CHAPTER 2

POLYNOMIALS

2.1 INTRODUCTION

In earlier classes, we have learnt about polynomials in one variable, their degrees, factors, multiples and zeros (or roots). In this chapter, we will study about the geometrical representation of linear and quadrat $p \mid b$ ic polynomials and geometrical meaning of their zeros. We will also study about the relationship between the zeros and coefficients of a polynomial. Let us first recall some useful definitions and results which we have studied in class IX.

2.2 RECAPITULATION

POLYNOMIAL *Let x be a variable, n be a positive integer and as,* $a_1, a_2, \dots, a_n$ *be constants (real numbers). Then,* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ *is called a polynomial in variable x.*

In the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$, $a_n x^n, a_{n-1}x^{n-1}, \dots, a_1 x$ and a_0 are known as the terms of the polynomial and $a_n, a_{n-1}, a_{n-2}, \dots, a_1$ and a_0 are their coefficients.

For example,

(i) $p(x) = 3x - 2$ is a polynomial in variable x.

(ii) $q(y) = 3y^2 - 2y + 4$ is a polynomial in variable y.

(iii) $f(u) = \frac{1}{2}u^3 - 3u^2 + 2u - 4$ is a polynomial in variable u.

Note that the expressions like $2x^2 - 3\sqrt{x} + 5, \frac{1}{x^2 - 2x + 5}, 2x^3 - \frac{3}{x} + 4$ etc. are not polynomials.

DEGREE OF A POLYNOMIAL *The exponent of the highest degree term in a polynomial is known as its degree.*

In other words, the highest power of x in a polynomial $f(x)$ is called the degree of the polynomial $f(x)$.

For example,

(i) $f(x) = 3x + \frac{1}{2}$ is a polynomial of degree 1 in the variable x.

(ii) $g(y) = 2y^2 - \frac{3}{2}y + 7$ is a polynomial of degree 2 in the variable y.

(iii) $p(x) = 5x^3 - 3x^2 + x - \frac{1}{\sqrt{2}}$ is a polynomial of degree 3 in the variable x.

(iv) $q(u) = 9u^5 - \frac{2}{3}u^4 + u^2 - \frac{1}{2}$ is a polynomial of degree 5 in the variable u.

CONSTANT POLYNOMIAL *A polynomial of degree zero is called a constant polynomial.*

For example, $f(x) = 7, g(x) = -\frac{3}{2}, h(y) = 2, p(t) = 1$ etc are constant polynomial.

The constant polynomial 0 or $f(x) = 0$ is called the zero polynomial. The degree of the zero polynomial is not defined, because

$$f(x) = 0, g(x) = 0x, h(x) = 0x^2, p(x) = 0x^3, q(x) = 0x^{12}$$

etc. are all equal to the zero polynomial.

LINEAR POLYNOMIAL *A polynomial of degree 1 is called a linear polynomial.*

For example, $p(x) = 4x - 3, q(y) = 3y, f(t) = \sqrt{3}\,t + 5$ and $g(u) = \frac{2}{3}u - \frac{5}{2}$ etc are all linear polynomials.

Polynomials such as $f(x) = 2x^2 + 3, g(x) = 3 - x^2$ etc are not linear polynomials.

More generally, any linear polynomial in variable x with real coefficients is of the form $f(x) = ax + b$, where a, b are real numbers and $a \neq 0$.

REMARK 1 *A linear polynomial may be a monomial or a binomial. Because, linear polynomial* $f(x) = \frac{2}{3}x - \frac{5}{2}$ *is a binomial whereas the linear polynomial* $g(x) = \frac{2}{5}x$ *is a monomial.*

QUADRATIC POLYNOMIAL *polynomial of degree 2 is called a quadratic polynomial.*

The name 'quadratic' has been derived from 'quadrate', which means 'square'.

For example,

$$f(x) = 2x^2 + 3x - \frac{4}{5}, g(y) = 2y^2 - 3, h(u) = 2 - u^2 + \sqrt{3}\,u,\ p(v) = \sqrt{3}\,v^2 - \frac{4}{3}v + \frac{1}{2},$$

$q(\alpha) = \frac{2}{3}\alpha^2 + 4\alpha$ etc. are quadratic polynomials with real coefficients.

More generally, any quadratic polynomial in variable x with real coefficients is of the form $f(x) = ax^2 + bx + c$, where a, b, c are real numbers and $a \neq 0$.

REMARK 2 *A quadratic polynomial may be a monomial or a binomial or a trinomial, because* $f(x) = \frac{1}{5}x^2$ *is a monomial,* $g(x) = 3x^2 - 5$ *is a binomial and* $h(x) = 3x^2 - 2x + 5$ *is a trinomial.*

CUBIC POLYNOMIAL *A polynomial of degree 3 is called a cubic polynomial.*

For example,

(i) $f(x) = \frac{9}{5}x^3 - 2x^2 + \frac{7}{3}x - \frac{1}{5}$ is a cubic polynomial in variable x.

(ii) $g(y) = 2y^3 + 5y - 7$ is a cubic polynomial in variable y.

(iii) $p(u) = \frac{\sqrt{2}}{3}u^3 + 1$ is a cubic polynomial in variable u.

The most general form of a cubic polynomial with coefficients as real numbers is

$$f(x) = ax^3 + bx^2 + cx + d, \text{ where } a \neq 0, b, c, d \text{ are real numbers.}$$

BI-QUADRATIC POLYNOMIAL *A fourth degree polynomial is called a biquadratic polynomial.*

For example,

(i) $f(x)=\frac{3}{5}x^4-2x^3+\frac{3}{2}x^2-\sqrt{2}x+\frac{1}{5}$ is a biquadratic polynomial with real coefficients in variable x.

(ii) $g(y)=2y^4+3$ is a biquadratic polynomial in variable y.

(iii) $h(u)=3u^4-5u^2+2$ is a biquadratic polynomial in variable u.

The most general form of a biquadratic polynomial with real coefficients in variable x is

$$f(x)=ax^4+bx^3+cx^2+dx+e,\text{ where } a\neq 0, b, c, d, e \text{ are real numbers.}$$

REMARK 3 *Throughout this chapter, we shall be studying polynomials with real coefficients.*

VALUE OF A POLYNOMIAL *If $f(x)$ is a polynomial and α is any real number, then the real number obtained by replacing x by α in $f(x)$, is called the value of $f(x)$ at $x=\alpha$ and is denoted by $f(\alpha)$.*

The values of the quadratic polynomial $f(x)=2x^2-3x-2$ at $x=1$ and $x=-2$ are given by

$$f(1)=2\times(1)^2-3\times 1-2=2-3-2=-3$$

and, $$f(-2)=2\times(-2)^2-3\times(-2)-2=8+6-2=12$$

If $f(x)=2x^3-13x^2+17x+12$, then its value at $x=-\frac{1}{2}$ is

$$f\left(-\frac{1}{2}\right)=2\times\left(-\frac{1}{2}\right)^3-13\times\left(-\frac{1}{2}\right)^2+17\times\left(-\frac{1}{2}\right)+12=-\frac{1}{4}-\frac{13}{4}-\frac{17}{2}+12=0$$

Consider the cubic polynomial $f(x)=x^3-6x^2+11x-6$. The value of this polynomial at $x=2$ is given by

$$f(2)=2^3-6\times 2^2+11\times 2-6=8-24+22-6=0$$

Also, $$f(1)=1^3-6\times 1^2+11\times 1-6=1-6+11-6=0$$

and, $$f(3)=3^3-6\times 3^2+11\times 3-6=27-54+33-6=0$$

Thus, we find that the values of $f(x)$ at $x=1, 2,$ and 3 are each equal to zero. So, 1, 2 and 3 are called zeros of the cubic polynomial $f(x)=x^3-6x^2+11x-6$.

Thus, we may define zeros of a polynomial as follows:

ZERO OF A POLYNOMIAL *A real number α is a zero of a polynomial $f(x)$, if $f(\alpha)=0$.*

Finding a zero of a polynomial $f(x)$ means solving the polynomial equation $f(x)=0$.

In class IX, we have learnt how to find the zero of a linear polynomial. We have studied that the linear polynomial $f(x)=ax+b, a\neq 0$ has only one zero α given by

$$\alpha=-\frac{b}{a}=-\frac{\text{Constant term}}{\text{Coefficient of } x}$$

We observe that the zero of a linear polynomial is related to its coefficients. In fact, zeros of any polynomial are related to its coefficients. We will study this in the susequent sections. Let us first discuss about the graphs of polynomials of degree 1, 2 and 3.

2.3 GRAPHS OF POLYNOMIALS

In algebraic or in set theoretic language the graph of a polynomial $f(x)$ is the collection (or set) of all points (x, y), where $y = f(x)$. In geometrical or in graphical language the graph of a polynomial $f(x)$ is a smooth free hand curve passing through points (x_1, y_1), (x_2, y_2), (x_3, y_3), ... etc, where $y_1, y_2, y_3, \ldots$ are the values of the polynomial $f(x)$ at $x_1, x_2, x_3 \ldots$ respectively.

