CHAPTER 11

TRIGONOMETRIC IDENTITIES

11.1 INTRODUCTION

In the previous chapter, we have learnt about various trigonometric ratios and relations between them. In this chapter, we will prove some trigonometric identities, and use them to prove other useful trigonometric identities.

11.2 TRIGONOMETRIC IDENTITIES

We know that an equation is called an identity if it is true for all values of the variable (s) involved. For example,

$$x^2 - 9 = (x-3)(x+3) \text{ and } \frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} = 1$$

are algebraic identities as they are satisfied by every value of the variable x.

In this section, we will discuss some trigonometric identities. We may define the term trigonometric identity as follows.

DEFINITION *An equation involving trigonometric ratios of an angle θ (say) is said to be a trigonometric identity if it is satisfied for all values of θ for which the given trigonometric ratios are defined.*

For example, $\cos^2\theta - \frac{1}{2}\cos\theta = \cos\theta\left(\cos\theta - \frac{1}{2}\right)$ is a trigonometric identity, whereas $\cos\theta\left(\cos\theta - \frac{1}{2}\right) = 0$ is an equation.

Also, $\sec\theta = \frac{1}{\cos\theta}$ is a trigonometric identity, because it holds for all values of q except for which $\cos\theta = 0$. For $\cos\theta = 0$, $\sec\theta$ is not defined.

In this section, we shall establish some fundamental trigonometric identities and use them to obtain some more identities.

THEOREM 1 *Prove that* $\sin^2\theta + \cos^2\theta = 1$.

PROOF The following three cases arise :

CASE I *When* $\theta = 0$

In this case, we have

$$\sin\theta = \sin 0^\circ = 0 \text{ and } \cos\theta = \cos 0^\circ = 1$$

$$\therefore \quad \sin^2\theta + \cos^2\theta = 0 + 1 = 1$$

CASE II *When* $\theta = 90°$

In this case, we have

$$\sin\theta = \sin 90° = 1 \text{ and } \cos\theta = \cos 90° = 0$$

$$\therefore \quad \sin^2\theta + \cos^2\theta = 1 + 0 = 1$$

CASE III *When* θ *is an acute angle*

Let $XAY = \theta$ be the given acute angle. Let P be any point on the terminal side AY other than A. Draw perpendicular PM from P on the initial side AX.

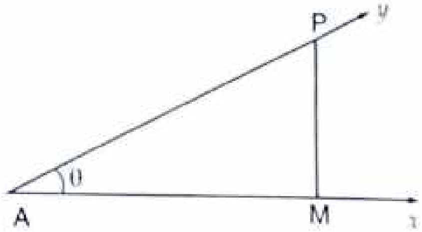

Fig. 11.1

In ΔAMP, we have

$$\sin\theta = \frac{PM}{AP} \text{ and } \cos\theta = \frac{AM}{AP}$$

$$\Rightarrow \quad (\sin\theta)^2 + (\cos\theta)^2 = \left(\frac{PM}{AP}\right)^2 + \left(\frac{AM}{AP}\right)^2$$

$$\Rightarrow \quad \sin^2\theta + \cos^2\theta = \frac{PM^2}{AP^2} + \frac{AM^2}{AP^2} \qquad [\because (\sin\theta)^2 = \sin^2\theta \text{ and } (\cos\theta)^2 = \cos^2\theta]$$

$$\Rightarrow \quad \sin^2\theta + \cos^2\theta = \frac{PM^2 + AM^2}{AP^2}$$

$$\Rightarrow \quad \sin^2\theta + \cos^2\theta = \frac{AP^2}{AP^2} \qquad \left[\begin{array}{l}\because \Delta AMP \text{ is a right angled triangle} \\ \therefore PM^2 + AM^2 = AP^2\end{array}\right]$$

$$\Rightarrow \quad \sin^2\theta + \cos^2\theta = 1$$

Thus, in all the three cases, we have

$$\sin^2\theta + \cos^2\theta = 1$$

Hence, $\sin^2\theta + \cos^2\theta = 1$ for all values of variable θ.

REMARK 1 *Note that* $\sin^2\theta$ *is the square of the sine of angle* θ. *Similarly* $\cos^2\theta$ *is the square of the cosine of the angle* θ.

REMARK 2 *It follows from the above identity that* $\sin^2\theta = 1 - \cos^2\theta$ *and* $\cos^2\theta = 1 - \sin^2\theta$

THEOREM 2 *Prove that* $\sec^2\theta = 1 + \tan^2\theta$

PROOF In the right angled triangle AMP (Fig. 11.1), we have

$$AM^2 + MP^2 = AP^2 \qquad \text{[By Pythagoras Theorem]}$$

$$\Rightarrow \quad \frac{AM^2 + MP^2}{AM^2} = \frac{AP^2}{AM^2} \qquad \left[\text{Dividing both sides by } AM^2\right]$$

$$\Rightarrow \quad \frac{AM^2}{AM^2} + \frac{MP^2}{AM^2} = \frac{AP^2}{AM^2}$$

$$\Rightarrow \quad 1 + \tan^2\theta = \sec^2\theta \qquad \left[\because \tan\theta = \frac{MP}{AM} \text{ and } \sec\theta = \frac{AP}{AM}\right]$$

Hence, $1 + \tan^2\theta = \sec^2\theta$

Q.E.D.

REMARK 1 *It follows from the above identity that:*

$$\sec^2\theta - \tan^2\theta = 1 \text{ and, } \sec^2\theta - 1 = \tan^2\theta$$

REMARK 2 $\sec^2\theta - \tan^2\theta = 1 \Rightarrow (\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = 1$

$$\therefore \quad \sec\theta + \tan\theta = \frac{1}{\sec\theta - \tan\theta} \text{ and, } \sec\theta - \tan\theta = \frac{1}{\sec\theta + \tan\theta}$$

THEOREM 3 *Prove that* $\text{cosec}^2\theta = 1 + \cot^2\theta$.

PROOF In the right-angled triangle AMP (Fig. 11.1), we have

$$AM^2 + MP^2 = AP^2 \qquad \text{[Using Pythagoras Theorem]}$$

$$\Rightarrow \quad \frac{AM^2 + MP^2}{MP^2} = \frac{AP^2}{MP^2} \qquad \text{[Dividing both sides by } MP^2\text{]}$$

$$\Rightarrow \quad \frac{AM^2}{MP^2} + \frac{MP^2}{MP^2} = \frac{AP^2}{MP^2}$$

$$\Rightarrow \quad \left(\frac{AM}{MP}\right)^2 + 1 = \left(\frac{AP}{MP}\right)^2$$

$$\Rightarrow \quad \cot^2\theta + 1 = \text{cosec}^2\theta \qquad \left[\because \cot\theta = \frac{AM}{MP} \text{ and } \text{cosec}\,\theta = \frac{AP}{MP}\right]$$

Hence, $1 + \cot^2\theta = \text{cosec}^2\theta$

Q.E.D.

REMARK 3 *It follows from the above identity that:*

$$\text{cosec}^2\theta - \cot^2\theta = 1 \text{ and, } \text{cosec}^2\theta - 1 = \cot^2\theta$$

REMARK 4 $\text{cosec}^2\theta - \cot^2\theta = 1 \Rightarrow (\text{cosec}\,\theta - \cot\theta)(\text{cosec}\,\theta + \cot\theta) = 1$

$$\therefore \quad \text{cosec}\,\theta + \cot\theta = \frac{1}{\text{cosec}\,\theta - \cot\theta} \text{ and, } \text{cosec}\,\theta - \cot\theta = \frac{1}{\text{cosec}\,\theta + \cot\theta}.$$

THEOREM 4 *For any acute angle* θ, *prove the following identities:*

(i) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ (ii) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

PROOF From Fig. 11.1, we have

$$\sin\theta = \frac{PM}{AP}, \cos\theta = \frac{AM}{AP}, \tan\theta = \frac{PM}{AM} \text{ and } \cot\theta = \frac{AM}{PM}$$

(i) We have,

$$\tan\theta = \frac{PM}{AM}$$

$$\Rightarrow \quad \tan\theta = \frac{(PM/AP)}{(AM/AP)} \qquad \text{[Dividing Numerator and Denominator by } AP\text{]}$$

$$\Rightarrow \quad \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \left[\because \sin\theta = \frac{PM}{AP} \text{ and } \cos\theta = \frac{AM}{AP}\right]$$

Q.E.D.

(ii) We have,

$$\cot\theta = \frac{AM}{PM}$$

$\Rightarrow \quad \cot\theta = \dfrac{\left(\dfrac{AM}{PM}\right)}{\left(\dfrac{PM}{AP}\right)}$ [Dividing Numerator and Denominator by AP]

$\Rightarrow \quad \cot\theta = \dfrac{\cos\theta}{\sin\theta}$ $\left[\because \cos\theta = \dfrac{AM}{AP} \text{ and } \sin\theta = \dfrac{PM}{AP}\right]$

Hence, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Q.E.D

The above identities and some more identities obtained from the above identities by performing simple algebraic operations like addition, subtraction are listed below for ready reference.

(i) $\sin^2\theta + \cos^2\theta = 1$ (ii) $\cos^2\theta = 1 - \sin^2\theta$

(iii) $\sin^2\theta = 1 - \cos^2\theta$ (iv) $1 + \tan^2\theta = \sec^2\theta$

(v) $\sec^2\theta - \tan^2\theta = 1$ (vi) $\sec^2\theta - 1 = \tan^2\theta$

(vii) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ (viii) $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

(ix) $\operatorname{cosec}^2\theta - 1 = \cot^2\theta$ (x) $\sec\theta + \tan\theta = \dfrac{1}{\sec\theta - \tan\theta}$

(xi) $\operatorname{cosec}\theta + \cot\theta = \dfrac{1}{\operatorname{cosec}\theta - \cot\theta}$

REMARK 5 *We have proved the above identities for an acute angle θ. But, these identities are true for any angle θ for which the trigonometric ratios are meaningful.*

REMARK 6 *In this book, we shall deal mainly with acute angles.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Prove the following trigonometric identiti4es:*

(i) $(1 - \sin^2\theta)\sec^2\theta = 1$ (ii) $\cos^2\theta(1 + \tan^2\theta) = 1$

(iii) $\cos^2\theta + \dfrac{1}{1 + \cot^2\theta} = 1$ (iv) $\dfrac{1}{1 + \sin\theta} + \dfrac{1}{1 - \sin\theta} = 2\sec^2\theta$

(v) $\operatorname{cosec}^2\theta + \sec^2\theta = \operatorname{cosec}^2\theta\sec^2\theta$ [CBSE 2001]

(vi) $\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta} = \tan\theta + \cot\theta$ [NCERT EXEMPLAR]

(vii) $(\sin^4\theta - \cos^4\theta + 1)\operatorname{cosec}^2\theta = 2$ [NCERT EXEMPLAR]

SOLUTION (i) We have,

$$\text{LHS} = (1 - \sin^2\theta)\sec^2\theta$$

$\Rightarrow \quad \text{LHS} = \cos^2\theta\sec^2\theta$ $\left[\because 1 - \sin^2\theta = \cos^2\theta\right]$

$\Rightarrow \quad \text{LHS} = \cos^2\theta\left(\dfrac{1}{\cos^2\theta}\right) = 1 = \text{RHS}$ $\left[\because \sec\theta = \dfrac{1}{\cos\theta} \therefore \sec^2\theta = \dfrac{1}{\cos^2\theta}\right]$

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$1 + \tan^2\theta = \sec^2\theta$

(ii) We have,

$$\text{LHS} = \cos^2\theta\,(1 + \tan^2\theta)$$

$$\Rightarrow \quad \text{LHS} = \cos^2\theta \sec^2\theta \qquad \left[\because 1 + \tan^2\theta = \sec^2\theta\right]$$

$$\Rightarrow \quad \text{LHS} = \cos^2\theta\left(\frac{1}{\cos^2\theta}\right) = 1 = \text{RHS} \qquad \left[\because \sec\theta = \frac{1}{\cos\theta}\right]$$

(iii) We have,

$$\text{LHS} = \cos^2\theta + \frac{1}{1 + \cot^2\theta}$$

$$\Rightarrow \quad \text{LHS} = \cos^2\theta + \frac{1}{\operatorname{cosec}^2\theta} \qquad \left[\because 1 + \cot^2\theta = \operatorname{cosec}^2\theta\right]$$

$$\Rightarrow \quad \text{LHS} = \cos^2\theta + \sin^2\theta = 1 = \text{RHS} \qquad \left[\because \frac{1}{\operatorname{cosec}\theta} = \sin\theta\right]$$

(iv) We have,

$$\text{LHS} = \frac{1}{1 + \sin\theta} + \frac{1}{1 - \sin\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{1 - \sin\theta + 1 + \sin\theta}{(1 + \sin\theta)(1 - \sin\theta)} \qquad \text{[On taking LCM]}$$

$$\Rightarrow \quad \text{LHS} = \frac{2}{1 - \sin^2\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{2}{\cos^2\theta} \qquad \left[\because 1 - \sin^2\theta = \cos^2\theta\right]$$

$$\Rightarrow \quad \text{LHS} = 2\sec^2\theta = \text{RHS} \qquad \left[\because \sec\theta = \frac{1}{\cos\theta}\right]$$

(v) We have,

$$\text{LHS} = \operatorname{cosec}^2\theta + \sec^2\theta$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta\sin^2\theta} = \frac{1}{\sin^2\theta\cos^2\theta} = \frac{1}{\sin^2\theta} \times \frac{1}{\cos^2\theta} = \operatorname{cosec}^2\theta\sec^2\theta = \text{RHS}$$

(vi) We have,

$$\text{LHS} = \sqrt{\sec^2\theta + \operatorname{cosec}^2\theta}$$

$$= \sqrt{(1 + \tan^2\theta) + (1 + \cot^2\theta)}$$

$$= \sqrt{\tan^2 + \cot^2\theta + 2)} = \sqrt{\tan^2\theta + \cot^2\theta + 2\tan\theta\cot\theta} \qquad [\because \tan\theta\cot\theta = 1]$$

$$= \sqrt{(\tan\theta + \cot\theta)^2} = \tan\theta + \cot\theta = \text{RHS}$$

(vii) We have,

$$\text{LHS} = (\sin^4\theta - \cos^4\theta + 1)\,\text{cosec}^2\theta$$
$$= \{(\sin^2\theta)^2 - (\cos^2\theta)^2 + 1\}\,\text{cosec}^2\theta$$
$$= \{(\sin^2\theta + \cos^2\theta)(\sin^2\theta - \cos^2\theta) + 1\}\,\text{cosec}^2\theta$$
$$= (\sin^2\theta - \cos^2\theta + 1)\,\text{cosec}^2\theta \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$
$$= (\sin^2\theta + \sin^2\theta)\,\text{cosec}^2\theta \qquad [1 - \cos^2\theta = \sin^2\theta]$$
$$= 2\sin^2\theta - \text{cosec}^2\theta = 2\sin^2\theta \times \frac{1}{\sin^2\theta} = 2 = \text{RHS}$$

EXAMPLE 2 *Prove that following trigonometric identities :*

(i) $\cot^2\theta - \dfrac{1}{\sin^2\theta} = -1$

(ii) $(1 + \tan^2\theta)(1 + \sin\theta)(1 - \sin\theta) = 1$ **[NCERT EXEMPLAR]**

(iii) $(1 + \cot^2\theta)(1 - \cos\theta)(1 + \cos\theta) = 1$ (iv) $\tan^2\theta - \dfrac{1}{\cos^2\theta} = -1$

SOLUTION (i) We have,

$$\text{LHS} = \cot^2\theta - \frac{1}{\sin^2\theta}$$
$$\Rightarrow \quad \text{LHS} = \cot^2\theta - \text{cosec}^2\theta \qquad \left[\because \frac{1}{\sin\theta} = \text{cosec}\,\theta \;\therefore \frac{1}{\sin^2\theta} = \text{cosec}^2\theta\right]$$
$$\Rightarrow \quad \text{LHS} = -(\text{cosec}^2\theta - \cot^2\theta) = -1 = \text{RHS}$$

(ii) We have,

$$\text{LHS} = (1 + \tan^2\theta)(1 + \sin\theta)(1 - \sin\theta)$$
$$\Rightarrow \quad \text{LHS} = (1 + \tan^2\theta)\{(1 + \sin\theta)(1 - \sin\theta)\}$$
$$\Rightarrow \quad \text{LHS} = (1 + \tan^2\theta)(1 - \sin^2\theta)$$
$$\Rightarrow \quad \text{LHS} = \sec^2\theta\cos^2\theta \qquad [\because 1 + \tan^2\theta = \sec^2\theta \text{ and } 1 - \sin^2\theta = \cos^2\theta]$$
$$\Rightarrow \quad \text{LHS} = \frac{1}{\cos^2\theta} \times \cos^2\theta = 1 = \text{RHS} \qquad \left[\because \sec\theta = \frac{1}{\cos\theta} \therefore \sec^2\theta = \frac{1}{\cos^2\theta}\right]$$

(iii) We have,

$$\text{LHS} = (1 + \cot^2\theta)(1 - \cos\theta)(1 + \cos\theta)$$
$$\Rightarrow \quad \text{LHS} = (1 + \cot^2\theta)(1 - \cos^2\theta)$$
$$\Rightarrow \quad \text{LHS} = \text{cosec}^2\theta\sin^2\theta \qquad [\because 1 + \cot^2\theta = \text{cosec}^2\theta \text{ and } 1 - \cos^2\theta = \sin^2\theta]$$
$$\Rightarrow \quad \text{LHS} = \text{cosec}^2\theta\sin^2\theta$$
$$\Rightarrow \quad \text{LHS} = \frac{1}{\sin^2\theta} \times \sin^2\theta = 1 = \text{RHS} \qquad \left[\because \text{cosec}\,\theta = \frac{1}{\sin\theta} \therefore \text{cosec}^2\theta = \frac{1}{\sin\theta}\right]$$

(iv) We have,

$$\text{LHS} = \tan^2\theta - \frac{1}{\cos^2\theta}$$

$$\Rightarrow \quad \text{LHS} = \tan^2\theta - \sec^2\theta \qquad \left[\because \frac{1}{\cos\theta} = \sec\theta \therefore \frac{1}{\cos^2\theta} = \sec^2\theta\right]$$

$$\Rightarrow \quad \text{LHS} = -(\sec^2\theta - \tan^2\theta) = -1 = \text{RHS}$$

EXAMPLE 3 *Prove the following trigonometric identities:*

(i) $\dfrac{\sin\theta}{1-\cos\theta} = \operatorname{cosec}\theta + \cot\theta$ (ii) $\dfrac{\tan\theta + \sin\theta}{\tan\theta - \sin\theta} = \dfrac{\sec\theta + 1}{\sec\theta - 1}$

(iii) $\cot\theta - \tan\theta = \dfrac{2\cos^2\theta - 1}{\sin\theta\cos\theta}$ (iv) $\tan\theta - \cot\theta = \dfrac{2\sin^2\theta - 1}{\sin\theta\cos\theta}$

SOLUTION (i) We have,

$$\text{LHS} = \frac{\sin\theta}{1-\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin\theta}{(1-\cos\theta)} \times \frac{(1+\cos\theta)}{(1+\cos\theta)} \qquad \left[\begin{array}{l}\text{Multiplying numerator and} \\ \text{denominator by } (1+\cos\theta)\end{array}\right]$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta} \qquad [\because (1+\cos\theta)(1-\cos\theta) = 1-\cos^2\theta]$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta} \qquad [\because 1-\cos^2\theta = \sin^2\theta]$$

$$\Rightarrow \quad \text{LHS} = \frac{1+\cos\theta}{\sin\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow \quad \text{LHS} = \operatorname{cosec}\theta + \cot\theta = \text{RHS} \qquad \left[\because \frac{1}{\sin\theta} = \operatorname{cosec}\theta \text{ and, } \frac{\cos\theta}{\sin\theta} = \cot\theta\right]$$

(ii) We have,

$$\text{LHS} = \frac{\tan\theta + \sin\theta}{\tan\theta - \sin\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\dfrac{\sin\theta}{\cos\theta} + \sin\theta}{\dfrac{\sin\theta}{\cos\theta} - \sin\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin\theta\left(\dfrac{1}{\cos\theta} + 1\right)}{\sin\theta\left(\dfrac{1}{\cos\theta} - 1\right)} = \frac{\dfrac{1}{\cos\theta} + 1}{\dfrac{1}{\cos\theta} - 1} = \frac{\sec\theta + 1}{\sec\theta - 1} = \text{RHS}$$

(iii) We have,

$$\text{LHS} = \cot\theta - \tan\theta$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta} = \frac{\cos^2\theta - (1-\cos^2\theta)}{\sin\theta\cos\theta} \qquad [\because \sin^2\theta = 1-\cos^2\theta]$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos^2\theta - 1 + \cos^2\theta}{\sin\theta\cos\theta} = \frac{2\cos^2\theta - 1}{\sin\theta\cos\theta} = \text{RHS}$$

(iv) We have,

$$\text{LHS} = \tan\theta - \cot\theta$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta} = \frac{\sin^2\theta - (1-\sin^2\theta)}{\sin\theta\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin^2\theta - 1 + \sin^2\theta}{\sin\theta\cos\theta} = \frac{2\sin^2\theta - 1}{\sin\theta\cos\theta} = \text{RHS}$$

EXAMPLE 4 *Prove the following trigonometric identities:*

(i) $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$ (ii) $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \text{cosec}\,\theta + \cot\theta$

SOLUTION (i) We have,

$$\text{LHS} = \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$$

$$\Rightarrow \quad \text{LHS} = \sqrt{\frac{1-\sin\theta}{1+\sin\theta} \times \frac{1-\sin\theta}{1-\sin\theta}} \qquad [\text{Multiplying and dividing by } (1-\sin\theta)]$$

$$\Rightarrow \quad \text{LHS} = \sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}} = \sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}}$$

$$\Rightarrow \quad \text{LHS} = \sqrt{\left(\frac{1-\sin\theta}{\cos\theta}\right)^2} = \frac{1-\sin\theta}{\cos\theta} = \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta} = \sec\theta - \tan\theta = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$$

$$\Rightarrow \quad \text{LHS} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta} \times \frac{1+\cos\theta}{1+\cos\theta}} \qquad \left[\begin{array}{l}\text{Multiplying and dividing within the}\\ \text{square root sign by } (1+\cos\theta)\end{array}\right]$$

$$\Rightarrow \quad \text{LHS} = \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}} = \sqrt{\frac{(1+\cos\theta)^2}{\sin^2\theta}}$$

$$\Rightarrow \quad \text{LHS} = \sqrt{\left(\frac{1+\cos\theta}{\sin\theta}\right)^2} = \frac{1+\cos\theta}{\sin\theta} = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} = \text{cosec}\,\theta + \cot\theta = \text{RHS}$$

