions:

(i) $2 \sin^2 30^\circ \tan 60^\circ - 3 \cos^2 60^\circ \sec^2 30^\circ$

(ii) $\operatorname{cosec}^2 30^\circ \sin^2 45^\circ - \sec^2 60^\circ$

(iii) $\tan 60^\circ \operatorname{cosec}^2 45^\circ + \sec^2 60^\circ \tan 45^\circ$

(iv) $4 \cot^2 45^\circ - \sec^2 60^\circ + \sin^2 60^\circ + \cos^2 90^\circ$

(v) $3 \cos^2 30^\circ + \sec^2 30^\circ + 2 \cos 0^\circ + 3 \sin 90^\circ - \tan^2 60^\circ$

SOLUTION (i) We have,

$$\begin{aligned} & 2 \sin^2 30^\circ \tan 60^\circ - 3 \cos^2 60^\circ \sec^2 30^\circ \\ &= 2 (\sin 30^\circ)^2 \tan 60^\circ - 3 (\cos 60^\circ)^2 (\sec 30^\circ)^2 \\ &= 2 \times \left(\frac{1}{2} \right)^2 \times \sqrt{3} - 3 \times \left(\frac{1}{2} \right)^2 \times \left(\frac{2}{\sqrt{3}} \right)^2 \\ &= 2 \times \frac{1}{4} \times \sqrt{3} - 3 \times \frac{1}{4} \times \frac{4}{3} = \frac{\sqrt{3}}{2} - 1 = \frac{\sqrt{3} - 2}{2} \end{aligned}$$

(ii) We have,

$$\begin{aligned} & \operatorname{cosec}^2 30^\circ \sin^2 45^\circ - \sec^2 60^\circ \\ &= (\operatorname{cosec} 30^\circ)^2 (\sin 45^\circ)^2 - (\sec 60^\circ)^2 = (2)^2 \times \left(\frac{1}{\sqrt{2}} \right)^2 - (2)^2 = 2 - 4 = -2 \end{aligned}$$

(iii) We have,

$$\tan 60^\circ \operatorname{cosec}^2 45^\circ + \sec^2 60^\circ \tan 45^\circ$$

$$\begin{aligned}
 &= \tan 60^\circ (\operatorname{cosec} 45^\circ)^2 + (\sec 60^\circ)^2 \tan 45^\circ \\
 &= \sqrt{3} \times (\sqrt{2})^2 + (2)^2 \times 1 = \sqrt{3} \times 2 + 4 = 4 + 2\sqrt{3}
 \end{aligned}$$

(iv) We have,

$$\begin{aligned}
 &4 \cot^2 45^\circ - \sec^2 60^\circ + \sin^2 60^\circ + \cos^2 90^\circ \\
 &= 4 (\cot 45^\circ)^2 - (\sec 60^\circ)^2 + (\sin 60^\circ)^2 + (\cos 90^\circ)^2 \\
 &= 4 \times (1)^2 - (2)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + 0 = 4 - 4 + \frac{3}{4} + 0 = \frac{3}{4}
 \end{aligned}$$

(v) We have,

$$\begin{aligned}
 &3 \cos^2 30^\circ + \sec^2 30^\circ + 2 \cos 0^\circ + 3 \sin 90^\circ - \tan^2 60^\circ \\
 &= 3 (\cos 30^\circ)^2 + (\sec 30^\circ)^2 + 2 \cos 0^\circ + 3 \sin 90^\circ - (\tan 60^\circ)^2 \\
 &= 3 \times \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{2}{\sqrt{3}}\right)^2 + 2 \times 1 + 3 \times 1 - (\sqrt{3})^2 = \frac{9}{4} + \frac{4}{3} + 2 + 3 - 3 = \frac{67}{12}
 \end{aligned}$$

EXAMPLE 4 Prove that: $\frac{\cos 30^\circ + \sin 60^\circ}{1 + \cos 60^\circ + \sin 30^\circ} = \frac{\sqrt{3}}{2}$.

SOLUTION We have,

$$\frac{\cos 30^\circ + \sin 60^\circ}{1 + \cos 60^\circ + \sin 30^\circ} = \frac{\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}}{1 + \frac{1}{2} + \frac{1}{2}} = \frac{2 \left(\frac{\sqrt{3}}{2}\right)}{2} = \frac{\sqrt{3}}{2}$$

EXAMPLE 5 Evaluate each of the following:

(i) $\frac{\sin^2 45^\circ + \cos^2 45^\circ}{\tan^2 60^\circ}$

(ii) $\frac{\sin 30^\circ - \sin 90^\circ + 2 \cos 0^\circ}{\tan 30^\circ \tan 60^\circ}$

(iii) $\frac{\sin 60^\circ}{\cos^2 45^\circ} - \cot 30^\circ + 15 \cos 90^\circ$

(iv) $\frac{5 \sin^2 30^\circ + \cos^2 45^\circ - 4 \tan^2 30^\circ}{2 \sin 30^\circ \cos 30^\circ + \tan 45^\circ}$

SOLUTION (i) We have,

$$\frac{\sin^2 45^\circ + \cos^2 45^\circ}{\tan^2 60^\circ} = \frac{(\sin 45^\circ)^2 + (\cos 45^\circ)^2}{(\tan 60^\circ)^2} = \frac{(1/\sqrt{2})^2 + (1/\sqrt{2})^2}{(\sqrt{3})^2} = \frac{\frac{1}{2} + \frac{1}{2}}{3} = \frac{1}{3}$$

(ii) We have,

$$\frac{\sin 30^\circ - \sin 90^\circ + 2 \cos 0^\circ}{\tan 30^\circ \tan 60^\circ} = \frac{\frac{1}{2} - 1 + 2 \times 1}{\frac{1}{\sqrt{3}} \times \sqrt{3}} = \frac{\frac{1}{2} - 1 + 2}{1} = \frac{3}{2}$$

(iii) We have,

$$\frac{\sin 60^\circ}{\cos^2 45^\circ} - \cot 30^\circ + 15 \cos 90^\circ = \frac{\sqrt{3}}{(1/\sqrt{2})^2} - \sqrt{3} + 15 \times 0 = \sqrt{3} - \sqrt{3} + 0 = 0$$

(iv) We have,

$$\begin{aligned} & \frac{5\sin^2 30^\circ + \cos^2 45^\circ - 4\tan^2 30^\circ}{2\sin 30^\circ \cos 30^\circ + \tan 45^\circ} \\ &= \frac{5 \times (1/2)^2 + (1/\sqrt{2})^2 - 4 \times (1/\sqrt{3})^2}{2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} + 1} \\ &= \frac{5 + \frac{1}{2} - \frac{4}{3}}{\frac{\sqrt{3}}{2} + 1} = \frac{\frac{5}{2} - \frac{4}{3}}{\frac{\sqrt{3} + 2}{2}} = \frac{5}{12} \times \frac{2}{\sqrt{3} + 2} = \frac{5}{6(\sqrt{3} + 2)} = \frac{5}{6}(2 - \sqrt{3}) \end{aligned}$$

EXAMPLE 6 Show that:

(i) $2(\cos^2 45^\circ + \tan^2 60^\circ) - 6(\sin^2 45^\circ - \tan^2 30^\circ) = 6$

(ii) $2(\cos^4 60^\circ + \sin^4 30^\circ) - (\tan^2 60^\circ + \cot^2 45^\circ) + 3\sec^2 30^\circ = \frac{1}{4}$

SOLUTION (i) We have,

$$\begin{aligned} & 2(\cos^2 45^\circ + \tan^2 60^\circ) - 6(\sin^2 45^\circ - \tan^2 30^\circ) \\ &= 2\left\{\left(\frac{1}{\sqrt{2}}\right)^2 + (\sqrt{3})^2\right\} - 6\left\{\left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^2\right\} \\ &= 2\left(\frac{1}{2} + 3\right) - 6\left(\frac{1}{2} - \frac{1}{3}\right) = 2\left(\frac{1+6}{2}\right) - 6\left(\frac{3-2}{6}\right) = 7 - 1 = 6 \end{aligned}$$

(ii) We have,

$$\begin{aligned} & 2(\cos^4 60^\circ + \sin^4 30^\circ) - (\tan^2 60^\circ + \cot^2 45^\circ) + 3\sec^2 30^\circ \\ &= 2\left\{\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4\right\} - \{(\sqrt{3})^2 + (1)^2\} + 3\left(\frac{2}{\sqrt{3}}\right)^2 \\ &= 2\left(\frac{1}{16} + \frac{1}{16}\right) - (3 + 1) + 3 \times \frac{4}{3} = 2 \times \frac{1}{8} - 4 + 4 = \frac{1}{4} \end{aligned}$$

EXAMPLE 7 Find the value of θ in each of the following:

(i) $2\sin 2\theta = \sqrt{3}$

(ii) $2\cos 3\theta = 1$

(iii) $\sqrt{3}\tan 2\theta - 3 = 0$

SOLUTION (i) We have,

$$2\sin 2\theta = \sqrt{3}$$

$$\Rightarrow \sin 2\theta = \frac{\sqrt{3}}{2} \Rightarrow \sin 2\theta = \sin 60^\circ \Rightarrow 2\theta = 60^\circ \Rightarrow \theta = 30^\circ$$

(ii) We have,

$$2 \cos 3\theta = 1$$

$$\Rightarrow \cos 3\theta = \frac{1}{2} \Rightarrow \cos 3\theta = \cos 60^\circ \Rightarrow 3\theta = 60^\circ \Rightarrow \theta = 20^\circ$$

(iii) We have,

$$\sqrt{3} \tan 2\theta - 3 = 0$$

$$\Rightarrow \sqrt{3} \tan 2\theta = 3$$

$$\Rightarrow \tan 2\theta = \frac{3}{\sqrt{3}} = \sqrt{3} \Rightarrow \tan 2\theta = \tan 60^\circ \Rightarrow 2\theta = 60^\circ \Rightarrow \theta = 30^\circ$$

EXAMPLE 8 Find the value of x in each of the following:

$$(i) \tan 3x = \sin 45^\circ \cos 45^\circ + \sin 30^\circ \quad (ii) \cos x = \cos 60^\circ \cos 30^\circ + \sin 60^\circ \sin 30^\circ$$

$$(iii) \sin 2x = \sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ$$

SOLUTION (i) We have,

$$\tan 3x = \sin 45^\circ \cos 45^\circ + \sin 30^\circ$$

$$\Rightarrow \tan 3x = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + \frac{1}{2}$$

$$\Rightarrow \tan 3x = \frac{1}{2} + \frac{1}{2}$$

$$\Rightarrow \tan 3x = 1 \Rightarrow \tan 3x = \tan 45^\circ \Rightarrow 3x = 45^\circ \Rightarrow x = 15^\circ$$

(ii) We have,

$$\cos x = \cos 60^\circ \cos 30^\circ + \sin 60^\circ \sin 30^\circ$$

$$\Rightarrow \cos x = \frac{1}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \times \frac{1}{2}$$

$$\Rightarrow \cos x = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}$$

$$\Rightarrow \cos x = \frac{\sqrt{3}}{2} \Rightarrow \cos x = \cos 30^\circ \Rightarrow x = 30^\circ$$

(iii) We have,

$$\sin 2x = \sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ$$

$$\Rightarrow \sin 2x = \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} - \frac{1}{2} \times \frac{1}{2}$$

$$\Rightarrow \sin 2x = \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\Rightarrow \sin 2x = \sin 30^\circ \Rightarrow 2x = 30^\circ \Rightarrow x = 15^\circ$$

EXAMPLE 9 Solve each of the following equations when $0^\circ < \theta < 90^\circ$.

$$(i) 2 \cos \theta = 1 \quad (ii) 2 \cos^2 \theta = \frac{1}{2} \quad (iii) 2 \sin^2 \theta = \frac{1}{2}$$

$$(iv) 3 \tan^2 \theta - 1 = 0$$

SOLUTION (i) We have,

$$2 \cos \theta = 1 \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = 60^\circ$$

(ii) We have,

$$2 \cos^2 \theta = \frac{1}{2}$$

$$\Rightarrow \cos^2 \theta = \frac{1}{4}$$

$$\Rightarrow \cos \theta = \frac{1}{2}$$

$[\because \cos \theta > 0 \text{ for } 0^\circ < \theta < 90^\circ]$

$$\Rightarrow \theta = 60^\circ$$

(iii) We have,

$$2 \sin^2 \theta = \frac{1}{2}$$

$$\Rightarrow \sin^2 \theta = \frac{1}{4}$$

$$\Rightarrow \sin \theta = \frac{1}{2}$$

$[\because \sin \theta > 0 \text{ for } 0^\circ < \theta < 90^\circ]$

$$\Rightarrow \theta = 30^\circ$$

(iv) We have,

$$3 \tan^2 \theta = 1$$

$$\Rightarrow \tan^2 \theta = \frac{1}{3}$$

$$\Rightarrow \tan \theta = \frac{1}{\sqrt{3}}$$

$[\because \tan \theta > 0 \text{ for } 0^\circ < \theta < 90^\circ]$

$$\Rightarrow \theta = 30^\circ$$

EXAMPLE 10 Solve each of the following equations for $0^\circ < \theta < 90^\circ$.