In this section, we will learn about the construction of graphs of linear, quadratic and cubic polynomials.

In order to draw the graph of a polynomial $f(x)$, we may follow the following algorithm.

ALGORITHM

STEP I *Find the values* $y_1, y_2, \ldots, y_n, \ldots$ *of polynomial* $f(x)$ *at different points* x_1 $x_2, \ldots, x_n, \ldots$ *and prepare a table that gives values of* y *or* $f(x)$ *for various values of* x.

x:	x_1	x_2	...	x_n	x_{n+1}	...
$y = f(x)$:	$y_1 = f(x_1)$	$y_2 = f(x_2)$	...	$y_n = f(x_n)$	$y_{n+1} = f(x_{n+1})$	...

STEP II *Plot the points* $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n), \ldots$ *on rectangular coordinate system. In plotting these points you may use different scales on the x and y-axes.*

STEP III *Draw a free hand smooth curve passing through points plotted in step II to get the graph of the polynomial* $f(x)$.

2.3.1 GRAPH OF A LINEAR POLYNOMIAL

Consider a linear polynomial $f(x) = ax + b, a \neq 0$. In class IX, we have learnt that the graph of $y = ax + b$ is a straight line. That is why $f(x) = ax + b$ is called a linear polynomial. Since two points determine a straight line, so only two points need to be plotted to draw the line $y = ax + b$. The line represented by $y = ax + b$ crosses the x-axis at exactly one point, namely $\left(-\frac{b}{a}, 0\right)$.

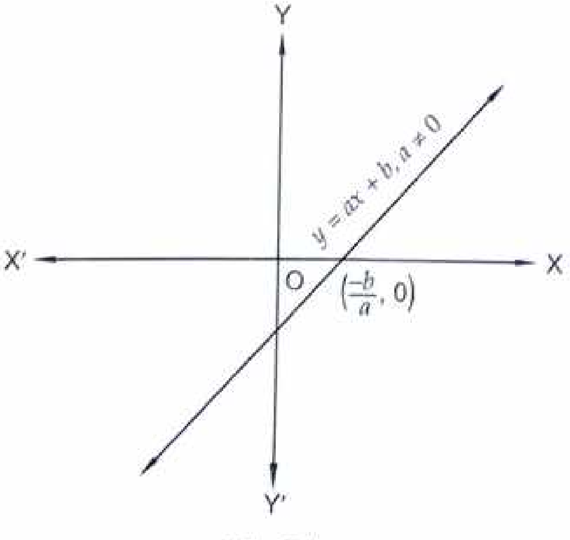

Fig. 2.1

ILLUSTRATION 1 *Draw the graph of the polynomial* $f(x) = 2x - 5$. *Also, find the coordinates of the point where it crosses X-axis.*

SOLUTION Let $y = 2x - 5$.

The following table lists the values of y corresponding to different values of x.

x	1	4
y	-3	3

The points $A(1, -3)$ and $B(4, 3)$ are plotted on the graph paper on a suitable scale. A line is drawn passing through these points to obtain the graph of the given polynomial.

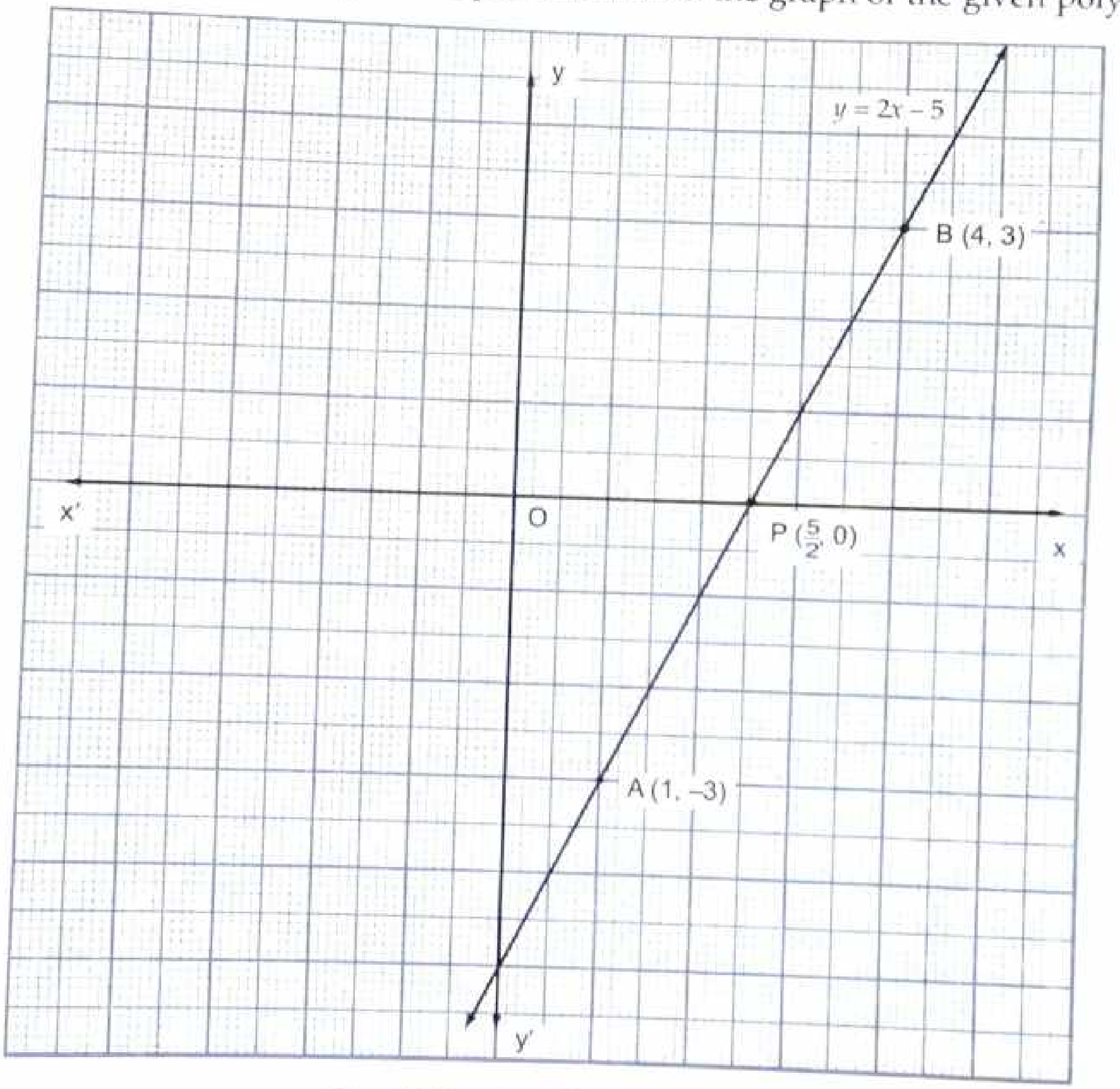

Fig. 2.2 Graph of $f(x) = 2x - 5$

2.3.2 GRAPH OF A QUADRATIC POLYNOMIAL

In this section, we will be interested to see what the graph of a quadratic polynomial $ax^2 + bx + c, a \neq 0$ looks like. We will also learn the construction of the graph of a quadratic polynomial without plotting many points on the graph paper.

ILLUSTRATION 1 *Draw the graph of the polynomial* $f(x) = x^2 - 2x - 8$.

SOLUTION Let $y = x^2 - 2x - 8$.

The following table gives the values of y or $f(x)$ for various values of x.

x	-4	-3	-2	-1	0	1	2	3	4	5	6
$y = x^2 - 2x - 8$	16	7	0	-5	-8	-9	-8	-5	0	7	16

Let us now plot the points $(-4, 16)$, $(-3, 7)$, $(-2, 0)$, $(-1, -5)$, $(0, -8)$, $(1, -9)$, $(2, -8)$, $(3, -5)$, $(4, 0)$, $(5, 7)$ and $(6, 16)$ on a graph paper and draw a smooth free hand curve passing through these points. The curve thus obtained represents the graph of the polynomial $f(x) = x^2 - 2x - 8$. This is called a parabola. The lowest point P, called a minimum point, is the vertex of the parabola.

Vertical line passing through P is called the axis of the parabola. Parabola is symmetric about the axis. So, it is also called the line of symmetry.