EXAMPLE 5 *Prove the following identities:*

(i) $\frac{1-\sin\theta}{1+\sin\theta}=(\sec\theta-\tan\theta)^2$ (ii) $\frac{1-\cos\theta}{1+\cos\theta}=(\text{cosec}\,\theta-\cot\theta)^2$

(iii) $\frac{\cos\theta}{1-\sin\theta}+\frac{\cos\theta}{1+\sin\theta}=2\sec\theta$

(iv) $\frac{\sin A+\cos A}{\sin A-\cos A}+\frac{\sin A-\cos A}{\sin A+\cos A}=\frac{2}{\sin^2 A-\cos^2 A}=\frac{2}{2\sin^2 A-1}=\frac{2}{1-2\cos^2 A}$

[CBSE 2000, 2000C]

(v) $(\text{cosec}\,\theta-\sin\theta)(\sec\theta-\cos\theta)(\tan\theta+\cot\theta)=1$

(vi) $\frac{\sin\theta-2\sin^3\theta}{2\cos^3\theta-\cos\theta}=\tan\theta$ [NCERT, CBSE 2000]

SOLUTION (i) We have,

$$\text{LHS}=\frac{1-\sin\theta}{1+\sin\theta}$$

$$\Rightarrow \text{LHS}=\frac{1-\sin\theta}{1+\sin\theta}\times\frac{1-\sin\theta}{1-\sin\theta}$$ [Multiplying numerator and denominator by $1-\sin\theta$]

$$\Rightarrow \text{LHS}=\frac{(1-\sin\theta)^2}{1-\sin^2\theta}$$

$$\Rightarrow \text{LHS}=\frac{(1-\sin\theta)^2}{\cos^2\theta}$$ [$\because 1-\sin^2\theta=\cos^2\theta$]

$$\Rightarrow \text{LHS}=\left(\frac{1-\sin\theta}{\cos\theta}\right)^2$$

$$\Rightarrow \text{LHS}=\left(\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}\right)^2=(\sec\theta-\tan\theta)^2=\text{RHS}$$

(ii) We have,

$$\text{LHS}=\frac{1-\cos\theta}{1+\cos\theta}$$

$$\Rightarrow \text{LHS}=\frac{1-\cos\theta}{1+\cos\theta}\times\frac{1-\cos\theta}{1-\cos\theta}$$ [Multiplying numerator and denominator by $1-\cos\theta$]

$$\Rightarrow \text{LHS}=\frac{(1-\cos\theta)^2}{1-\cos^2\theta}=\frac{(1-\cos\theta)^2}{\sin^2\theta}$$ [$\because 1-\cos^2\theta=\sin^2\theta$]

$$\Rightarrow \text{LHS}=\left(\frac{1-\cos\theta}{\sin\theta}\right)^2=\left(\frac{1}{\sin\theta}-\frac{\cos\theta}{\sin\theta}\right)^2=(\text{cosec}\,\theta-\cot\theta)^2=\text{RHS}$$

(iii) We have,

$$\text{LHS}=\frac{\cos\theta}{1-\sin\theta}+\frac{\cos\theta}{1+\sin\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos\theta(1+\sin\theta)+\cos\theta(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos\theta+\cos\theta\sin\theta+\cos\theta-\cos\theta\sin\theta}{1-\sin^2\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\cos\theta}{\cos^2\theta} \qquad [\because 1-\sin^2\theta$$

$$\Rightarrow \quad \text{LHS} = \frac{2}{\cos\theta} = 2\sec\theta = \text{RHS}$$

(iv) We have,

$$\text{LHS} = \frac{\sin A+\cos A}{\sin A-\cos A} + \frac{\sin A-\cos A}{\sin A+\cos A}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sin A+\cos A)^2+(\sin A-\cos A)^2}{(\sin A-\cos A)(\sin A+\cos A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sin^2 A+\cos^2 A+2\sin A\cos A)+(\sin^2 A+\cos^2 A-2\sin A\cos A)}{\sin^2 A-\cos^2 A}$$

$$\Rightarrow \quad \text{LHS} = \frac{(1+2\sin A\cos A)+(1-2\sin A\cos A)}{\sin^2 A-\cos^2 A} \qquad [\because \sin^2 A+\cos$$

$$\Rightarrow \quad \text{LHS} = \frac{2}{\sin^2 A-\cos^2 A} = \frac{2}{\sin^2 A-(1-\sin^2 A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{2}{2\sin^2 A-1} = \frac{2}{2(1-\cos^2 A)-1} = \frac{2}{1-2\cos^2 A} = \text{RHS}$$

(v) We have,

$$\text{LHS} = (\operatorname{cosec}\theta-\sin\theta)(\sec\theta-\cos\theta)(\tan\theta+\cot\theta)$$

$$\Rightarrow \quad \text{LHS} = \left(\frac{1}{\sin\theta}-\sin\theta\right)\left(\frac{1}{\cos\theta}-\cos\theta\right)\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow \quad \text{LHS} = \left(\frac{1-\sin^2\theta}{\sin\theta}\right)\left(\frac{1-\cos^2\theta}{\cos\theta}\right)\left(\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}\right)$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos^2\theta}{\sin\theta}\times\frac{\sin^2\theta}{\cos\theta}\times\frac{1}{\sin\theta\cos\theta} \qquad [\because \sin^2\theta+\cos^2$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin^2\theta\cos^2\theta}{\sin^2\theta\cos^2\theta} = 1 = \text{RHS}$$

(vi) We have,

$$\text{LHS} = \frac{\sin\theta-2\sin^3\theta}{2\cos^3\theta-\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin\theta(1-2\sin^2\theta)}{\cos\theta(2\cos^2\theta-1)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin\theta\{1 - 2(1-\cos^2\theta)\}}{\cos\theta(2\cos^2\theta - 1)} = \frac{\sin\theta(2\cos^2\theta - 1)}{\cos\theta(2\cos^2\theta - 1)} = \tan\theta = \text{RHS}$$

EXAMPLE 6 *Prove the following identities:*

(i) $(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 = 7 + \tan^2\theta + \cot^2\theta$ **[NCERT, CBSE 2000]**

(ii) $(\sin\theta + \sec\theta)^2 + (\cos\theta + \operatorname{cosec}\theta)^2 = (1 + \sec\theta\operatorname{cosec}\theta)^2$ **[CBSE 2000C]**

(iii) $(\operatorname{cosec}\theta - \cot\theta)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$ **[NCERT, CBSE 2000C]**

(iv) $\sec^4\theta - \sec^2\theta = \tan^4\theta + \tan^2\theta$ **[NCERT EXEMPLAR]**

(v) $2\sec^2\theta - \sec^4\theta - 2\operatorname{cosec}^2\theta + \operatorname{cosec}^4\theta = \cot^4\theta - \tan^4\theta$ **[CBSE 2000]**

(vi) $(\sin\theta - \sec\theta)^2 + (\cos\theta - \operatorname{cosec}\theta)^2 = (1 - \sec\theta\operatorname{cosec}\theta)^2$

SOLUTION (i) We have,

$$\text{LHS} = (\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2$$

$$\Rightarrow \quad \text{LHS} = (\sin^2\theta + \operatorname{cosec}^2\theta + 2\sin\theta\operatorname{cosec}\theta) + (\cos^2\theta + \sec^2\theta + 2\cos\theta\sec\theta)$$

$$\Rightarrow \quad \text{LHS} = \left(\sin^2\theta + \operatorname{cosec}^2\theta + 2\sin\theta\frac{1}{\sin\theta}\right) + \left(\cos^2\theta + \sec^2\theta + 2\cos\theta\frac{1}{\cos\theta}\right)$$

$$\Rightarrow \quad \text{LHS} = (\sin^2\theta + \operatorname{cosec}^2\theta + 2) + (\cos^2\theta + \sec^2\theta + 2)$$

$$\Rightarrow \quad \text{LHS} = \sin^2\theta + \cos^2\theta + \operatorname{cosec}^2\theta + \sec^2\theta + 4$$

$$\Rightarrow \quad \text{LHS} = 1 + (1 + \cot^2\theta) + (1 + \tan^2\theta) + 4 \qquad \left[\because \operatorname{cosec}^2\theta = 1 + \cot^2\theta, \; \sec^2\theta = 1 + \tan^2\theta\right]$$

$$\Rightarrow \quad \text{LHS} = 7 + \tan^2\theta + \cot^2\theta = \text{RHS}$$

(ii) We have,

$$\text{LHS} = (\sin\theta + \sec\theta)^2 + (\cos\theta + \operatorname{cosec}\theta)^2$$

$$\Rightarrow \quad \text{LHS} = \left(\sin\theta + \frac{1}{\cos\theta}\right)^2 + \left(\cos\theta + \frac{1}{\sin\theta}\right)^2$$

$$\Rightarrow \quad \text{LHS} = \sin^2\theta + \frac{1}{\cos^2\theta} + \frac{2\sin\theta}{\cos\theta} + \cos^2\theta + \frac{1}{\sin^2\theta} + \frac{2\cos\theta}{\sin\theta}$$

$$\Rightarrow \quad \text{LHS} = (\sin^2\theta + \cos^2\theta) + \left(\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}\right) + 2\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow \quad \text{LHS} = (\sin^2\theta + \cos^2\theta) + \left(\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta\cos^2\theta}\right) + \frac{2(\sin^2\theta + \cos^2\theta)}{\sin\theta\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = 1 + \frac{1}{\sin^2\theta\cos^2\theta} + \frac{2}{\sin\theta\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = \left(1 + \frac{1}{\sin\theta\cos\theta}\right)^2 = (1 + \sec\theta\,\text{cosec}\,\theta)^2 = \text{RHS}$$

(iii) We have,

$$\text{LHS} = (\text{cosec}\,\theta - \cot\theta)^2$$

$$\Rightarrow \quad \text{LHS} = \left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)^2$$

$$\Rightarrow \quad \text{LHS} = \left(\frac{1-\cos\theta}{\sin\theta}\right)^2$$

$$\Rightarrow \quad \text{LHS} = \frac{(1-\cos\theta)^2}{\sin^2\theta} = \frac{(1-\cos\theta)^2}{1-\cos^2\theta} \qquad [\because \sin^2\theta = 1 -$$

$$\Rightarrow \quad \text{LHS} = \frac{(1-\cos\theta)^2}{(1-\cos\theta)(1+\cos\theta)} = \frac{1-\cos\theta}{1+\cos\theta} = \text{RHS}$$

(iv) We have,

$$\text{LHS} = \sec^4\theta - \sec^2\theta$$

$$\Rightarrow \quad \text{LHS} = \sec^2\theta(\sec^2\theta - 1)$$

$$\Rightarrow \quad \text{LHS} = (1+\tan^2\theta)(1+\tan^2\theta - 1) \qquad [\because \sec^2\theta = 1 +$$

$$\Rightarrow \quad \text{LHS} = (1+\tan^2\theta)\tan^2\theta = \tan^2\theta + \tan^4\theta = \text{RHS}$$

(v) We have,

$$\text{LHS} = 2\sec^2\theta - \sec^4\theta - 2\,\text{cosec}^2\theta + \text{cosec}^4\theta$$

$$\Rightarrow \quad \text{LHS} = (\text{cosec}^4\theta - 2\,\text{cosec}^2\theta) - (\sec^4\theta - 2\sec^2\theta)$$

$$\Rightarrow \quad \text{LHS} = (\text{cosec}^4\theta - 2\,\text{cosec}^2\theta + 1) - (\sec^4\theta - 2\sec^2\theta + 1)$$

$$\Rightarrow \quad \text{LHS} = (\text{cosec}^2\theta - 1)^2 - (\sec^2\theta - 1)^2$$

$$\Rightarrow \quad \text{LHS} = (\cot^2\theta)^2 - (\tan^2\theta)^2$$

$$\Rightarrow \quad \text{LHS} = \cot^4\theta - \tan^4\theta = \text{RHS}$$

(vi) Proceed as in (ii) part.

EXAMPLE 7 *Prove the following identities:*

(i) $\dfrac{\sin\theta}{1-\cos\theta} + \dfrac{\tan\theta}{1+\cos\theta} = \sec\theta\,\text{cosec}\,\theta + \cot\theta$

(ii) $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta} = 2\,\text{cosec}\,\theta$

(iii) $\dfrac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \sec^2\theta - \text{cosec}^2\theta = \tan^2\theta - \cot^2\theta$ **[NCERT EXEMPLAR, CBSE 20**

(iv) $\frac{1}{\sec\theta - \tan\theta} = \sec\theta + \tan\theta$

(v) $\frac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta} = 1 - 2\sec\theta\tan\theta + 2\tan^2\theta$

SOLUTION (i) We have,

$$\text{LHS} = \frac{\sin\theta}{1-\cos\theta} + \frac{\tan\theta}{1+\cos\theta}$$

$$\text{LHS} = \frac{\sin\theta(1+\cos\theta)}{(1+\cos\theta)(1-\cos\theta)} + \frac{\tan\theta(1-\cos\theta)}{(1+\cos\theta)\ (1-\cos\theta)}$$

$$= \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta} + \frac{\tan\theta(1-\cos\theta)}{1-\cos^2\theta}$$

$$= \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta} + \frac{\tan\theta(1-\cos\theta)}{\sin^2\theta} = \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta} + \frac{\sin\theta(1-\cos\theta)}{\cos\theta\sin^2\theta}$$

$$= \frac{1+\cos\theta}{\sin\theta} + \frac{1-\cos\theta}{\cos\theta\sin\theta} = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} + \frac{1}{\cos\theta\sin\theta} - \frac{\cos\theta}{\cos\theta\sin\theta}$$

$$= \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} + \frac{1}{\cos\theta\sin\theta} - \frac{1}{\sin\theta} = \cot\theta + \sec\theta\,\text{cosec}\,\theta = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin^2\theta + (1+\cos\theta)^2}{\sin\theta(1+\cos\theta)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}{\sin\theta(1+\cos\theta)}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sin^2\theta + \cos^2\theta) + 1 + 2\cos\theta}{\sin\theta(1+\cos\theta)}$$

$$\Rightarrow \quad \text{LHS} = \frac{1+1+2\cos\theta}{\sin\theta(1+\cos\theta)} \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$\Rightarrow \quad \text{LHS} = \frac{2+2\cos\theta}{\sin\theta(1+\cos\theta)} = \frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)} = \frac{2}{\sin\theta} = 2\,\text{cosec}\,\theta = \text{RHS}$$

(iii) We have,

$$\text{LHS} = \frac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \frac{\frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta}}{\sin\theta\cos\theta} = \frac{\frac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta}}{\cos\theta\sin\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin^2\theta - \cos^2\theta}{\sin^2\theta\cos^2\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin^2\theta}{\sin^2\theta\cos^2\theta} - \frac{\cos^2\theta}{\sin^2\theta\cos^2\theta}$$

$\Rightarrow \quad \text{LHS} = \frac{1}{\cos^2\theta} - \frac{1}{\sin^2\theta}$

$\Rightarrow \quad \text{LHS} = \sec^2\theta - \text{cosec}^2\theta = (1+\tan^2\theta) - (1+\cot^2\theta) = \tan^2\theta - \cot^2\theta = \text{RHS}$

(iv) We have,

$\text{LHS} = \frac{1}{\sec\theta - \tan\theta}$

$\Rightarrow \quad \text{LHS} = \frac{1}{\sec\theta - \tan\theta} \times \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}$

$\Rightarrow \quad \text{LHS} = \frac{\sec\theta + \tan\theta}{\sec^2\theta - \tan^2\theta} = \frac{\sec\theta + \tan\theta}{1}$ $\quad [\because \sec^2\theta - \tan^2\theta = 1]$

$\Rightarrow \quad \text{LHS} = \sec\theta + \tan\theta = \text{RHS}$

(v) We have,

$\text{LHS} = \frac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta}$

$\Rightarrow \quad \text{LHS} = \frac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta} \times \frac{\sec\theta - \tan\theta}{\sec\theta - \tan\theta}$

$\Rightarrow \quad \text{LHS} = \frac{(\sec\theta - \tan\theta)^2}{\sec^2\theta - \tan^2\theta} = \frac{(\sec\theta - \tan\theta)^2}{1}$ $\quad [\because \sec^2\theta - \tan^2\theta = 1]$

$\Rightarrow \quad \text{LHS} = \sec^2\theta - 2\sec\theta\tan\theta + \tan^2\theta$

$\Rightarrow \quad \text{LHS} = (1+\tan^2\theta) - 2\sec\theta\tan\theta + \tan^2\theta$ $\quad [\because \sec^2\theta = 1 + \tan^2\theta]$

$\Rightarrow \quad \text{LHS} = 1 - 2\sec\theta\tan\theta + 2\tan^2\theta = \text{RHS}$

EXAMPLE 8 *Prove the following identities:*

(i) $(1 + \cot\theta - \text{cosec}\,\theta)(1 + \tan\theta + \sec\theta) = 2$

(ii) $\tan^2\theta + \cot^2\theta + 2 = \sec^2\theta\,\text{cosec}^2\theta$ **[CBSE 2008]**

(iii) $\sqrt{\sec^2\theta + \text{cosec}^2\theta} = \tan\theta + \cot\theta$ **[NCERT EXEMPLAR]**

SOLUTION (i) We have,

$\text{LHS} = (1 + \cot\theta - \text{cosec}\,\theta)(1 + \tan\theta + \sec\theta)$

$\Rightarrow \quad \text{LHS} = \left(1 + \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}\right)\left(1 + \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)$

$\Rightarrow \quad \text{LHS} = \left(\frac{\sin\theta + \cos\theta - 1}{\sin\theta}\right)\left(\frac{\cos\theta + \sin\theta + 1}{\cos\theta}\right)$

$\Rightarrow \quad \text{LHS} = \frac{(\sin\theta + \cos\theta)^2 - 1^2}{\sin\theta\cos\theta}$

$\Rightarrow \quad \text{LHS} = \frac{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 1}{\sin\theta\cos\theta}$

$\Rightarrow \quad \text{LHS} = \dfrac{1 + 2\sin\theta\cos\theta - 1}{\sin\theta\cos\theta} = \dfrac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} = 2 = \text{RHS}$

(ii) We have,

$\text{LHS} = \tan^2\theta + \cot^2\theta + 2$

$\Rightarrow \quad \text{LHS} = \tan^2\theta + \cot^2\theta + 2\tan\theta\cot\theta \qquad [\because \tan\theta \cdot \cot\theta = 1]$

$\Rightarrow \quad \text{LHS} = (\tan\theta + \cot\theta)^2$

$\Rightarrow \quad \text{LHS} = \left(\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}\right)^2$

$\Rightarrow \quad \text{LHS} = \left(\dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\right)^2$

$\Rightarrow \quad \text{LHS} = \left(\dfrac{1}{\sin\theta\cos\theta}\right)^2 = \dfrac{1}{\sin^2\theta\cos^2\theta} = \text{cosec}^2\theta\sec^2\theta = \text{RHS}$

ALITER 1 We have,

$\text{LHS} = \tan^2\theta + \cot^2\theta + 2 = (1 + \tan^2\theta) + (1 + \cot^2\theta)$

$\Rightarrow \quad \text{LHS} = \sec^2\theta + \text{cosec}^2\theta = \dfrac{1}{\cos^2\theta} + \dfrac{1}{\sin^2\theta} = \dfrac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin^2\theta}$

$\Rightarrow \quad \text{LHS} = \dfrac{1}{\cos^2\theta\sin^2\theta} = \text{cosec}^2\theta\sec^2\theta = \text{RHS}$

ALITER 2 $\text{LHS} = \tan^2\theta + \cot^2\theta + 2$

$\Rightarrow \quad \text{LHS} = 1 + \tan^2\theta + \cot^2\theta + 1$

$\Rightarrow \quad \text{LHS} = 1 + \tan^2\theta + \cot^2\theta + \tan^2\theta\cot^2\theta \qquad [\because \tan^2\theta\cot^2\theta = 1]$

$\Rightarrow \quad \text{LHS} = (1 + \tan^2\theta) + \cot^2\theta(1 + \tan^2\theta)$

$\Rightarrow \quad \text{LHS} = (1 + \tan^2\theta)(1 + \cot^2\theta) = \sec^2\theta\,\text{cosec}^2\theta = \text{RHS}$

ALITER 3 $\text{RHS} = \sec^2\theta\,\text{cosec}^2\theta$

$\Rightarrow \quad \text{RHS} = (1 + \tan^2\theta)(1 + \cot^2\theta)$

$\Rightarrow \quad \text{RHS} = 1 + \tan^2\theta + \cot^2\theta + \tan^2\theta\cot^2\theta$

$\Rightarrow \quad \text{RHS} = 1 + \tan^2\theta + \cot^2\theta + 1 = \tan^2\theta + \cot^2\theta + 2 = \text{LHS}$

(iii) We have,

$\text{LHS} = \sqrt{\sec^2\theta + \text{cosec}^2\theta}$

$\Rightarrow \quad \text{LHS} = \sqrt{(1 + \tan^2\theta) + (1 + \cot^2\theta)} = \sqrt{2 + \tan^2\theta + \cot^2\theta}$

$\Rightarrow \quad \text{LHS} = \sqrt{\tan^2\theta + \cot^2\theta + 2\tan\theta\cot\theta} \qquad [\because \tan\theta\cot\theta = 1]$

$\Rightarrow \quad \text{LHS} = \sqrt{(\tan\theta + \cot\theta)^2} = \tan\theta + \cot\theta = \text{RHS}$

EXAMPLE 9 *Prove the following identities:*

(i) $\dfrac{1}{\operatorname{cosec} A - \cot A} - \dfrac{1}{\sin A} = \dfrac{1}{\sin A} - \dfrac{1}{\operatorname{cosec} A + \cot A}$

(ii) $\dfrac{\cos A}{1 - \tan A} + \dfrac{\sin A}{1 - \cot A} = \cos A + \sin A$ [C

(iii) $\dfrac{\tan A}{1 - \cot A} + \dfrac{\cot A}{1 - \tan A} = 1 + \tan A + \cot A = 1 + \sec A \operatorname{cosec} A$ [NCEI

SOLUTION (i) We have,

$$\text{LHS} = \frac{1}{\operatorname{cosec} A - \cot A} - \frac{1}{\sin A}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{(\operatorname{cosec} A - \cot A)} \times \frac{(\operatorname{cosec} A + \cot A)}{(\operatorname{cosec} A + \cot A)} - \frac{1}{\sin A}$$

$$\Rightarrow \quad \text{LHS} = \frac{\operatorname{cosec} A + \cot A}{\operatorname{cosec}^2 A - \cot^2 A} - \operatorname{cosec} A$$

$$\Rightarrow \quad \text{LHS} = \operatorname{cosec} A + \cot A - \operatorname{cosec} A \qquad [\because \operatorname{cosec}$$

$$\Rightarrow \quad \text{LHS} = \cot A$$

and, $$\text{RHS} = \frac{1}{\sin A} - \frac{1}{\operatorname{cosec} A + \cot A}$$

$$\Rightarrow \quad \text{RHS} = \frac{1}{\sin A} - \frac{\operatorname{cosec} A - \cot A}{(\operatorname{cosec} A + \cot A)(\operatorname{cosec} A - \cot A)}$$

$$\Rightarrow \quad \text{RHS} = \frac{1}{\sin A} - \frac{\operatorname{cosec} A - \cot A}{\operatorname{cosec}^2 A - \cot^2 A}$$

$$\Rightarrow \quad \text{RHS} = \operatorname{cosec} A - (\operatorname{cosec} A - \cot A)$$

$$\Rightarrow \quad \text{RHS} = \cot A \qquad [\because \operatorname{cosec}^2$$

From (i) and (ii) we find that LHS = RHS.