(i) $2 \cos 3\theta = 1$

(ii) $2 \sin 2\theta = \sqrt{3}$

(iii) $\tan 5\theta = 1$

SOLUTION (i) We have,

$$2 \cos 3\theta = 1 \Rightarrow \cos 3\theta = \frac{1}{2} \Rightarrow 3\theta = 60^\circ \Rightarrow \theta = 20^\circ$$

(ii) We have,

$$2 \sin 2\theta = \sqrt{3} \Rightarrow \sin \theta = \frac{\sqrt{3}}{2} \Rightarrow 2\theta = 60^\circ \Rightarrow \theta = 30^\circ$$

(iii) We have,

$$\tan 5\theta = 1 \Rightarrow 5\theta = 45^\circ \Rightarrow \theta = 9^\circ$$

EXAMPLE 11 If $x = 30^\circ$, verify that

(i) $\sin 3x = 3 \sin x - 4 \sin^3 x$

(ii) $\cos 3x = 4 \cos^3 x - 3 \cos x$

(iii) $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$

(iv) $\sin x = \sqrt{\frac{1 - \cos 2x}{2}}$

SOLUTION (i) We have,

$$x = 30^\circ \Rightarrow 3x = 90^\circ$$

$$\therefore \sin 3x = \sin 90^\circ = 1$$

$$\text{and, } 3 \sin x - 4 \sin^3 x = 3 \sin 30^\circ - 4 \sin^3 30^\circ = 3 \times \frac{1}{2} - 4 \left(\frac{1}{2}\right)^3 = \frac{3}{2} - \frac{1}{2} = 1.$$

$$\therefore \sin 3x = 3 \sin x - 4 \sin^3 x$$

(ii) We have,

$$x = 30^\circ \Rightarrow 3x = 90^\circ$$

$$\therefore \cos 3x = \cos 90^\circ = 0$$

$$\text{and, } 4 \cos^3 x - 3 \cos x = 4 \cos^3 30^\circ - 3 \cos 30^\circ$$

$$\Rightarrow 4 \cos^3 x - 3 \cos x = 4 \left(\frac{\sqrt{3}}{2}\right)^3 - 3 \left(\frac{\sqrt{3}}{2}\right) = 4 \times \frac{3\sqrt{3}}{8} - \frac{3\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{2} = 0$$

$$\therefore \cos 3x = 4 \cos^3 x - 3 \cos x$$

(iii) We have,

$$x = 30^\circ \Rightarrow 2x = 60^\circ$$

$$\therefore \tan 2x = \tan 60^\circ = \sqrt{3}$$

$$\text{and, } \frac{2 \tan x}{1 - \tan^2 x} = \frac{2 \tan 30^\circ}{1 - \tan^2 30^\circ} = \frac{2 \times \frac{1}{\sqrt{3}}}{1 - \left(\frac{1}{\sqrt{3}}\right)^2} = \frac{2/\sqrt{3}}{1 - \frac{1}{3}} = \frac{2/\sqrt{3}}{2/3} = \frac{2}{\sqrt{3}} \times \frac{3}{2} = \sqrt{3}$$

$$\therefore \tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

(iv) We have,

$$x = 30^\circ \Rightarrow 2x = 60^\circ$$

$$\therefore \sqrt{\frac{1 - \cos 2x}{2}} = \sqrt{\frac{1 - \cos 60^\circ}{2}} = \sqrt{\frac{1 - \frac{1}{2}}{2}} = \sqrt{\frac{1}{4}} = \frac{1}{2} \text{ and, } \sin x = \sin 30^\circ = \frac{1}{2}$$

$$\therefore \sin x = \sqrt{\frac{1 - \cos 2x}{2}}$$

EXAMPLE 12 Verify that:

$$(i) \sin 60^\circ = \frac{2 \tan 30^\circ}{1 + \tan^2 30^\circ} = \frac{\sqrt{3}}{2}$$

$$(ii) \cos 60^\circ = \frac{1 - \tan^2 30^\circ}{1 + \tan^2 30^\circ} = \frac{1}{2}$$

$$(iii) \cos 60^\circ = \cos^2 30^\circ - \sin^2 30^\circ = \frac{1}{2}$$

$$(iv) 4(\sin^4 30^\circ + \cos^4 60^\circ) - 3(\cos^2 45^\circ - \sin^2 90^\circ) = 2$$

SOLUTION (i) We have,

$$\text{LHS} = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\text{RHS} = \frac{2 \tan 30^\circ}{1 + \tan^2 30^\circ} = \frac{2 \times \frac{1}{\sqrt{3}}}{1 + (1/\sqrt{3})^2} = \frac{2/\sqrt{3}}{1 + \frac{1}{3}} = \frac{2/\sqrt{3}}{4/3} = \frac{2}{\sqrt{3}} \times \frac{3}{4} = \frac{\sqrt{3}}{2}$$

$$\therefore \text{LHS} = \text{RHS} \text{ i.e., } \sin 60^\circ = \frac{2 \tan 30^\circ}{1 + \tan^2 30^\circ} = \frac{\sqrt{3}}{2}$$

(ii) We have,

$$\text{RHS} = \frac{1 - \tan^2 30^\circ}{1 + \tan^2 30^\circ} = \frac{1 - (1/\sqrt{3})^2}{1 + (1/\sqrt{3})^2} = \frac{1 - \frac{1}{3}}{1 + \frac{1}{3}} = \frac{2/3}{4/3} = \frac{1}{2} \text{ and, LHS} = \cos$$

$$\therefore \text{LHS} = \text{RHS} \text{ i.e., } \cos 60^\circ = \frac{1 - \tan^2 30^\circ}{1 + \tan^2 30^\circ} = \frac{1}{2}$$

(iii) We have,

$$\text{RHS} = \cos^2 30^\circ - \sin^2 30^\circ = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\text{and, LHS} = \cos 60^\circ = \frac{1}{2}$$

$$\therefore \text{LHS} = \text{RHS} \text{ i.e., } \cos 60^\circ = \cos^2 30^\circ - \sin^2 30^\circ = \frac{1}{2}$$

(iv) We have,

$$\begin{aligned} \text{LHS} &= 4(\sin^4 30^\circ + \cos^4 60^\circ) - 3(\cos^2 45^\circ - \sin^2 90^\circ) \\ &= 4\left\{\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4\right\} - 3\left\{\left(\frac{1}{\sqrt{2}}\right)^2 - (1)^2\right\} \\ &= 4\left(\frac{1}{16} + \frac{1}{16}\right) - 3\left(\frac{1}{2} - 1\right) = 4 \times \frac{2}{16} - 3\left(-\frac{1}{2}\right) = \frac{1}{2} + \frac{3}{2} = 2 = \text{RHS} \end{aligned}$$

LEVEL-2

EXAMPLE 13 If θ is an acute angle and $\tan \theta + \cot \theta = 2$, find the value of $\tan^7 \theta + \cot^7 \theta$.

SOLUTION We have,

$$\tan \theta + \cot \theta = 2$$

$$\Rightarrow \tan \theta + \frac{1}{\tan \theta} = 2$$

$$\Rightarrow \frac{\tan^2 \theta + 1}{\tan \theta} = 2$$

$$\Rightarrow \tan^2 \theta - 2 \tan \theta + 1 = 0$$

$$\Rightarrow (\tan \theta - 1)^2 = 0$$

$$\Rightarrow \tan \theta - 1 = 0 \Rightarrow \tan \theta = 1 \Rightarrow \tan \theta = \tan 45^\circ \Rightarrow \theta = 45$$

$$\therefore \tan^7 \theta + \cot^7 \theta = \tan^7 45^\circ + \cot^7 45^\circ = (\tan 45^\circ)^7 + (\cot 45^\circ)^7 = (1)^7 + (1)^7 = 2$$

EXAMPLE 14 Find an acute angle θ , when $\frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} = \frac{1 - \sqrt{3}}{1 + \sqrt{3}}$

SOLUTION We have,

$$\frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} = \frac{1 - \sqrt{3}}{1 + \sqrt{3}}$$

$$\Rightarrow \frac{(\cos \theta - \sin \theta) + (\cos \theta + \sin \theta)}{(\cos \theta - \sin \theta) - (\cos \theta + \sin \theta)} = \frac{(1 - \sqrt{3}) + (1 + \sqrt{3})}{(1 - \sqrt{3}) - (1 + \sqrt{3})} \quad \left[\text{Applying componendo and dividendo} \right]$$

$$\Rightarrow \frac{2 \cos \theta}{-2 \sin \theta} = \frac{2}{-2\sqrt{3}}$$

$$\Rightarrow \cot \theta = \frac{1}{\sqrt{3}} \Rightarrow \tan \theta = \sqrt{3} \Rightarrow \tan \theta = \tan 60^\circ \Rightarrow \theta = 60^\circ$$

ALITER We have,

$$\frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} = \frac{1 - \sqrt{3}}{1 + \sqrt{3}}$$

$$\Rightarrow \frac{\frac{\cos \theta - \sin \theta}{\cos \theta}}{\frac{\cos \theta + \sin \theta}{\cos \theta}} = \frac{1 - \sqrt{3}}{1 + \sqrt{3}} \quad \left[\text{Dividing Numerator and Denominator by } \cos \theta \right]$$

$$\Rightarrow \frac{1 - \tan \theta}{1 + \tan \theta} = \frac{1 - \sqrt{3}}{1 + \sqrt{3}}$$

$$\Rightarrow \tan \theta = \sqrt{3}$$

$$\Rightarrow \tan \theta = \tan 60^\circ$$

$$\Rightarrow \theta = 60^\circ$$

[On comparing two sides]

EXAMPLE 15 If $\sin(A + B) = 1$ and $\cos(A - B) = \frac{\sqrt{3}}{2}$, $0^\circ < A + B \leq 90^\circ$, $A > B$ then find A and B .

SOLUTION We have,

$$\sin(A + B) = 1$$

$$\Rightarrow \sin(A + B) = \sin 90^\circ$$

$$\Rightarrow A + B = 90^\circ \quad \dots(i)$$

$$\text{and, } \cos(A - B) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \cos(A - B) = \cos 30^\circ$$

$$\Rightarrow A - B = 30^\circ \quad \dots(ii)$$

Adding (i) and (ii), we get

$$(A + B) + (A - B) = 90^\circ + 30^\circ \Rightarrow 2A = 120^\circ \Rightarrow A = 60^\circ$$

Putting $A = 60^\circ$ in (i), we get

$$60^\circ + B = 90^\circ \Rightarrow B = 30^\circ$$

Hence, $A = 60^\circ$ and $B = 30^\circ$

EXAMPLE 16 If θ is an acute angle and $\sin \theta = \cos \theta$, find the value of $2 \tan^2 \theta + \sin^2 \theta - 1$.

SOLUTION We have,

$$\sin \theta = \cos \theta$$

$$\Rightarrow \frac{\sin \theta}{\cos \theta} = \frac{\cos \theta}{\cos \theta}$$

[Dividing both sides by $\cos \theta$]

$$\Rightarrow \tan \theta = 1 \Rightarrow \tan \theta = \tan 45^\circ \Rightarrow \theta = 45^\circ$$

$$\therefore 2 \tan^2 \theta + \sin^2 \theta - 1$$

$$= 2 \tan^2 45^\circ + \sin^2 45^\circ - 1 = 2(1)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 - 1 = 2 + \frac{1}{2} - 1 = \frac{5}{2} - 1 = \frac{3}{2}$$

EXAMPLE 17 Given that $\sin(A+B) = \sin A \cos B + \cos A \sin B$, find the value of $\sin 75^\circ$.

SOLUTION Putting $A = 45^\circ$ and $B = 30^\circ$ in $\sin(A+B) = \sin A \cos B + \cos A \sin B$, we get

$$\sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$\Rightarrow \sin 75^\circ = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3} + 1}{2\sqrt{2}}$$

EXAMPLE 18 ABC is a right triangle, right angled at C . If $A = 30^\circ$ and $AB = 40$ units, find the remaining two sides and $\angle B$ of ΔABC .

SOLUTION We have,

$$\angle A + \angle B + \angle C = 180^\circ$$

$$\Rightarrow 30^\circ + \angle B + 90^\circ = 180^\circ$$

$$\Rightarrow \angle B = 180^\circ - 120^\circ = 60^\circ$$

[$\because \angle A = 30^\circ$ and $\angle C = 90^\circ$]

$$\text{Now, } \cos A = \frac{AC}{AB}$$

$$\Rightarrow \cos 30^\circ = \frac{AC}{40}$$

$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{AC}{40}$$

$$\Rightarrow AC = \frac{\sqrt{3}}{2} \times 40 \Rightarrow AC = 20\sqrt{3} \text{ units}$$

$$\text{and, } \sin A = \frac{BC}{AB}$$

$$\Rightarrow \sin 30^\circ = \frac{BC}{40}$$

$$\Rightarrow \frac{1}{2} = \frac{BC}{40} \Rightarrow BC = 40 \times \frac{1}{2} = 20 \text{ units.}$$

Hence, $AC = 20\sqrt{3}$ units, $BC = 20$ units and $\angle B = 60^\circ$.