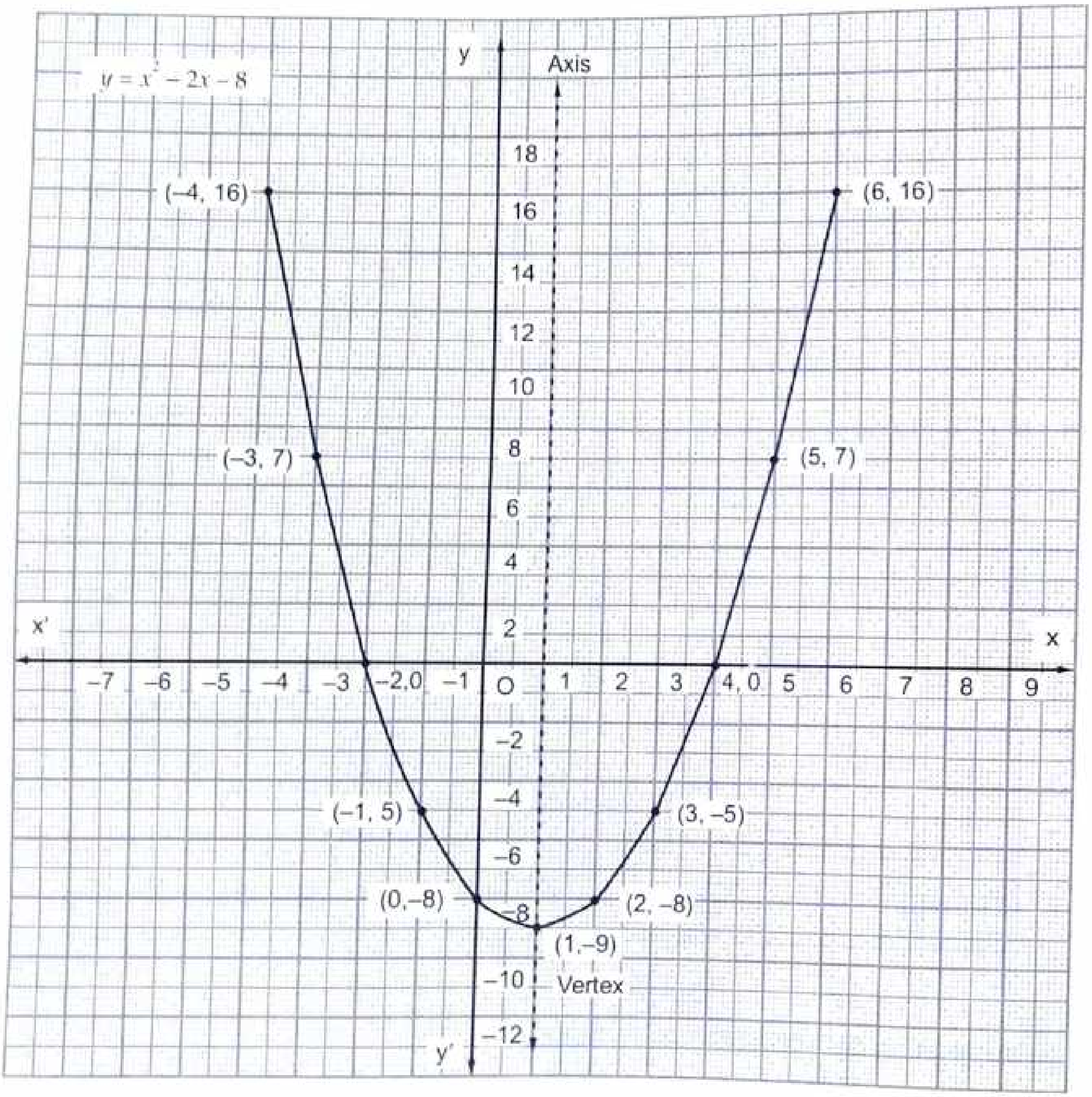

Fig. 2.3 Graph of $f(x) = x^2 - 2x - 8$

Observations: From the graph of the polynomial $f(x) = x^2 - 2x - 8$, we make the following observations:

(i) The coefficient of x^2 in $f(x) = x^2 - 2x - 8$ is 1 (a positive real number) and so the parabola opens upwards.

(ii) The polynomial $f(x) = x^2 - 2x - 8 = (x-4)(x+2)$ is factorizable into two distinct linear factors $(x-4)$ and $(x+2)$. So, the parabola cuts X-axis at two distinct points $(4, 0)$ and $(-2, 0)$. The x-coordinates of these points are zeros of $f(x)$.

(iii) The polynomial $f(x) = x^2 - 2x - 8$ has two distinct zeros namely 4 and -2. So, the parabola cuts X-axis at $(4, 0)$ and $(-2, 0)$.

(iv) On comparing the polynomial $x^2 - 2x - 8$ with $ax^2 + bx + c$, we get $a = 1$, $b = -2$ and $c = -8$. The vertex of the parabola has coordinates $(1, -9)$ i.e. $\left(-b/2a, -D/4a\right)$, where $D = b^2 - 4ac$.

(v) $D = b^2 - 4ac = 4 + 32 = 36 > 0$. So, the parabola cuts X-axis at two distinct points.

ILLUSTRATION 2 *Draw the graph of the quadratic polynomial* $f(x) = 3 - 2x - x^2$.

SOLUTION Let $y = f(x)$ or, $y = 3 - 2x - x^2$.

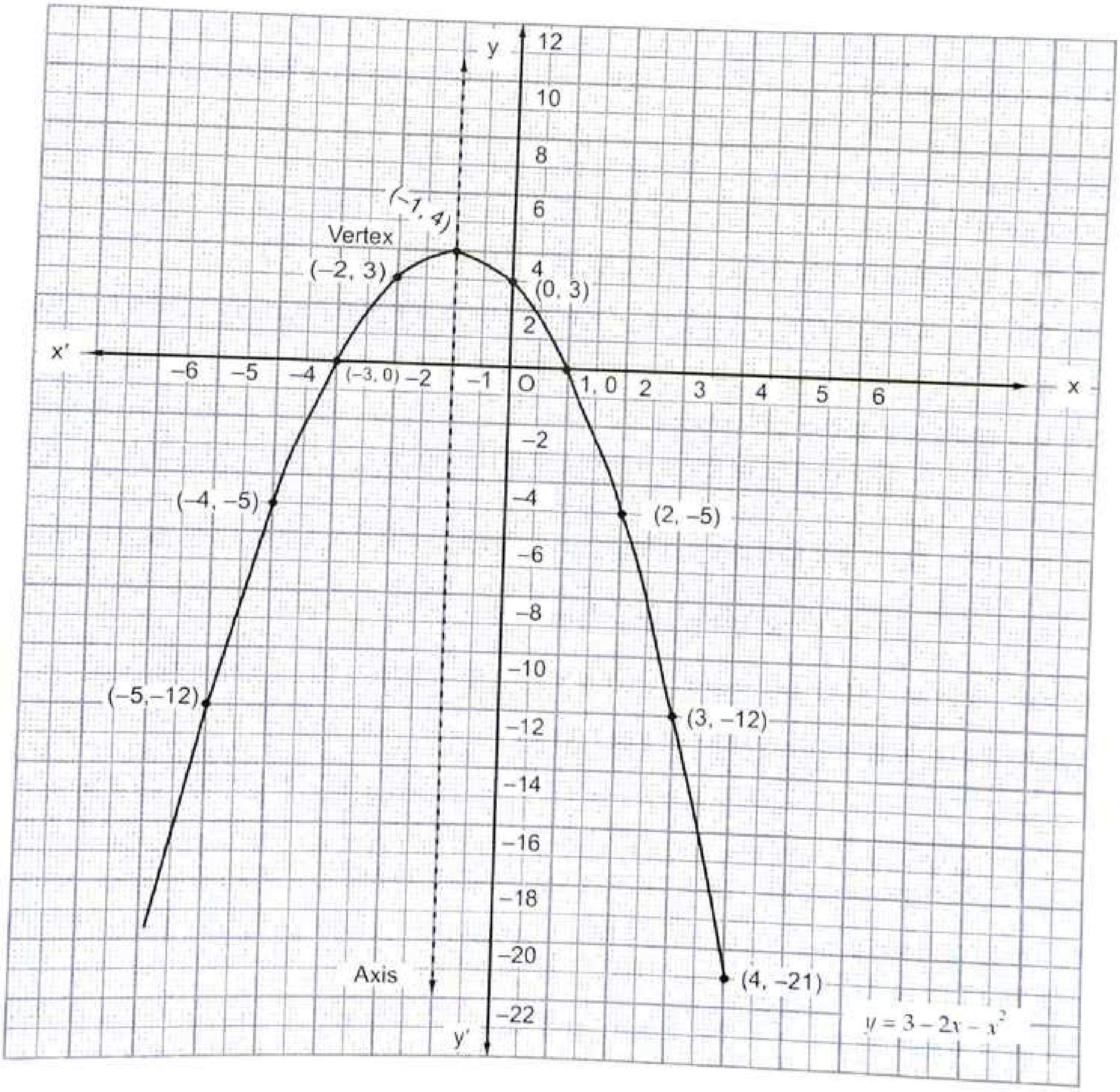

Fig. 2.4 Graph of $f(x) = 3 - 2x - x^2$

Let us list a few values of $y = 3 - 2x - x^2$ corresponding to a few values of x as follows:

x	-5	-4	-3	-2	-1	0	1	2	3	4
$y = 3 - 2x - x^2$	-12	-5	0	3	4	3	0	-5	-12	-21

Thus, the following points lie on the graph of the polynomial $y = 3 - 2x - x^2$:

$(-5, -12), (-4, -5), (-3, 0), (-2, 3), (-1, 4), (0, 3), (1, 0), (2, -5), (3, -12)$ and $(4, -21)$.