ALITER $$\text{LHS} = \frac{1}{\operatorname{cosec} A - \cot A} - \frac{1}{\sin A}$$

$$\Rightarrow \quad \text{LHS} = (\operatorname{cosec} A + \cot A) - \operatorname{cosec} A \qquad \left[\because \frac{1}{\operatorname{cosec} A - \cot A} = \text{co}\right.$$

$$\Rightarrow \quad \text{LHS} = \operatorname{cosec} A + (\cot A - \operatorname{cosec} A)$$

$$\Rightarrow \quad \text{LHS} = \operatorname{cosec} A - (\operatorname{cosec} A - \cot A)$$

$$\Rightarrow \quad \text{LHS} = \operatorname{cosec} A - \frac{1}{\operatorname{cosec} A + \cot A} \qquad \left[\because \operatorname{cosec} A - \cot A = \frac{}{\text{cos}}\right.$$

$$\text{LHS} = \frac{1}{\sin A} - \frac{1}{\operatorname{cosec} A + \cot A} = \text{RHS}$$

We have,

$$\text{LHS} = \frac{\cos A}{1 - \tan A} + \frac{\sin A}{1 - \cot A}$$

$$\text{LHS} = \frac{\cos A}{1 - \frac{\sin A}{\cos A}} + \frac{\sin A}{1 - \frac{\cos A}{\sin A}}$$

$$\text{LHS} = \frac{\cos A}{\frac{\cos A - \sin A}{\cos A}} + \frac{\sin A}{\frac{\sin A - \cos A}{\sin A}}$$

$$\text{LHS} = \frac{\cos^2 A}{\cos A - \sin A} + \frac{\sin^2 A}{\sin A - \cos A}$$

$$\text{LHS} = \frac{\cos^2 A}{\cos A - \sin A} - \frac{\sin^2 A}{\cos A - \sin A}$$

$$\text{LHS} = \frac{\cos^2 A - \sin^2 A}{\cos A - \sin A} = \frac{(\cos A - \sin A)(\cos A + \sin A)}{\cos A - \sin A} = \cos A + \sin A = \text{RHS}$$

We have,

$$\text{LHS} = \frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A}$$

$$\text{LHS} = \frac{\tan A}{1 - \frac{1}{\tan A}} + \frac{\frac{1}{\tan A}}{1 - \tan A}$$

$$\text{LHS} = \frac{\tan A}{\frac{\tan A - 1}{\tan A}} + \frac{1}{\tan A\,(1 - \tan A)}$$

$$\text{LHS} = \frac{\tan^2 A}{\tan A - 1} + \frac{1}{\tan A\,(1 - \tan A)}$$

$$\text{LHS} = \frac{\tan^2 A}{\tan A - 1} - \frac{1}{\tan A\,(\tan A - 1)}$$

$$\text{LHS} = \frac{\tan^3 A - 1}{\tan A\,(\tan A - 1)} \qquad \text{[Taking LCM]}$$

$$\text{LHS} = \frac{(\tan A - 1)(\tan^2 A + \tan A + 1)}{\tan A\,(\tan A - 1)} \qquad [\because a^3 - b^3 = (a - b)(a^2 + ab + b^2)]$$

$\Rightarrow \quad LHS = \dfrac{\tan^2 A + \tan A + 1}{\tan A}$

$\Rightarrow \quad LHS = \dfrac{\tan^2 A}{\tan A} + \dfrac{\tan A}{\tan A} + \dfrac{1}{\tan A}$

$\Rightarrow \quad LHS = \tan A + 1 + \cot A = (1 + \tan A + \cot A)$

$\therefore \quad \dfrac{\tan A}{1 - \cot A} + \dfrac{\cot A}{1 - \tan A} = 1 + \tan A + \cot A$

Now, $\quad 1 + \tan A + \cot A$

$= 1 + \dfrac{\sin A}{\cos A} + \dfrac{\cos A}{\sin A}$

$= 1 + \dfrac{\sin^2 A + \cos^2 A}{\sin A \cos A} = 1 + \dfrac{1}{\sin A \cos A} = 1 + \operatorname{cosec} A \sec A$

From (i) and (ii), we obtain

$\dfrac{\tan A}{1 - \cot A} + \dfrac{\cot A}{1 - \tan A} = 1 + \tan A + \cot A = 1 + \operatorname{cosec} A \sec A$

EXAMPLE 10 *Prove the following identities:*

(i) $\cos^4 A - \cos^2 A = \sin^4 A - \sin^2 A$

(ii) $\cot^4 A - 1 = \operatorname{cosec}^4 A - 2\operatorname{cosec}^2 A$

(iii) $\sin^4 A + \cos^4 A = 1 - 2\sin^2 A \cos^2 A$

(iv) $\sin^4 A - \cos^4 A = \sin^2 A - \cos^2 A = 2\sin^2 A - 1 = 1 - 2\cos^2 A$

(v) $\sin^6 A + \cos^6 A = 1 - 3\sin^2 A \cos^2 A$

(vi) $\sec^4 A - \sec^2 A = \tan^4 A + \tan^2 A$

SOLUTION (i) We have, **[NCERT EXEMPI**

$LHS = \cos^4 A - \cos^2 A$

$\Rightarrow \quad LHS = \cos^2 A(\cos^2 A - 1)$

$\Rightarrow \quad LHS = -\cos^2 A(1 - \cos^2 A)$

$\Rightarrow \quad LHS = -\cos^2 A \sin^2 A = -(1 - \sin^2 A)\sin^2 A = -\sin^2 A + \sin^4 A$

$\Rightarrow \quad LHS = \sin^4 A - \sin^2 A = RHS$

(ii) We have,

$LHS = \cot^4 A - 1$

$\Rightarrow \quad LHS = (\operatorname{cosec}^2 A - 1)^2 - 1 \quad \left[\because \cot^2 A = \operatorname{cosec}^2 A - 1 \therefore \cot^4 A = (\operatorname{cosec}^2 A - 1\right.$

$\Rightarrow \quad LHS = \operatorname{cosec}^4 A - 2\operatorname{cosec}^2 A + 1 - 1 = \operatorname{cosec}^4 A - 2\operatorname{cosec}^2 A = RHS$

(iii) We have,

$$\text{LHS} = \sin^4 A + \cos^4 A$$

$\Rightarrow$ $\text{LHS} = (\sin^2 A)^2 + (\cos^2 A)^2 + 2\sin^2 A\cos^2 A - 2\sin^2 A\cos^2 A$

[Adding and subtracting $2\sin^2 A\cos^2 A$]

$\Rightarrow$ $\text{LHS} = (\sin^2 A + \cos^2 A)^2 - 2\sin^2 A\cos^2 A = 1 - 2\sin^2 A\cos^2 A = \text{RHS}$

(iv) We have,

$$\text{LHS} = \sin^4 A - \cos^4 A$$

$\Rightarrow$ $\text{LHS} = (\sin^2 A)^2 - (\cos^2 A)^2$

$\Rightarrow$ $\text{LHS} = (\sin^2 A + \cos^2 A)(\sin^2 A - \cos^2 A)$

$\Rightarrow$ $\text{LHS} = \sin^2 A - \cos^2 A$ $[\because \sin^2 A + \cos^2 A = 1]$

$\Rightarrow$ $\text{LHS} = \sin^2 A - (1 - \sin^2 A) = 2\sin^2 A - 1$

$\Rightarrow$ $\text{LHS} = 2(1 - \cos^2 A) - 1 = 1 - 2\cos^2 A = \text{RHS}$

(v) We have,

$$\text{LHS} = \sin^6 A + \cos^6 A$$

$\Rightarrow$ $\text{LHS} = (\sin^2 A)^3 + (\cos^2 A)^3$

$\Rightarrow$ $\text{LHS} = (\sin^2 A + \cos^2 A)\left\{(\sin^2 A)^2 + (\cos^2 A)^2 - \sin^2 A\cos^2 A\right\}$

$[\because a^3 + b^3 = (a+b)(a^2 - ab + b^2)]$

$\Rightarrow$ $\text{LHS} = \left\{(\sin^2 A)^2 + (\cos^2 A)^2 + 2\sin^2 A\cos^2 A - 2\sin^2 A\cos^2 A - \sin^2 A\cos^2 A\right\}$

$\Rightarrow$ $\text{LHS} = \left\{(\sin^2 A + \cos^2 A)^2 - 3\sin^2 A\cos^2 A\right\} = 1 - 3\sin^2 A\cos^2 A = \text{RHS}$

ALITER $\text{LHS} = (\sin^2 A)^3 + (\cos^2 A)^3$

$\Rightarrow$ $\text{LHS} = (\sin^2 A + \cos^2 A) - 3\sin^2 A\cos^2 A(\sin^2 A + \cos^2 A)$

$[\because a^3 + b^3 = (a+b)^3 - 3ab(a+b)]$

$\Rightarrow$ $\text{LHS} = 1 - 3\sin^2 A\cos^2 A = \text{RHS}$

(vi) We have,

$\Rightarrow$ $\text{LHS} = \sec^4 A - \sec^2 A = \sec^2 A(\sec^2 A - 1)$

$\Rightarrow$ $\text{LHS} = (1 + \tan^2 A)(1 + \tan^2 A - 1) = (1 + \tan^2 A)\tan^2 A$

$\Rightarrow$ $\text{LHS} = \tan^2 A + \tan^4 A = \text{RHS}$

EXAMPLE 11 *Prove the following identities :*

(i) $\dfrac{\sin^2 A}{\cos^2 A} + \dfrac{\cos^2 A}{\sin^2 A} = \dfrac{1}{\sin^2 A\cos^2 A} - 2$

(ii) $\frac{\cos A}{1-\tan A}+\frac{\sin^2 A}{\sin A-\cos A}=\sin A+\cos A$

(iii) $\frac{(1+\sin\theta)^2+(1-\sin\theta)^2}{\cos^2\theta}=2\left(\frac{1+\sin^2\theta}{1-\sin^2\theta}\right)$

(iv) $\frac{\cos^2\theta}{1-\tan\theta}+\frac{\sin^3\theta}{\sin\theta-\cos\theta}=1+\sin\theta\cos\theta$ [C

(v) $\frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}+\sin\theta\cos\theta=1$ [C

SOLUTION (i) We have,

$$\text{LHS}=\frac{\sin^2 A}{\cos^2 A}+\frac{\cos^2 A}{\sin^2 A}=\frac{\sin^4 A+\cos^4 A}{\sin^2 A\cos^2 A}$$ [On t

$$\Rightarrow \quad \text{LHS}=\frac{(\sin^2 A)^2+(\cos^2 A)^2+2\sin^2 A\cos^2 A-2\sin^2 A\cos^2 A}{\sin^2 A\cos^2 A}$$

$$\Rightarrow \quad \text{LHS}=\frac{(\sin^2 A+\cos^2 A)^2-2\sin^2 A\cos^2 A}{\sin^2 A\cos^2 A}$$

$$\Rightarrow \quad \text{LHS}=\frac{1-2\sin^2 A\cos^2 A}{\sin^2 A\cos^2 A}$$

$$\Rightarrow \quad \text{LHS}=\frac{1}{\sin^2 A\cos^2 A}-\frac{2\sin^2 A\cos^2 A}{\sin^2 A\cos^2 A}=\frac{1}{\sin^2 A\cos^2 A}-2=\text{RHS}$$

(ii) We have,

$$\text{LHS}=\frac{\cos A}{1-\tan A}+\frac{\sin^2 A}{\sin A-\cos A}$$

$$\Rightarrow \quad \text{LHS}=\frac{\cos A}{1-\frac{\sin A}{\cos A}}+\frac{\sin^2 A}{\sin A-\cos A}$$

$$\Rightarrow \quad \text{LHS}=\frac{\cos A}{\frac{\cos A-\sin A}{\cos A}}+\frac{\sin^2 A}{\sin A-\cos A}$$

$$\Rightarrow \quad \text{LHS}=\frac{\cos^2 A}{\cos A-\sin A}+\frac{\sin^2 A}{\sin A-\cos A}$$

$$\Rightarrow \quad \text{LHS}=\frac{\cos^2 A}{\cos A-\sin A}-\frac{\sin^2 A}{\cos A-\sin A}$$

$$\Rightarrow \quad \text{LHS}=\frac{\cos^2 A-\sin^2 A}{\cos A-\sin A}$$

$$\Rightarrow \quad \text{LHS}=\frac{(\cos A+\sin A)(\cos A-\sin A)}{\cos A-\sin A}=\cos A+\sin A=\text{RHS}$$

(iii) We have,

$$\text{LHS} = \frac{(1+\sin\theta)^2 + (1-\sin\theta)^2}{\cos^2\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{(1+2\sin\theta+\sin^2\theta)+(1-2\sin\theta+\sin^2\theta)}{\cos^2\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{2+2\sin^2\theta}{\cos^2\theta} = \frac{2(1+\sin^2\theta)}{1-\sin^2\theta} = 2\left(\frac{1+\sin^2\theta}{1-\sin^2\theta}\right) = \text{RHS}$$

(iv) We have,

$$\text{LHS} = \frac{\cos^2\theta}{1-\tan\theta} + \frac{\sin^3\theta}{\sin\theta-\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos^3\theta}{\cos\theta-\sin\theta} - \frac{\sin^3\theta}{\cos\theta-\sin\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos^3\theta-\sin^3\theta}{\cos\theta-\sin\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\cos\theta-\sin\theta)(\cos^2\theta+\sin^2\theta+\cos\theta\sin\theta)}{\cos\theta-\sin\theta} = 1+\sin\theta\cos\theta = \text{LHS}$$

(v) We have,

$$\text{LHS} = \frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta} + \sin\theta\cos\theta$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sin\theta+\cos\theta)(\sin^2\theta+\cos^2\theta-\sin\theta\cos\theta)}{\sin\theta+\cos\theta} + \sin\theta\cos\theta$$

$$\Rightarrow \quad \text{LHS} = 1-\sin\theta\cos\theta+\sin\theta\cos\theta = 1 = \text{RHS}$$

EXAMPLE 12 *Prove the following identities :*

(i) $\tan^2 A - \tan^2 B = \dfrac{\cos^2 B - \cos^2 A}{\cos^2 B\cos^2 A} = \dfrac{\sin^2 A - \sin^2 B}{\cos^2 A\cos^2 B}$ **[CBSE 2005]**

(ii) $\dfrac{\sin A - \sin B}{\cos A + \cos B} + \dfrac{\cos A - \cos B}{\sin A + \sin B} = 0$

SOLUTION We have,

$$\text{LHS} = \tan^2 A - \tan^2 B$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin^2 A}{\cos^2 A} - \frac{\sin^2 B}{\cos^2 B}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin^2 A\cos^2 B - \cos^2 A\sin^2 B}{\cos^2 A\ \cos^2 B}$$

$$\Rightarrow \quad \text{LHS} = \frac{(1-\cos^2 A)\cos^2 B - \cos^2 A(1-\cos^2 B)}{\cos^2 A \cos^2 B}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos^2 B - \cos^2 A\cos^2 B - \cos^2 A + \cos^2 A\cos^2 B}{\cos^2 A \cos^2 B}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos^2 B - \cos^2 A}{\cos^2 A \cos^2 B}$$

$$\Rightarrow \quad \text{LHS} = \frac{(1-\sin^2 B)-(1-\sin^2 A)}{\cos^2 A \cos^2 B} = \frac{\sin^2 A - \sin^2 B}{\cos^2 A \cos^2 B} = \text{RHS}$$

(ii) We have,

$$\Rightarrow \quad \text{LHS} = \frac{\sin A - \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A + \sin B}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sin A - \sin B)(\sin A + \sin B) + (\cos A + \cos B)(\cos A - \cos B)}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin^2 A - \sin^2 B + \cos^2 A - \cos^2 B}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sin^2 A + \cos^2 A) - (\sin^2 B + \cos^2 B)}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$\Rightarrow \quad \text{LHS} = \frac{1-1}{(\cos A + \cos B)(\sin A + \sin B)} = 0 = \text{RHS}$$

EXAMPLE 13 *Prove that:* $(1-\sin\theta+\cos\theta)^2 = 2(1+\cos\theta)(1-\sin\theta)$ **[CBS**

SOLUTION We know that $(a-b+c)^2 = a^2+b^2+c^2-2ab-2bc+2ac$

$$\text{LHS} = (1-\sin\theta+\cos\theta)^2$$

$$\Rightarrow \quad \text{LHS} = 1+\sin^2\theta+\cos^2\theta-2\sin\theta+2\cos\theta-2\sin\theta\cos\theta$$

$$\Rightarrow \quad \text{LHS} = 2-2\sin\theta+2\cos\theta-2\sin\theta\cos\theta$$

$$\Rightarrow \quad \text{LHS} = 2(1-\sin\theta)+2\cos\theta(1-\sin\theta) = 2(1-\sin\theta)(1+\cos\theta) = \text{RHS}$$

EXAMPLE 14 *If* $\sin\theta+\sin^2\theta = 1$, *prove that* $\cos^2\theta+\cos^4\theta = 1$

SOLUTION We have, **[CBSE 2002 C, NCERT EXE**

$$\sin\theta+\sin^2\theta = 1 \Rightarrow \sin\theta = 1-\sin^2\theta \Rightarrow \sin\theta = \cos^2\theta$$

Now, $\cos^2\theta+\cos^4\theta = \cos^2\theta+(\cos^2\theta)^2 = \cos^2\theta+\sin^2\theta = 1$ $[\because \cos^2\theta$

Type II **ON PROVING RESULTS INVOLVING TRIGONOMETRIC RATIOS**

EXAMPLE 15 *If* $\tan\theta+\sin\theta = m$ *and* $\tan\theta-\sin\theta = n$, *show that* $m^2-n^2 = 4\sqrt{mn}$

[CBS

SOLUTION We have, $m = \tan\theta + \sin\theta$ and, $n = \tan\theta - \sin\theta$.

$$\therefore \quad \text{LHS} = m^2 - n^2 = (m+n)(m-n)$$
$$= (\tan\theta + \sin\theta + \tan\theta - \sin\theta)(\tan\theta + \sin\theta - \tan\theta + \sin\theta)$$
$$= (2\tan\theta)(2\sin\theta) = 4\tan\theta\sin\theta = 4\sqrt{\tan^2\theta\sin^2\theta}$$
$$= 4\sqrt{\tan^2\theta(1-\cos^2\theta)} = 4\sqrt{\tan^2\theta - \tan^2\theta\cos^2\theta}$$
$$= 4\sqrt{\tan^2\theta - \sin^2\theta} = 4\sqrt{(\tan\theta+\sin\theta)(\tan\theta-\sin\theta)} = 4\sqrt{mn} = \text{RHS}$$

EXAMPLE 16 *If* $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, *show that* $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$ **[CBSE 2002 C]**

SOLUTION We have,

$$\cos\theta + \sin\theta = \sqrt{2}\cos\theta$$
$$\Rightarrow \quad (\cos\theta + \sin\theta)^2 = 2\cos^2\theta$$
$$\Rightarrow \quad \cos^2\theta + \sin^2\theta + 2\cos\theta\sin\theta = 2\cos^2\theta$$
$$\Rightarrow \quad \cos^2\theta - 2\cos\theta\sin\theta = \sin^2\theta$$
$$\Rightarrow \quad \cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta = 2\sin^2\theta$$
$$\Rightarrow \quad (\cos\theta - \sin\theta)^2 = 2\sin^2\theta$$
$$\Rightarrow \quad \cos\theta - \sin\theta = \sqrt{2}\sin\theta$$

ALITER We have,

$$\cos\theta + \sin\theta = \sqrt{2}\cos\theta$$
$$\Rightarrow \quad (\cos\theta + \sin\theta)^2 = (\sqrt{2}\cos\theta)^2$$
$$\Rightarrow \quad \cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta = 2\cos^2\theta$$
$$\Rightarrow \quad \cos^2\theta - \sin^2\theta = 2\sin\theta\cos\theta$$
$$\Rightarrow \quad (\cos\theta + \sin\theta)(\cos\theta - \sin\theta) = 2\sin\theta\cos\theta$$
$$\Rightarrow \quad \cos\theta - \sin\theta = \frac{2\sin\theta\cos\theta}{\cos\theta + \sin\theta}$$
$$\Rightarrow \quad \cos\theta - \sin\theta = \frac{2\sin\theta\cos\theta}{\sqrt{2}\cos\theta} \qquad [\because \cos\theta + \sin\theta = \sqrt{2}\cos\theta]$$
$$\Rightarrow \quad \cos\theta - \sin\theta = \sqrt{2}\sin\theta$$

EXAMPLE 17 *If* $x = a\sin\theta$ *and* $y = b\tan\theta$, *then prove that* $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$

SOLUTION We have, $x = a\sin\theta$ and $y = b\tan\theta$

$$\therefore \quad \text{LHS} = \frac{a^2}{x^2} - \frac{b^2}{y^2}$$

$\Rightarrow$ $\quad$ $\text{LHS} = \dfrac{a^2}{a^2 \sin^2\theta} - \dfrac{b^2}{b^2 \tan^2\theta}$ $\quad$ $[\because x = a\sin\theta,\ y = b\tan\theta]$

$\Rightarrow$ $\quad$ $\text{LHS} = \dfrac{1}{\sin^2\theta} - \dfrac{1}{\tan^2\theta}$

$\Rightarrow$ $\quad$ $\text{LHS} = \text{cosec}^2\theta - \cot^2\theta = 1 = \text{RHS}$ $\quad$ $\left[\because 1 + \cot^2\theta = \text{cosec}^2\theta \therefore \text{cosec}^2\theta - \cot^2\theta = 1\right]$

EXAMPLE 18 *If* $x = r\sin A\cos C$, $y = r\sin A\sin C$ *and* $z = r\cos A$, *prove that* $r^2 = x^2 + y^2 + z^2$.