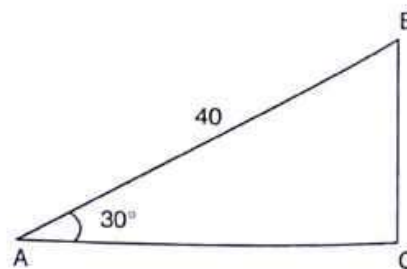


Fig. 10.41

EXAMPLE 19 In a rectangle $ABCD$, $AB = 20$ cm, $\angle BAC = 60^\circ$. Calculate side BC .

SOLUTION In $\triangle ABC$, we have

$$AB = 20, \angle BAC = 60^\circ$$

$$\therefore \tan \angle BAC = \frac{BC}{AB}$$

$$\Rightarrow \tan 60^\circ = \frac{BC}{20}$$

$$\Rightarrow \sqrt{3} = \frac{BC}{20}$$

$$\Rightarrow BC = 20\sqrt{3} \text{ cm}$$

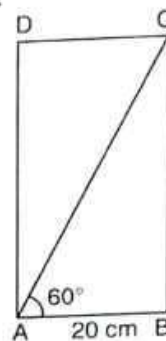


Fig. 10.42

EXAMPLE 20 A rhombus of side 20 cm has two angles of 60° each. Find the length of the diagonals.

SOLUTION Let $ABCD$ be a rhombus of side 20 cm and $\angle BAD = \angle BCD = 60^\circ$.

Note that the diagonals of a rhombus are perpendicular bisectors of each other and diagonals AC and BD are bisectors of $\angle BAD$ and $\angle ABC$ respectively.

So, $\triangle AOB$ is a right triangle such that $\angle BAO = 30^\circ$, $\angle AOB = 90^\circ$ and $AB = 20$ cm.

$$\therefore \cos \angle BAO = \frac{OA}{AB}$$

$$\Rightarrow \cos 30^\circ = \frac{OA}{20}$$

$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{OA}{20}$$

$$\Rightarrow OA = \frac{\sqrt{3}}{2} \times 20 = 10\sqrt{3}$$

Also, $\sin \angle BAO = \frac{OB}{AB}$

$$\Rightarrow \sin 30^\circ = \frac{BO}{20}$$

$$\Rightarrow \frac{1}{2} = \frac{BO}{20}$$

$$\Rightarrow BO = \frac{20}{2} = 10$$

$$\therefore AC = 2 AO = 2 \times 10\sqrt{3} = 20\sqrt{3} \text{ cm}, BD = 2 BO = 2 \times 10 = 20 \text{ cm}$$

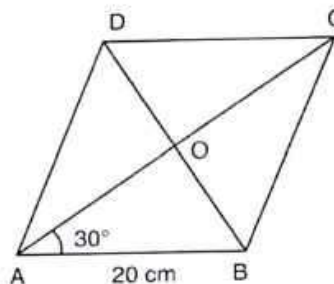


Fig. 10.43

EXAMPLE 21 The altitude AD of a $\triangle ABC$, in which $\angle A$ obtuse and, $AD = 10$ cm. If $BD = 10$ cm and $CD = 10\sqrt{3}$ cm, determine $\angle A$.

SOLUTION $\triangle ABC$ is a right triangle, right angled at D such that $AD = 10$ cm and $BD = 10$ cm.

$$\therefore \tan \angle BAD = \frac{BD}{AD}$$

$$\Rightarrow \tan \angle BAD = \frac{10}{10} = 1$$

$$\Rightarrow \tan \angle BAD = \tan 45^\circ$$

$$\Rightarrow \angle BAD = 45^\circ$$

ΔACD is a right triangle right angled at D such that $AD = 10$ cm and $DC = 10\sqrt{3}$ cm.

$$\therefore \tan \angle CAD = \frac{CD}{AD}$$

$$\Rightarrow \tan \angle CAD = \frac{10\sqrt{3}}{10} = \sqrt{3}$$

$$\Rightarrow \tan \angle CAD = \tan 60^\circ$$

$$\therefore \angle CAD = 60^\circ$$

From (i) and (ii), we have

$$\angle BAC = \angle BAD + \angle CAD = 45^\circ + 60^\circ = 105^\circ$$

EXAMPLE 22 An equilateral triangle is inscribed in a circle of radius 6 cm. Find its side.

SOLUTION Let ABC be an equilateral triangle inscribed in a circle of radius 6 cm. Let O be the centre of the circle. Then,

$$OA = OB = OC = 6 \text{ cm.}$$

Let OD be perpendicular from O on side BC . Then, D is the mid-point of BC and OB and OC are bisectors of $\angle B$ and $\angle C$ respectively.

$$\therefore \angle OBD = 30^\circ$$

In ΔOBD , right angled at D , we have

$$\angle OBD = 30^\circ \text{ and } OB = 6 \text{ cm.}$$

$$\therefore \cos \angle OBD = \frac{BD}{OB}$$

$$\Rightarrow \cos 30^\circ = \frac{BD}{6}$$

$$\Rightarrow BD = 6 \cos 30^\circ = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3} \text{ cm}$$

$$\Rightarrow BC = 2BD = 2(3\sqrt{3}) = 6\sqrt{3} \text{ cm}$$

Hence, the side of the equilateral triangle is $6\sqrt{3}$ cm.

EXAMPLE 23 If each of α , β and γ is a positive acute angle such that

$$\sin(\alpha + \beta - \gamma) = \frac{1}{2}, \cos(\beta + \gamma - \alpha) = \frac{1}{2} \text{ and } \tan(\gamma + \alpha - \beta) = 1,$$

find the values of α , β and γ .

SOLUTION We have,

$$\sin(\alpha + \beta - \gamma) = \frac{1}{2}, \cos(\beta + \gamma - \alpha) = \frac{1}{2} \text{ and } \tan(\gamma + \alpha - \beta) = 1,$$

$$\Rightarrow \sin(\alpha + \beta - \gamma) = \sin 30^\circ, \cos(\beta + \gamma - \alpha) = \cos 60^\circ \text{ and } \tan(\gamma + \alpha - \beta) = \tan 45^\circ$$

$$\Rightarrow \alpha + \beta - \gamma = 30^\circ \quad \dots\text{(i)}$$

$$\beta + \gamma - \alpha = 60^\circ \quad \dots\text{(ii)}$$

$$\gamma + \alpha - \beta = 45^\circ \quad \dots\text{(iii)}$$

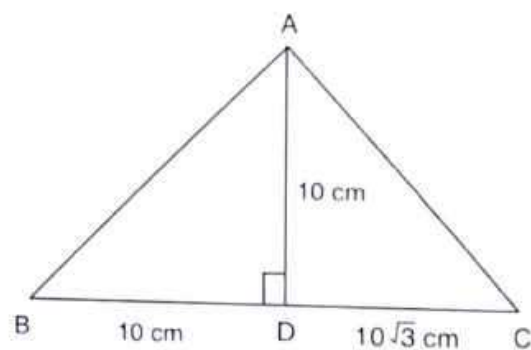


Fig. 10.44

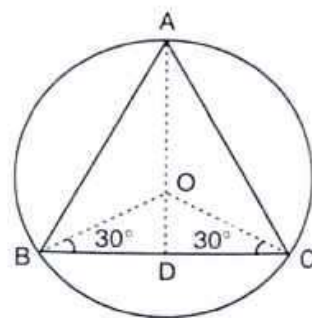


Fig. 10.45

Adding (i) to (ii) and (iii) respectively, we get

$$2\beta = 90^\circ \text{ and } 2\alpha = 75^\circ \Rightarrow \beta = 45^\circ \text{ and } \alpha = 37\frac{1}{2}^\circ$$

Putting the values of α and β in (i), we get,

$$37\frac{1}{2}^\circ + 45^\circ - \gamma = 30^\circ \Rightarrow \gamma = 52\frac{1}{2}^\circ$$

Hence, $\alpha = 37\frac{1}{2}^\circ$, $\beta = 45^\circ$ and $\gamma = 52\frac{1}{2}^\circ$

EXAMPLE 24 In an acute angled triangle ABC , if $\tan(A+B-C) = 1$ and $\sec(B+C-A) = 2$, find the value of A , B and C .

SOLUTION We have,

$$\begin{aligned} & \tan(A+B-C) = 1 \text{ and } \sec(B+C-A) = 2 \\ \Rightarrow & \tan(A+B-C) = \tan 45^\circ \text{ and } \sec(B+C-A) = \sec 60^\circ \\ \Rightarrow & A+B-C = 45^\circ \text{ and } B+C-A = 60^\circ \\ \Rightarrow & (A+B-C) + (B+C-A) = 45^\circ + 60^\circ \\ \Rightarrow & 2B = 105^\circ \\ \Rightarrow & B = 52\frac{1}{2}^\circ \end{aligned}$$

Putting $B = 52\frac{1}{2}^\circ$ in $B+C-A = 60^\circ$, we get

$$52\frac{1}{2}^\circ + C - A = 60^\circ \Rightarrow C - A = 7\frac{1}{2}^\circ \quad \dots(i)$$

Also, in ΔABC , we have

$$\begin{aligned} & A + B + C = 180^\circ \\ \Rightarrow & A + 52\frac{1}{2}^\circ + C = 180^\circ \quad \left[\because B = 52\frac{1}{2}^\circ \right] \\ \Rightarrow & C + A = 127\frac{1}{2}^\circ \quad \dots(ii) \end{aligned}$$

Adding and subtracting (i) and (ii), we get

$$2C = 135^\circ \text{ and } 2A = 120^\circ \Rightarrow C = 67\frac{1}{2}^\circ \text{ and } A = 60^\circ$$

Hence, $A = 60^\circ$, $B = 52\frac{1}{2}^\circ$ and $C = 67\frac{1}{2}^\circ$.

EXERCISE 10.2

LEVEL-1

Evaluate each of the following (1-19):

- $\sin 45^\circ \sin 30^\circ + \cos 45^\circ \cos 30^\circ$
- $\sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ$
- $\cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$
- $\sin^2 30^\circ + \sin^2 45^\circ + \sin^2 60^\circ + \sin^2 90^\circ$
- $\cos^2 30^\circ + \cos^2 45^\circ + \cos^2 60^\circ + \cos^2 90^\circ$
- $\tan^2 30^\circ + \tan^2 60^\circ + \tan^2 45^\circ$
- $2\sin^2 30^\circ - 3\cos^2 45^\circ + \tan^2 60^\circ$
- $\sin^2 30^\circ \cos^2 45^\circ + 4\tan^2 30^\circ + \frac{1}{2}\sin^2 90^\circ - 2\cos^2 90^\circ + \frac{1}{24}\cos^2 0^\circ$

9. $4(\sin^4 60^\circ + \cos^4 30^\circ) - 3(\tan^2 60^\circ - \tan^2 45^\circ) + 5\cos^2 45^\circ$
10. $(\operatorname{cosec}^2 45^\circ \sec^2 30^\circ)(\sin^2 30^\circ + 4\cot^2 45^\circ - \sec^2 60^\circ)$
11. $\operatorname{cosec}^3 30^\circ \cos 60^\circ \tan^3 45^\circ \sin^2 90^\circ \sec^2 45^\circ \cot 30^\circ$
12. $\cot^2 30^\circ - 2\cos^2 60^\circ - \frac{3}{4}\sec^2 45^\circ - 4\sec^2 30^\circ$
13. $(\cos 0^\circ + \sin 45^\circ + \sin 30^\circ)(\sin 90^\circ - \cos 45^\circ + \cos 60^\circ)$
14. $\frac{\sin 30^\circ - \sin 90^\circ + 2\cos 0^\circ}{\tan 30^\circ \tan 60^\circ}$
15. $\frac{4}{\cot^2 30^\circ} + \frac{1}{\sin^2 60^\circ} - \cos^2 45^\circ$
16. $4(\sin^4 30^\circ + \cos^2 60^\circ) - 3(\cos^2 45^\circ - \sin^2 90^\circ) - \sin^2 60^\circ$
17. $\frac{\tan^2 60^\circ + 4\cos^2 45^\circ + 3\sec^2 30^\circ + 5\cos^2 90^\circ}{\operatorname{cosec} 30^\circ + \sec 60^\circ - \cot^2 30^\circ}$
18. $\frac{\sin 30^\circ}{\sin 45^\circ} + \frac{\tan 45^\circ}{\sec 60^\circ} - \frac{\sin 60^\circ}{\cot 45^\circ} - \frac{\cos 30^\circ}{\sin 90^\circ}$
19. $\frac{\tan 45^\circ}{\operatorname{cosec} 30^\circ} + \frac{\sec 60^\circ}{\cot 45^\circ} - \frac{5\sin 90^\circ}{2\cos 0^\circ}$

Find the value of x in each of the following: (20-25)

20. $2\sin 3x = \sqrt{3}$
21. $2\sin \frac{x}{2} = 1$
22. $\sqrt{3}\sin x = \cos x$
23. $\tan x = \sin 45^\circ \cos 45^\circ + \sin 30^\circ$
24. $\sqrt{3}\tan 2x = \cos 60^\circ + \sin 45^\circ \cos 45^\circ$
25. $\cos 2x = \cos 60^\circ \cos 30^\circ + \sin 60^\circ \sin 30^\circ$

26. If $\theta = 30^\circ$, verify that:

- (i) $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$
- (ii) $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$
- (iii) $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$
- (iii) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

27. If $A = B = 60^\circ$, verify that

- (i) $\cos(A - B) = \cos A \cos B + \sin A \sin B$
- (ii) $\sin(A - B) = \sin A \cos B - \cos A \sin B$
- (iii) $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

28. If $A = 30^\circ$ and $B = 60^\circ$, verify that

- (i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$
- (ii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

29. If $\sin(A + B) = 1$ and $\cos(A - B) = 1$, $0^\circ < A + B \leq 90^\circ$, $A > B$ find A and B .

30. If $\tan(A - B) = \frac{1}{\sqrt{3}}$ and $\tan(A + B) = \sqrt{3}$, $0^\circ < A + B \leq 90^\circ$, $A > B$ find A and B .