Let us plot these points on a graph paper and draw a smooth free hand curve passing through these points to obtain the graph of $y = 3 - 2x - x^2$. The curve thus obtained represents a parabola, as shown in Fig. 2.4. The highest point $P(-1, 4)$, is called a maximum point, is the vertex of the parabola. Vertical line through P is the axis of the parabola. Clearly, parabola is symmetric about the axis.

Observations: We make the following observations from the graph of the polynomial $f(x) = 3 - 2x - x^2$.

(i) The coefficient of x^2 in $f(x) = 3 - 2x - x^2$ is -1 i.e. a negative real number and so the parabola opens downwards.

(ii) The polynomial $f(x) = 3 - 2x - x^2 = (1 - x)(x + 3)$ is factorizable into two distinct linear factors $(1 - x)$ and $(x + 3)$. So, the parabola cuts X-axis at two distinct points $(1, 0)$ and $(-3, 0)$. The x-coordinates of these points are zeros of $f(x)$.

(iii) The polynomial $f(x) = 3 - 2x - x^2$ has two distinct roots namely -3 and 1. So, the parabola $y = 3 - 2x - x^2$ cuts X-axis at two distinct points.

(iv) On comparing the polynomial $3 - 2x - x^2$ with $ax^2 + bx + c$, we have $a = -1$, $b = -2$ and $c = 3$. The vertex of the parabola is at the point $(-1, 4)$ i.e. at $(-b/2a, -D/4a)$, where $D = b^2 - 4ac$.

(v) $D = b^2 - 4ac = 4 + 12 = 16 > 0$. So, the parabola cuts x-axis at two distinct points.

ILLUSTRATION 3 *Draw the graph of the polynomial* $f(x) = x^2 - 6x + 9$.

SOLUTION Let $y = f(x)$ or, $y = x^2 - 6x + 9$.

The following table gives the values of y or $f(x)$ for various values of x.

x	-2	-1	0	1	2	3	4	5	6	7	8
$y = x^2 - 6x + 9$	25	16	9	4	1	0	1	4	9	16	25

Thus, the graph of $y = x^2 - 6x - 9$ passes through the points $(-2, 25)$, $(-1, 16)$, $(0, 9)$, $(1, 4), (2, 1), (3, 0), (4, 1), (5, 4), (6, 9), (7, 16)$ and $(8, 25)$.

Let us plot these points on the graph and draw a free hand smooth curve passing through these points. We observe that the vertex of the parabola is at point $P(3, 0)$ as shown in Fig. 2.5.

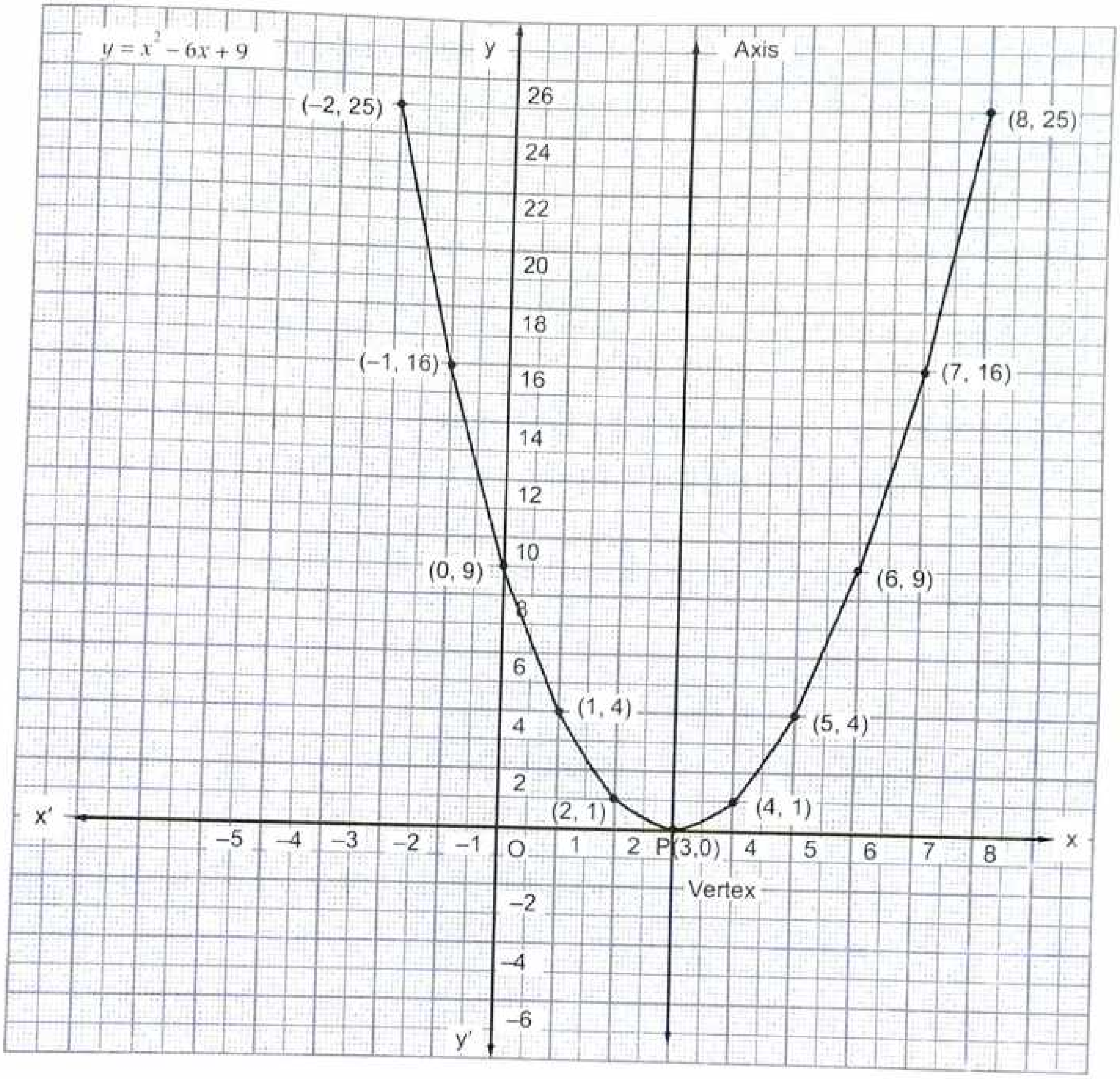

Fig. 2.5 Graph of $f(x) = x^2 - 6x + 9$

Observations: From the graph of the polynomial $f(x) = x^2 - 6x + 9$, we make the following observations:

(i) The coefficient of x^2 in $f(x) = x^2 - 6x + 9$ is 1, a positive real number, and so the parabola opens upwards.

(ii) The polynomial $f(x) = x^2 - 6x + 9 = (x - 3)^2$ is factorizable into two equal factors each equal to $(x - 3)$. So, the parabola $y = x^2 - 6x + 9$ touches X-axis at one point $(3, 0)$. In other words, $y = x^2 - 6x + 9$ touches X-axis at one point $(3, 0)$. In other words, $y = x^2 - 6x + 9$ cuts X-axis at coincident points. The x-coordinate of this point gives two equal roots of the polynomial.

(iii) The polynomial $f(x) = x^2 - 6x + 9$ has two equal roots each equal to 3. So, the parabola $y = x^2 - 6x + 9$ touches X-axis at $(3, 0)$ i.e. it cuts X-axis at coincident points.

(iv) On comparing the polynomial $x^2 - 6x + 9$ with $ax^2 + bx + c$, we get $a = 1$, $b = -6$ and $c = 9$. The vertex of the parabola is at $(3, 0)$ i.e., at $\left(-b/2a, -D/4a\right)$, where $D = b^2 - 4ac$.

(v) $D = b^2 - 4ac = 36 - 36 = 0$. So, the parabola touches X-axis.