SOLUTION We have,

$$x^2 + y^2 + z^2 = r^2\sin^2 A\cos^2 C + r^2\sin^2 A\sin^2 C + r^2\cos^2 A$$
$$= r^2\sin^2 A(\cos^2 C + \sin^2 C) + r^2\cos^2 A$$
$$= r^2\sin^2 A + r^2\cos^2 A \qquad [\because \cos^2 C + \sin^2 C = 1]$$
$$= r^2(\sin^2 A + \cos^2 A) = r^2$$

Hence, $\quad r^2 = x^2 + y^2 + z^2$

EXAMPLE 19 *If* $a\cos\theta + b\sin\theta = m$ *and* $a\sin\theta - b\cos\theta = n$, *prove that* $a^2 + b^2 = m^2 + n^2$.

[CBSE 2001 C]

SOLUTION We have, $m = a\cos\theta + b\sin\theta$ and $n = a\sin\theta - b\cos\theta$

$\therefore$ $\quad$ $\text{RHS} = m^2 + n^2$

$$= (a\cos\theta + b\sin\theta)^2 + (a\sin\theta - b\cos\theta)^2$$
$$= (a^2\cos^2\theta + b^2\sin^2\theta - 2ab\cos\theta\sin\theta) + (a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\cos\theta)$$
$$= a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta) = a^2 + b^2 = \text{LHS}$$

EXAMPLE 20 *If* $a\cos\theta - b\sin\theta = c$, *prove that* $a\sin\theta + b\cos\theta = \pm\sqrt{a^2 + b^2 - c^2}$

[CBSE 2001 C]

SOLUTION $\quad (a\cos\theta - b\sin\theta)^2 + (a\sin\theta + b\cos\theta)^2$

$$= (a^2\cos^2\theta + b^2\sin^2\theta - 2ab\sin\theta\cos\theta) + (a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\cos\theta)$$
$$= a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta) = a^2 + b^2$$

$\therefore$ $\quad$ $c^2 + (a\sin\theta + b\cos\theta)^2 = a^2 + b^2$ $\quad$ $[\because a\cos\theta - b\sin\theta = c]$

$\Rightarrow$ $\quad$ $(a\sin\theta + b\cos\theta)^2 = a^2 + b^2 - c^2$

$\Rightarrow$ $\quad$ $a\sin\theta + b\cos\theta = \pm\sqrt{a^2 + b^2 - c^2}$

ALITER We have,

$a\cos\theta - b\sin\theta = c$

$\Rightarrow$ $\quad$ $(a\cos\theta - b\sin\theta)^2 = c^2$ $\quad$ [Squaring both sides]

$\Rightarrow$ $\quad$ $a^2\cos^2\theta + b^2\sin^2\theta - 2ab\sin\theta\cos\theta = c^2$

$\Rightarrow \quad a^2(1-\sin^2\theta)+b^2(1-\cos^2\theta)-2ab\sin\theta\cos\theta=c^2$

$\Rightarrow \quad a^2\sin^2\theta+b^2\cos^2\theta+2ab\sin\theta\cos\theta=a^2+b^2-c^2$

$\Rightarrow \quad (a\sin\theta+b\cos\theta)^2=a^2+b^2-c^2$

$\Rightarrow \quad a\sin\theta+b\cos\theta=\pm\sqrt{a^2+b^2-c^2}$

Type III ON PROVING TRIGONOMETRIC IDENTITIES INVOLVING TRIGONOMETRIC RATIOS OF COMPLEMENTARY ANGLES

EXAMPLE 21 *Prove the following identities:*

(i) $\cos\theta\sin(90^\circ-\theta)+\sin\theta\cos(90^\circ-\theta)=1$ (ii) $\dfrac{\sin(90^\circ-\theta)\sin\theta}{\tan\theta}-1=-\sin^2\theta$

(iii) $\dfrac{\sin(90^\circ-\theta)\cos(90^\circ-\theta)}{\tan\theta}=1-\sin^2\theta$

SOLUTION (i) We know that $\sin(90^\circ-\theta)=\cos\theta$ and $\cos(90^\circ-\theta)=\sin\theta$

$$\text{LHS}=\cos\theta\sin(90^\circ-\theta)+\sin\theta\cos(90^\circ-\theta)$$

$$=\cos\theta\cos\theta+\sin\theta\sin\theta=\cos^2\theta+\sin^2\theta=1$$

(ii) We know that $\sin(90^\circ-\theta)=\cos\theta$

$$\therefore \quad \text{LHS}=\frac{\sin(90^\circ-\theta)\sin\theta}{\tan\theta}-1$$

$$=\frac{\cos\theta\sin\theta}{\sin\theta/\cos\theta}-1=\cos^2\theta-1=-(1-\cos^2\theta)=-\sin^2\theta=\text{RHS}$$

(iii) We know that $\sin(90^\circ-\theta)=\cos\theta$ and $\cos(90^\circ-\theta)=\sin\theta$

$$\therefore \quad \text{LHS}=\frac{\sin(90^\circ-\theta)\cos(90^\circ-\theta)}{\tan\theta}=\frac{\cos\theta\sin\theta}{\sin\theta/\cos\theta}=\cos^2\theta=1-\sin^2\theta=\text{RHS}$$

EXAMPLE 22 *Prove that*

(i) $\dfrac{\sin\theta\cos(90^\circ-\theta)\cos\theta}{\sin(90^\circ-\theta)}+\dfrac{\cos\theta\sin(90^\circ-\theta)\sin\theta}{\cos(90^\circ-\theta)}=1$

(ii) $\operatorname{cosec}^2(90^\circ-\theta)-\tan^2\theta=\cos^2(90^\circ-\theta)+\cos^2\theta$

SOLUTION (i) We know that $\sin(90^\circ-\theta)=\cos\theta$ and $\cos(90^\circ-\theta)=\sin\theta$

$$\therefore \quad \text{LHS}=\frac{\sin\theta\cos(90^\circ-\theta)\cos\theta}{\sin(90^\circ-\theta)}+\frac{\cos\theta\sin(90^\circ-\theta)\sin\theta}{\cos(90^\circ-\theta)}$$

$$=\frac{\sin\theta\sin\theta\cos\theta}{\cos\theta}+\frac{\cos\theta\cos\theta\sin\theta}{\sin\theta}=\sin^2\theta+\cos^2\theta=1=\text{RHS}$$

(ii) We have,

$$\text{LHS}=\operatorname{cosec}^2(90^\circ-\theta)-\tan^2\theta \qquad [\because \operatorname{cosec}(90^\circ-\theta)=\sec\theta]$$

$$=\sec^2\theta-\tan^2\theta=1 \qquad [\because \sec^2\theta-\tan^2\theta=1]$$

and,

$$\text{RHS} = \cos^2(90^\circ - \theta) + \cos^2\theta = \sin^2\theta + \cos^2\theta = 1 \qquad [\because \cos(90^\circ - \theta) = \sin\theta$$

Hence, LHS = RHS

EXAMPLE 23 *Without using trigonometric tables, evaluate each of the following:*

(i) $\dfrac{\sin^2 20^\circ + \sin^2 70^\circ}{\cos^2 20^\circ + \cos^2 70^\circ} + \dfrac{\sin(90^\circ - \theta)\sin\theta}{\tan\theta} + \dfrac{\cos(90^\circ - \theta)\cos\theta}{\cot\theta}$ **[CBSE 2002C**

(ii) $\cos(40^\circ + \theta) - \sin(50^\circ - \theta) + \dfrac{\cos^2 40^\circ + \cos^2 50^\circ}{\sin^2 40^\circ + \sin^2 50^\circ}$ **[CBSE 2002**

SOLUTION (i) We have,

$$\frac{\sin^2 20^\circ + \sin^2 70^\circ}{\cos^2 20^\circ + \cos^2 70^\circ} + \frac{\sin(90^\circ - \theta)\sin\theta}{\tan\theta} + \frac{\cos(90^\circ - \theta)\cos\theta}{\cot\theta}$$

$$= \frac{\sin^2 20^\circ + \sin^2(90^\circ - 20^\circ)}{\cos^2 20^\circ + \cos^2(90^\circ - 20^\circ)} + \frac{\sin(90^\circ - \theta)\sin\theta}{\tan\theta} + \frac{\cos(90^\circ - \theta)\cos\theta}{\cot\theta}$$

$$= \frac{\sin^2 20^\circ + \cos^2 20^\circ}{\cos^2 20^\circ + \sin^2 20^\circ} + \frac{\cos\theta\sin\theta}{\frac{\sin\theta}{\cos\theta}} + \frac{\sin\theta\cos\theta}{\frac{\cos\theta}{\sin\theta}} \qquad \begin{bmatrix} \sin(90^\circ - \theta) = \cos\theta \\ \text{and} \\ \cos(90^\circ - \theta) = \sin\theta \end{bmatrix}$$

$$= \frac{1}{1} + \cos^2\theta + \sin^2\theta = 1 + 1 = 2$$

(ii) We have,

$$\cos(40^\circ + \theta) - \sin(50^\circ - \theta) + \frac{\cos^2 40^\circ + \cos^2 50^\circ}{\sin^2 40^\circ + \sin^2 50^\circ}$$

$$= \sin[90^\circ - (40^\circ + \theta)] - \sin(50^\circ - \theta) + \frac{\cos^2 40 + \cos^2(90^\circ - 40^\circ)}{\sin^2 40^\circ + \sin^2(90^\circ - 40^\circ)}$$

$$= \sin(50^\circ - \theta) - \sin(50^\circ - \theta) + \frac{\cos^2 40^\circ + \sin^2 40^\circ}{\sin^2 40^\circ + \cos^2 40^\circ} = 0 + \frac{1}{1} = 1$$

EXAMPLE 24 *Without using trigonometric tables, evaluate each of the following:*

(i) $\dfrac{\cos^2 20^\circ + \cos^2 70^\circ}{\sec^2 50^\circ - \cot^2 40^\circ} + 2\,\text{cosec}^2 58^\circ - 2\cot 58^\circ \tan 32^\circ - 4\tan 13^\circ \tan 37^\circ \tan 45^\circ \tan 53^\circ \tan 77^\circ$

[CBSE 2006C]

(ii) $\dfrac{\sec 39^\circ}{\text{cosec}\, 51^\circ} + \dfrac{2}{\sqrt{3}}\tan 17^\circ \tan 38^\circ \tan 60^\circ \tan 52^\circ \tan 73^\circ - 3(\sin^2 31^\circ + \sin^2 59^\circ)$

[CBSE 2006C]

(iii) $\dfrac{-\tan\theta\cot(90^\circ - \theta) + \sec\theta\,\text{cosec}(90^\circ - \theta) + \sin^2 35^\circ + \sin^2 55^\circ}{\tan 10^\circ \tan 20^\circ \tan 30^\circ \tan 70^\circ \tan 80^\circ}$

[CBSE 2005]

(iv) $\dfrac{\sec^2 54^\circ - \cot^2 36^\circ}{\text{cosec}^2 57^\circ - \tan^2 33^\circ} + 2\sin^2 38^\circ \sec^2 52^\circ - \sin^2 45^\circ$

[CBSE 2005]

(v) $\frac{2}{3}\operatorname{cosec}^2 58° - \frac{2}{3}\cot 58° \tan 32° - \frac{5}{3}\tan 13° \tan 37° \tan 45° \tan 53° \tan 77°$

[CBSE 2009]

SOLUTION (i) We have,

$$\frac{\cos^2 20° + \cos^2 70°}{\sec^2 50° - \cot^2 40°} + 2\operatorname{cosec}^2 58° - 2\cot 58° \tan 32° - 4\tan 13° \tan 37° \tan 45° \tan 53° \tan 77°$$

$$= \frac{\cos^2 20° + \cos^2(90° - 20°)}{\sec^2 50° - \cot^2(90° - 50°)} + 2\operatorname{cosec}^2 58° - 2\cot 58° \tan(90° - 58°)$$

$$- 4\tan 13° \tan 37° \tan 45° \tan(90° - 37°) \tan(90° - 13°)$$

$$= \frac{\cos^2 20° + \sin^2 20°}{\sec^2 50° - \tan^2 50°} + 2\operatorname{cosec}^2 58° - 2\cot^2 58° - 4\tan 13° \tan 37° \tan 45° \cot 37° \cot 13°$$

$$= \frac{1}{1} + 2(\operatorname{cosec}^2 58° - \cot^2 58°) - 4(\tan 13° \cot 13°)(\tan 37° \cot 37°) \tan 45°$$

$$= 1 + 2 - 4 \times 1 \times 1 \times 1 = 3 - 4 = -1$$

(ii) We have,

$$\frac{\sec 39°}{\operatorname{cosec} 51°} + \frac{2}{\sqrt{3}}\tan 17° \tan 38° \tan 60° \tan 52° \tan 73° - 3(\sin^2 31° + \sin^2 59°)$$

$$= \frac{\sec 39°}{\operatorname{cosec}(90° - 39°)} + \frac{2}{\sqrt{3}}\tan 17° \tan 38° \tan 60° \tan(90° - 38°) \tan(90° - 17°)$$

$$-3\left(\sin^2 31° + \sin^2(90° - 31°)\right)$$

$$= \frac{\sec 39°}{\sec 39°} + \frac{2}{\sqrt{3}}\tan 17° \tan 38° \times \sqrt{3} \times \cot 38° \times \cot 17° - 3(\sin^2 31° + \cos^2 31°)$$

$$= 1 + \frac{2}{\sqrt{3}} \times \sqrt{3} - 3 \times 1 = 1 + 2 - 3 = 0$$

(iii) We have,

$$\frac{-\tan\theta \cot(90° - \theta) + \sec\theta \operatorname{cosec}(90° - \theta) + \sin^2 35° + \sin^2 55°}{\tan 10° \tan 20° \tan 30° \tan 70° \tan 80°}$$

$$= \frac{-\tan\theta \tan\theta + \sec\theta \sec\theta + \sin^2 35° + \sin^2(90° - 35°)}{\tan 10° \tan 20° \tan 30° \tan(90° - 20°) \tan(90° - 10°)}$$

$$= \frac{-\tan^2\theta + \sec^2\theta + \sin^2 35° + \cos^2 35°}{\tan 10° \tan 20° \tan 30° \cot 20° \cot 10°}$$

$$= \frac{(\sec^2\theta - \tan^2\theta) + (\cos^2 35° + \sin^2 35°)}{(\tan 10° \cot 10°)(\tan 20° \cot 20°) \times \frac{1}{\sqrt{3}}} = \frac{1 + 1}{1 \times 1 \times \frac{1}{\sqrt{3}}} = 2\sqrt{3}$$

(iv) We have,

$$\frac{\sec^2 54° - \cot^2 36°}{\operatorname{cosec}^2 57° - \tan^2 33°} + 2\sin^2 38° \sec^2 52° - \sin^2 45°$$

$$= \frac{\sec^2 54^\circ - \cot^2(90^\circ - 54^\circ)}{\operatorname{cosec}^2(90^\circ - 33^\circ) - \tan^2 33^\circ} + 2\sin^2 38^\circ \sec^2(90^\circ - 38^\circ) - \sin^2 45^\circ$$

$$= \frac{\sec^2 54^\circ - \tan^2 54^\circ}{\sec^2 33^\circ - \tan^2 33^\circ} + 2\sin^2 38^\circ \operatorname{cosec}^2 38^\circ - \sin^2 45^\circ = \frac{1}{1} + 2\times 1 - \left(\frac{1}{\sqrt{2}}\right)^2 = 3 - \frac{1}{2}$$

(v) We have,

$$\frac{2}{3}\operatorname{cosec}^2 58^\circ - \frac{2}{3}\cot 58^\circ \tan 32^\circ - \frac{5}{3}\tan 13^\circ \tan 37^\circ \tan 45^\circ \tan 53^\circ \tan 77^\circ$$

$$= \frac{2}{3}\operatorname{cosec}^2 58^\circ - \frac{2}{3}\cot 58^\circ \tan(90^\circ - 58^\circ) - \frac{5}{3}\tan 13^\circ \tan 37^\circ \tan 45^\circ \tan(90^\circ - 37^\circ)\tan(90^\circ - 13^\circ)$$

$$= \frac{2}{3}\operatorname{cosec}^2 58 - \frac{2}{3}\cot^2 58 - \frac{5}{3}\tan 13^\circ \tan 37^\circ \tan 45^\circ \cot 37^\circ \cot 13^\circ$$

$$= \frac{2}{3}(\operatorname{cosec}^2 58^\circ - \cot^2 58^\circ) - \frac{5}{3}\tan 13^\circ \tan 37^\circ \times 1 \times \frac{1}{\tan 37^\circ} \times \frac{1}{\tan 13^\circ}$$

$$= \frac{2}{3}\times 1 - \frac{5}{3} = \frac{2}{3} - \frac{5}{3} = -1.$$

LEVEL-2

Type I **ON PROVING TRIGONOMETRIC IDENTITIES**

EXAMPLE 25 *Prove the following identities :*

(i) $\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \frac{1+\sin\theta}{\cos\theta}$ [NCERT EXEMPLAR, CBSE 20

(ii) $\frac{\cot A + \operatorname{cosec} A - 1}{\cot A - \operatorname{cosec} A + 1} = \frac{1+\cos A}{\sin A}$

(iii) $\frac{\sin\theta}{\cot\theta + \operatorname{cosec}\theta} = 2 + \frac{\sin\theta}{\cot\theta - \operatorname{cosec}\theta}$ [CBSE 20

(iv) $(\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta) = \frac{1}{\tan\theta + \cot\theta}$ [NCERT, CBSE 20

SOLUTION (i) We have,

$$\text{LHS} = \frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\tan\theta + \sec\theta) - 1}{(\tan\theta - \sec\theta) + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sec\theta + \tan\theta) - (\sec^2\theta - \tan^2\theta)}{\tan\theta - \sec\theta + 1} \quad [\because \sec^2\theta - \tan^2\theta =$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sec\theta + \tan\theta) - (\sec\theta + \tan\theta)(\sec\theta - \tan\theta)}{\tan\theta - \sec\theta + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sec\theta + \tan\theta)[1 - (\sec\theta - \tan\theta)]}{\tan\theta - \sec\theta + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sec\theta + \tan\theta)(1 - \sec\theta + \tan\theta)}{(\tan\theta - \sec\theta + 1)}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sec\theta + \tan\theta)(\tan\theta - \sec\theta + 1)}{(\tan\theta - \sec\theta + 1)}$$

$$\Rightarrow \quad \text{LHS} = \sec\theta + \tan\theta = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = \frac{1 + \sin\theta}{\cos\theta} = \text{RHS}$$

ALITER We have,

$$\text{LHS} = \frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sec\theta + \tan\theta) - 1}{\tan\theta - \sec\theta + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{\dfrac{1}{\sec\theta - \tan\theta} - 1}{\tan\theta - \sec\theta + 1} \qquad \left[\because \sec\theta + \tan\theta = \frac{1}{\sec\theta - \tan\theta}\right]$$

$$\Rightarrow \quad \text{LHS} = \frac{\dfrac{1 - (\sec\theta - \tan\theta)}{\sec\theta - \tan\theta}}{\tan\theta - \sec\theta + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{1 - \sec\theta + \tan\theta}{\tan\theta - \sec\theta + 1} \times \frac{1}{\sec\theta - \tan\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{\sec\theta - \tan\theta} = \sec\theta + \tan\theta \qquad \left[\because \frac{1}{\sec\theta - \tan\theta} = \sec\theta + \tan\theta\right]$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = \frac{1 + \sin\theta}{\cos\theta} = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \frac{\cot A + \operatorname{cosec} A - 1}{\cot A - \operatorname{cosec} A + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\cot A + \operatorname{cosec} A) - (\operatorname{cosec}^2 A - \cot^2 A)}{(\cot A - \operatorname{cosec} A) + 1} \qquad [\because \operatorname{cosec}^2 A - \cot^2 A = 1]$$

$$\Rightarrow \quad \text{LHS} = \frac{(\operatorname{cosec} A + \cot A) - (\operatorname{cosec} A + \cot A)(\operatorname{cosec} A - \cot A)}{\cot A - \operatorname{cosec} A + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\operatorname{cosec} A + \cot A)[1 - (\operatorname{cosec} A - \cot A)]}{\cot A - \operatorname{cosec} A + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\operatorname{cosec} A + \cot A)(\cot A - \operatorname{cosec} A + 1)}{(\cot A - \operatorname{cosec} A + 1)}$$

$$\Rightarrow \quad \text{LHS} = \operatorname{cosec} A + \cot A = \frac{1}{\sin A} + \frac{\cos A}{\sin A} = \frac{1 + \cos A}{\sin A} = \text{RHS}$$

ALITER We have,

$$\text{LHS} = \frac{\cot A + \operatorname{cosec} A - 1}{\cot A - \operatorname{cosec} A + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{\dfrac{1}{\operatorname{cosec} A - \cot A} - 1}{\cot A - \operatorname{cosec} A + 1} \qquad \left[\because \operatorname{cosec} A + \cot A = \frac{1}{\operatorname{cosec} A - \cot A}\right]$$

$\Rightarrow$ $\text{LHS} = \dfrac{\dfrac{1-(\text{cosec } A - \cot A)}{\text{cosec } A - \cot A}}{\cot A - \text{cosec } A + 1}$

$\Rightarrow$ $\text{LHS} = \dfrac{\cot A - \text{cosec } A + 1}{\cot A - \text{cosec } A + 1} \times \dfrac{1}{\text{cosec } A - \cot A}$

$\Rightarrow$ $\text{LHS} = \dfrac{1}{\text{cosec } A - \cot A} = \text{cosec } A + \cot A$ $\left[\because \dfrac{1}{\text{cosec } A - \cot A} = \text{cosec } A + \cot A\right]$

$\Rightarrow$ $\text{LHS} = \dfrac{1}{\sin A} + \dfrac{\cos A}{\sin A} = \dfrac{1+\cos A}{\sin A} = \text{RHS}$

(iii) We have to prove that

$$\frac{\sin\theta}{\cot\theta + \text{cosec}\,\theta} = 2 + \frac{\sin\theta}{\cot\theta - \text{cosec}\,\theta} \text{ or, } \frac{\sin\theta}{\cot\theta + \text{cosec}\,\theta} - \frac{\sin\theta}{\cot\theta - \text{cosec}\,\theta} = 2$$