[NCERT]

31. If $\sin(A - B) = \frac{1}{2}$ and $\cos(A + B) = \frac{1}{2}$, $0^\circ < A + B \leq 90^\circ$, $A < B$ find A and B .

[NCERT]

32. In a ΔABC right angled at B , $\angle A = \angle C$. Find the values of

(i) $\sin A \cos C + \cos A \sin C$ (ii) $\sin A \sin B + \cos A \cos B$

33. Find acute angles A and B , if $\sin(A + 2B) = \frac{\sqrt{3}}{2}$ and $\cos(A + 4B) = 0$, $A > B$.

34. In ΔPQR , right-angled at Q , $PQ = 3$ cm and $PR = 6$ cm. Determine $\angle P$ and $\angle R$.

[NCERT]

LEVEL-2

35. If $\sin(A - B) = \sin A \cos B - \cos A \sin B$ and $\cos(A - B) = \cos A \cos B + \sin A \sin B$, find the values of $\sin 15^\circ$ and $\cos 15^\circ$.

36. In a right triangle ABC , right angled at C , if $\angle B = 60^\circ$ and $AB = 15$ units. Find the remaining angles and sides.

37. If ΔABC is a right triangle such that $\angle C = 90^\circ$, $\angle A = 45^\circ$ and $BC = 7$ units. Find $\angle B$, AB and AC .

38. In a rectangle $ABCD$, $AB = 20$ cm, $\angle BAC = 60^\circ$, calculate side BC and diagonals AC and BD .

39. If A and B are acute angles such that $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$ and

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}, \text{ find } A + B.$$

40. Prove that $(\sqrt{3} + 1)(3 - \cot 30^\circ) = \tan^3 60^\circ - 2 \sin 60^\circ$.

[NCERT EXEMPLAR]

ANSWERS

- | | | | | |
|-------------------------------------|-------------------------------------|--|--------------------------------------|--------------------|
| 1. $\frac{\sqrt{3} + 1}{2\sqrt{2}}$ | 2. 1 | 3. $\frac{1 - \sqrt{3}}{2\sqrt{2}}$ | 4. $\frac{5}{2}$ | 5. $\frac{3}{2}$ |
| 6. $\frac{13}{3}$ | 7. 2 | 8. 2 | 9. 1 | 10. $\frac{2}{3}$ |
| 11. $8\sqrt{3}$ | 12. $\frac{-13}{3}$ | 13. $\frac{7}{4}$ | 14. $\frac{3}{2}$ | 15. $\frac{13}{6}$ |
| 16. 2 | 17. 9 | 18. $\frac{\sqrt{2} + 1 - 2\sqrt{3}}{2}$ | 19. 0 | 20. 20° |
| 21. 60° | 22. 30° | 23. 45° | 24. 15° | 25. 15° |
| 29. $A = B = 45^\circ$ | 30. $A = 45^\circ$, $B = 15^\circ$ | 31. $A = 45^\circ$, $B = 115^\circ$ | 32. (i) 1, (ii) $\frac{1}{\sqrt{2}}$ | |

$$33. A = 30^\circ, B = 15^\circ \quad 34. \angle P = 60^\circ, \angle R = 30^\circ \quad 35. \sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}, \cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$36. \angle A = 30^\circ, BC = 7.5 \text{ units and } AC = \frac{15}{2}\sqrt{3} \text{ units}$$

$$37. \angle B = 45^\circ, AC = 7, AB = 7\sqrt{2} \quad 38. BC = 20\sqrt{3} \text{ cm}, AC = 40 \text{ cm}, BD = 40 \text{ cm} \quad 39. 45^\circ$$

10.6 TRIGONOMETRIC RATIOS OF COMPLEMENTARY ANGLES

In this section, we will obtain the trigonometric ratios of complementary angles in terms of the trigonometric ratios of the given angles.

COMPLEMENTARY ANGLE Two angles are said to be complementary, if their sum is 90° .

It follows from the above definition that θ and $(90^\circ - \theta)$ are complementary angles for an acute angle θ .

THEOREM If θ is an acute angle, then prove that

$$\begin{aligned} \sin(90^\circ - \theta) &= \cos \theta, & \cos(90^\circ - \theta) &= \sin \theta, \\ \tan(90^\circ - \theta) &= \cot \theta, & \cot(90^\circ - \theta) &= \tan \theta, \\ \sec(90^\circ - \theta) &= \operatorname{cosec} \theta & \text{and, } \operatorname{cosec}(90^\circ - \theta) &= \sec \theta \end{aligned}$$

PROOF Consider a right triangle OPM , right angled at M as shown in Fig. 10.46.

Let $\angle MOP = \theta$, then $\angle OPM = (90^\circ - \theta)$

For the reference angle θ , we have

$$\left. \begin{aligned} \sin \theta &= \frac{PM}{OP}, \cos \theta = \frac{OM}{OP}, \tan \theta = \frac{PM}{OM} \\ \operatorname{cosec} \theta &= \frac{OP}{PM}, \cot \theta = \frac{OM}{PM} \text{ and, } \sec \theta = \frac{OP}{OM} \end{aligned} \right\} \dots(i)$$

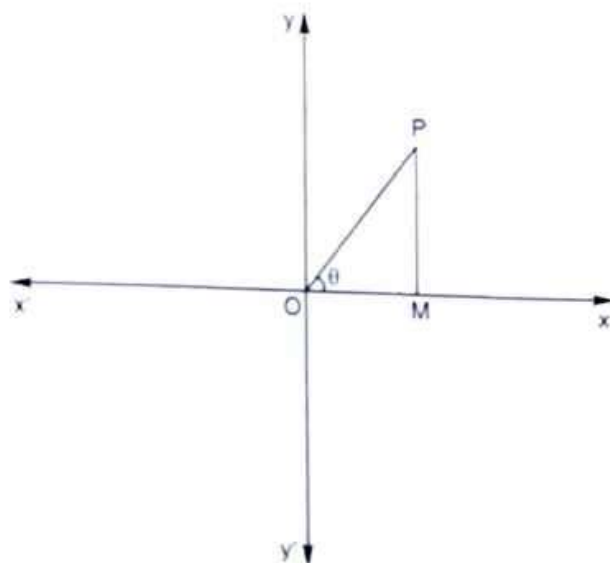


Fig. 10.46

For the reference angle $\angle OPM = (90^\circ - \theta)$, we have

Base = PM , Perpendicular = OM and, Hypotenuse = OP

$$\therefore \left. \begin{aligned} \sin(90^\circ - \theta) &= \frac{OM}{OP}, \cos(90^\circ - \theta) = \frac{PM}{OP}, \tan(90^\circ - \theta) = \frac{OM}{PM} \\ \operatorname{cosec}(90^\circ - \theta) &= \frac{OP}{OM}, \sec(90^\circ - \theta) = \frac{OP}{PM} \text{ and } \cot(90^\circ - \theta) = \frac{PM}{OM} \end{aligned} \right\} \dots(ii)$$

From (i) and (ii), we obtain

$$\begin{aligned} \sin(90^\circ - \theta) &= \cos \theta, \cos(90^\circ - \theta) = \sin \theta, \tan(90^\circ - \theta) = \cot \theta \\ \operatorname{cosec}(90^\circ - \theta) &= \sec \theta, \sec(90^\circ - \theta) = \operatorname{cosec} \theta \text{ and } \cot(90^\circ - \theta) = \tan \theta \end{aligned}$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I ON EVALUATING EXPRESSIONS INVOLVING TRIGONOMETRIC RATIOS OF COMPLEMENTARY ANGLES

EXAMPLE 1 Evaluate the following:

$$(i) \frac{\cos 37^\circ}{\sin 53^\circ} \quad (ii) \frac{\sin 41^\circ}{\cos 49^\circ} \quad (iii) \frac{\tan 54^\circ}{\cot 36^\circ} \quad (iv) \frac{\operatorname{cosec} 32^\circ}{\sec 58^\circ}$$

SOLUTION (i) We have,

$$\frac{\cos 37^\circ}{\sin 53^\circ} = \frac{\cos(90^\circ - 53^\circ)}{\sin 53^\circ} = \frac{\sin 53^\circ}{\sin 53^\circ} = 1 \quad [\because \cos(90^\circ - \theta) = \sin \theta]$$

(ii) We have,

$$\frac{\sin 41^\circ}{\cos 49^\circ} = \frac{\sin(90^\circ - 49^\circ)}{\cos 49^\circ} = \frac{\cos 49^\circ}{\cos 49^\circ} = 1 \quad [\because \sin(90^\circ - \theta) = \cos \theta]$$

(iii) We have,

$$\frac{\tan 54^\circ}{\cot 36^\circ} = \frac{\tan(90^\circ - 36^\circ)}{\cot 36^\circ} = \frac{\cot 36^\circ}{\cot 36^\circ} = 1 \quad [\because \tan(90^\circ - \theta) = \cot \theta]$$

(iv) We have,

$$\frac{\operatorname{cosec} 32^\circ}{\sec 58^\circ} = \frac{\operatorname{cosec}(90^\circ - 58^\circ)}{\sec 58^\circ} = \frac{\sec 58^\circ}{\sec 58^\circ} = 1 \quad [\because \operatorname{cosec}(90^\circ - \theta) = \sec \theta]$$

EXAMPLE 2 Evaluate the following:

$$(i) \sin 39^\circ - \cos 51^\circ \quad (ii) \operatorname{cosec} 25^\circ - \sec 65^\circ \quad (iii) \cot 34^\circ - \tan 56^\circ$$

$$(iv) \frac{\sin 36^\circ}{\cos 54^\circ} - \frac{\sin 54^\circ}{\cos 36^\circ} \quad (v) \cos^2 13^\circ - \sin^2 77^\circ$$

SOLUTION (i) We have,

$$\begin{aligned} \sin 39^\circ - \cos 51^\circ &= \sin(90^\circ - 51^\circ) - \cos 51^\circ \\ &= \cos 51^\circ - \cos 51^\circ = 0 \quad [\because \sin(90^\circ - \theta) = \cos \theta] \end{aligned}$$

(ii) We have,

$$\begin{aligned} \operatorname{cosec} 25^\circ - \sec 65^\circ &= \operatorname{cosec}(90^\circ - 65^\circ) - \sec 65^\circ \\ &= \sec 65^\circ - \sec 65^\circ = 0 \quad [\because \operatorname{cosec}(90^\circ - \theta) = \sec \theta] \end{aligned}$$

(iii) We have,

$$\begin{aligned} \cot 34^\circ - \tan 56^\circ &= \cot(90^\circ - 56^\circ) - \tan 56^\circ \\ &= \tan 56^\circ - \tan 56^\circ = 0 \quad [\because \cot(90^\circ - \theta) = \tan \theta] \end{aligned}$$

(iv) We have,

$$\begin{aligned} \frac{\sin 36^\circ}{\cos 54^\circ} - \frac{\sin 54^\circ}{\cos 36^\circ} &= \frac{\sin (90^\circ - 54^\circ)}{\cos 54^\circ} - \frac{\sin (90^\circ - 36^\circ)}{\cos 36^\circ} \\ &= \frac{\cos 54^\circ}{\cos 54^\circ} - \frac{\cos 36^\circ}{\cos 36^\circ} = 1 - 1 = 0 \end{aligned} \quad \left[\begin{array}{l} \because \sin(90^\circ - \theta) = \cos \theta, \\ \cos(90^\circ - \theta) = \sin \theta \end{array} \right]$$

(v) We have,

$$\begin{aligned} \cos^2 13^\circ - \sin^2 77^\circ &= \cos^2 (90^\circ - 77^\circ) - \sin^2 77^\circ \\ &= \sin^2 77^\circ - \sin^2 77^\circ = 0 \end{aligned} \quad \left[\because \cos(90^\circ - \theta) = \sin \theta \right]$$

EXAMPLE 1 Evaluate the following:

- (i) $\frac{\cos 80^\circ}{\sin 10^\circ} + \cos 59^\circ \operatorname{cosec} 31^\circ$ (ii) $\frac{\cot 54^\circ}{\tan 36^\circ} + \frac{\tan 20^\circ}{\cot 70^\circ} - 2$
- (iii) $\frac{2 \tan 53^\circ}{\cot 37^\circ} - \frac{\cot 80^\circ}{\tan 10^\circ}$ (iv) $\sec 50^\circ \sin 40^\circ + \cos 40^\circ \operatorname{cosec} 50^\circ$
- (v) $\sec 70^\circ \sin 20^\circ - \cos 20^\circ \operatorname{cosec} 70^\circ$

SOLUTION (i) We have,

$$\begin{aligned} &\frac{\cos 80^\circ}{\sin 10^\circ} + \cos 59^\circ \operatorname{cosec} 31^\circ \\ &= \frac{\cos (90^\circ - 10^\circ)}{\sin 10^\circ} + \cos 59^\circ \operatorname{cosec} (90^\circ - 59^\circ) \\ &= \frac{\sin 10^\circ}{\sin 10^\circ} + \cos 59^\circ \sec 59^\circ = \frac{\sin 10^\circ}{\sin 10^\circ} + \frac{\cos 59^\circ}{\cos 59^\circ} = 1 + 1 = 2 \end{aligned}$$

(ii) We have,

$$\begin{aligned} &\frac{\cot 54^\circ}{\tan 36^\circ} + \frac{\tan 20^\circ}{\cot 70^\circ} - 2 \\ &= \frac{\cot (90^\circ - 36^\circ)}{\tan 36^\circ} + \frac{\tan 20^\circ}{\cot (90^\circ - 20^\circ)} - 2 = \frac{\tan 36^\circ}{\tan 36^\circ} + \frac{\tan 20^\circ}{\tan 20^\circ} - 2 = 1 + 1 - 2 = 0 \end{aligned}$$

(iii) We have,

$$\begin{aligned} &\frac{2 \tan 53^\circ}{\cot 37^\circ} - \frac{\cot 80^\circ}{\tan 10^\circ} \\ &= \frac{2 \tan 53^\circ}{\tan (90^\circ - 37^\circ)} - \frac{\cot (90^\circ - 10^\circ)}{\tan 10^\circ} = \frac{2 \tan 53^\circ}{\tan 53^\circ} - \frac{\tan 10^\circ}{\tan 10^\circ} = 2 - 1 = 1 \end{aligned}$$

(iv) We have,

$$\begin{aligned} &\sec 50^\circ \sin 40^\circ + \cos 40^\circ \operatorname{cosec} 50^\circ \\ &= \sec (90^\circ - 40^\circ) \sin 40^\circ + \cos 40^\circ \operatorname{cosec} (90^\circ - 40^\circ) \\ &= \operatorname{cosec} 40^\circ \sin 40^\circ + \cos 40^\circ \sec 40^\circ = \frac{\sin 40^\circ}{\sin 40^\circ} + \frac{\cos 40^\circ}{\cos 40^\circ} = 1 + 1 = 2 \end{aligned}$$

(v) We have,

$$\sec 70^\circ \sin 20^\circ - \cos 20^\circ \operatorname{cosec} 70^\circ$$

$$\begin{aligned}
 &= \sec(90^\circ - 20^\circ) \sin 20^\circ - \cos 20^\circ \operatorname{cosec}(90^\circ - 20^\circ) \\
 &= \operatorname{cosec} 20^\circ \sin 20^\circ - \cos 20^\circ \sec 20^\circ \\
 &= \frac{\sin 20^\circ}{\sin 20^\circ} - \frac{\cos 20^\circ}{\cos 20^\circ} = 1 - 1 = 0
 \end{aligned}
 \quad \left[\begin{array}{l} \because \operatorname{cosec}(90^\circ - \theta) = \sec \theta \\ \text{and, } \sec(90^\circ - \theta) = \operatorname{cosec} \theta \end{array} \right]$$

EXAMPLE 4 Prove that:

$$\textcircled{1} \sin 35^\circ \sin 55^\circ - \cos 35^\circ \cos 55^\circ = 0 \quad \text{(ii) } \frac{\cos 70^\circ}{\sin 20^\circ} + \frac{\cos 59^\circ}{\sin 31^\circ} - 8 \sin^2 30^\circ = 0$$

SOLUTION (i) We have,

$$\begin{aligned}
 \text{LHS} &= \sin 35^\circ \sin 55^\circ - \cos 35^\circ \cos 55^\circ \\
 \Rightarrow \text{LHS} &= \sin(90^\circ - 55^\circ) \sin(90^\circ - 35^\circ) - \cos 35^\circ \cos 55^\circ \\
 \Rightarrow \text{LHS} &= \cos 55^\circ \cos 35^\circ - \cos 35^\circ \cos 55^\circ && [\because \sin(90^\circ - \theta) = \cos \theta] \\
 \Rightarrow \text{LHS} &= 0 = \text{RHS}
 \end{aligned}$$

(ii) We have,

$$\begin{aligned}
 \text{LHS} &= \frac{\cos 70^\circ}{\sin 20^\circ} + \frac{\cos 59^\circ}{\sin 31^\circ} - 8 \sin^2 30^\circ \\
 \Rightarrow \text{LHS} &= \frac{\cos(90^\circ - 20^\circ)}{\sin 20^\circ} + \frac{\cos(90^\circ - 31^\circ)}{\sin 31^\circ} - 8 \sin^2 30^\circ \\
 \Rightarrow \text{LHS} &= \frac{\sin 20^\circ}{\sin 20^\circ} + \frac{\sin 31^\circ}{\sin 31^\circ} - 8 \sin^2 30^\circ && [\because \cos(90^\circ - \theta) = \sin \theta] \\
 \Rightarrow &= 1 + 1 - 8 \left(\frac{1}{2} \right)^2 = 2 - 2 = 0 = \text{RHS} && \left[\because \sin 30^\circ = \frac{1}{2} \right]
 \end{aligned}$$

EXAMPLE 5 Express each of the following in terms of trigonometric ratios of angles between 0° and 45° :

$$\begin{array}{lll}
 \text{(i) } \sin 85^\circ + \operatorname{cosec} 85^\circ & \text{(ii) } \tan 68^\circ + \sec 68^\circ & \text{(iii) } \operatorname{cosec} 69^\circ + \cot 69^\circ \\
 \text{(iv) } \sin 81^\circ + \tan 81^\circ & \text{(v) } \sin 72^\circ + \cot 72^\circ &
 \end{array}$$

SOLUTION (i) We have,

$$\begin{aligned}
 &\sin 85^\circ + \operatorname{cosec} 85^\circ \\
 &= \sin(90^\circ - 5^\circ) + \operatorname{cosec}(90^\circ - 5^\circ) \\
 &= \cos 5^\circ + \sec 5^\circ && [\because \sin(90^\circ - \theta) = \cos \theta \text{ and } \operatorname{cosec}(90^\circ - \theta) = \sec \theta]
 \end{aligned}$$

(ii) We have,

$$\begin{aligned}
 &\tan 68^\circ + \sec 68^\circ \\
 &= \tan(90^\circ - 22^\circ) + \sec(90^\circ - 22^\circ) \\
 &= \cot 22^\circ + \operatorname{cosec} 22^\circ && [\because \tan(90^\circ - \theta) = \cot \theta, \sec(90^\circ - \theta) = \operatorname{cosec} \theta]
 \end{aligned}$$

(iii) We have,

$$\begin{aligned}
 &\operatorname{cosec} 69^\circ + \cot 69^\circ \\
 &= \operatorname{cosec}(90^\circ - 21^\circ) + \cot(90^\circ - 21^\circ) \\
 &= \sec 21^\circ + \tan 21^\circ && [\because \operatorname{cosec}(90^\circ - \theta) = \sec \theta \text{ and } \cot(90^\circ - \theta) = \tan \theta]
 \end{aligned}$$

(iv) We have,

$$\begin{aligned} & \sin 81^\circ + \tan 81^\circ \\ &= \sin (90^\circ - 9^\circ) + \tan (90^\circ - 9^\circ) \\ &= \cos 9^\circ + \cot 9^\circ \quad [\because \sin (90^\circ - \theta) = \cos \theta \text{ and, } \tan (90^\circ - \theta) = \cot \theta] \end{aligned}$$

(v) We have,

$$\begin{aligned} & \sin 72^\circ + \cot 72^\circ \\ &= \sin (90^\circ - 18^\circ) + \cot (90^\circ - 18^\circ) \\ &= \cos 18^\circ + \tan 18^\circ \quad [\because \sin (90^\circ - \theta) = \cos \theta \text{ and, } \tan (90^\circ - \theta) = \cot \theta] \end{aligned}$$

EXAMPLE 6 Evaluate the following:

$$\begin{aligned} \text{(i)} \quad & \left(\frac{\sin 35^\circ}{\cos 55^\circ} \right)^2 + \left(\frac{\cos 55^\circ}{\sin 35^\circ} \right)^2 - 2 \cos 60^\circ & \text{(ii)} \quad \frac{2 \cos 67^\circ}{\sin 23^\circ} - \frac{\tan 40^\circ}{\cot 50^\circ} - \cos 0^\circ \\ \text{(iii)} \quad & \left(\frac{\sin 47^\circ}{\cos 43^\circ} \right)^2 + \left(\frac{\cos 43^\circ}{\sin 47^\circ} \right)^2 - 4 \cos^2 45^\circ \end{aligned}$$

SOLUTION (i) We have,

$$\begin{aligned} & \left(\frac{\sin 35^\circ}{\cos 55^\circ} \right)^2 + \left(\frac{\cos 55^\circ}{\sin 35^\circ} \right)^2 - 2 \cos 60^\circ \\ &= \left(\frac{\sin (90^\circ - 55^\circ)}{\cos 55^\circ} \right)^2 + \left(\frac{\cos (90^\circ - 35^\circ)}{\sin 35^\circ} \right)^2 - 2 \cos 60^\circ \\ &= \left(\frac{\cos 55^\circ}{\cos 55^\circ} \right)^2 + \left(\frac{\sin 35^\circ}{\sin 35^\circ} \right)^2 - 2 \cos 60^\circ \quad \left[\because \cos (90^\circ - \theta) = \sin \theta \right. \\ & \quad \left. \text{and, } \sin (90^\circ - \theta) = \cos \theta \right] \\ &= 1^2 + 1^2 - 2 \times \frac{1}{2} = 1 \quad \left[\because \cos 60^\circ = \frac{1}{2} \right] \end{aligned}$$

(ii) We have,

$$\begin{aligned} & \frac{2 \cos 67^\circ}{\sin 23^\circ} - \frac{\tan 40^\circ}{\cot 50^\circ} - \cos 0^\circ \\ &= \frac{2 \cos (90^\circ - 23^\circ)}{\sin 23^\circ} - \frac{\tan (90^\circ - 50^\circ)}{\cot 50^\circ} - \cos 0^\circ \\ &= \frac{2 \sin 23^\circ}{\sin 23^\circ} - \frac{\cot 50^\circ}{\cot 50^\circ} - \cos 0^\circ \quad \left[\because \cos (90^\circ - \theta) = \sin \theta \right. \\ & \quad \left. \text{and, } \tan (90^\circ - \theta) = \cot \theta \right] \\ &= 2 \times 1 - 1 - 1 = 0 \quad [\because \cos 0^\circ = 1] \end{aligned}$$

(iii) We have,

$$\begin{aligned} & \left(\frac{\sin 47^\circ}{\cos 43^\circ} \right)^2 + \left(\frac{\cos 43^\circ}{\sin 47^\circ} \right)^2 - 4 \cos^2 45^\circ \\ &= \left(\frac{\sin (90^\circ - 45^\circ)}{\cos 43^\circ} \right)^2 + \left(\frac{\cos (90^\circ - 47^\circ)}{\sin 47^\circ} \right)^2 - 4 \cos^2 45^\circ \end{aligned}$$

$$= \left(\frac{\cos 43^\circ}{\cos 43^\circ} \right)^2 + \left(\frac{\sin 47^\circ}{\sin 47^\circ} \right)^2 - 4 \cos^2 45^\circ$$

$$\left[\begin{array}{l} \because \sin(90^\circ - \theta) = \cos \theta \\ \text{and, } \cos(90^\circ - \theta) = \sin \theta \end{array} \right]$$

$$= 1^2 + 1^2 - 4 \left(\frac{1}{\sqrt{2}} \right)^2 = 0$$

$$\left[\because \cos 45^\circ = \frac{1}{\sqrt{2}} \right]$$

EXAMPLE 7 Evaluate each of the following:

(i) $\cot 12^\circ \cot 38^\circ \cot 52^\circ \cot 60^\circ \cot 78^\circ$

[CBSE 2001 C]

(ii) $\tan 5^\circ \tan 25^\circ \tan 30^\circ \tan 65^\circ \tan 85^\circ$

[CBSE 2001 C]

SOLUTION (i) Grouping terms in such a way that the angles involved are complementary i.e., their sum is 90° , we have

$$\cot 12^\circ \cot 38^\circ \cot 52^\circ \cot 60^\circ \cot 78^\circ$$

$$= (\cot 12^\circ \cot 78^\circ) (\cot 38^\circ \cot 52^\circ) \cot 60^\circ$$

$$= (\cot 12^\circ \tan 12^\circ) (\cot 38^\circ \tan 38^\circ) (\cot 60^\circ)$$

$$\left[\because \cot 78^\circ = \cot(90^\circ - 12^\circ) = \tan 12^\circ \cot 52^\circ = \cot(50^\circ - 38^\circ) = \tan 38^\circ \right]$$

$$= 1 \times 1 \times \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

(ii) We have,

$$\tan 5^\circ \tan 25^\circ \tan 30^\circ \tan 65^\circ \tan 85^\circ$$

$$= (\tan 5^\circ \tan 85^\circ) (\tan 25^\circ \tan 65^\circ) \tan 30^\circ$$

$$= (\tan 5^\circ \cot 5^\circ) (\tan 25^\circ \cot 25^\circ) \tan 30^\circ$$

$$= 1 \times 1 \times \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\left[\begin{array}{l} \because \tan 85^\circ = \tan(90^\circ - 5^\circ) = \cot 5^\circ \\ \tan 65^\circ = \tan(90^\circ - 25^\circ) = \cot 25^\circ \end{array} \right]$$

EXAMPLE 8 If A, B, C are the interior angles of a triangle ABC , prove that: $\tan \frac{B+C}{2} = \cot \frac{A}{2}$.