ILLUSTRATION 4 *Draw the graph of the polynomial* $f(x) = -4x^2 + 4x - 1$. *Also, find the vertex of this parabola.*

SOLUTION Let $y = f(x)$ or, $y = -4x^2 + 4x - 1$

The following table gives the values of y for various values of x.

x	-2	$-3/2$	-1	$-1/2$	0	$1/2$	1	$3/2$	2	$5/2$	3
$y = -4x^2 + 4x - 1$	-25	-16	-9	-4	-1	0	-1	-4	-9	-16	-25

Thus, the following points lie on the graph of $y = -4x^2 + 4x - 1$: $(-2, -25)$, $(-3/2, -16)$, $(-1, -9)$, $(-1/2, -4)$, $(0, -1)$, $(1/2, 0)$, $(1, -1)$, $(3/2, -4)$, $(2, -9)$, $(5/2, -16)$, $(3, -25)$ etc.

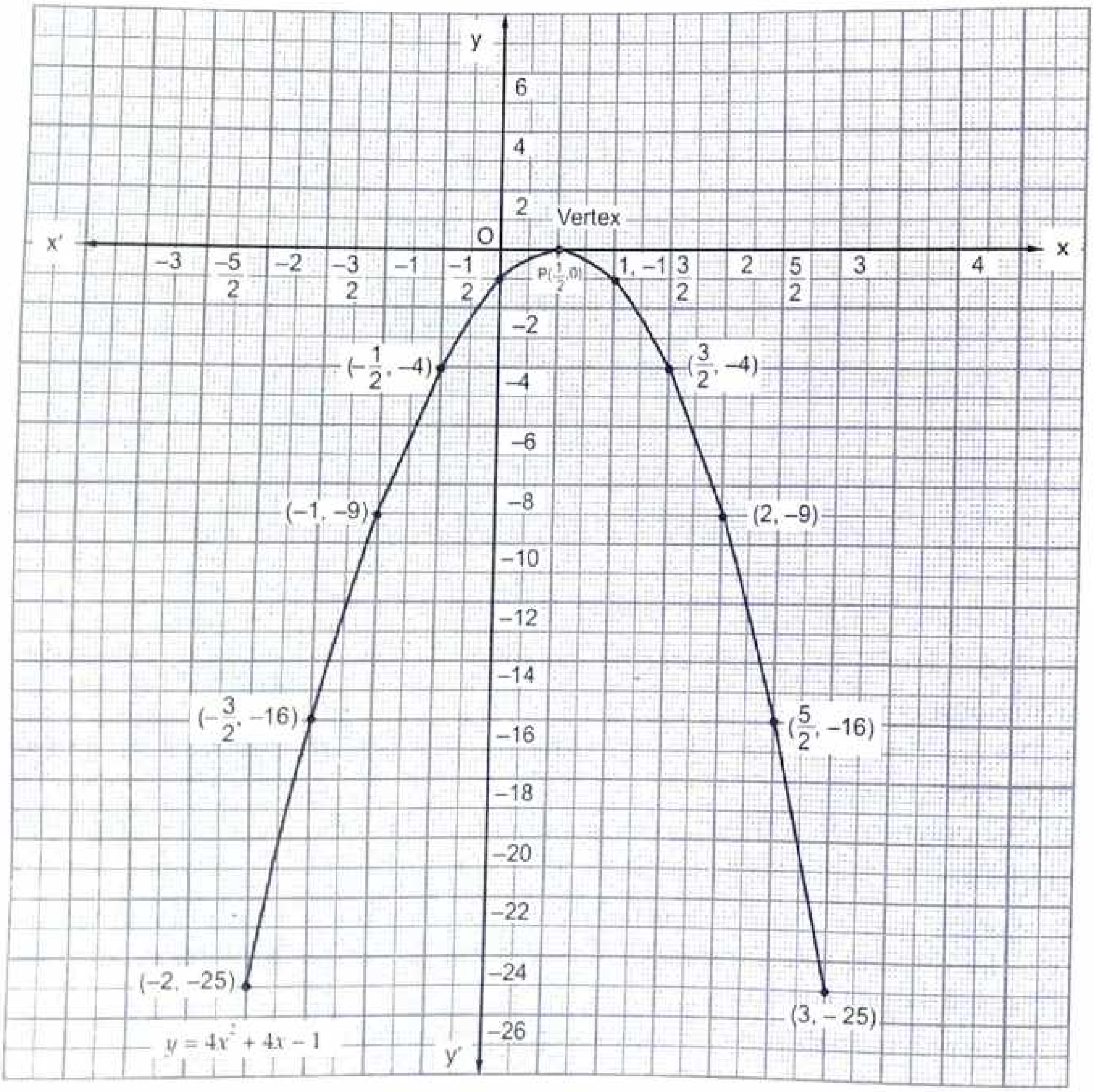

Fig. 2.6 Graph of $f(x) = -4x^2 + 4x - 1$

Let us plot these points on a graph paper and draw a free hand smooth curve passing through these points. The shape of the curve is shown in Fig. 2.6. It is a parabola opening downward having its vertex at (1/2, 0).

Observations: From the graph of the polynomial $f(x) = -4x^2 + 4x - 1$, we make the following observations:

(i) The coefficient of x^2 in $f(x) = -4x^2 + 4x - 1$ is -4, a negative real number and so the parabola opens downwards.

(ii) The polynomial $f(x) = -4x^2 + 4x - 1 = -(2x-1)^2$ is factorizable into two equal factors each equal to $2x - 1$. So, the parabola cuts X-axis at two coincident points having coordinates $(1/2, 0)$.

(iii) The polynomial $f(x) = -4x^2 + 4x - 1$ has two equal roots each equal to $1/2$. So, the parabola touches X-axis at one point $(1/2, 0)$ only i.e. it cuts X-axis at coincident points.

(iv) On comparing the polynomial $-4x^2 + 4x - 1$ with $ax^2 + bx + c$, we get $a = -4$, $a = -4, b = 4$ and $c = -1$. The vertex of the parabola has the coordinates $(1/2, 0)$ i.e. $(-b/2a, -D/4a)$, where $D = b^2 - 4ac$.

(v) $d = b^2 - 4ac = 4 - 4 = 0$. So, the parabola touches X-axis.

ILLUSTRATION 5 *Draw the graph of the polynomial* $f(x) = 2x^2 - 4x + 5$.

SOLUTION Let $y = f(x)$ or, $y = 2x^2 - 4x + 5$.

The following table gives the values of y for various values of x:

x	-3	-2	-1	0	1	2	3	4	5
$y = 2x^2 - 4x + 5$	35	21	11	5	3	5	11	21	35

Thus, the graph of $y = 2x^2 - 4x + 5$ passes through the following points:

$$(-3, 35), (-2, 21), (-1, 11), (0, 5), (1, 3), (2, 5), (3, 11), (4, 21), (5, 35) \text{ etc.}$$

Let us plot these points on a graph paper and draw a smooth free hand curve passing through these points to obtain the graph of $y = 2x^2 - 4x + 5$ as shown in Fig. 2.7.

Observations: From the graph of the polynomial $f(x) = 2x^2 - 4x + 5$, we make following observations:

(i) The coefficient of x^2 in $f(x) = 2x^2 - 4x + 5$ is 2 i.e. a positive real number and so the parabola opens upwards.

(ii) The polynomial $f(x) = 2x^2 - 4x + 5$ is not factorizable into linear factors and so the parabola $y = 2x^2 - 4x + 5$ does not cross or touch X-axis.

(iii) The polynomial $f(x) = 2x^2 - 4x + 5$ does not have any real zero and so the parabola does not cut X-axis.