Now,

$$\text{LHS} = \frac{\sin\theta}{\cot\theta + \text{cosec}\,\theta} - \frac{\sin\theta}{\cot\theta - \text{cosec}\,\theta}$$

$\Rightarrow$ $\text{LHS} = \dfrac{\sin\theta}{\text{cosec}\,\theta + \cot\theta} + \dfrac{\sin\theta}{\text{cosec}\,\theta - \cot\theta}$

$\Rightarrow$ $\text{LHS} = \sin\theta\left\{\dfrac{1}{\text{cosec}\,\theta + \cot\theta} + \dfrac{1}{\text{cosec}\,\theta - \cot\theta}\right\}$

$\Rightarrow$ $\text{LHS} = \sin\theta\left\{\dfrac{\text{cosec}\,\theta - \cot\theta + \text{cosec}\,\theta + \cot\theta}{\text{cosec}^2\theta - \cot^2\theta}\right\} = \sin\theta\left(\dfrac{2\,\text{cosec}\,\theta}{1}\right)$

$\Rightarrow$ $\text{LHS} = \sin\theta\,(2\,\text{cosec}\,\theta) = 2\sin\theta \times \dfrac{1}{\sin\theta} = 2 = \text{RHS}$

ALITER $\text{LHS} = \dfrac{\sin\theta}{\cot\theta + \text{cosec}\,\theta}$

$\Rightarrow$ $\text{LHS} = \sin\theta\,(\text{cosec}\,\theta - \cot\theta)$ $\left[\because \dfrac{1}{\text{cosec}\,\theta + \cot\theta} = \text{cosec}\,\theta - \cot\theta\right]$

$\Rightarrow$ $\text{LHS} = \sin\theta\left(\dfrac{1}{\sin\theta} - \dfrac{\cos\theta}{\sin\theta}\right) = \sin\theta\left(\dfrac{1-\cos\theta}{\sin\theta}\right)$

$\Rightarrow$ $\text{LHS} = 1 - \cos\theta$

$\Rightarrow$ $\text{LHS} = 2 - (1 + \cos\theta)$

$\Rightarrow$ $\text{LHS} = 2 - \dfrac{(1+\cos\theta)(1-\cos\theta)}{1-\cos\theta}$

$\Rightarrow$ $\text{LHS} = 2 - \dfrac{(1-\cos^2\theta)}{1-\cos\theta}$

$\Rightarrow$ $\text{LHS} = 2 - \dfrac{\sin^2\theta}{1-\cos\theta}$

$$\Rightarrow \quad \text{LHS} = 2 - \frac{\sin\theta}{\frac{1-\cos\theta}{\sin\theta}} = 2 - \frac{\sin\theta}{\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}} = 2 - \frac{\sin\theta}{\operatorname{cosec}\theta - \cot\theta} = \text{RHS}$$

(iv) We have,

$$\text{LHS} = (\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta)$$

$$\Rightarrow \quad \text{LHS} = \left(\frac{1}{\sin\theta} - \sin\theta\right)\left(\frac{1}{\cos\theta} - \cos\theta\right)$$

$$\Rightarrow \quad \text{LHS} = \frac{1-\sin^2\theta}{\sin\theta} \times \frac{1-\cos^2\theta}{\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos^2\theta}{\sin\theta} \times \frac{\sin^2\theta}{\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin\theta\cos\theta}{1} = \frac{\sin\theta\cos\theta}{\sin^2\theta + \cos^2\theta} = \frac{1}{\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{\frac{\sin^2\theta}{\sin\theta\cos\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta}} = \frac{1}{\tan\theta + \cot\theta} = \text{RHS}$$

EXAMPLE 26 *Prove the following identities :*

(i) $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1 = 0$

(ii) $\sin^6\theta + \cos^6\theta + 3\sin^2\theta\cos^2\theta = 1$

(iii) $(\sin^8\theta - \cos^8\theta) = (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta)$

SOLUTION (i) We have,

$$\text{LHS} = 2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1$$

$$\Rightarrow \quad \text{LHS} = 2\left\{(\sin^2\theta)^3 + (\cos^2\theta)^3\right\} - 3(\sin^4\theta + \cos^4\theta) + 1$$

Using $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$ and $a^2 + b^2 = (a+b)^2 - 2ab$, we obtain

$$\text{LHS} = 2\left\{(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)\right\}$$
$$-3\left\{(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta + 1\right\}$$

$$\Rightarrow \quad \text{LHS} = 2(1 - 3\sin^2\theta\cos^2\theta) - 3(1 - 2\sin^2\theta\cos^2\theta) + 1$$

$$\Rightarrow \quad \text{LHS} = 2 - 6\sin^2\theta\cos^2\theta - 3 + 6\sin^2\theta\cos^2\theta + 1 = 0 = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \sin^6\theta + \cos^6\theta + 3\sin^2\theta\cos^2\theta$$

$$\Rightarrow \quad \text{LHS} = (\sin^2\theta)^3 + (\cos^2\theta)^3 + 3\sin^2\theta\cos^2\theta$$

$$\Rightarrow \quad \text{LHS} = (\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta) + 3\sin^2\theta\cos^2\theta$$

$$[\because a^3 + b^3 = (a+b)^3 - 3ab(a+b)]$$

$\Rightarrow$ LHS $= 1 - 3\sin^2\theta\cos^2\theta + 3\sin^2\theta\cos^2\theta = 1 =$ RHS

(iii) We have,

LHS $= \sin^8\theta - \cos^8\theta = (\sin^4\theta)^2 - (\cos^4\theta)^2 = (\sin^4\theta - \cos^4\theta)(\sin^4\theta + \cos^4\theta)$

$\Rightarrow$ LHS $= (\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)(\sin^4\theta + \cos^4\theta)$

$\Rightarrow$ LHS $= (\sin^2\theta - \cos^2\theta)\left\{(\sin^2\theta)^2 + (\cos^2\theta)^2 + 2\sin^2\theta\cos^2\theta - 2\sin^2\theta\cos^2\theta\right\}$

$\Rightarrow$ LHS $= (\sin^2\theta - \cos^2\theta)\left\{(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta\right\}$

$\Rightarrow$ LHS $= (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta) =$ RHS

EXAMPLE 27 *Prove the following identities:*

(i) $(1 + \tan A \tan B)^2 + (\tan A - \tan B)^2 = \sec^2 A \sec^2 B$

(ii) $(\tan A + \operatorname{cosec} B)^2 - (\cot B - \sec A)^2 = 2\tan A \cot B(\operatorname{cosec} A + \sec B)$

SOLUTION (i) We have,

LHS $= (1 + \tan A \tan B)^2 + (\tan A - \tan B)^2$

$\Rightarrow$ LHS $= 1 + 2\tan A \tan B + \tan^2 A \tan^2 B + \tan^2 A + \tan^2 B - 2\tan A \tan B$

$\Rightarrow$ LHS $= 1 + \tan^2 A \tan^2 B + \tan^2 A + \tan^2 B$

$\Rightarrow$ LHS $= (1 + \tan^2 A) + (\tan^2 B + \tan^2 A \tan^2 B)$

$\Rightarrow$ LHS $= (1 + \tan^2 A) + \tan^2 B(1 + \tan^2 A)$

$\Rightarrow$ LHS $= (1 + \tan^2 A)(1 + \tan^2 B) = \sec^2 A \sec^2 B =$ RHS

(ii) We have,

LHS $= (\tan A + \operatorname{cosec} B)^2 - (\cot B - \sec A)^2$

$\Rightarrow$ LHS $= (\tan^2 A + \operatorname{cosec}^2 B + 2\tan A \operatorname{cosec} B) - (\cot^2 B + \sec^2 A - 2\cot B \sec A)$

$\Rightarrow$ LHS $= (\tan^2 A - \sec^2 A) + (\operatorname{cosec}^2 B - \cot^2 B) + 2\tan A \operatorname{cosec} B + 2\cot B \sec A$

$\Rightarrow$ LHS $= -1 + 1 + 2\tan A \operatorname{cosec} B + 2\cot B \sec A$

$\Rightarrow$ LHS $= 2(\tan A \operatorname{cosec} B + \cot B \sec A)$

$\Rightarrow$ LHS $= 2\tan A \cot B\left(\dfrac{\operatorname{cosec} B}{\cot B} + \dfrac{\sec A}{\tan A}\right)$ [Dividing and multiplying by $\tan A \cot B$]

$\Rightarrow$ LHS $= 2\tan A \cot B\left\{\dfrac{\frac{1}{\sin B}}{\frac{\cos B}{\sin B}} + \dfrac{\frac{1}{\cos A}}{\frac{\sin A}{\cos A}}\right\}$

$\Rightarrow$ LHS $= 2\tan A \cot B\left(\dfrac{1}{\cos B} + \dfrac{1}{\sin A}\right) = 2\tan A \cot B(\sec B + \operatorname{cosec} A) =$ RHS

EXAMPLE 28 *Prove the following identities:*

(i) $(\sin A + \sec A)^2 + (\cos A + \operatorname{cosec} A)^2 = (1 + \sec A \operatorname{cosec} A)^2$

(ii) $\cot^2 A\left(\dfrac{\sec A - 1}{1 + \sin A}\right) + \sec^2 A\left(\dfrac{\sin A - 1}{1 + \sec A}\right) = 0$

SOLUTION (i) We have,

$$\text{LHS} = (\sin A + \sec A)^2 + (\cos A + \operatorname{cosec} A)^2$$

$$\Rightarrow \quad \text{LHS} = \sin^2 A + \sec^2 A + 2\sin A \sec A + \cos^2 A + \operatorname{cosec}^2 A + 2\cos A \operatorname{cosec} A$$

$$\Rightarrow \quad \text{LHS} = (\sin^2 A + \cos^2 A) + (\sec^2 A + \operatorname{cosec}^2 A) + 2\sin A \sec A + 2\cos A \operatorname{cosec} A$$

$$\Rightarrow \quad \text{LHS} = 1 + \left(\frac{1}{\cos^2 A} + \frac{1}{\sin^2 A}\right) + 2\left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right)$$

$$\Rightarrow \quad \text{LHS} = 1 + \left(\frac{\sin^2 A + \cos^2 A}{\sin^2 A \cos^2 A}\right) + 2\left(\frac{\sin^2 A + \cos^2 A}{\sin A \cos A}\right)$$

$$\Rightarrow \quad \text{LHS} = 1 + \frac{1}{\sin^2 A \cos^2 A} + \frac{2}{\sin A \cos A}$$

$$\Rightarrow \quad \text{LHS} = 1 + \operatorname{cosec}^2 A \sec^2 A + 2\operatorname{cosec} A \sec A = (1 + \sec A \operatorname{cosec} A)^2 = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \cot^2 A\left(\frac{\sec A - 1}{1 + \sin A}\right) + \sec^2 A\left(\frac{\sin A - 1}{1 + \sec A}\right)$$

$$\Rightarrow \quad \text{LHS} = \frac{\cot^2 A(\sec A - 1)(\sec A + 1) + \sec^2 A(\sin A - 1)(1 + \sin A)}{(1 + \sin A)(1 + \sec A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cot^2 A(\sec^2 A - 1) + \sec^2 A(\sin^2 A - 1)}{(1 + \sin A)(1 + \sec A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cot^2 A(\sec^2 A - 1) - \sec^2 A(1 - \sin^2 A)}{(1 + \sin A)(1 + \sec A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cot^2 A \tan^2 A - \sec^2 A \cos^2 A}{(1 + \sin A)(1 + \sec A)} = \frac{1 - 1}{(1 + \sin A)(1 + \sec A)} = 0 = \text{RHS}$$

EXAMPLE 29 *Prove the following identities:*

(i) $\dfrac{\cos A}{1 - \sin A} + \dfrac{\sin A}{1 - \cos A} + 1 = \dfrac{\sin A \cos A}{(1 - \sin A)(1 - \cos A)}$

(ii) $\dfrac{(1 + \cot A + \tan A)(\sin A - \cos A)}{\sec^3 A - \operatorname{cosec}^3 A} = \sin^2 A \cos^2 A$

SOLUTION (i) We have,

$$\Rightarrow \quad \text{LHS} = \frac{\cos A}{1-\sin A} + \frac{\sin A}{1-\cos A} + 1$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos A\,(1-\cos A) + \sin A\,(1-\sin A) + (1-\sin A)(1-\cos A)}{(1-\sin A)(1-\cos A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos A - \cos^2 A + \sin A - \sin^2 A + 1 - \sin A - \cos A + \sin A\cos A}{(1-\sin A)(1-\cos A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\cos A + \sin A) - (\cos^2 A + \sin^2 A) + 1 - (\cos A + \sin A) + \sin A\cos A}{(1-\sin A)(1-\cos A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\cos A + \sin A) - 1 + 1 - (\cos A + \sin A) + \sin A\cos A}{(1-\sin A)(1-\cos A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin A\cos A}{(1-\sin A)(1-\cos A)} = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \frac{\left(1 + \frac{\cos A}{\sin A} + \frac{\sin A}{\cos A}\right)(\sin A - \cos A)}{\left(\frac{1}{\cos^3 A} - \frac{1}{\sin^3 A}\right)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\left(1 + \frac{\cos^2 A + \sin^2 A}{\sin A\cos A}\right)(\sin A - \cos A)}{\left(\frac{\sin^3 A - \cos^3 A}{\sin^3 A\cos^3 A}\right)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\left(1 + \frac{1}{\sin A\cos A}\right)(\sin A - \cos A)\sin^3 A\cos^3 A}{(\sin^3 A - \cos^3 A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sin A\cos A + 1)(\sin A - \cos A)\sin^2 A\cos^2 A}{(\sin A - \cos A)(\sin^2 A + \cos^2 A + \sin A\cos A)}$$

$$\left[\because a^3 - b^3 = (a-b)(a^2 + b^2 + ab)\right]$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sin A\cos A + 1)\sin^2 A\cos^2 A}{(1 + \sin A\cos A)} = \sin^2 A\cos^2 A = \text{RHS}$$

EXAMPLE 30 *If*

$$(\sec A + \tan A)(\sec B + \tan B)(\sec C + \tan C) = (\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C)$$

Prove that each of the side is equal to ± 1.

SOLUTION We have,

$$(\sec A + \tan A)(\sec B + \tan B)(\sec C + \tan C) = (\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C)$$

Multiplying both sides by $(\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C)$, we get

$$(\sec A + \tan A)(\sec B + \tan B)(\sec C + \tan C) \times (\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C)$$

$$= (\sec A - \tan A)^2 (\sec B - \tan B)^2 (\sec C - \tan C)^2$$

$$\Rightarrow \quad (\sec^2 A - \tan^2 A)(\sec^2 B - \tan^2 B)(\sec^2 C - \tan^2 C)$$

$$= (\sec A - \tan A)^2 (\sec B - \tan B)^2 (\sec C - \tan C)^2$$

$$\Rightarrow \quad 1 = \left[(\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C)\right]^2$$

$$\Rightarrow \quad (\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C) = \pm 1$$

Similarly, multiplying both sides by $(\sec A + \tan A)(\sec B + \tan B)(\sec C + \tan C)$, we get

$$(\sec A + \tan A)(\sec B + \tan B)(\sec C + \tan C) = \pm 1$$

EXAMPLE 31 *If* $\sin\theta + \cos\theta = p$ *and* $\sec\theta + \operatorname{cosec}\theta = q$, *show that* $q(p^2 - 1) = 2p$.

[NCERT EXEMPLAR]

SOLUTION We have, $p = \sin\theta + \cos\theta$ and $q = \sec\theta + \operatorname{cosec}\theta$

$$\therefore \quad \text{LHS} = q(p^2 - 1)$$

$$= (\sec\theta + \operatorname{cosec}\theta)\left\{(\sin\theta + \cos\theta)^2 - 1\right\}$$

$$= \left(\frac{1}{\cos\theta} + \frac{1}{\sin\theta}\right)\left\{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 1\right\}$$

$$= \left(\frac{\sin\theta + \cos\theta}{\cos\theta\sin\theta}\right)(1 + 2\sin\theta\cos\theta - 1)$$

$$= \left(\frac{\sin\theta + \cos\theta}{\cos\theta\sin\theta}\right)(2\sin\theta\cos\theta) = 2(\sin\theta + \cos\theta) = 2p = \text{RHS}$$

ALITER We have, $\sec\theta + \operatorname{cosec}\theta = q$

$$\Rightarrow \quad \frac{1}{\cos\theta} + \frac{1}{\sin\theta} = q \Rightarrow \frac{\sin\theta + \cos\theta}{\sin\theta\cos\theta} = q \Rightarrow \frac{p}{\sin\theta\cos\theta} = q \Rightarrow \sin\theta\cos\theta = \frac{p}{q}$$

$$\Rightarrow \quad 2\sin\theta\cos\theta = \frac{2p}{q} \Rightarrow 1 + 2\sin\theta\cos\theta = 1 + \frac{2p}{q} \Rightarrow (\sin\theta + \cos\theta)^2 = 1 + \frac{2p}{q}$$

$$\Rightarrow \quad p^2 = 1 + \frac{2p}{q} \Rightarrow (p^2 - 1) = \frac{2p}{q} \Rightarrow q(p^2 - 1) = 2p$$

EXAMPLE 32 *If* $\sec\theta + \tan\theta = p$, *obtain the values of* $\sec\theta$, $\tan\theta$ *and* $\sin\theta$ *in terms of* p.

[NCERT EXEMPLAR]

SOLUTION We have,

$$\sec\theta + \tan\theta = p \quad \text{...(i)}$$

Now, $\sec^2\theta - \tan^2\theta = 1$

$\Rightarrow$ $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$

$\Rightarrow$ $p(\sec\theta - \tan\theta) = 1 \Rightarrow \sec\theta - \tan\theta = \dfrac{1}{p}$ ...(ii)

Adding and subtracting (i) and (ii), we get

$$(\sec\theta + \tan\theta) + (\sec\theta - \tan\theta) = p + \frac{1}{p} \text{ and, } (\sec\theta + \tan\theta) - (\sec\theta - \tan\theta) = p - \frac{1}{p}$$

$$\Rightarrow \quad 2\sec\theta = p + \frac{1}{p} \text{ and, } 2\tan\theta = p - \frac{1}{p}$$

$$\Rightarrow \quad \sec\theta = \frac{1}{2}\left(p + \frac{1}{p}\right) \text{ and, } \tan\theta = \frac{1}{2}\left(p - \frac{1}{p}\right)$$

Now, $$\sin\theta = \frac{\tan\theta}{\sec\theta} \Rightarrow \sin\theta = \frac{\frac{1}{2}\left(p - \frac{1}{p}\right)}{\frac{1}{2}\left(p + \frac{1}{p}\right)} = \frac{p^2 - 1}{p^2 + 1}$$

EXAMPLE 33 *If* $\sec\theta + \tan\theta = p$, *show that* $\dfrac{p^2 - 1}{p^2 + 1} = \sin\theta$. **[CBSE 2004]**

SOLUTION We have,

$$\text{LHS} = \frac{p^2 - 1}{p^2 + 1} = \frac{(\sec\theta + \tan\theta)^2 - 1}{(\sec\theta + \tan\theta)^2 + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sec^2\theta + \tan^2\theta + 2\sec\theta\tan\theta - 1}{\sec^2\theta + \tan^2\theta + 2\sec\theta\tan\theta + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sec^2\theta - 1) + \tan^2\theta + 2\sec\theta\tan\theta}{\sec^2\theta + 2\sec\theta\tan\theta + (1 + \tan^2\theta)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\tan^2\theta + \tan^2\theta + 2\sec\theta\tan\theta}{\sec^2\theta + 2\sec\theta\tan\theta + \sec^2\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\tan^2\theta + 2\tan\theta\sec\theta}{2\sec^2\theta + 2\sec\theta\tan\theta} = \frac{2\tan\theta(\tan\theta + \sec\theta)}{2\sec\theta(\sec\theta + \tan\theta)} = \frac{\tan\theta}{\sec\theta} = \frac{\sin\theta}{\cos\theta . \sec\theta} = \sin\theta = \text{RHS}$$

ALITER We have, $\sec\theta + \tan\theta = p$ ...(i)

$\therefore$ $(\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = 1 \Rightarrow (\sec\theta - \tan\theta)p = 1 \Rightarrow \sec\theta - \tan\theta = \dfrac{1}{p}$ ...(ii)

Adding (i) and (ii), we obtain

$$2\sec\theta = p + \frac{1}{p} \Rightarrow \sec\theta = \frac{p^2 + 1}{2p} \Rightarrow \cos\theta = \frac{2p}{p^2 + 1}$$

Subtracting (ii) from (i), we obtain

$$2\tan\theta = p - \frac{1}{p}$$

$$\Rightarrow \quad \tan\theta = \frac{p^2-1}{2p}$$

$$\Rightarrow \quad \frac{\sin\theta}{\cos\theta} = \frac{p^2-1}{2p} \Rightarrow \frac{\sin\theta}{\frac{2p}{p^2+1}} = \frac{p^2-1}{2p} \Rightarrow \sin\theta = \frac{p^2-1}{p^2+1}$$

EXAMPLE 34 *If* $\operatorname{cosec}\theta + \cot\theta = p$, *then prove that* $\cos\theta = \frac{p^2-1}{p^2+1}$.

SOLUTION We know that $\operatorname{cosec} - \cot\theta = \frac{1}{\operatorname{cosec}\theta + \cot\theta}$

$$\therefore \quad \operatorname{cosec}\theta + \cot\theta = p \quad \text{...(i)}$$

$$\Rightarrow \quad \operatorname{cosec}\theta - \cot\theta = \frac{1}{p} \quad \text{...(ii)}$$

Adding (i) and (ii), we obtain

$$2\operatorname{cosec}\theta = p + \frac{1}{p} \Rightarrow \operatorname{cosec}\theta = \frac{p^2+1}{2p} \Rightarrow \sin\theta = \frac{2p}{p^2+1} \quad \text{...(iii)}$$

Subtracting (ii) from (i), we obtain

$$2\cot\theta = p - \frac{1}{p} \Rightarrow \cot\theta = \frac{p^2-1}{2p} \quad \text{...(iv)}$$

Now, $\cos\theta = \cot\theta \times \sin\theta$

$$\Rightarrow \quad \cos\theta = \frac{p^2-1}{2p^2} \times \frac{2p}{p^2+1} = \frac{p^2-1}{p^2+1}.$$

EXAMPLE 35 *If* $\frac{\cos\alpha}{\cos\beta} = m$ *and* $\frac{\cos\alpha}{\sin\beta} = n$ *show that* $(m^2+n^2)\cos^2\beta = n^2$.