SOLUTION In ΔABC , we have

$$A + B + C = 180^\circ$$

$$\Rightarrow B + C = 180^\circ - A$$

$$\Rightarrow \frac{B+C}{2} = 90^\circ - \frac{A}{2}$$

$$\Rightarrow \tan \left(\frac{B+C}{2} \right) = \tan \left(90^\circ - \frac{A}{2} \right) \Rightarrow \tan \left(\frac{B+C}{2} \right) = \cot \frac{A}{2}$$

Type II ON FINDING THE UNKNOWN VARIABLE

EXAMPLE 9 Find θ , if $\sin(\theta + 36^\circ) = \cos \theta$, where $\theta + 36^\circ$ is an acute angle

SOLUTION We have,

$$\sin(\theta + 36^\circ) = \cos \theta$$

$$\Rightarrow \cos[90^\circ - (\theta + 36^\circ)] = \cos \theta$$

$$\Rightarrow 90^\circ - (\theta + 36^\circ) = \theta \Rightarrow 2\theta = 54^\circ \Rightarrow \theta = 27^\circ$$

EXAMPLE 10 If $\tan 2\theta = \cot(\theta + 6^\circ)$, where 2θ and $\theta + 6^\circ$ are acute angles, find the value of θ .

SOLUTION We have,

$$\tan 2\theta = \cot(\theta + 6^\circ)$$

$$\Rightarrow \cot(90^\circ - 2\theta) = \cot(\theta + 6^\circ)$$

$$\Rightarrow 90^\circ - 2\theta = \theta + 6^\circ \Rightarrow 3\theta = 84^\circ \Rightarrow \theta = 28^\circ$$

EXAMPLE 11 If $\sin 5\theta = \cos 4\theta$, where 5θ and 4θ are acute angles, find the value of θ .

SOLUTION We have,

$$\sin 5\theta = \cos 4\theta$$

$$\Rightarrow \sin 5\theta = \sin(90^\circ - 4\theta)$$

$$\Rightarrow 5\theta = 90^\circ - 4\theta$$

$$\Rightarrow 9\theta = 90^\circ$$

$$\Rightarrow \theta = 10^\circ$$

EXAMPLE 12 If $\tan 2A = \cot(A - 18^\circ)$, where $2A$ is an acute angle, find the value of A .

SOLUTION We have,

$$\tan 2A = \cot(A - 18^\circ)$$

$$\Rightarrow \tan 2A = \tan\{90^\circ - (A - 18^\circ)\}$$

$$\Rightarrow \tan 2A = \tan(108^\circ - A)$$

$$\Rightarrow 2A = 108^\circ - A \Rightarrow 3A = 108^\circ \Rightarrow A = 36^\circ$$

[NCERT]

EXAMPLE 13 If $\tan A = \cot B$, Prove that $A + B = 90^\circ$

SOLUTION We have,

$$\tan A = \cot B$$

$$\Rightarrow \tan A = \tan(90^\circ - B) \Rightarrow A = 90^\circ - B \Rightarrow A + B = 90^\circ$$

[NCERT]

EXAMPLE 14 If $\sec 5A = \operatorname{cosec}(A + 36^\circ)$, where $5A$ is an acute angle, find the value of A

SOLUTION We have,

$$\sec 5A = \operatorname{cosec}(A + 36^\circ)$$

$$\Rightarrow \sec 5A = \sec(90^\circ - (A + 36^\circ))$$

$$\Rightarrow \sec 5A = \sec(54^\circ - A)$$

$$\Rightarrow 5A = 54^\circ - A \Rightarrow 6A = 54^\circ \Rightarrow A = 9^\circ$$

EXAMPLE 15 If $\sec 4A = \operatorname{cosec}(A - 20^\circ)$, where $4A$ is an acute angle, find the value of A .

SOLUTION We have,

$$\sec 4A = \operatorname{cosec}(A - 20^\circ)$$

$$\Rightarrow \sec 4A = \sec\{90^\circ - (A - 20^\circ)\}$$

$$\Rightarrow \sec 4A = \sec(110^\circ - A)$$

$$\Rightarrow 4A = 110^\circ - A \Rightarrow 5A = 110^\circ \Rightarrow A = 22^\circ$$

[NCERT, CBSE 2008]

LEVEL-2

EXAMPLE 16: Evaluate each of the following:

$$(i) \cos(40^\circ - 0) - \sin(50^\circ + 0) + \frac{\cos^2 40^\circ + \cos^2 50^\circ}{\sin^2 40^\circ + \sin^2 50^\circ} \quad [\text{CBSE 2002}]$$

$$(ii) \frac{\cos 70^\circ}{\sin 20^\circ} + \frac{\cos 55^\circ \operatorname{cosec} 35^\circ}{\tan 5^\circ \tan 25^\circ \tan 45^\circ \tan 65^\circ \tan 85^\circ} \quad [\text{CBSE 2002}]$$

$$(iii) 2 \left(\frac{\cos 58^\circ}{\sin 32^\circ} \right) - \sqrt{3} \left(\frac{\cos 38^\circ \operatorname{cosec} 52^\circ}{\tan 15^\circ \tan 60^\circ \tan 75^\circ} \right) \quad [\text{CBSE 2008}]$$

SOLUTION (i) We have,

$$\begin{aligned} & \cos(40^\circ - 0) - \sin(50^\circ + 0) + \frac{\cos^2 40^\circ + \cos^2 50^\circ}{\sin^2 40^\circ + \sin^2 50^\circ} \\ &= \sin[90^\circ - (40^\circ - 0)] - \sin(50^\circ + 0) + \frac{\cos^2 40^\circ + \cos^2(90^\circ - 40^\circ)}{\sin^2 40^\circ + \sin^2(90^\circ - 40^\circ)} \\ & \quad [\because \cos \alpha = \sin(90^\circ - \alpha)] \\ &= \sin(50^\circ + 0) - \sin(50^\circ + 0) + \frac{\cos^2 40^\circ + \sin^2 40^\circ}{\sin^2 40^\circ + \cos^2 40^\circ} = 0 + 1 = 1 \end{aligned}$$

(ii) We have,

$$\begin{aligned} & \frac{\cos 70^\circ}{\sin 20^\circ} + \frac{\cos 55^\circ \operatorname{cosec} 35^\circ}{\tan 5^\circ \tan 25^\circ \tan 45^\circ \tan 65^\circ \tan 85^\circ} \\ &= \frac{\cos(90^\circ - 20^\circ)}{\sin 20^\circ} + \frac{\cos 55^\circ \operatorname{cosec}(90^\circ - 55^\circ)}{\tan 5^\circ \tan 25^\circ \tan 45^\circ \tan(90^\circ - 25^\circ) \tan(90^\circ - 5^\circ)} \\ &= \frac{\sin 20^\circ}{\sin 20^\circ} + \frac{\cos 55^\circ \sec 55^\circ}{\tan 5^\circ \tan 25^\circ \tan 45^\circ \cot 25^\circ \cot 5^\circ} \\ &= 1 + \frac{1}{\tan 45^\circ} = 1 + 1 = 2 \quad [\because \tan 5^\circ \cot 5^\circ = 1 \text{ and } \tan 25^\circ \cot 25^\circ = 1] \end{aligned}$$

(iii) We have,

$$\begin{aligned} & 2 \left(\frac{\cos 58^\circ}{\sin 32^\circ} \right) - \sqrt{3} \left(\frac{\cos 38^\circ \operatorname{cosec} 52^\circ}{\tan 15^\circ \tan 60^\circ \tan 75^\circ} \right) \\ &= 2 \left\{ \frac{\cos(90^\circ - 32^\circ)}{\sin 32^\circ} \right\} - \sqrt{3} \left\{ \frac{\cos 38^\circ \operatorname{cosec}(90^\circ - 38^\circ)}{\tan 15^\circ \tan 60^\circ \tan(90^\circ - 15^\circ)} \right\} \\ &= 2 \left(\frac{\sin 32^\circ}{\sin 32^\circ} \right) - \sqrt{3} \left\{ \frac{\cos 38^\circ \sec 38^\circ}{\tan 15^\circ \times \sqrt{3} \times \cot 15^\circ} \right\} \\ &= 2 - \sqrt{3} \left\{ \frac{\cos 38^\circ \times \frac{1}{\cos 38^\circ}}{\tan 15^\circ \times \sqrt{3} \times \frac{1}{\tan 15^\circ}} \right\} = 2 - \frac{\sqrt{3}}{\sqrt{3}} = 2 - 1 = 1 \end{aligned}$$

EXAMPLE 17 Prove that:

(i) $\tan 10^\circ \tan 15^\circ \tan 75^\circ \tan 80^\circ = 1$

(ii) $\tan 1^\circ \tan 2^\circ \tan 3^\circ \dots \tan 89^\circ = 1$

(iii) $\cos 1^\circ \cos 2^\circ \cos 3^\circ \dots \cos 180^\circ = 0$

SOLUTION (i) We have,

$$\begin{aligned} \text{LHS} &= \tan 10^\circ \tan 15^\circ \tan 75^\circ \tan 80^\circ \\ &= \tan (90^\circ - 80^\circ) \tan (90^\circ - 75^\circ) \tan 75^\circ \tan 80^\circ \\ &= \cot 80^\circ \cot 75^\circ \tan 75^\circ \tan 80^\circ && [\because \tan (90^\circ - \theta) = \cot \theta] \\ &= (\cot 80^\circ \tan 80^\circ) (\cot 75^\circ \tan 75^\circ) = 1 \times 1 = 1 = \text{RHS} && [\because \cot \theta \cdot \tan \theta = 1] \end{aligned}$$

(ii) We have,

$$\begin{aligned} \text{LHS} &= \tan 1^\circ \tan 2^\circ \tan 3^\circ \dots \tan 89^\circ \\ &= \tan (90^\circ - 89^\circ) \tan (90^\circ - 88^\circ) \tan (90^\circ - 87^\circ) \dots \tan 87^\circ \tan 88^\circ \tan 89^\circ \\ &= \cot 89^\circ \cot 88^\circ \cot 87^\circ \dots \tan 87^\circ \tan 88^\circ \tan 89^\circ \\ &= (\cot 89^\circ \tan 89^\circ) (\cot 88^\circ \tan 88^\circ) (\cot 87^\circ \tan 87^\circ) \dots (\cot 44^\circ \tan 44^\circ) \cdot \tan 45^\circ \\ &= 1 \times 1 \times 1 \dots \times 1 = 1 = \text{RHS} && [\because \cot \theta \tan \theta = 1 \text{ and } \tan 45^\circ = 1] \end{aligned}$$

(iii) We have,

$$\begin{aligned} \text{LHS} &= \cos 1^\circ \cos 2^\circ \cos 3^\circ \dots \cos 180^\circ \\ &= \cos 1^\circ \cos 2^\circ \cos 3^\circ \dots \cos 89^\circ \cos 90^\circ \cos 91^\circ \dots \cos 180^\circ \\ &= \cos 1^\circ \cos 2^\circ \cos 3^\circ \dots \times \cos 89^\circ \times 0 \times \cos 91^\circ \times \dots \cos 180^\circ = 0 = \text{RHS} \\ &&& [\because \cos 90^\circ = 0] \end{aligned}$$

EXAMPLE 18 If $A + B = 90^\circ$, prove that

$$\sqrt{\frac{\tan A \tan B + \tan A \cot B}{\sin A \sec B} - \frac{\sin^2 B}{\cos^2 A}} = \tan A$$