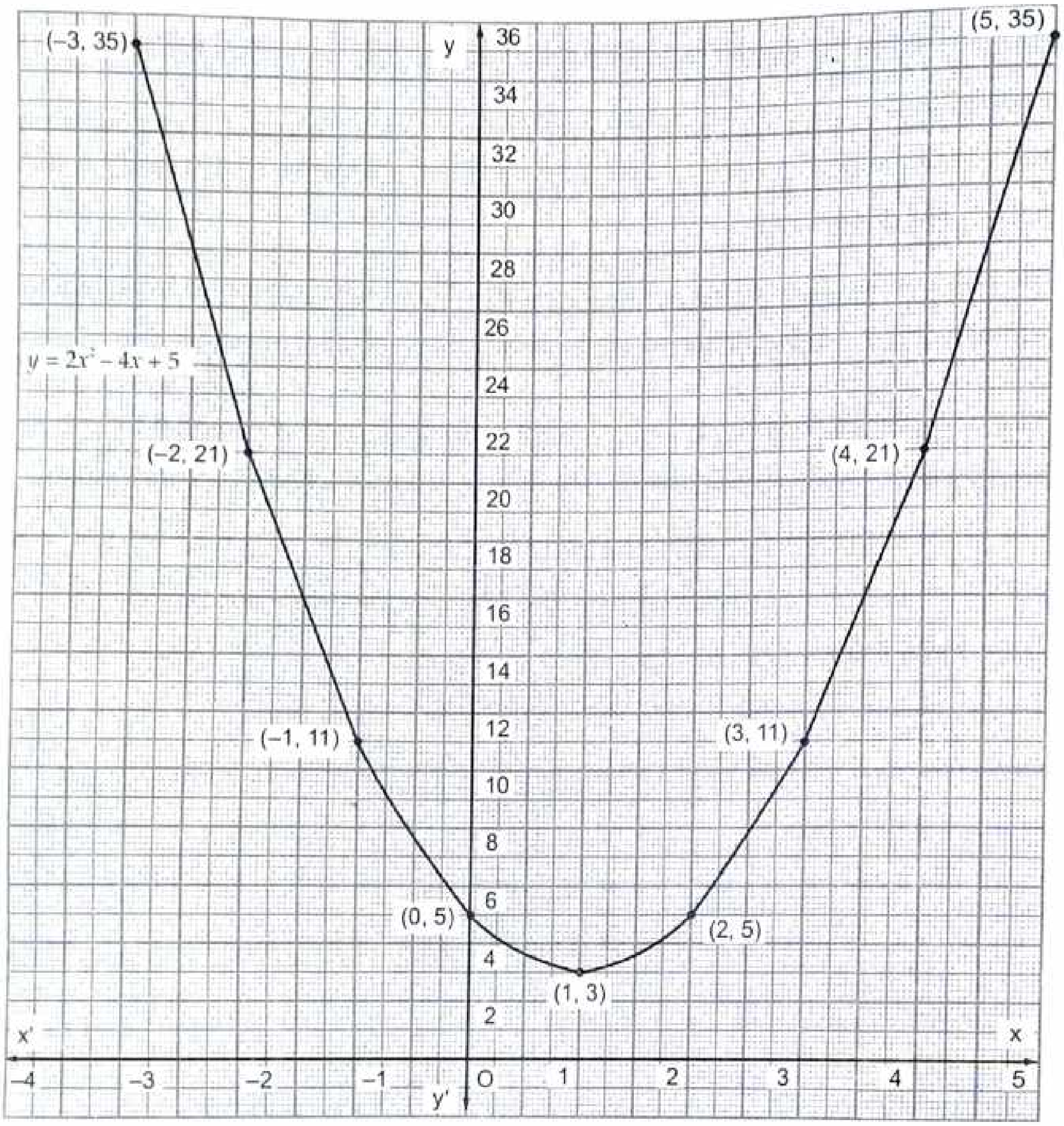

Fig. 2.7 Graph of $f(x) = 2x^2 - 4x + 5$

(iv) On comparing the polynomial $f(x) = 2x^2 - 4x + 5$ with $ax^2 + bx + c$, we get $a = 2, b = -4$ and $c = 5$. The vertex of the parabola is at $(-b/2a, -D/4a)$ i.e. at $(1, 3)$, where $D = b^2 - 4ac$.

(v) All values of $f(x)$ are positive as the parabola remains above X-axis.

(vi) $D = b^2 - 4ac = 16 - 40 < 0$. So, the parabola does not cross X-axis.

ILLUSTRATION 6 *Draw the graph of the polynomial* $f(x) = -3x^2 + 2x - 1$.

SOLUTION Let $y = f(x)$ or, $y = -3x^2 + 2x - 1$.

The values of y for various values of x are listed in the following table:

x	−4	−3	−2	−1	0	1	2	3	4
$y = 3x^2 + 2x - 1$	−57	−34	−17	−6	−1	−2	−9	−26	−41

Thus, the graph of $y = -3x^2 + 2x - 1$ passes through the points: $(-4, -57)$, $(-3, -34)$, $(-2, -17)$, $(-1, -6)$, $(0, -1)$, $(1, -2)$, $(2, -9)$, $(3, -26)$, $(4, -41)$ etc.

Let us plot these points on a graph paper and draw a smooth free hand curve passing through these points. The curve thus obtained is shown in Fig. 2.8.

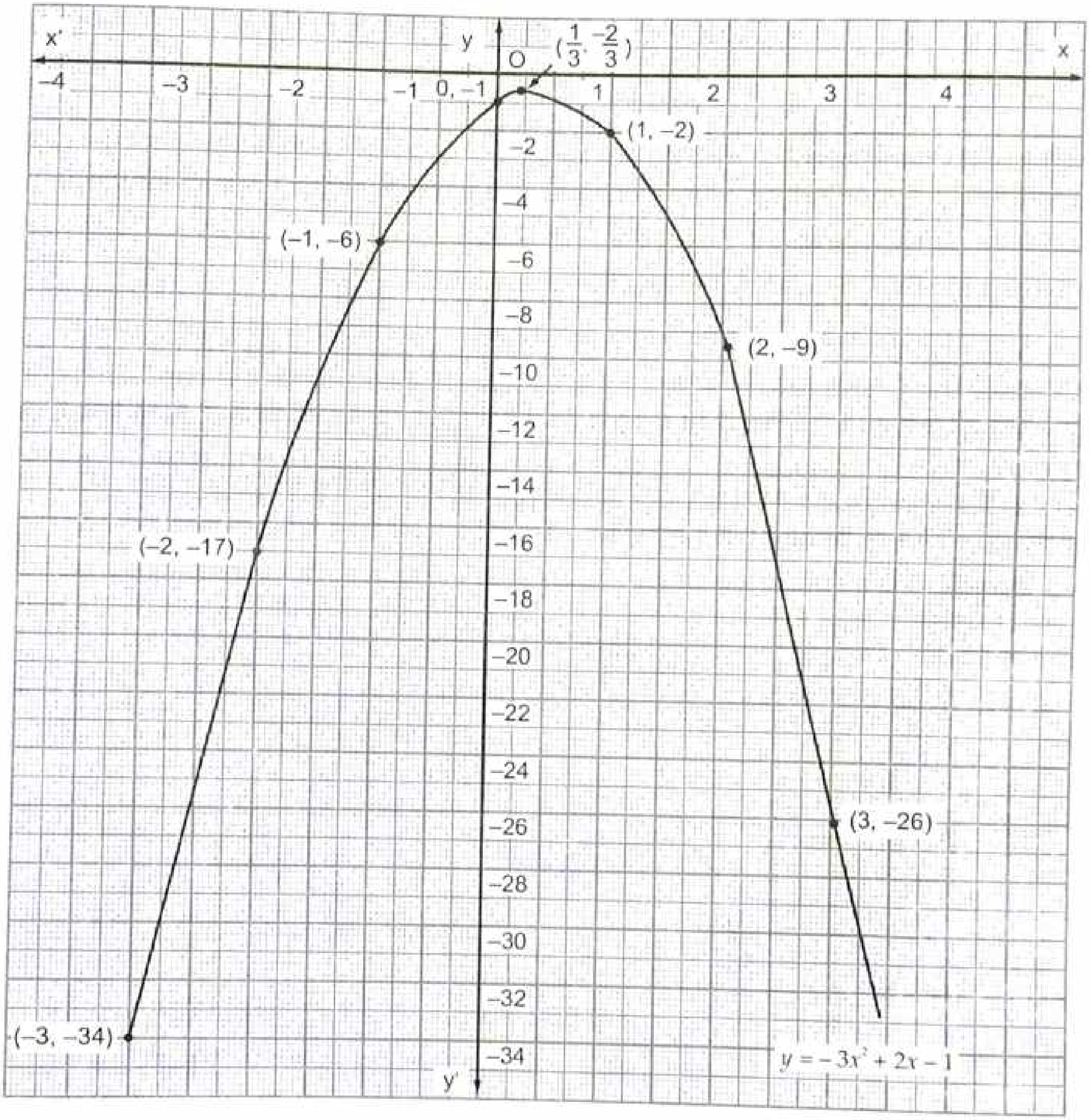

Fig. 2.8 Graph of $f(x) = -3x^2 + 2x - 1$

Observations: We make the following observations from the graph of the polynomial $f(x) = -3x^2 + 2x - 1$:

(i) The coefficient of x^2 in $f(x) = -3x^2 + 2x - 1$ is -3 i.e. a negative real number and so the parabola opens downwards.

(ii) The polynomial $f(x) = -3x^2 + 2x - 1$ is not factorizable into real factors and so the parabola does not cross or touch x-axis.

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(iii) The polynomial $f(x) = -3x^2 + 2x - 1$ does not have any real zero and the parabola does not cross or touch x-axis.

(iv) All values of $f(x)$ are negative as the parabola opens downwards and remains below x-axis.