SOLUTION We have, $m = \frac{\cos\alpha}{\cos\beta}$ and $n = \frac{\cos\alpha}{\sin\beta}$.

$$\therefore \quad \text{LHS} = (m^2+n^2)\cos^2\beta$$

$$\Rightarrow \quad \text{LHS} = \left(\frac{\cos^2\alpha}{\cos^2\beta} + \frac{\cos^2\alpha}{\sin^2\beta}\right)\cos^2\beta \quad \left[\because m = \frac{\cos\alpha}{\cos\beta} \text{ and } n = \frac{\cos\alpha}{\sin\beta}\right]$$

$$\Rightarrow \quad \text{LHS} = \left(\frac{\cos^2\alpha\sin^2\beta + \cos^2\alpha\cos^2\beta}{\cos^2\beta\sin^2\beta}\right)\cos^2\beta$$

$$\Rightarrow \quad \text{LHS} = \cos^2\alpha\left(\frac{\sin^2\beta + \cos^2\beta}{\cos^2\beta\sin^2\beta}\right)\cos^2\beta$$

$$\Rightarrow \quad \text{LHS} = \cos^2\alpha\left(\frac{1}{\cos^2\beta\sin^2\beta}\right)\cos^2\beta = \frac{\cos^2\alpha}{\sin^2\beta} = \left(\frac{\cos\alpha}{\sin\beta}\right)^2 = n^2 = \text{RHS}$$

ALITER We have, $m = \dfrac{\cos\alpha}{\cos\beta}$ and $n = \dfrac{\cos\alpha}{\sin\beta}$.

$$\Rightarrow \quad \cos\beta = \frac{1}{m}\cos\alpha \text{ and } \sin\beta = \frac{1}{n}\cos\alpha$$

$$\Rightarrow \quad \cos^2\beta + \sin^2\beta = \frac{1}{m^2}\cos^2\alpha + \frac{1}{n^2}\cos^2\alpha$$

$$\Rightarrow \quad 1 = \cos^2\alpha\left(\frac{1}{m^2} + \frac{1}{n^2}\right)$$

$$\Rightarrow \quad 1 = \left(\frac{m^2+n^2}{m^2n^2}\right)\cos^2\alpha$$

$$\Rightarrow \quad 1 = \left(\frac{m^2+n^2}{n^2}\right)\left(\frac{\cos\alpha}{m}\right)^2$$

$$\Rightarrow \quad 1 = \left(\frac{m^2+n^2}{n^2}\right)(\cos\beta)^2 \qquad \left[\because m = \frac{\cos\alpha}{\cos\beta} \therefore \frac{\cos\alpha}{m} = \cos\beta\right]$$

$$\Rightarrow \quad (m^2+n^2)\cos^2\beta = n^2.$$

EXAMPLE 36 *If* $\operatorname{cosec}\theta - \sin\theta = l$ *and* $\sec\theta - \cos\theta = m$, *prove that* $l^2m^2(l^2 + m^2 + 3) = 1$.

SOLUTION We have, $l = \operatorname{cosec}\theta - \sin\theta$ and $m = \sec\theta - \cos\theta$

$$\therefore \quad \text{LHS} = l^2m^2(l^2 + m^2 + 3)$$

$$\Rightarrow \quad \text{LHS} = (\operatorname{cosec}\theta - \sin\theta)^2(\sec\theta - \cos\theta)^2\left\{(\operatorname{cosec}\theta - \sin\theta)^2 + (\sec\theta - \cos\theta)^2 + 3\right\}$$

$$\Rightarrow \quad \text{LHS} = \left(\frac{1}{\sin\theta} - \sin\theta\right)^2\left(\frac{1}{\cos\theta} - \cos\theta\right)^2\left\{\left(\frac{1}{\sin\theta} - \sin\theta\right)^2 + \left(\frac{1}{\cos\theta} - \cos\theta\right)^2 + 3\right\}$$

$$\Rightarrow \quad \text{LHS} = \left(\frac{1-\sin^2\theta}{\sin\theta}\right)^2\left(\frac{1-\cos^2\theta}{\cos\theta}\right)^2\left\{\left(\frac{1-\sin^2\theta}{\sin\theta}\right)^2 + \left(\frac{1-\cos^2\theta}{\cos\theta}\right)^2 + 3\right\}$$

$$\Rightarrow \quad \text{LHS} = \left(\frac{\cos^2\theta}{\sin\theta}\right)^2 \left(\frac{\sin^2\theta}{\cos\theta}\right)^2 \left\{\left(\frac{\cos^2\theta}{\sin\theta}\right)^2 + \left(\frac{\sin^2\theta}{\cos\theta}\right)^2 + 3\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos^4\theta}{\sin^2\theta} \times \frac{\sin^4\theta}{\cos^2\theta} \left\{\frac{\cos^4\theta}{\sin^2\theta} + \frac{\sin^4\theta}{\cos^2\theta} + 3\right\}$$

$$\Rightarrow \quad \text{LHS} = \cos^2\theta \times \sin^2\theta \left\{\frac{\cos^6\theta + \sin^6\theta + 3\cos^2\theta\sin^2\theta}{\cos^2\theta\sin^2\theta}\right\}$$

$$\Rightarrow \quad \text{LHS} = \cos^6\theta + \sin^6\theta + 3\cos^2\theta\sin^2\theta$$

$$\Rightarrow \quad \text{LHS} = \left\{(\cos^2\theta)^3 + (\sin^2\theta)^3\right\} + 3\cos^2\theta\sin^2\theta$$

$$\Rightarrow \quad \text{LHS} = \left\{(\cos^2\theta + \sin^2\theta)^3 - 3\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)\right\} + 3\sin^2\theta\cos^2\theta$$

$$[\because a^3 + b^3 = (a+b)^3 - 3ab(a+b)]$$

$$\Rightarrow \quad \text{LHS} = \left\{1 - 3\cos^2\theta\sin^2\theta\right\} + 3\cos^2\theta\sin^2\theta = \text{RHS} \qquad [\because \cos^2\theta + \sin^2\theta = 1]$$

EXAMPLE 37 *If* $\tan A = n\tan B$ and $\sin A = m\sin B$, *prove that* $\cos^2 A = \dfrac{m^2-1}{n^2-1}$.

SOLUTION We have to find $\cos^2 A$ in terms of m and n. This means that the angle B is to be eliminated from the given relations.
Now,

$$\tan A = n\tan B \Rightarrow \tan B = \frac{1}{n}\tan A \Rightarrow \cot B = \frac{n}{\tan A}$$

and,
$$\sin A = m\sin B \Rightarrow \sin B = \frac{1}{m}\sin A \Rightarrow \operatorname{cosec} B = \frac{m}{\sin A}$$

Substituting the values of cot B and cosec B in $\operatorname{cosec}^2 B - \cot^2 B = 1$, we get

$$\Rightarrow \quad \frac{m^2}{\sin^2 A} - \frac{n^2}{\tan^2 A} = 1$$

$$\Rightarrow \quad \frac{m^2}{\sin^2 A} - \frac{n^2\cos^2 A}{\sin^2 A} = 1$$

$$\Rightarrow \quad \frac{m^2 - n^2\cos^2 A}{\sin^2 A} = 1$$

$$\Rightarrow \quad m^2 - n^2\cos^2 A = \sin^2 A$$

$$\Rightarrow \quad m^2 - n^2\cos^2 A = 1 - \cos^2 A$$

$$\Rightarrow \quad m^2 - 1 = n^2\cos^2 A - \cos^2 A$$

$$\Rightarrow \quad m^2 - 1 = (n^2 - 1)\cos^2 A$$

$$\Rightarrow \quad \frac{m^2-1}{n^2-1} = \cos^2 A$$

EXAMPLE 38 *If* $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$ *and* $x\sin\theta = y\cos\theta$, *prove that* $x^2 + y^2 = 1$.

SOLUTION We have,

$$x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$$

$\Rightarrow$ $(x\sin\theta)\sin^2\theta + (y\cos\theta)\cos^2\theta = \sin\theta\cos\theta$

$\Rightarrow$ $x\sin\theta(\sin^2\theta) + (x\sin\theta)\cos^2\theta = \sin\theta\cos\theta$ $[\because x\sin\theta = y\cos\theta]$

$\Rightarrow$ $x\sin\theta(\sin^2\theta + \cos^2\theta) = \sin\theta\cos\theta$

$\Rightarrow$ $x\sin\theta = \sin\theta\cos\theta$

$\Rightarrow$ $x = \cos\theta$

Now,

$$x\sin\theta = y\cos\theta$$

$\Rightarrow$ $\cos\theta\sin\theta = y\cos\theta$ $[\because x = \cos\theta]$

$\Rightarrow$ $y = \sin\theta$

Hence, $x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1$

EXAMPLE 39 *If* $\operatorname{cosec}\theta - \sin\theta = m$ *and* $\sec\theta - \cos\theta = n$, *prove that* $(m^2n)^{2/3} + (mn^2)^{2/3} = 1$

SOLUTION We have,

$$\operatorname{cosec}\theta - \sin\theta = m \text{ and } \sec\theta - \cos\theta = n$$

$\Rightarrow$ $\dfrac{1}{\sin\theta} - \sin\theta = m$ and $\dfrac{1}{\cos\theta} - \cos\theta = n$

$\Rightarrow$ $\dfrac{1-\sin^2\theta}{\sin\theta} = m$ and $\dfrac{1-\cos^2\theta}{\cos\theta} = n$

$\Rightarrow$ $\dfrac{\cos^2\theta}{\sin\theta} = m$ and $\dfrac{\sin^2\theta}{\cos\theta} = n$

$\therefore$ $(m^2n)^{2/3} + (mn^2)^{2/3} = \left(\dfrac{\cos^4\theta}{\sin^2\theta} \times \dfrac{\sin^2\theta}{\cos\theta}\right)^{2/3} + \left(\dfrac{\cos^2\theta}{\sin\theta} \times \dfrac{\sin^4\theta}{\cos^2\theta}\right)^{2/3}$

$$= (\cos^3\theta)^{2/3} + (\sin^3\theta)^{2/3} = \cos^2\theta + \sin^2\theta = 1$$

Hence, $(m^2n)^{2/3} + (mn^2)^{2/3} = 1$

EXAMPLE 40 *If* $\cot\theta + \tan\theta = x$ *and* $\sec\theta - \cos\theta = y$, *prove that* $(x^2y)^{2/3} - (xy^2)^{2/3} = 1$.

SOLUTION We have,

$$\cot\theta + \tan\theta = x \text{ and } \sec\theta - \cos\theta = y$$

$\Rightarrow$ $\dfrac{1}{\tan\theta} + \tan\theta = x$ and $\dfrac{1}{\cos\theta} - \cos\theta = y$

$\Rightarrow$ $\dfrac{1+\tan^2\theta}{\tan\theta} = x$ and $\dfrac{1-\cos^2\theta}{\cos\theta} = y$

$$\Rightarrow \quad \frac{\sec^2\theta}{\tan\theta} = x \text{ and } \frac{\sin^2\theta}{\cos\theta} = y$$

$$\Rightarrow \quad \frac{1}{\cos^2\theta \times \dfrac{\sin\theta}{\cos\theta}} = x \text{ and } \frac{\sin^2\theta}{\cos\theta} = y$$

$$\Rightarrow \quad \frac{1}{\cos\theta\sin\theta} = x \text{ and, } \frac{\sin^2\theta}{\cos\theta} = y$$

$$\therefore \quad (x^2y)^{2/3} - (xy^2)^{2/3} = \left\{\frac{1}{\cos^2\theta\sin^2\theta} \times \frac{\sin^2\theta}{\cos\theta}\right\}^{2/3} - \left\{\frac{1}{\cos\theta\sin\theta} \times \frac{\sin^4\theta}{\cos^2\theta}\right\}^{2/3}$$

$$= \left(\frac{1}{\cos^3\theta}\right)^{2/3} - \left(\frac{\sin^3\theta}{\cos^3\theta}\right)^{2/3} = \frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta} = \sec^2\theta - \tan^2\theta = 1$$

Hence, $(x^2y)^{2/3} - (xy^2)^{2/3} = 1$

EXAMPLE 41 *If* $\sin\theta + \sin^2\theta = 1$, *find the value of*

$$\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta + 2\cos^4\theta + 2\cos^2\theta - 2$$

SOLUTION We have,

$$\sin\theta + \sin^2\theta = 1 \Rightarrow \sin\theta = 1 - \sin^2\theta \Rightarrow \sin\theta = \cos^2\theta$$

$$\therefore \quad \cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta + 2\cos^4\theta + 2\cos^2\theta - 2$$

$$= (\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta) + 2(\cos^4\theta + \cos^2\theta - 1)$$

$$= (\cos^4\theta + \cos^2\theta)^3 + 2(\cos^4\theta + \cos^2\theta - 1)$$

$$= (\sin^2\theta + \cos^2\theta)^3 + 2(\sin^2\theta + \cos^2\theta - 1) \quad \left[\because \cos^2\theta = \sin\theta \therefore \cos^4\theta = \sin^2\theta\right]$$

$$= 1 + 2(1-1) = 1$$

EXAMPLE 42 *If* $a\sec\theta + b\tan\theta + c = 0$ *and* $p\sec\theta + q\tan\theta + r = 0$, *prove that*

$$(br - qc)^2 - (pc - ar)^2 = (aq - bp)^2$$

SOLUTION We have,

$$a\sec\theta + b\tan\theta + c = 0$$

and, $p\sec\theta + q\tan\theta + r = 0$

Solving these two equations by the cross-multiplication for $\sec\theta$ and $\tan\theta$, we get

$$\frac{\sec\theta}{br - qc} = \frac{\tan\theta}{cp - ar} = \frac{1}{aq - bp} \Rightarrow \sec\theta = \frac{br - cq}{aq - bp} \text{ and } \tan\theta = \frac{cp - ar}{aq - bp}$$

$$\therefore \quad \sec^2\theta - \tan^2\theta = 1$$

$$\Rightarrow \quad \left(\frac{br - cq}{aq - bp}\right)^2 - \left(\frac{cp - ar}{aq - bp}\right)^2 = 1 \Rightarrow (br - cq)^2 - (cp - ar)^2 = (aq - bp)^2$$

EXAMPLE 43 *If* $\tan^2\theta = 1 - a^2$, *prove that* $\sec\theta + \tan^3\theta \operatorname{cosec}\theta = (2-a^2)^{3/2}$.

SOLUTION We have,

$$\sec\theta + \tan^3\theta \operatorname{cosec}\theta$$

$$= \sec\theta\left\{\frac{\sec\theta + \tan^3\theta \operatorname{cosec}\theta}{\sec\theta}\right\}$$ [Multiplying and dividing by sec θ]

$$= \sec\theta\left\{1 + \tan^3\theta \cdot \frac{\cos\theta}{\sin\theta}\right\}$$

$$= \sec\theta\left\{1 + \tan^3\theta \times \cot\theta\right\}$$

$$= \sqrt{1+\tan^2\theta}\left\{1+\tan^2\theta\right\}$$

$$= \left\{1+\tan^2\theta\right\}^{3/2} = \left\{1+(1-a^2)\right\}^{3/2} = (2-a^2)^{3/2}$$ $[\because \tan^2\theta = 1 - a^2]$

EXAMPLE 44 *If* $\sin\theta + \sin^2\theta + \sin^3\theta = 1$, *then prove that* $\cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4$

SOLUTION We have,

$$\sin\theta + \sin^2\theta + \sin^3\theta = 1$$

$$\Rightarrow \sin\theta + \sin^3\theta = 1 - \sin^2\theta$$

$$\Rightarrow \sin\theta(1+\sin^2\theta) = \cos^2\theta$$

$$\Rightarrow \sin^2\theta(1+\sin^2\theta)^2 = \cos^4\theta$$

$$\Rightarrow (1-\cos^2\theta)\left\{1+(1-\cos^2\theta)\right\}^2 = \cos^4\theta$$

$$\Rightarrow (1-\cos^2\theta)(2-\cos^2\theta)^2 = \cos^4\theta$$

$$\Rightarrow (1-\cos^2\theta)(4-4\cos^2\theta+\cos^4\theta) = \cos^4\theta$$

$$\Rightarrow 4 - 4\cos^2\theta + \cos^4\theta - 4\cos^2\theta + 4\cos^4\theta - \cos^6\theta = \cos^4\theta$$

$$\Rightarrow -\cos^6\theta + 4\cos^4\theta - 8\cos^2\theta + 4 = 0 \Rightarrow \cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4$$

EXAMPLE 45 *If* $\sin\theta + \cos\theta = \sqrt{2}$, *then prove that* $\tan\theta + \cot\theta = 2$.

SOLUTION We have, [NCERT EXEMPLAR]

$$\sin\theta + \cos\theta = \sqrt{2}$$

$$\Rightarrow (\sin\theta + \cos\theta)^2 = (\sqrt{2})^2$$

$$\Rightarrow \sin^2 + \cos^2\theta + 2\sin\theta\cos\theta = 2$$

$$\Rightarrow 1 + 2\sin\theta\cos\theta = 2$$

$$\Rightarrow 2\sin\theta\cos\theta = 1$$

$$\Rightarrow 2\sin\theta\cos\theta = \sin^2\theta + \cos^2\theta$$ $\left[\because 1 = \sin^2\theta + \cos^2\theta\right]$

$$\Rightarrow \quad \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$$

$$\Rightarrow \quad 2 = \frac{\sin^2\theta}{\sin\theta\cos\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta} \Rightarrow 2 = \tan\theta + \cot\theta$$

EXAMPLE 46 *If* $1 + \sin^2\theta = 3\sin\theta\cos\theta$, *prove that* $\tan\theta = 1$ *or* $\frac{1}{2}$.

[NCERT EXEMPLAR]

SOLUTION We have, $1 + \sin^2\theta = 3\sin\theta\cos\theta$

Dividing both sides by $\cos^2\theta$, we obtain

$$\frac{1 + \sin^2\theta}{\cos^2\theta} = \frac{3\sin\theta\cos\theta}{\cos^2\theta}$$

$$\Rightarrow \quad \frac{1}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{3\sin\theta}{\cos\theta}$$

$$\Rightarrow \quad \sec^2\theta + \tan^2\theta = 3\tan\theta$$

$$\Rightarrow \quad 1 + \tan^2\theta + \tan^2\theta = 3\tan\theta \qquad [\because \sec^2\theta = 1 + \tan^2\theta]$$

$$\Rightarrow \quad 2\tan^2\theta - 3\tan\theta + 1 = 0$$

$$\Rightarrow \quad 2\tan^2\theta - 2\tan\theta - \tan\theta + 1 = 0$$

$$\Rightarrow \quad 2\tan\theta(\tan\theta - 1) - (\tan\theta - 1) = 0$$

$$\Rightarrow \quad (2\tan\theta - 1)(\tan\theta - 1) = 0$$

$$\Rightarrow \quad 2\tan\theta - 1 = 0 \text{ or } \tan\theta - 1 = 0 \Rightarrow 2\tan\theta = 1 \text{ or } \tan\theta = 1 \Rightarrow \tan\theta = \frac{1}{2} \text{ or } \tan\theta = 1$$

EXERCISE 11.1

$1 + \cot^2 A = \text{cosec}^2 A$

LEVEL-1

Prove the following trigonometric identities:

1. $(1 - \cos^2 A)\,\text{cosec}^2 A = 1$
2. $(1 + \cot^2 A)\sin^2 A = 1$ $\quad 1 + \tan^2\theta =$
3. $\tan^2\theta\cos^2\theta = 1 - \cos^2\theta$
4. $\text{cosec}\,\theta\sqrt{1 - \cos^2\theta} = 1$
5. $(\sec^2\theta - 1)(\text{cosec}^2\theta - 1) = 1$
6. $\tan\theta + \frac{1}{\tan\theta} = \sec\theta\,\text{cosec}\,\theta$
7. $\frac{\cos\theta}{1 - \sin\theta} = \frac{1 + \sin\theta}{\cos\theta}$
8. $\frac{\cos\theta}{1 + \sin\theta} = \frac{1 - \sin\theta}{\cos\theta}$
9. $\cos^2 A + \frac{1}{1 + \cot^2 A} = 1$
10. $\sin^2 A + \frac{1}{1 + \tan^2 A} = 1$
11. $\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} = \text{cosec}\,\theta - \cot\theta$
12. $\frac{1 - \cos\theta}{\sin\theta} = \frac{\sin\theta}{1 + \cos\theta}$

13. $\dfrac{\sin\theta}{1-\cos\theta} = \operatorname{cosec}\theta + \cot\theta$

14. $\dfrac{1-\sin\theta}{1+\sin\theta} = (\sec\theta - \tan\theta)^2$

15. $\dfrac{(1+\cot^2\theta)\tan\theta}{\sec^2\theta} = \cot\theta$

16. $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$

17. $(\sec\theta + \cos\theta)(\sec\theta - \cos\theta) = \tan^2\theta + \sin^2\theta$

18. $(\operatorname{cosec}\theta + \sin\theta)(\operatorname{cosec}\theta - \sin\theta) = \cot^2\theta + \cos^2\theta$

19. $\sec A(1-\sin A)(\sec A + \tan A) = 1$

20. $(\operatorname{cosec} A - \sin A)(\sec A - \cos A)(\tan A + \cot A) = 1$

21. $(1+\tan^2\theta)(1-\sin\theta)(1+\sin\theta) = 1$

22. $\sin^2 A\cot^2 A + \cos^2 A\tan^2 A = 1$

23. (i) $\cot\theta - \tan\theta = \dfrac{2\cos^2\theta - 1}{\sin\theta\cos\theta}$

(ii) $\tan\theta - \cot\theta = \dfrac{2\sin^2\theta - 1}{\sin\theta\cos\theta}$

(iii) $\dfrac{\sin A - 2\sin^3 A}{2\cos^3 A - \cos A} = \tan A$ [CBSE 2018]

24. $\dfrac{\cos^2\theta}{\sin\theta} - \operatorname{cosec}\theta + \sin\theta = 0$

25. $\dfrac{1}{1+\sin A} + \dfrac{1}{1-\sin A} = 2\sec^2 A$

26. $\dfrac{1+\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1+\sin\theta} = 2\sec\theta$ [NCERT]

27. $\dfrac{(1+\sin\theta)^2 + (1-\sin\theta)^2}{2\cos^2\theta} = \dfrac{1+\sin^2\theta}{1-\sin^2\theta}$