SOLUTION We have,

$$A + B = 90^\circ \Rightarrow B = 90^\circ - A$$

$$\begin{aligned} \therefore \text{LHS} &= \sqrt{\frac{\tan A \tan B + \tan A \cot B}{\sin A \sec B} - \frac{\sin^2 B}{\cos^2 A}} \\ \Rightarrow \text{LHS} &= \sqrt{\frac{\tan A \tan (90^\circ - A) + \tan A \cot (90^\circ - A)}{\sin A \sec (90^\circ - A)} - \frac{\sin^2 (90^\circ - A)}{\cos^2 A}} \\ \Rightarrow \text{LHS} &= \sqrt{\frac{\tan A \cot A + \tan A \tan A}{\sin A \operatorname{cosec} A} - \frac{\cos^2 A}{\cos^2 A}} \\ \Rightarrow \text{LHS} &= \sqrt{1 + \tan^2 A - 1} = \sqrt{\tan^2 A} = \tan A = \text{RHS} \end{aligned}$$

LEVEL-1

EXERCISE 10.3

1. Evaluate the following:

(i) $\frac{\sin 20^\circ}{\cos 70^\circ}$

(ii) $\frac{\cos 19^\circ}{\sin 71^\circ}$

(iii) $\frac{\sin 21^\circ}{\cos 69^\circ}$

$$(iv) \frac{\tan 10^\circ}{\cot 80^\circ} \quad (v) \frac{\sec 11^\circ}{\operatorname{cosec} 79^\circ}$$

2. Evaluate the following:

$$(i) \left(\frac{\sin 49^\circ}{\cos 41^\circ} \right)^2 + \left(\frac{\cos 41^\circ}{\sin 49^\circ} \right)^2$$

$$(ii) \cos 48^\circ - \sin 42^\circ$$

$$(iii) \frac{\cot 40^\circ}{\tan 50^\circ} - \frac{1}{2} \left(\frac{\cos 35^\circ}{\sin 55^\circ} \right)$$

$$(iv) \left(\frac{\sin 27^\circ}{\cos 63^\circ} \right)^2 - \left(\frac{\cos 63^\circ}{\sin 27^\circ} \right)^2$$

$$(v) \frac{\tan 35^\circ}{\cot 55^\circ} + \frac{\cot 78^\circ}{\tan 12^\circ} - 1$$

$$(vi) \frac{\sec 70^\circ}{\operatorname{cosec} 20^\circ} + \frac{\sin 59^\circ}{\cos 31^\circ}$$

$$(vii) \operatorname{cosec} 31^\circ - \sec 59^\circ$$

$$(viii) (\sin 72^\circ + \cos 18^\circ)(\sin 72^\circ - \cos 18^\circ)$$

$$(ix) \sin 35^\circ \sin 55^\circ - \cos 35^\circ \cos 55^\circ$$

$$(x) \tan 48^\circ \tan 23^\circ \tan 42^\circ \tan 67^\circ$$

$$(xi) \sec 50^\circ \sin 40^\circ + \cos 40^\circ \operatorname{cosec} 50^\circ$$

3. Express each one of the following in terms of trigonometric ratios of angles lying between 0° and 45°

$$(i) \sin 59^\circ + \cos 56^\circ \quad (ii) \tan 65^\circ + \cot 49^\circ \quad (iii) \sec 76^\circ + \operatorname{cosec} 52^\circ$$

$$(iv) \cos 78^\circ + \sec 78^\circ \quad (v) \operatorname{cosec} 54^\circ + \sin 72^\circ \quad (vi) \cot 85^\circ + \cos 75^\circ \quad \text{[NCERT]}$$

$$(vii) \sin 67^\circ + \cos 75^\circ \quad \text{[NCERT]}$$

4. Express $\cos 75^\circ + \cot 75^\circ$ in terms of angles between 0° and 30° .

5. If $\sin 3A = \cos (A - 26^\circ)$, where $3A$ is an acute angle, find the value of A . [NCERT]

6. If A, B, C , are the interior angles of a triangle ABC , prove that

$$(i) \tan \left(\frac{C+A}{2} \right) = \cot \frac{B}{2} \quad (ii) \sin \left(\frac{B+C}{2} \right) = \cos \frac{A}{2} \quad \text{[NCERT]}$$

7. Prove that:

$$(i) \tan 20^\circ \tan 35^\circ \tan 45^\circ \tan 55^\circ \tan 70^\circ = 1$$

$$(ii) \sin 48^\circ \sec 42^\circ + \cos 48^\circ \operatorname{cosec} 42^\circ = 2$$

$$(iii) \frac{\sin 70^\circ}{\cos 20^\circ} + \frac{\operatorname{cosec} 20^\circ}{\sec 70^\circ} - 2 \cos 70^\circ \operatorname{cosec} 20^\circ = 0$$

$$(iv) \frac{\cos 80^\circ}{\sin 10^\circ} + \cos 59^\circ \operatorname{cosec} 31^\circ = 2$$

8. Prove the following:

$$(i) \sin \theta \sin(90^\circ - \theta) - \cos \theta \cos(90^\circ - \theta) = 0$$

$$(ii) \frac{\cos(90^\circ - \theta) \sec(90^\circ - \theta) \tan \theta}{\operatorname{cosec}(90^\circ - \theta) \sin(90^\circ - \theta) \cot(90^\circ - \theta)} + \frac{\tan(90^\circ - \theta)}{\cot \theta} = 2$$

$$(iii) \frac{\tan(90^\circ - A) \cot A}{\operatorname{cosec}^2 A} - \cos^2 A = 0$$

$$(iv) \frac{\cos(90^\circ - A) \sin(90^\circ - A)}{\tan(90^\circ - A)} = \sin^2 A$$

$$(v) \sin(50^\circ + 0) - \cos(40^\circ - 0) + \tan 1^\circ \tan 10^\circ \tan 20^\circ \tan 70^\circ \tan 80^\circ \tan 89^\circ = 1$$

9. Evaluate:

[CBSE 2002]

$$(i) \frac{2}{3} (\cos^4 30^\circ - \sin^4 45^\circ) - 3 (\sin^2 60^\circ - \sec^2 45^\circ) + \frac{1}{4} \cot^2 30^\circ$$

[CBSE 2001 C]

$$(ii) 4 (\sin^4 30^\circ + \cos^4 60^\circ) - \frac{2}{3} (\sin^2 60^\circ - \cos^2 45^\circ) + \frac{1}{2} \tan^2 60^\circ$$

[CBSE 2001 C]

$$(iii) \frac{\sin 50^\circ}{\cos 40^\circ} + \frac{\operatorname{cosec} 40^\circ}{\sec 50^\circ} - 4 \cos 50^\circ \operatorname{cosec} 40^\circ$$

[CBSE 2001]

$$(iv) \tan 35^\circ \tan 40^\circ \tan 45^\circ \tan 50^\circ \tan 55^\circ$$

[CBSE 2000]

$$(v) \operatorname{cosec}(65^\circ + \theta) - \sec(25^\circ - \theta) - \tan(55^\circ - \theta) + \cot(35^\circ + \theta)$$

[CBSE 2000]

$$(vi) \tan 7^\circ \tan 23^\circ \tan 60^\circ \tan 67^\circ \tan 83^\circ$$

[CBSE 2000]

$$(vii) \frac{2 \sin 68^\circ}{\cos 22^\circ} - \frac{2 \cot 15^\circ}{5 \tan 75^\circ} - \frac{3 \tan 45^\circ \tan 20^\circ \tan 40^\circ \tan 50^\circ \tan 70^\circ}{5}$$

[CBSE 2004]

$$(viii) \frac{3 \cos 55^\circ}{7 \sin 35^\circ} - \frac{4 (\cos 70^\circ \operatorname{cosec} 20^\circ)}{7 (\tan 5^\circ \tan 25^\circ \tan 45^\circ \tan 65^\circ \tan 85^\circ)}$$

[CBSE 2007]

$$(ix) \frac{\sin 18^\circ}{\cos 72^\circ} + \sqrt{3} \{ \tan 10^\circ \tan 30^\circ \tan 40^\circ \tan 50^\circ \tan 80^\circ \}$$

[CBSE 2008]

$$(x) \frac{\cos 58^\circ}{\sin 32^\circ} + \frac{\sin 22^\circ}{\cos 68^\circ} - \frac{\cos 38^\circ \operatorname{cosec} 52^\circ}{\tan 18^\circ \tan 35^\circ \tan 60^\circ \tan 72^\circ \tan 55^\circ} = 1$$

[CBSE 2009]

10. If $\sin \theta = \cos(\theta - 45^\circ)$, where θ and $\theta - 45^\circ$ are acute angles, find the degree measure of θ .

11. If A, B, C are the interior angles of a ΔABC , show that:

$$(i) \sin \frac{B+C}{2} = \cos \frac{A}{2} \quad (ii) \cos \frac{B+C}{2} = \sin \frac{A}{2}$$

12. If $20 + 45^\circ$ and $30^\circ - \theta$ are acute angles, find the degree measure of θ satisfying

$$\sin(20 + 45^\circ) = \cos(30^\circ - \theta)$$

13. If θ is a positive acute angle such that $\sec \theta = \operatorname{cosec} 60^\circ$, find the value of $2 \cos^2 \theta - 1$.

14. If $\cos 2\theta = \sin 4\theta$, where 2θ and 4θ are acute angles, find the value of θ .

15. If $\sin 3\theta = \cos(\theta - 6^\circ)$, where 3θ and $\theta - 6^\circ$ are acute angles, find the value of θ .

16. If $\sec 4A = \operatorname{cosec}(A - 20^\circ)$, where $4A$ is an acute angle, find the value of A .

17. If $\sec 2A = \operatorname{cosec}(A - 42^\circ)$, where $2A$ is an acute angle, find the value of A .

[CBSE 2008]

18. If $\tan 2A = \cot(A - 18^\circ)$, where $2A$ is an acute angle, find the value of A .

[CBSE 2018]

ANSWERS

1. (i) 1 (ii) 1 (iii) 1 (iv) 1 (v) 1
 2. (i) 2 (ii) 0 (iii) $1/2$ (iv) 0
 (v) 1 (vi) 2 (vii) 0 (viii) 0
 (ix) 0 (x) 1 (xi) 2
 3. (i) $\cos 31^\circ + \sin 34^\circ$ (ii) $\cot 25^\circ + \tan 41^\circ$
 (iii) $\operatorname{cosec} 14^\circ + \sec 38^\circ$ (iv) $\sin 12^\circ + \operatorname{cosec} 12^\circ$
 (v) $\sec 36^\circ + \cos 18^\circ$ (vi) $\tan 5^\circ + \sin 15^\circ$
 (vii) $\cos 23^\circ + \sin 15^\circ$ 4. $\sin 15^\circ + \tan 15^\circ$ 5. 29°
 9. (i) $\frac{113}{24}$ (ii) $\frac{11}{6}$ (iii) -2 (iv) 1 (v) 0
 (vi) $\sqrt{3}$ (vii) 1 (viii) $-\frac{1}{7}$ (ix) 2 (x) $\frac{6-\sqrt{3}}{3}$
 10. $67\frac{1}{2}^\circ$ 12. 15° 13. $\frac{1}{2}$ 14. 15°
 15. 24° 16. 22° 17. 44° 18. 36°

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

Answer each of the following questions either in one word or one sentence or as per requirement of the questions:

- Write the maximum and minimum values of $\sin \theta$.
- Write the maximum and minimum values of $\cos \theta$.
- What is the maximum value of $\frac{1}{\sec \theta}$?
- What is the maximum value of $\frac{1}{\operatorname{cosec} \theta}$?
- If $\tan \theta = \frac{4}{5}$, find the value of $\frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta}$.
- If $\cos \theta = \frac{2}{3}$, find the value of $\frac{\sec \theta - 1}{\sec \theta + 1}$.
- If $3 \cot \theta = 4$, find the value of $\frac{4 \cos \theta - \sin \theta}{2 \cos \theta + \sin \theta}$.
- Given $\tan \theta = \frac{1}{\sqrt{5}}$, what is the value of $\frac{\operatorname{cosec}^2 \theta - \sec^2 \theta}{\operatorname{cosec}^2 \theta + \sec^2 \theta}$?
- If $\cot \theta = \frac{1}{\sqrt{3}}$, write the value of $\frac{1 - \cos^2 \theta}{2 - \sin^2 \theta}$.
- If $\tan A = \frac{3}{4}$ and $A + B = 90^\circ$, then what is the value of $\cot B$?

11. If $A + B = 90^\circ$ and $\cos B = \frac{3}{5}$, what is the value of $\sin A$?
12. Write the acute angle θ satisfying $\sqrt{3} \sin \theta = \cos \theta$.
13. Write the value of $\cos 1^\circ \cos 2^\circ \cos 3^\circ \dots \cos 179^\circ \cos 180^\circ$.
14. Write the value of $\tan 10^\circ \tan 15^\circ \tan 75^\circ \tan 80^\circ$.
15. If $A + B = 90^\circ$ and $\tan A = \frac{3}{4}$, what is $\cot B$?
16. If $\tan A = \frac{5}{12}$, find the value of $(\sin A + \cos A) \sec A$.