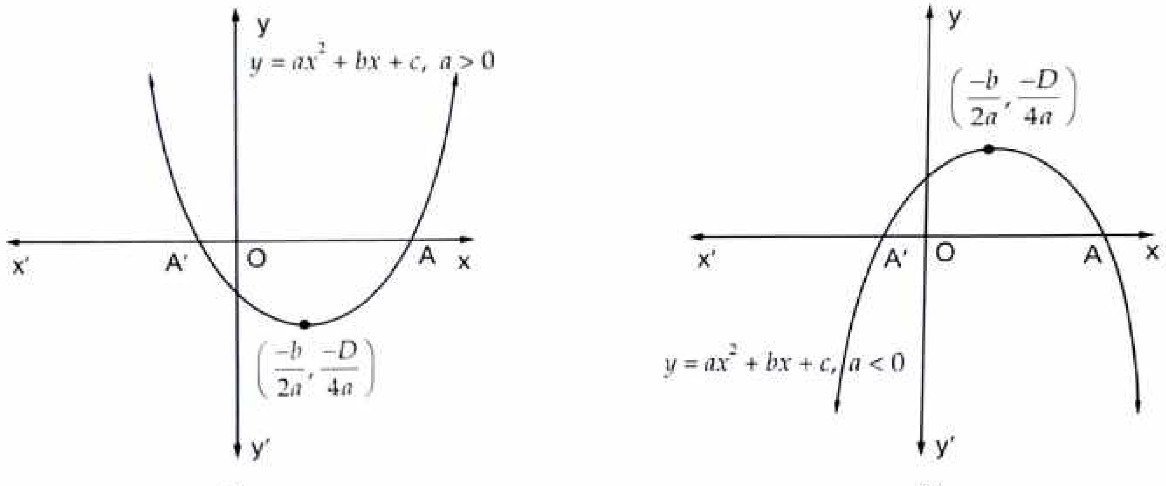
(v) On comparing the polynomial $f(x) = -3x^2 + 2x - 1$ with $ax^2 + bx + c$, we get $a = -3, b = +2$ and $c = -1$. The vertex of the parabola is at $(-b/2a, -D/4a)$, i.e. at $(1/3, -2/3)$, where $D = b^2 - 4ac$.

It follows from the above discussion on the graph of a quadratic polynomial that the graph of the quadratic polynomial $ax^2 + bx + c, a \neq 0$ is a parabola which opens upwards (⋃) or downwards (⋂) according as $a > 0$ or $a < 0$.

We also observe that there are following three possibilities:

CASE I *When polynomial* $ax^2 + bx + c$ *is factorizable into two distinct linear factors:*

In this case, the graph of $ax^2 + bx + c$ or the curve $y = ax^2 + bx + c$ cuts x-axis at two distinct points A and A'. The x-coordinates of these points are the two zeros of the polynomial $ax^2 + bx + c$. The coordinates of the vertex of the parabola $y = ax^2 + bx + c$ are $(-b/2a, -D/4a)$, where $D = b^2 - 4ac$.

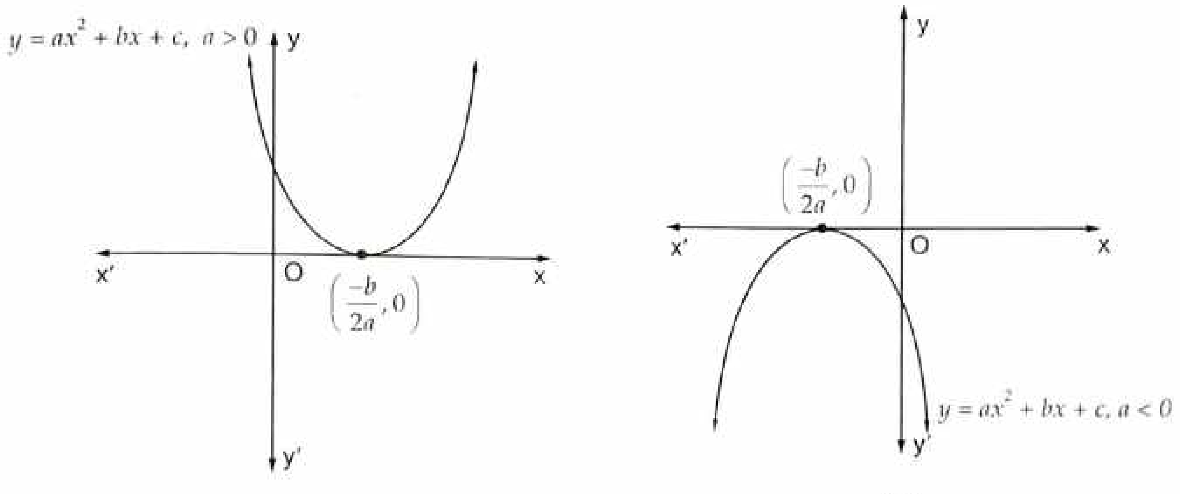


(i) (ii)

Fig. 2.9

CASE II *When polynomial* $ax^2 + bx + c$ *is factorizable into two equal factors:*

In this case, the graph of the polynomial $ax^2 + bx + c$ or the curve $y = ax^2 + bx + c$ touches x-axis at point $(-b/2a, 0)$. The x-coordinate of this point gives two equal zeros of the polynomial.

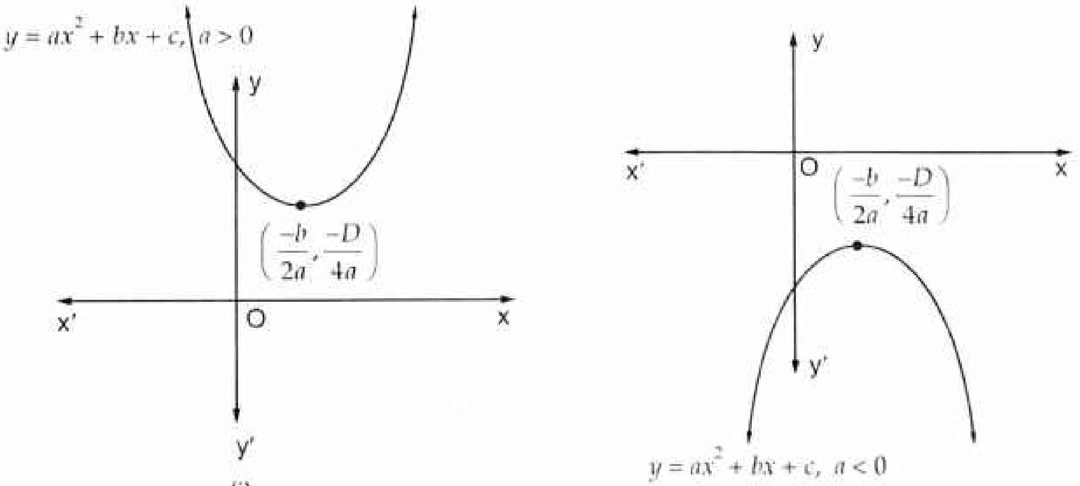


(i) (ii)

Fig. 2.10 Graph of $y = ax^2 + bx + c$

CASE III *When the polynomial* $ax^2 + bx + c$ *is not factorizable:*

In this case, the graph of the polynomial $ax^2 + bx + c$ or the curve $y = ax^2 + bx + c$ does not cut or touch x-axis. The parabola $y = ax^2 + bx + c$ opens upwards and remains completely above x-axis, if $a > 0$. The parabola opens downward and remains completely below x-axis, if $a < 0$.



(i) (ii)

Fig. 2.11 Graph of $y = ax^2 + bx + c$

2.3.3 GRAPH OF A CUBIC POLYNOMIAL

In the previous section, we have seen that the graph of a quadratic polynomial is always a parabola either opening upwards or opening downwards. In this section, we will see that the graph of a cubic polynomial does not have a fixed standard shape. We have also seen that the graph of a quadratic polynomial may or may not cut or touch x-axis. But, in case of a cubic polynomial the graph will always cross x-axis at least once and at most thrice.

ILLUSTRATION 1 *Draw the graph of the polynomial* $f(x) = x^3 - 4x$.

SOLUTION Let $y = f(x)$ or, $y = x^3 - 4x$.

The values of y for various values of x are listed in the following table:

x	-4	-3	-2	-1	0	1	2	3	4
$y = x^3 - 4x$	-48	-15	0	3	0	-3	0	15	48

Thus, the curve $y = x^3 - 4x$ passes through the points $(-4, -48), (-3, -15), (-2, 0)$, $(-1, 3)$ $(0, 0)$, $(1, -3)$, $(2, 0)$, $(3, 15)$, $(4, 48)$ etc.

By plotting these points on a graph paper and drawing a free hand smooth curve throug these points, we obtain the graph of the given polynomial as shown in Fig. 2.12.