28. $\dfrac{1+\tan^2\theta}{1+\cot^2\theta} = \left(\dfrac{1-\tan\theta}{1-\cot\theta}\right)^2 = \tan^2\theta$ [NCERT]

29. $\dfrac{1+\sec\theta}{\sec\theta} = \dfrac{\sin^2\theta}{1-\cos\theta}$ [NCERT]

30. $\dfrac{\tan\theta}{1-\cot\theta} + \dfrac{\cot\theta}{1-\tan\theta} = 1 + \tan\theta + \cot\theta$

31. $\sec^6\theta = \tan^6\theta + 3\tan^2\theta\sec^2\theta + 1$

32. $\operatorname{cosec}^6\theta = \cot^6\theta + 3\cot^2\theta\operatorname{cosec}^2\theta + 1$

33. $\dfrac{(1+\tan^2\theta)\cot\theta}{\operatorname{cosec}^2\theta} = \tan\theta$

34. $\dfrac{1+\cos A}{\sin^2 A} = \dfrac{1}{1-\cos A}$

35. $\dfrac{\sec A - \tan A}{\sec A + \tan A} = \dfrac{\cos^2 A}{(1+\sin A)^2}$

36. $\dfrac{1+\cos A}{\sin A} = \dfrac{\sin A}{1-\cos A}$

37. (i) $\sqrt{\dfrac{1+\sin A}{1-\sin A}} = \sec A + \tan A$ [NCERT]

(ii) $\sqrt{\dfrac{1-\cos A}{1+\cos A}} + \sqrt{\dfrac{1+\cos A}{1-\cos A}} = 2\operatorname{cosec} A$

38. Prove that :

(i) $\sqrt{\dfrac{\sec\theta - 1}{\sec\theta + 1}} + \sqrt{\dfrac{\sec\theta + 1}{\sec\theta - 1}} = 2\operatorname{cosec}\theta$ [CBSE 2001, 2006C]

(ii) $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} + \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = 2\sec\theta$ [CBSE 2001]

(iii) $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} + \sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = 2\operatorname{cosec}\theta$

(iv) $\dfrac{\sec\theta - 1}{\sec\theta + 1} = \left(\dfrac{\sin\theta}{1+\cos\theta}\right)^2$ [CBSE 2001 C]

39. $(\sec A - \tan A)^2 = \dfrac{1-\sin A}{1+\sin A}$

40. $\dfrac{1-\cos A}{1+\cos A} = (\cot A - \operatorname{cosec} A)^2$

41. $\dfrac{1}{\sec A - 1} + \dfrac{1}{\sec A + 1} = 2\operatorname{cosec} A \cot A$

42. $\dfrac{\cos A}{1-\tan A} + \dfrac{\sin A}{1-\cot A} = \sin A + \cos A$

43. $\dfrac{\operatorname{cosec} A}{\operatorname{cosec} A - 1} + \dfrac{\operatorname{cosec} A}{\operatorname{cosec} A + 1} = 2\sec^2 A$

44. $\dfrac{\tan^2 A}{1+\tan^2 A} + \dfrac{\cot^2 A}{1+\cot^2 A} = 1$

45. $\dfrac{\cot A - \cos A}{\cot A + \cos A} = \dfrac{\operatorname{cosec} A - 1}{\operatorname{cosec} A + 1}$

[NCERT, CBSE 2008]

46. $\dfrac{1+\cos\theta - \sin^2\theta}{\sin\theta(1+\cos\theta)} = \cot\theta$

47. (i) $\dfrac{1+\cos\theta+\sin\theta}{1+\cos\theta-\sin\theta} = \dfrac{1+\sin\theta}{\cos\theta}$

(ii) $\dfrac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \dfrac{1}{\sec\theta - \tan\theta}$ [CBSE 2001, NCERT]

(iii) $\dfrac{\cos\theta - \sin\theta + 1}{\cos\theta + \sin\theta - 1} = \operatorname{cosec}\theta + \cot\theta$

(iv) $(\sin\theta + \cos\theta)(\tan\theta + \cot\theta) = \sec\theta + \operatorname{cosec}\theta$ [NCERT EXEMPLAR]

48. $\dfrac{1}{\sec A + \tan A} - \dfrac{1}{\cos A} = \dfrac{1}{\cos A} - \dfrac{1}{\sec A - \tan A}$ [CBSE 2005]

49. $\tan^2 A + \cot^2 A = \sec^2 A \operatorname{cosec}^2 A - 2$

50. $\dfrac{\tan A}{1+\sec A} - \dfrac{\tan A}{1-\sec A} = 2\operatorname{cosec} A$ [NCERT EXEMPLAR]

51. $1 + \dfrac{\cot^2\theta}{1+\operatorname{cosec}\theta} = \operatorname{cosec}\theta$ [NCERT EXEMPLAR]

52. $\dfrac{\cos\theta}{\operatorname{cosec}\theta + 1} + \dfrac{\cos\theta}{\operatorname{cosec}\theta - 1} = 2\tan\theta$

53. $\left(1+\tan^2 A\right) + \left(1 + \dfrac{1}{\tan^2 A}\right) = \dfrac{1}{\sin^2 A - \sin^4 A}$ [CBSE 2006C]

54. $\sin^2 A \cos^2 B - \cos^2 A \sin^2 B = \sin^2 A - \sin^2 B$

55. (i) $\dfrac{\cot A + \tan B}{\cot B + \tan A} = \cot A \tan B$

(ii) $\dfrac{\tan A + \tan B}{\cot A + \cot B} = \tan A \tan B$

56. $\cot^2 A \operatorname{cosec}^2 B - \cot^2 B \operatorname{cosec}^2 A = \cot^2 A - \cot^2 B$

57. $\tan^2 A \sec^2 B - \sec^2 A \tan^2 B = \tan^2 A - \tan^2 B$

LEVEL-2

Prove the following identities: (58-75)

58. If $x = a\sec\theta + b\tan\theta$ and $y = a\tan\theta + b\sec\theta$, prove that $x^2 - y^2 = a^2 - b^2$ [CBSE 2001, 2002C]

59. If $3\sin\theta + 5\cos\theta = 5$, prove that $5\sin\theta - 3\cos\theta = \pm 3$.

60. If $\operatorname{cosec}\theta + \cot\theta = m$ and $\operatorname{cosec}\theta - \cot\theta = n$, prove that $mn = 1$

61. $\dfrac{\tan^3\theta}{1+\tan^2\theta} + \dfrac{\cot^3\theta}{1+\cot^2\theta} = \sec\theta\operatorname{cosec}\theta - 2\sin\theta\cos\theta$

62. If $T_n = \sin^n\theta + \cos^n\theta$, prove that $\dfrac{T_3 - T_5}{T_1} = \dfrac{T_5 - T_7}{T_3}$

63. $\left(\tan\theta + \dfrac{1}{\cos\theta}\right)^2 + \left(\tan\theta - \dfrac{1}{\cos\theta}\right)^2 = 2\left(\dfrac{1+\sin^2\theta}{1-\sin^2\theta}\right)$

64. $\left(\dfrac{1}{\sec^2\theta - \cos^2\theta} + \dfrac{1}{\operatorname{cosec}^2\theta - \sin^2\theta}\right)\sin^2\theta\cos^2\theta = \dfrac{1-\sin^2\theta\cos^2\theta}{2+\sin^2\theta\cos^2\theta}$

65. (i) $\left(\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}\right)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$ (ii) $\dfrac{1+\sec\theta-\tan\theta}{1+\sec\theta+\tan\theta} = \dfrac{1-\sin\theta}{\cos\theta}$

[NCERT EXEMPLAR]

66. $(\sec A + \tan A - 1)(\sec A - \tan A + 1) = 2\tan A$

67. $(1 + \cot A - \operatorname{cosec} A)(1 + \tan A + \sec A) = 2$

68. $(\operatorname{cosec}\theta - \sec\theta)(\cot\theta - \tan\theta) = (\operatorname{cosec}\theta + \sec\theta)(\sec\theta\operatorname{cosec}\theta - 2)$

69. $(\sec A - \operatorname{cosec} A)(1 + \tan A + \cot A) = \tan A\sec A - \cot A\operatorname{cosec} A$

70. $\dfrac{\cos A\operatorname{cosec} A - \sin A\sec A}{\cos A + \sin A} = \operatorname{cosec} A - \sec A$

71. $\dfrac{\sin A}{\sec A + \tan A - 1} + \dfrac{\cos A}{\operatorname{cosec} A + \cot A - 1} = 1$

72. $\dfrac{\tan A}{(1+\tan^2 A)^2} + \dfrac{\cot A}{(1+\cot^2 A)^2} = \sin A\cos A$

73. $\sec^4 A(1 - \sin^4 A) - 2\tan^2 A = 1$

74. $\dfrac{\cot^2 A(\sec A - 1)}{1+\sin A} = \sec^2 A\left(\dfrac{1-\sin A}{1+\sec A}\right)$

75. $(1 + \cot A + \tan A)(\sin A - \cos A) = \dfrac{\sec A}{\operatorname{cosec}^2 A} - \dfrac{\operatorname{cosec} A}{\sec^2 A} = \sin A\tan A - \cot A\cos A$

[CBSE 2008]

76. If $\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1$ and $\dfrac{x}{a}\sin\theta - \dfrac{y}{b}\cos\theta = 1$, prove that $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2$

77. If $\operatorname{cosec}\theta - \sin\theta = a^3$, $\sec\theta - \cos\theta = b^3$, prove that $a^2 b^2(a^2 + b^2) = 1$

78. If $a\cos^3\theta + 3a\cos\theta\sin^2\theta = m$, $a\sin^3\theta + 3a\cos^2\theta\sin\theta = n$, prove that $(m+n)^{2/3} + (m-n)^{2/3} = 2a^{2/3}$

79. If $x = a\cos^3\theta$, $y = b\sin^3\theta$, prove that $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$.

80. If $a\cos\theta + b\sin\theta = m$ and $a\sin\theta - b\cos\theta = n$, prove that $a^2 + b^2 = m^2 + n^2$

81. If $\cos A + \cos^2 A = 1$, prove that $\sin^2 A + \sin^4 A = 1$

82. If $\cos\theta + \cos^2\theta = 1$, prove that
$\sin^{12}\theta + 3\sin^{10}\theta + 3\sin^8\theta + \sin^6\theta + 2\sin^4\theta + 2\sin^2\theta - 2 = 1$

83. Given that: $(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma) = (1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)$
Show that one of the values of each member of this equality is $\sin\alpha\sin\beta\sin\gamma$

84. If $\sin\theta + \cos\theta = x$, prove that $\sin^6\theta + \cos^6\theta = \frac{4 - 3(x^2-1)^2}{4}$

85. If $x = a\sec\theta\cos\phi$, $y = b\sec\theta\sin\phi$ and $z = c\tan\theta$, show that $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

86. If $\sin\theta + 2\cos\theta = 1$ prove that $2\sin\theta - \cos\theta = 2$. [NCERT EXEMPLAR]

11.3 VALUES OF TRIGONOMETRIC RATIOS IN TERMS OF THE VALUE OF ONE OF THEM

In this section, we will find the remaining trigonometric ratios of an angle when one of the trigonometric ratios of the same angle is given. We will also find the values of trigonometric expressions for the given value of one of the trigonometric ratios.

Finding all other trigonometric ratios of angle θ when the value of $\sin\theta$ is given:

Let $\sin\theta = x$. Then,

$$\cos\theta = \sqrt{1-\sin^2\theta} \Rightarrow \cos\theta = \sqrt{1-x^2}$$

$$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = \frac{x}{\sqrt{1-x^2}},\ \cot\theta = \frac{1}{\tan\theta} = \frac{\sqrt{1-x^2}}{x},$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{x} \text{ and } \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-x^2}}$$

Finding all other trigonometric ratios of angle θ when the value of $\cos\theta$ is given:

Let $\cos\theta = x$. Then,

$$\sin\theta = \sqrt{1-\cos^2\theta} \Rightarrow \sin\theta = \sqrt{1-x^2}$$

$$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1-x^2}}{x},\ \cot\theta = \frac{1}{\tan\theta} = \frac{x}{\sqrt{1-x^2}}$$

$$\text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{1}{\sqrt{1-x^2}},\ \sec\theta = \frac{1}{\cos\theta} = \frac{1}{x}$$

Finding all other trigonometric ratios of an angle when the value of is given:

Let $\tan\theta = x$. Then,

$$\sec\theta = \sqrt{1+\tan^2\theta} \Rightarrow \sec\theta = \sqrt{1+x^2}$$

$$\therefore \quad \cot\theta = \frac{1}{\tan\theta} = \frac{1}{x},\ \cos\theta = \frac{1}{\sec\theta} = \frac{1}{\sqrt{1+x^2}}$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{1+x^2}} = \frac{x}{\sqrt{1+x^2}} \text{ and, } \text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{\sqrt{1+x^2}}{x}$$

If the values of cosec, are given. From the given values we obtain the values of respectively and then proceed as discussed above.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *If* $\sin\theta = \frac{3}{5}$, *find the values of other trigonometric ratios.*

SOLUTION We have, $\sin\theta = \frac{3}{5}$

$$\therefore \quad \cos\theta = \sqrt{1-\sin^2\theta} \Rightarrow \cos\theta = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\therefore \quad \text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{5}{3},\ \sec\theta = \frac{1}{\cos\theta} = \frac{5}{4}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{3/5}{4/5} = \frac{3}{4} \text{ and } \cot\theta = \frac{1}{\tan\theta} = \frac{4}{3}$$

EXAMPLE 2 *If* $\cot\theta = \frac{9}{40}$, *find the values of* cosec θ *and* sec θ.

SOLUTION We have, $\cot\theta = \frac{9}{40}$

$$\therefore \quad \text{cosec}\,\theta = \sqrt{1+\cot^2\theta} \Rightarrow \text{cosec}\,\theta = \sqrt{1+\left(\frac{9}{40}\right)^2} = \sqrt{1+\frac{81}{1600}} = \sqrt{\frac{1681}{1600}} = \frac{41}{40}$$

Again, $\cot\theta = \frac{9}{40} \Rightarrow \tan\theta = \frac{1}{\cot\theta} = \frac{40}{9}$

$$\therefore \quad \sec\theta = \sqrt{1+\tan^2\theta} \Rightarrow \sec\theta = \sqrt{1+\left(\frac{40}{9}\right)^2} = \sqrt{\frac{1681}{81}} = \frac{41}{9}$$

EXAMPLE 3 *If* $\cos\theta = \frac{1}{2}$, *find the value of* $\frac{2\sec\theta}{1+\tan^2\theta}$.

SOLUTION We have, $\cos\theta = \dfrac{1}{2}$

$$\therefore \quad \sec\theta = \frac{1}{\cos\theta} = 2$$

$$\therefore \quad \frac{2\sec\theta}{1+\tan^2\theta} = \frac{2\sec\theta}{\sec^2\theta} = \frac{2}{\sec\theta} = \frac{2}{2} = 1$$

EXAMPLE 4 *If* $\tan\theta = \dfrac{12}{5}$, *find the value of* $\dfrac{1+\sin\theta}{1-\sin\theta}$.

SOLUTION We have, $\tan\theta = \dfrac{12}{5}$

$$\therefore \quad \sec\theta = \sqrt{1+\tan^2\theta} \Rightarrow \sec\theta = \sqrt{1+\left(\frac{12}{5}\right)^2} = \sqrt{\frac{169}{25}} = \frac{13}{5}$$

$$\therefore \quad \cos\theta = \frac{1}{\sec\theta} \Rightarrow \cos\theta = \frac{5}{13}$$

Now, $\sin\theta = \sqrt{1-\cos^2\theta} \Rightarrow \sin\theta = \sqrt{1-\left(\dfrac{5}{13}\right)^2} = \sqrt{\dfrac{144}{169}} = \dfrac{12}{13}$

Hence, $\dfrac{1+\sin\theta}{1-\sin\theta} = \dfrac{1+\frac{12}{13}}{1-\frac{12}{13}} = \dfrac{\frac{25}{13}}{\frac{1}{13}} = 25$

EXAMPLE 5 *If* $\sin\theta = \dfrac{3}{5}$, *find the value of* $(\tan\theta+\sec\theta)^2$.

SOLUTION We have, $\sin\theta = \dfrac{3}{5}$

$$\therefore \quad \cos\theta = \sqrt{1-\sin^2\theta} \Rightarrow \cos\theta = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4} \text{ and, } \sec\theta = \frac{1}{\cos\theta} \Rightarrow \sec\theta = \frac{5}{4}$$

Hence, $(\tan\theta+\sec\theta)^2 = \left(\dfrac{3}{4}+\dfrac{5}{4}\right)^2 = \left(\dfrac{8}{4}\right)^2 = 4$

EXAMPLE 6 *If* $\tan\theta = \dfrac{3}{4}$, *find the value of* $\dfrac{1-\cos\theta}{1+\cos\theta}$.

SOLUTION We have, $\tan\theta = \dfrac{3}{4}$

$$\therefore \quad \sec\theta = \sqrt{1+\tan^2\theta} \Rightarrow \sec\theta = \sqrt{1+\left(\frac{3}{4}\right)^2} = \sqrt{1+\frac{9}{16}} = \frac{5}{4}$$

and, $\cos\theta = \dfrac{1}{\sec\theta} \Rightarrow \cos\theta = \dfrac{4}{5}$

$\therefore$ $\dfrac{1-\cos\theta}{1+\cos\theta} = \dfrac{1-\dfrac{4}{5}}{1+\dfrac{4}{5}} = \dfrac{\dfrac{1}{5}}{\dfrac{9}{5}} = \dfrac{1}{9}$

EXAMPLE 7 *If* $\cos\theta = \dfrac{3}{5}$, *find the value of* $\cot\theta + \text{cosec}\,\theta$.

SOLUTION We have, $\cos\theta = \dfrac{3}{5}$

$\therefore$ $\sin\theta = \sqrt{1-\cos^2\theta} \Rightarrow \sin\theta = \sqrt{1-\left(\dfrac{3}{5}\right)^2} = \sqrt{1-\dfrac{9}{25}} = \dfrac{4}{5}$

and, $\cot\theta = \dfrac{\cos\theta}{\sin\theta} \Rightarrow \cot\theta = \dfrac{3/5}{4/5} = \dfrac{3}{4}$, $\text{cosec}\,\theta = \dfrac{1}{\sin\theta} \Rightarrow \text{cosec}\,\theta = \dfrac{5}{4}$

$\therefore$ $\cot\theta + \text{cosec}\,\theta = \dfrac{3}{4} + \dfrac{5}{4} = \dfrac{8}{4} = 2$

EXAMPLE 8 *If* $\tan\theta = \dfrac{1}{\sqrt{7}}$, *find the value of* $\dfrac{\text{cosec}^2\theta - \sec^2\theta}{\text{cosec}^2\theta + \sec^2\theta}$.

SOLUTION We have, $\tan\theta = \dfrac{1}{\sqrt{7}}$

$\therefore$ $\cot\theta = \dfrac{1}{\tan\theta} = \sqrt{7}$

Now,

$$\sec^2\theta = 1 + \tan^2\theta \Rightarrow \sec^2\theta = 1 + \left(\dfrac{1}{\sqrt{7}}\right)^2 = 1 + \dfrac{1}{7} = \dfrac{8}{7}$$

and, $\text{cosec}^2\theta = 1 + \cot^2\theta \Rightarrow \text{cosec}^2\theta = 1 + (\sqrt{7})^2 = 1 + 7 = 8.$

$\therefore$ $\dfrac{\text{cosec}^2\theta - \sec^2\theta}{\text{cosec}^2\theta + \sec^2\theta} = \dfrac{8-\dfrac{8}{7}}{8+\dfrac{8}{7}} = \dfrac{48/7}{64/7} = \dfrac{48}{64} = \dfrac{3}{4}$

EXAMPLE 9 *If* $\text{cosec}\,A = \sqrt{2}$, *find the value of* $\dfrac{2\sin^2 A + 3\cot^2 A}{4\tan^2 A - \cos^2 A}$.

SOLUTION We have, $\text{cosec}\,A = \sqrt{2}$

$\therefore$ $\sin A = \dfrac{1}{\text{cosec}\,A} \Rightarrow \sin A = \dfrac{1}{\sqrt{2}}$

$$\cos A = \sqrt{1-\sin^2 A} \Rightarrow \cos A = \sqrt{1-\left(\dfrac{1}{\sqrt{2}}\right)^2} = \dfrac{1}{\sqrt{2}}$$

$$\tan A = \frac{\sin A}{\cos A} \Rightarrow \tan A = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 1 \text{ and, } \cot A = \frac{1}{\tan A} \Rightarrow \cot A = \frac{1}{1} = 1$$

Hence, $$\frac{2\sin^2 A + 3\cot^2 A}{4\tan^2 A - \cos^2 A} = \frac{2\times\left(\frac{1}{\sqrt{2}}\right)^2 + 3\,(1)^2}{4\,(1)^2 - \left(\frac{1}{\sqrt{2}}\right)^2} = \frac{2\times\frac{1}{2}+3}{4-\frac{1}{2}} = \frac{1+3}{7/2} = \frac{8}{7}$$

EXAMPLE 10 *If* $\cot\theta = \frac{15}{8}$, *then evaluate:* $\frac{(2+2\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(2-2\cos\theta)}$ **[CBSE 2009]**

SOLUTION We have,

$$\frac{(2+2\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(2-2\cos\theta)}$$

$$= \frac{2\,(1+\sin\theta)(1-\sin\theta)}{2\,(1+\cos\theta)(1-\cos\theta)}$$

$$= \frac{2\,(1-\sin^2\theta)}{2\,(1-\cos^2\theta)} = \frac{\cos^2\theta}{\sin^2\theta} = \left(\frac{\cos\theta}{\sin\theta}\right)^2 = (\cot\theta)^2$$

$\therefore$ $$\cot\theta = \frac{15}{8} \Rightarrow \frac{(2+2\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(2-2\cos\theta)} = (\cot\theta)^2 = \left(\frac{15}{8}\right)^2 = \frac{225}{64}$$

EXAMPLE 11 *If* $\sin\theta = \frac{a}{\sqrt{a^2+b^2}}, 0 < \theta < 90°$, *find the values of* $\cos\theta$ *and* $\tan\theta$.

SOLUTION We have, $\sin\theta = \frac{a}{\sqrt{a^2+b^2}}$

$\therefore$ $$\cos\theta = \sqrt{1-\sin^2\theta} \Rightarrow \cos\theta = \sqrt{1-\frac{a^2}{a^2+b^2}} = \sqrt{\frac{b^2}{a^2+b^2}} = \frac{b}{\sqrt{a^2+b^2}}$$

and, $$\tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = \frac{a/\sqrt{a^2+b^2}}{b/\sqrt{a^2+b^2}} = \frac{a}{b}$$

LEVEL-2

EXAMPLE 12 *If* $\sin\theta + \cos\theta = \sqrt{2}\sin(90° - \theta)$, *determine* $\cot\theta$.