[CBSE 2008]

ANSWERS

1. -1 and 1 2. -1 and 1 3. 1 4. 1 5. $\frac{1}{9}$
6. $\frac{1}{5}$ 7. $\frac{13}{11}$ 8. $\frac{2}{3}$ 9. $\frac{3}{5}$ 10. $\frac{3}{4}$ 11. $\frac{3}{5}$ 12. 30°
13. 0 14. 1 15. $\frac{3}{4}$ 16. $\frac{17}{12}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If θ is an acute angle such that $\cos \theta = \frac{3}{5}$, then $\frac{\sin \theta \tan \theta - 1}{2 \tan^2 \theta} =$
- (a) $\frac{16}{625}$ (b) $\frac{1}{36}$ (c) $\frac{3}{160}$ (d) $\frac{160}{3}$
2. If $\tan \theta = \frac{a}{b}$, then $\frac{a \sin \theta + b \cos \theta}{a \sin \theta - b \cos \theta}$ is equal to
- (a) $\frac{a^2 + b^2}{a^2 - b^2}$ (b) $\frac{a^2 - b^2}{a^2 + b^2}$ (c) $\frac{a + b}{a - b}$ (d) $\frac{a - b}{a + b}$
3. If $5 \tan \theta - 4 = 0$, then the value of $\frac{5 \sin \theta - 4 \cos \theta}{5 \sin \theta + 4 \cos \theta}$ is
- (a) $\frac{5}{3}$ (b) $\frac{5}{6}$ (c) 0 (d) $\frac{1}{6}$
4. If $16 \cot x = 12$, then $\frac{\sin x - \cos x}{\sin x + \cos x}$ equals
- (a) $\frac{1}{7}$ (b) $\frac{3}{7}$ (c) $\frac{2}{7}$ (d) 0
5. If $8 \tan x = 15$, then $\sin x - \cos x$ is equal to
- (a) $\frac{8}{17}$ (b) $\frac{17}{7}$ (c) $\frac{1}{17}$ (d) $\frac{7}{17}$
6. If $\tan \theta = \frac{1}{\sqrt{7}}$, then $\frac{\operatorname{cosec}^2 \theta - \sec^2 \theta}{\operatorname{cosec}^2 \theta + \sec^2 \theta} =$

- (a) $\frac{5}{7}$ (b) $\frac{3}{7}$ (c) $\frac{1}{12}$ (d) $\frac{3}{4}$
7. If $\tan \theta = \frac{3}{4}$, then $\cos^2 \theta - \sin^2 \theta =$
 (a) $\frac{7}{25}$ (b) 1 (c) $-\frac{7}{25}$ (d) $\frac{4}{25}$
8. If θ is an acute angle such that $\tan^2 \theta = \frac{8}{7}$, then the value of $\frac{(1 + \sin \theta)(1 - \sin \theta)}{(1 + \cos \theta)(1 - \cos \theta)}$ is
 (a) $\frac{7}{8}$ (b) $\frac{8}{7}$ (c) $\frac{7}{4}$ (d) $\frac{64}{49}$
9. If $3 \cos \theta = 5 \sin \theta$, then the value of $\frac{5 \sin \theta - 2 \sec^3 \theta + 2 \cos \theta}{5 \sin \theta + 2 \sec^3 \theta - 2 \cos \theta}$ is
 (a) $\frac{271}{979}$ (b) $\frac{316}{2937}$ (c) $\frac{542}{2937}$ (d) None of these
10. If $\tan^2 45^\circ - \cos^2 30^\circ = x \sin 45^\circ \cos 45^\circ$, then $x =$
 (a) 2 (b) -2 (c) $-\frac{1}{2}$ (d) $\frac{1}{2}$
11. The value of $\cos^2 17^\circ - \sin^2 73^\circ$ is
 (a) 1 (b) $\frac{1}{3}$ (c) 0 (d) -1
12. The value of $\frac{\cos^3 20^\circ - \cos^3 70^\circ}{\sin^3 70^\circ - \sin^3 20^\circ}$ is
 (a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) 1 (d) 2
13. If $\frac{x \operatorname{cosec}^2 30^\circ \sec^2 45^\circ}{8 \cos^2 45^\circ \sin^2 60^\circ} = \tan^2 60^\circ - \tan^2 30^\circ$, then $x =$
 (a) 1 (b) -1 (c) 2 (d) 0
14. If A and B are complementary angles, then
 (a) $\sin A = \sin B$ (b) $\cos A = \cos B$ (c) $\tan A = \tan B$ (d) $\sec A = \operatorname{cosec} B$
15. If $x \sin (90^\circ - \theta) \cot (90^\circ - \theta) = \cos (90^\circ - \theta)$, then $x =$
 (a) 0 (b) 1 (c) -1 (d) 2
16. If $x \tan 45^\circ \cos 60^\circ = \sin 60^\circ \cot 60^\circ$, then x is equal to
 (a) 1 (b) $\sqrt{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{2}}$
17. If angles A, B, C of a ΔABC form an increasing AP, then $\sin B =$
 (a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) 1 (d) $\frac{1}{\sqrt{2}}$
18. If θ is an acute angle such that $\sec^2 \theta = 3$, then the value of $\frac{\tan^2 \theta - \operatorname{cosec}^2 \theta}{\tan^2 \theta + \operatorname{cosec}^2 \theta}$ is
 (a) $\frac{4}{7}$ (b) $\frac{3}{7}$ (c) $\frac{2}{7}$ (d) $\frac{1}{7}$

19. The value of $\tan 1^\circ \tan 2^\circ \tan 3^\circ \dots \tan 89^\circ$ is
 (a) 1 (b) -1 (c) 0 (d) None of these
20. The value of $\cos 1^\circ \cos 2^\circ \cos 3^\circ \dots \cos 180^\circ$ is
 (a) 1 (b) 0 (c) -1 (d) None of these
21. The value of $\tan 10^\circ \tan 15^\circ \tan 75^\circ \tan 80^\circ$ is
 (a) -1 (b) 0 (c) 1 (d) None of these
22. The value of $\frac{\cos(90^\circ - \theta) \sec(90^\circ - \theta) \tan \theta}{\operatorname{cosec}(90^\circ - \theta) \sin(90^\circ - \theta) \cot(90^\circ - \theta)} + \frac{\tan(90^\circ - \theta)}{\cot \theta}$ is
 (a) 1 (b) -1 (c) 2 (d) -2
23. If θ and $2\theta - 45^\circ$ are acute angles such that $\sin \theta = \cos(2\theta - 45^\circ)$, then $\tan \theta$ is equal to
 (a) 1 (b) -1 (c) $\sqrt{3}$ (d) $\frac{1}{\sqrt{3}}$
24. If 5θ and 4θ are acute angles satisfying $\sin 5\theta = \cos 4\theta$, then $2 \sin 3\theta - \sqrt{3} \tan 3\theta$ is equal to
 (a) 1 (b) 0 (c) -1 (d) $1 + \sqrt{3}$
25. If $A + B = 90^\circ$, then $\frac{\tan A \tan B + \tan A \cot B}{\sin A \sec B} - \frac{\sin^2 B}{\cos^2 A}$ is equal to
 (a) $\cot^2 A$ (b) $\cot^2 B$ (c) $-\tan^2 A$ (d) $-\cot^2 A$
26. $\frac{2 \tan 30^\circ}{1 + \tan^2 30^\circ}$ is equal to
 (a) $\sin 60^\circ$ (b) $\cos 60^\circ$ (c) $\tan 60^\circ$ (d) $\sin 30^\circ$ [NCERT]
27. $\frac{1 - \tan^2 45^\circ}{1 + \tan^2 45^\circ}$ is equal to
 (a) $\tan 90^\circ$ (b) 1 (c) $\sin 45^\circ$ (d) $\sin 0^\circ$ [NCERT]
28. $\sin 2A = 2 \sin A$ is true when $A =$
 (a) 0° (b) 30° (c) 45° (d) 60° [NCERT]
29. $\frac{2 \tan 30^\circ}{1 - \tan^2 30^\circ}$ is equal to
 (a) $\cos 60^\circ$ (b) $\sin 60^\circ$ (c) $\tan 60^\circ$ (d) $\sin 30^\circ$ [NCERT]
30. If A, B and C are interior angles of a triangle ABC , then $\sin\left(\frac{B+C}{2}\right) =$
 (a) $\sin \frac{A}{2}$ (b) $\cos \frac{A}{2}$ (c) $-\sin \frac{A}{2}$ (d) $-\cos \frac{A}{2}$
31. If $\cos \theta = \frac{2}{3}$, then $2 \sec^2 \theta + 2 \tan^2 \theta - 7$ is equal to
 (a) 1 (b) 0 (c) 3 (d) 4
32. $\tan 5^\circ \times \tan 30^\circ \times 4 \tan 85^\circ$ is equal to
 (a) $\frac{4}{\sqrt{3}}$ (b) $4\sqrt{3}$ (c) 1 (d) 4

33. The value of $\frac{\tan 55^\circ}{\cot 35^\circ} + \cot 1^\circ \cot 2^\circ \cot 3^\circ \dots \cot 90^\circ$, is

- (a) -2 (b) 2 (c) 1 (d) 0

34. In Fig. 10.47, the value of $\cos \phi$ is

- (a) $\frac{5}{4}$ (b) $\frac{5}{3}$ (c) $\frac{3}{5}$ (d) $\frac{4}{5}$

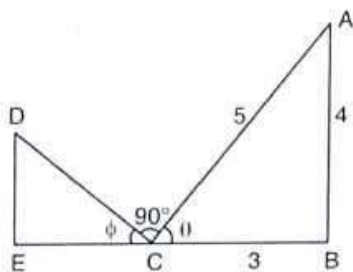


Fig. 10.47

35. In Fig. 10.48, if $AD = 4$ cm $BD = 3$ cm and $CB = 12$ cm, then $\cot \theta =$

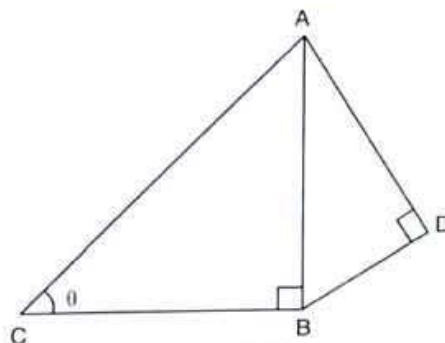


Fig. 10.48

- (a) $\frac{12}{5}$ (b) $\frac{5}{12}$ (c) $\frac{13}{12}$ (d) $\frac{12}{13}$

[CBSE 2008]

ANSWERS

1. (c)	2. (a)	3. (c)	4. (a)	5. (d)	6. (d)
7. (a)	8. (a)	9. (a)	10. (d)	11. (c)	12. (c)
13. (a)	14. (d)	15. (b)	16. (a)	17. (b)	18. (d)
19. (a)	20. (b)	21. (c)	22. (c)	23. (a)	24. (b)
25. (b)	26. (a)	27. (d)	28. (a)	29. (c)	30. (b)
31. (b)	32. (a)	33. (c)	34. (d)	35. (a)	

SUMMARY

- An angle is considered as the figure obtained by rotating a given ray about its end-point. The revolving ray is called the generating line of the angle. The initial position OA is called the initial side and the final position OB is called the terminal side of the angle.
- The measure of an angle is the amount of rotation from the initial side to the terminal side.

3. If ABC is a right triangle right angled at B and $\angle BAC = \theta$, then with reference to angle θ , we have

Base = AB , Perpendicular = BC and, Hypotenuse = AC

Also,

$$\sin \theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}}$$

$$\cos \theta = \frac{\text{Base}}{\text{Hypotenuse}}$$

$$\tan \theta = \frac{\text{Perpendicular}}{\text{Base}}$$

$$\operatorname{cosec} \theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}}$$

$$\sec \theta = \frac{\text{Hypotenuse}}{\text{Base}}$$

$$\cot \theta = \frac{\text{Base}}{\text{Perpendicular}}$$

4. We have, $\operatorname{cosec} \theta = \frac{1}{\sin \theta}$, $\sec \theta = \frac{1}{\cos \theta}$ and $\cot \theta = \frac{1}{\tan \theta}$

Also, $\tan \theta = \frac{\sin \theta}{\cos \theta}$ and $\cot \theta = \frac{\cos \theta}{\sin \theta}$

5. The trigonometric ratios for angles 0° , 30° , 45° , 60° and 90° are given in table given on page 10.30.
6. The values of $\sin \theta$ and $\cos \theta$ never exceed 1, whereas the values of $\sec \theta$ and $\operatorname{cosec} \theta$ are always greater than or equal to 1.
7. If θ is an acute angle, then
- $$\sin (90^\circ - \theta) = \cos \theta, \cos (90^\circ - \theta) = \sin \theta$$
- $$\tan (90^\circ - \theta) = \cot \theta, \cot (90^\circ - \theta) = \tan \theta$$
- $$\sec (90^\circ - \theta) = \operatorname{cosec} \theta, \operatorname{cosec} (90^\circ - \theta) = \sec \theta$$