SOLUTION We have,

$$\sin\theta + \cos\theta = \sqrt{2}\sin(90° - \theta)$$

$$\Rightarrow \sin\theta + \cos\theta = \sqrt{2}\cos\theta$$

$$\Rightarrow \sin\theta = \sqrt{2}\cos\theta - \cos\theta$$

$$\Rightarrow \sin\theta = (\sqrt{2}-1)\cos\theta$$

$\Rightarrow \quad \dfrac{\sin\theta}{\cos\theta} = \dfrac{(\sqrt{2}-1)\cos\theta}{\cos\theta}$ [Dividing throughout by $\cos\theta$

$\Rightarrow \quad \tan\theta = (\sqrt{2}-1)$

$\Rightarrow \quad \cot\theta = \dfrac{1}{\sqrt{2}-1}$ $\left[\because \cot\theta = \dfrac{1}{\tan\theta}\right]$

$\Rightarrow \quad \cot\theta = \dfrac{(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} = \dfrac{\sqrt{2}+1}{2-1} = \sqrt{2}+1$

EXAMPLE 13 *If* $\tan\theta + \cot\theta = 2$, *find the value of* $\tan^2\theta + \cot^2\theta$.

SOLUTION We have,

$\tan\theta + \cot\theta = 2$

$\Rightarrow \quad (\tan\theta + \cot\theta)^2 = 4$ [On squaring both sides]

$\Rightarrow \quad \tan^2\theta + \cot^2\theta + 2\tan\theta\cot\theta = 4$

$\Rightarrow \quad \tan^2\theta + \cot^2\theta + 2 = 4$ $[\because \tan\theta\cot\theta = 1]$

$\Rightarrow \quad \tan^2\theta + \cot^2\theta = 2$

EXAMPLE 14 *Prove that* $\tan\theta + \tan(90^\circ - \theta) = \sec\theta\sec(90^\circ - \theta)$. [NCERT EXEMPLAR

SOLUTION LHS $= \tan\theta + \tan(90^\circ - \theta)$

$= \tan\theta + \cot\theta = \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = \dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \dfrac{1}{\sin\theta\cos\theta}$

$= \sec\theta\,\mathrm{cosec}\,\theta = \sec\theta\sec(90^\circ - \theta) =$ RHS.

EXAMPLE 15 *Show that* $\dfrac{\cos^2(45^\circ + \theta) + \cos^2(45^\circ - \theta)}{\tan(60^\circ + \theta)\tan(30^\circ - \theta)} = 1$. [NCERT EXEMPLAR

SOLUTION LHS $= \dfrac{\cos^2(45^\circ + \theta) + \cos^2(45^\circ - \theta)}{\tan(60^\circ + \theta)\tan(30^\circ - \theta)}$

$= \dfrac{\cos^2(45^\circ + \theta) + \sin^2(90^\circ - (45^\circ - \theta))}{\tan(60^\circ + \theta)\cot(90^\circ - (30^\circ - \theta))}$

$= \dfrac{\cos^2(45^\circ + \theta) + \sin^2(45^\circ + \theta)}{\tan(60^\circ + \theta)\cot(60^\circ + \theta)} = \dfrac{1}{1} = 1 =$ RHS

EXAMPLE 16 *Given that* $\alpha + \beta = 90^\circ$, *show that* $\sqrt{\cos\alpha\,\mathrm{cosec}\,\beta - \cos\alpha\sin\beta} = \sin\alpha$.

[NCERT EXEMPLAR

SOLUTION We have, $\alpha + \beta = 90^\circ$. Therefore, $\beta = 90^\circ - \alpha$.

$\therefore \quad \mathrm{cosec}\,\beta = \mathrm{cosec}(90^\circ - \alpha) = \sec\alpha$ and, $\sin\beta = \sin(90^\circ - \alpha) = \cos\alpha$

Now, $\sqrt{\cos\alpha\,\mathrm{cosec}\,\beta - \cos\alpha\sin\beta}$

$= \sqrt{\cos\alpha\sec\alpha - \cos\alpha\cos\alpha} = \sqrt{1 - \cos^2\alpha} = \sqrt{\sin^2\alpha} = \sin\alpha$

EXAMPLE 17 *If* $\sec\theta = x + \frac{1}{4x}$, *prove that :* $\sec\theta + \tan\theta = 2x$ *or,* $\frac{1}{2x}$ **[CBSE 2001]**

SOLUTION We have,

$$\sec\theta = x + \frac{1}{4x}$$

$$\therefore \quad \tan^2\theta = \sec^2\theta - 1$$

$$\Rightarrow \quad \tan^2\theta = \left(x + \frac{1}{4x}\right)^2 - 1 = x^2 + \frac{1}{16x^2} + \frac{1}{2} - 1 = x^2 + \frac{1}{16x^2} - \frac{1}{2} = \left(x - \frac{1}{4x}\right)^2$$

$$\Rightarrow \quad \tan\theta = \pm\left(x - \frac{1}{4x}\right)$$

$$\Rightarrow \quad \tan\theta = \left(x - \frac{1}{4x}\right) \text{ or, } \tan\theta = -\left(x - \frac{1}{4x}\right)$$

CASE I When $\tan\theta = -\left(x - \frac{1}{4x}\right)$: In this case,

$$\sec\theta + \tan\theta = x + \frac{1}{4x} + x - \frac{1}{4x} = 2x$$

CASE II When $\tan\theta = -\left(x - \frac{1}{4x}\right)$: In this case,

$$\sec\theta + \tan\theta = \left(x + \frac{1}{4x}\right) - \left(x - \frac{1}{4x}\right) = \frac{2}{4x} = \frac{1}{2x}$$

Hence, $\sec\theta + \tan\theta = 2x$ or, $\frac{1}{2x}$

ALITER Let $\sec\theta + \tan\theta = \lambda$ ...(i)

Then,

$$\Rightarrow \quad \sec^2\theta - \tan^2\theta = 1$$

$$\Rightarrow \quad (\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = 1$$

$$\Rightarrow \quad \lambda(\sec\theta - \tan\theta) = 1 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \sec\theta - \tan\theta = \frac{1}{\lambda} \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2\sec\theta = \lambda + \frac{1}{\lambda}$$

$$\Rightarrow \quad 2\left(x + \frac{1}{4x}\right) = \lambda + \frac{1}{\lambda} \qquad \left[\because \sec\theta = x + \frac{1}{4x} \text{ (Given)}\right]$$

$$\Rightarrow \quad 2x + \frac{1}{2x} = \lambda + \frac{1}{\lambda}$$

$$\Rightarrow \quad \lambda = 2x \text{ or, } \lambda = \frac{1}{2x} \qquad \text{[On comparing two sides]}$$

$$\Rightarrow \quad \sec\theta + \tan\theta = 2x, \frac{1}{2x} \qquad \text{[Using (i)]}$$

EXERCISE 11.2

LEVEL-1

1. If $\cos\theta = \frac{4}{5}$, find all other trigonometric ratios of angle θ

2. If $\sin\theta = \frac{1}{\sqrt{2}}$, find all other trigonometric ratios of angle θ

3. If $\tan\theta = \frac{1}{\sqrt{2}}$, find the value of $\frac{\operatorname{cosec}^2\theta - \sec^2\theta}{\operatorname{cosec}^2\theta + \cot^2\theta}$

4. If $4\tan\theta = 3$, evaluate $\frac{4\sin\theta - \cos\theta + 1}{4\sin\theta + \cos\theta - 1}$ [CBSE 2018]

5. If $\tan\theta = \frac{12}{5}$, find the value of $\frac{1+\sin\theta}{1-\sin\theta}$

6. If $\cot\theta = \frac{1}{\sqrt{3}}$, find the value of $\frac{1-\cos^2\theta}{2-\sin^2\theta}$

7. If $\operatorname{cosec} A = \sqrt{2}$, find the value of $\frac{2\sin^2 A + 3\cot^2 A}{4(\tan^2 A - \cos^2 A)}$

8. If $\cot\theta = \sqrt{3}$, find the value of $\frac{\operatorname{cosec}^2\theta + \cot^2\theta}{\operatorname{cosec}^2\theta - \sec^2\theta}$

9. If $3\cos\theta = 1$, find the value of $\frac{6\sin^2\theta + \tan^2\theta}{4\cos\theta}$

10. If $\sqrt{3}\tan\theta = 3\sin\theta$, find the value of $\sin^2\theta - \cos^2\theta$ [CBSE 2001]

11. If $\operatorname{cosec}\theta = \frac{13}{12}$, find the value of $\frac{2\sin\theta - 3\cos\theta}{4\sin\theta - 9\cos\theta}$ [CBSE 2001]

LEVEL-2

12. If $\sin\theta + \cos\theta = \sqrt{2}\cos(90° - \theta)$, find $\cot\theta$.

13. If $2\sin^2\theta - \cos^2\theta = 2$, then find the value of θ. [NCERT EXEMPLAR]

13. If $\sqrt{3}\tan\theta - 1 = 0$, find the value of $\sin^2 - \cos^2\theta$. [NCERT EXEMPLAR]

ANSWERS

1. $\sin\theta = 3/5$, $\tan\theta = 3/4$, $\sec\theta = 5/4$, $\operatorname{cosec}\theta = 5/3$, $\cot\theta = 4/3$
2. $\cos\theta = \frac{1}{\sqrt{2}}$, $\tan\theta = 1$, $\sec\theta = \sqrt{2}$, $\operatorname{cosec}\theta = \sqrt{2}$, $\cot\theta = 1$
3. $\frac{3}{10}$ 4. $\frac{13}{11}$ 5. 25 6. $\frac{3}{5}$ 7. 2 8. $\frac{21}{8}$
9. 10 10. $\frac{1}{3}$ 11. 3 12. $\sqrt{2}-1$ 13. $90°$ 14. $-\frac{1}{2}$

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

Answer each of the following questions either in one word or one sentence or as per requirement of the questions:

1. Define an identity.
2. What is the value of $(1-\cos^2\theta)\ \operatorname{cosec}^2\theta$?
3. What is the value of $(1+\cot^2\theta)\ \sin^2\theta$?
4. What is the value of $\sin^2\theta + \frac{1}{1+\tan^2\theta}$?
5. If $\sec\theta + \tan\theta = x$, write the value of $\sec\theta - \tan\theta$ in terms of x.
6. If $\operatorname{cosec}\theta - \cot\theta = \alpha$, write the value of $\operatorname{cosec}\theta + \cot\alpha$.
7. Write the value of $\operatorname{cosec}^2(90^\circ-\theta) - \tan^2\theta$.
8. Write the value of $\sin A\cos(90^\circ - A) + \cos A\sin(90^\circ - A)$.
9. Write the value of $\cot^2\theta - \frac{1}{\sin^2\theta}$.
10. If $x = a\sin\theta$ and $y = b\cos\theta$, what is the value of $b^2x^2 + a^2y^2$?
11. If $\sin\theta = \frac{4}{5}$, what is the value of $\cot\theta + \operatorname{cosec}\theta$?
12. What is the value of $9\cot^2\theta - 9\operatorname{cosec}^2\theta$?
13. What is the value of $6\tan^2\theta - \frac{6}{\cos^2\theta}$?
14. What is the value of $\frac{\tan^2\theta - \sec^2\theta}{\cot^2\theta - \operatorname{cosec}^2\theta}$?
15. What is the value of $(1+\tan^2\theta)(1-\sin\theta)(1+\sin\theta)$?
16. If $\cos A = \frac{7}{25}$, find the value of $\tan A + \cot A$. [CBSE 2008]
17. If $\sin\theta = \frac{1}{3}$, then find the value of $2\cot^2\theta + 2$. [CBSE 2009]
18. If $\cos\theta = \frac{3}{4}$, then find the value of $9\tan^2\theta + 9$.
19. If $\sec^2\theta(1+\sin\theta)(1-\sin\theta) = k$, then find the value of k. [CBSE 2009]
20. If $\operatorname{cosec}^2\theta(1+\cos\theta)(1-\cos\theta) = \lambda$, then find the value of λ.
21. If $\sin^2\theta\cos^2\theta(1+\tan^2\theta)(1+\cot^2\theta) = \lambda$, then find the value of λ.
22. If $5x = \sec\theta$ and $\frac{5}{x} = \tan\theta$, find the value of $5\left(x^2 - \frac{1}{x^2}\right)$. [CBSE 2010]
23. If $\operatorname{cosec}\theta = 2x$ and $\cot\theta = \frac{2}{x}$, find the value of $2\left(x^2 - \frac{1}{x^2}\right)$ [CBSE 2010]

24. Write 'True' or 'False' and justify your answer in each of the following:

(i) The value of $\sin\theta$ is $x + \frac{1}{x}$, where 'x' is a positive real number.

(ii) $\cos\theta = \frac{a^2 + b^2}{2ab}$, where a and b are two distinct numbers such that $ab > 0$.

(iii) The value of $\cos^2 23 - \sin^2 67$ is positive.

(iv) The value of the expression $\sin 80° - \cos 80°$ is negative.

(v) The value of $\sin\theta + \cos\theta$ is always greater than 1.

25. What is the value of $\cos^2 67° - \sin^2 23°$? [CBSE 2018]

ANSWERS

2. 1	3. 1	4. 1	5. $\frac{1}{x}$	6. $\frac{1}{\alpha}$	7. 1	8. 1
9. −1	10. a^2b^2	11. 2	12. −9	13. −6	14. 1	15. 1
16. $\frac{625}{168}$	17. 18	18. 16	19. 1	20. 1	21. 1	22. $\frac{1}{5}$
23. $\frac{1}{2}$	24. (i) False	(ii) False	(iii) False	(iv) False	(v) False	25. 0

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If $\sec\theta + \tan\theta = x$, then $\sec\theta =$

(a) $\frac{x^2+1}{x}$ (b) $\frac{x^2+1}{2x}$ (c) $\frac{x^2-1}{2x}$ (d) $\frac{x^2-1}{x}$

2. If $\sec\theta + \tan\theta = x$, then $\tan\theta =$

(a) $\frac{x^2+1}{x}$ (b) $\frac{x^2-1}{x}$ (c) $\frac{x^2+1}{2x}$ (d) $\frac{x^2-1}{2x}$

3. $\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$ is equal to

(a) $\sec\theta + \tan\theta$ (b) $\sec\theta - \tan\theta$ (c) $\sec^2\theta + \tan^2\theta$ (d) $\sec^2\theta - \tan^2\theta$

4. The value of $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$ is

(a) $\cot\theta - \operatorname{cosec}\theta$ (b) $\operatorname{cosec}\theta + \cot\theta$ (c) $\operatorname{cosec}^2\theta + \cot^2\theta$ (d) $(\cot\theta + \operatorname{cosec}\theta)^2$

5. $\sec^4 A - \sec^2 A$ is equal to

(a) $\tan^2 A - \tan^4 A$ (b) $\tan^4 A - \tan^2 A$ (c) $\tan^4 A + \tan^2 A$ (d) $\tan^2 A + \tan^4 A$

6. $\cos^4 A - \sin^4 A$ is equal to

(a) $2\cos^2 A + 1$ (b) $2\cos^2 A - 1$ (c) $2\sin^2 A - 1$ (d) $2\sin^2 A + 1$

7. $\dfrac{\sin\theta}{1+\cos\theta}$ is equal to

(a) $\dfrac{1+\cos\theta}{\sin\theta}$ (b) $\dfrac{1-\cos\theta}{\cos\theta}$ (c) $\dfrac{1-\cos\theta}{\sin\theta}$ (d) $\dfrac{1-\sin\theta}{\cos\theta}$

8. $\dfrac{\sin\theta}{1-\cot\theta}+\dfrac{\cos\theta}{1-\tan\theta}$ is equal to

(a) 0 (b) 1 (c) $\sin\theta+\cos\theta$ (d) $\sin\theta-\cos\theta$

9. The value of $(1+\cot\theta-\operatorname{cosec}\theta)(1+\tan\theta+\sec\theta)$ is

(a) 1 (b) 2 (c) 4 (d) 0

10. $\dfrac{\tan\theta}{\sec\theta-1}+\dfrac{\tan\theta}{\sec\theta+1}$ is equal to

(a) $2\tan\theta$ (b) $2\sec\theta$ (c) $2\operatorname{cosec}\theta$ (d) $2\tan\theta\sec\theta$

11. $(\operatorname{cosec}\theta-\sin\theta)(\sec\theta-\cos\theta)(\tan\theta+\cot\theta)$ is equal

(a) 0 (b) 1 (c) -1 (d) none of these

12. If $x=a\cos\theta$ and $y=b\sin\theta$, then $b^2x^2+a^2y^2=$

(a) a^2b^2 (b) ab (c) a^4b^4 (d) a^2+b^2

13. If $x=a\sec\theta$ and $y=b\tan\theta$, then $b^2x^2-a^2y^2=$

(a) ab (b) a^2-b^2 (c) a^2+b^2 (d) a^2b^2

14. $\dfrac{\cot\theta}{\cot\theta-\cot 3\theta}+\dfrac{\tan\theta}{\tan\theta-\tan 3\theta}$ is equal to

(a) 0 (b) 1 (c) -1 (d) 2

15. $2(\sin^6\theta+\cos^6\theta)-3(\sin^4\theta+\cos^4\theta)$ is equal to

(a) 0 (b) 1 (c) -1 (d) none of these

16. If $a\cos\theta+b\sin\theta=4$ and $a\sin\theta-b\cos\theta=3$, then $a^2+b^2=$

(a) 7 (b) 12 (c) 25 (d) none of these

17. If $a\cot\theta+b\operatorname{cosec}\theta=p$ and $b\cot\theta+a\operatorname{cosec}\theta=q$, then $p^2-q^2=$

(a) a^2-b^2 (b) b^2-a^2 (c) a^2+b^2 (d) $b-a$

18. The value of $\sin^2 29°+\sin^2 61°$ is

(a) 1 (b) 0 (c) $2\sin^2 29°$ (d) $2\cos^2 61$

19. If $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$ and $z=r\cos\theta$, then

(a) $x^2+y^2+z^2=r^2$ (b) $x^2+y^2-z^2=r^2$

(c) $x^2-y^2+z^2=r^2$ (d) $z^2+y^2-x^2=r^2$

20. If $\sin\theta + \sin^2\theta = 1$, then $\cos^2\theta + \cos^4\theta =$
(a) -1 (b) 1 (c) 0 (d) none of these

21. If $a\cos\theta + b\sin\theta = m$ and $a\sin\theta - b\cos\theta = n$, then $a^2 + b^2 =$
(a) $m^2 - n^2$ (b) m^2n^2 (c) $n^2 - m^2$ (d) $m^2 + n^2$

22. If $\cos A + \cos^2 A = 1$, then $\sin^2 A + \sin^4 A =$
(a) -1 (b) 0 (c) 1 (d) none of these

23. If $x = a\sec\theta\cos\phi$, $y = b\sec\theta\sin\phi$ and $z = c\tan\theta$, then $\frac{x^2}{a^2} + \frac{y^2}{b^2} =$
(a) $\frac{z^2}{c^2}$ (b) $1 - \frac{z^2}{c^2}$ (c) $\frac{z^2}{c^2} - 1$ (d) $1 + \frac{z^2}{c^2}$

24. If $a\cos\theta - b\sin\theta = c$, then $a\sin\theta + b\cos\theta =$
(a) $\pm\sqrt{a^2 + b^2 + c^2}$ (b) $\pm\sqrt{a^2 + b^2 - c^2}$ (c) $\pm\sqrt{c^2 - a^2 - b^2}$ (d) none of thes

25. $9\sec^2 A - 9\tan^2 A$ is equal to
(a) 1 (b) 9 (c) 8 (d) 0
[NCER

26. $(1 + \tan\theta + \sec\theta)(1 + \cot\theta - \text{cosec}\,\theta) =$
(a) 0 (b) 1 (c) 1 (d) -1

27. $(\sec A + \tan A)(1 - \sin A) =$
(a) $\sec A$ (b) $\sin A$ (c) $\text{cosec}\,A$ (d) $\cos A$
[NCER

28. $\frac{1 + \tan^2 A}{1 + \cot^2 A}$ is equal to
(a) $\sec^2 A$ (b) -1 (c) $\cot^2 A$ (d) $\tan^2 A$
[NCER

29. If $\sin\theta - \cos\theta = 0$, then the value of $\sin^4\theta + \cos^4\theta$ is
(a) 1 (b) $\frac{3}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

30. The value of $\sin(45° + \theta) - \cos(45° - \theta)$ is equal to
(a) $2\cos\theta$ (b) 0 (c) $2\sin\theta$ (d) 1

31. If ΔABC is right angled at C, then the value of $\cos(A + B)$ is
(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) $\frac{\sqrt{3}}{2}$

32. If $\cos 9\theta = \sin\theta$ and $9\theta < 90°$, then the value of $\tan 6\theta$ is

(a) $1/\sqrt{3}$ (b) $\sqrt{3}$ (c) 1 (d) 0

33. If $\cos(\alpha+\beta) = 0$, then $\sin(\alpha-\beta)$ can be reduced to

(a) $\cos\beta$ (b) $\cos 2\beta$ (c) $\sin\alpha$ (d) $\sin 2\alpha$

ANSWERS

1. (b)	2. (d)	3. (a)	4. (b)	5. (c)	6. (b)
7. (c)	8. (c)	9. (b)	10. (c)	11. (b)	12. (a)
13. (d)	14. (b)	15. (c)	16. (c)	17. (b)	18. (a)
19. (a)	20. (b)	21. (d)	22. (c)	23. (d)	24. (b)
25. (b)	26. (c)	27. (d)	28. (d)	29. (c)	30. (b)
31. (a)	32. (b)	33. (b)			

SUMMARY

1. An equation is called an identity if it is true for all values of the variable (s) involved.
2. An equation involving trigonometric ratios of an angle is called a trigonometric identity if it is true for all values of the angle.
3. Following are some trigonometric identities:

(i) $\sin^2\theta + \cos^2\theta = 1$ or, $1 - \cos^2\theta = \sin^2\theta$ or, $1 - \sin^2\theta = \cos^2\theta$

(ii) $1 + \tan^2\theta = \sec^2\theta$ or, $\sec^2\theta - \tan^2\theta = 1$

(iii) $1 + \cot^2\theta = \text{cosec}^2\theta$ or, $\text{cosec}^2\theta - \cot^2\theta = 1$.