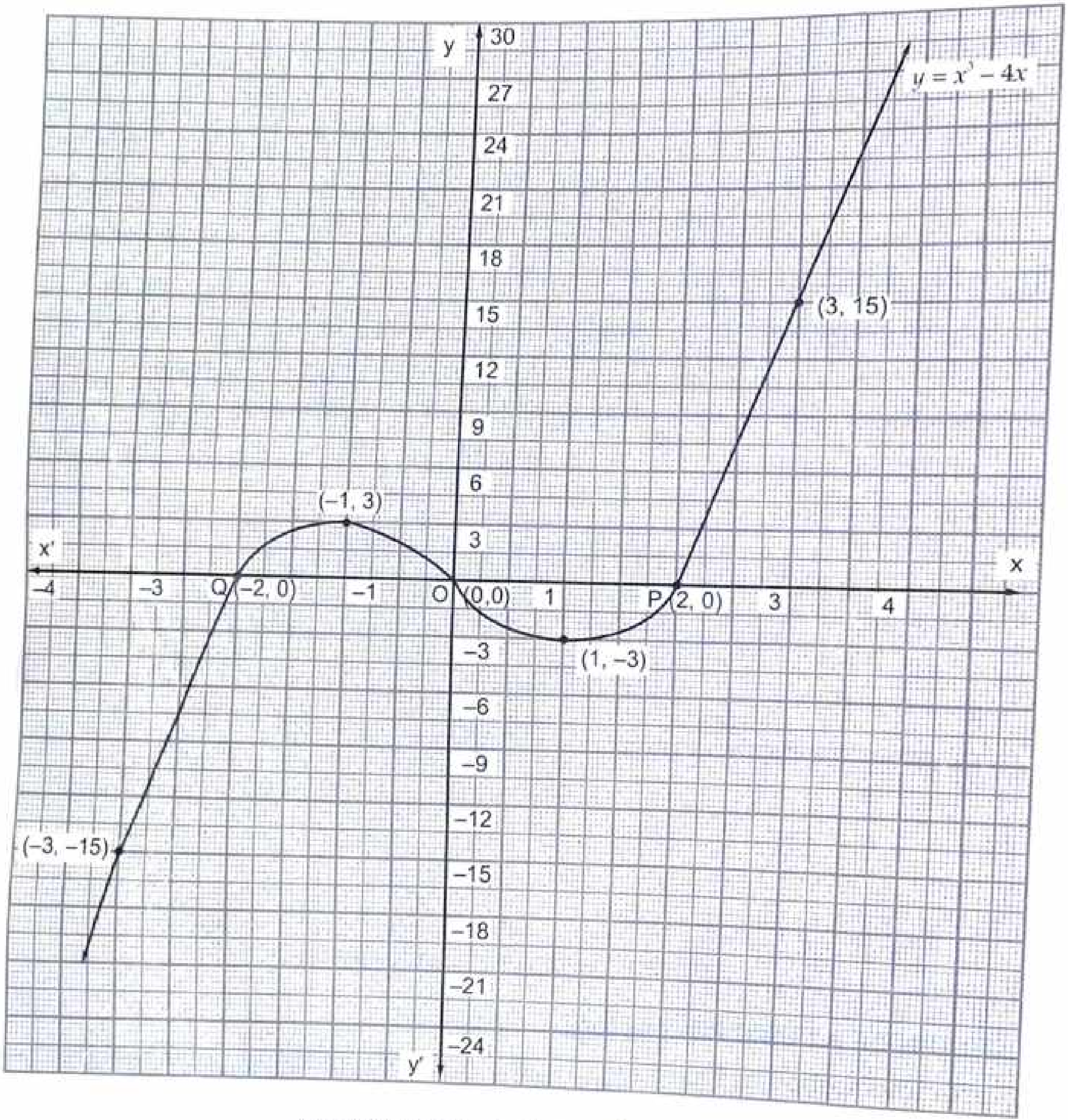


Fig. 2.12 Graph of $f(x) = x^3 - 4x$

Observations: From the graph of the polynomial $f(x) = x^3 - 4x$, we make the following observations:

(i) The polynomial $f(x) = x^3 - 4x = x(x^2 - 4) = x(x-2)(x+2)$ is factorizable into three distinct linear factors. The curve $y = f(x)$ also cuts x-axis at three distinct ponits.

(ii) We have, $f(x) = x(x-2)(x+2)$

Therefore, 0, 2 and -2 are three zeros of $f(x)$. The curve $y = f(x)$ cuts x-axis at three points O (0, 0), P (2, 0) and Q (-2, 0) whose x-coordinates are the zeros of the polynomial $f(x)$.

ILLUSTRATION 2 *Draw the graph of the cubic polynomial* $f(x) = x^3 - 2x^2$.

SOLUTION Let $y = f(x)$ or, $y = x^3 - 2x^2$.

The following table gives values of y for various values of x:

x	-3	-2	-1	0	1	2	3	4
$y = x^3 - 2x^2$	-45	-16	-3	0	-1	0	9	32

Thus, the curve $y = x^3 - 2x^2$ passes through the points $(-3, -45), (-2, -16), (-1, -3), (0, 0), (1, -1), (2, 0), (3, 9), (4, 32)$ etc. By plotting these points on a graph paper and drawing a free hand smooth curve passing through these points, we obtain the graph of the polynomial as shown in Fig. 2.13.

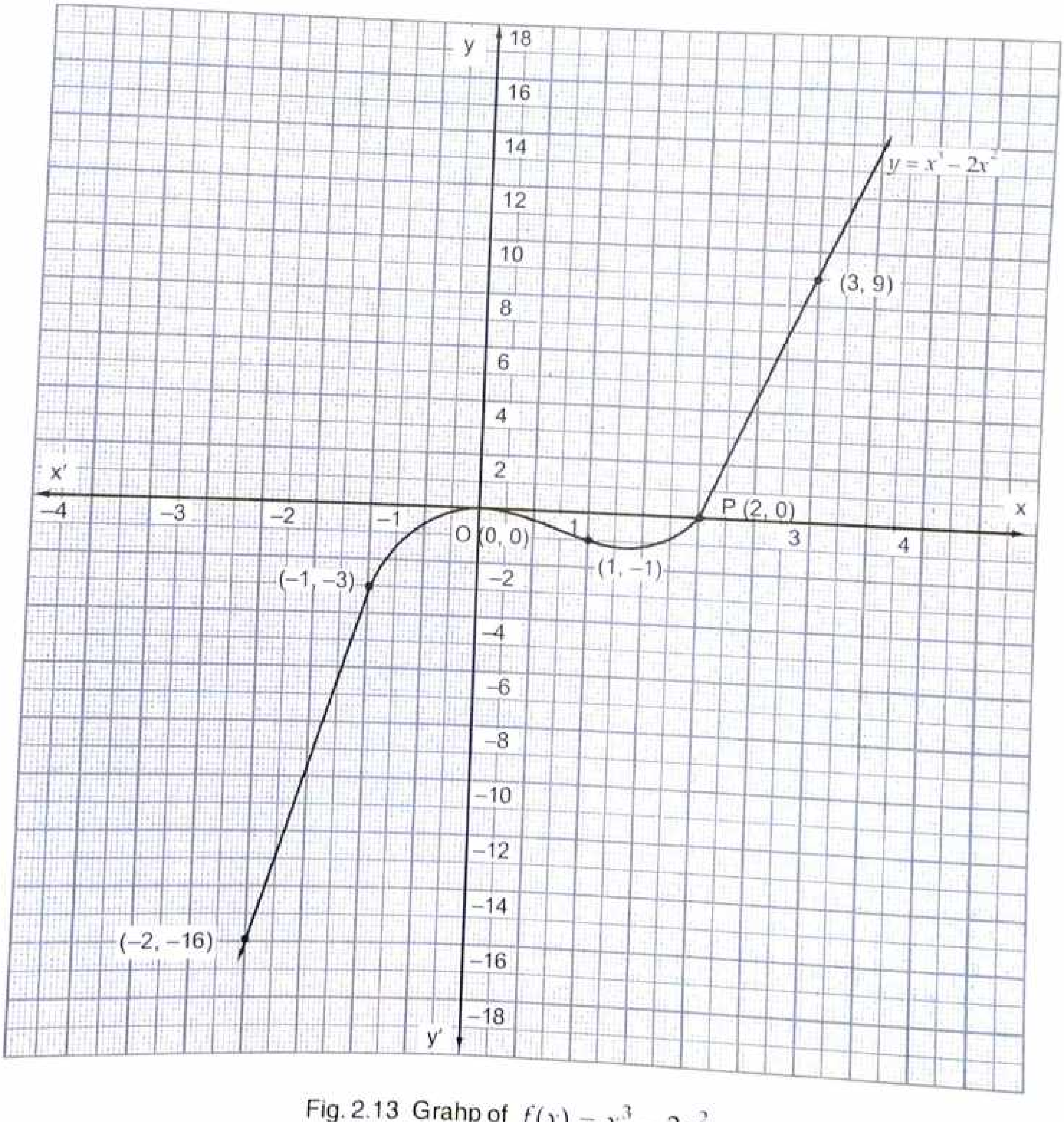


Fig. 2.13 Grahp of $f(x) = x^3 - 2x^2$

Observations: We make the following observations from the graph of the polynomial $f(x) = x^3 - 2x^2$.

(i) The polynomial $f(x) = x^3 - 2x^2 = x^2(x-2) = (x-0)(x-0)(x-2)$ is factorizable into two identical factors each equal to x and a linear factor $(x-2)$. The curve $y = x^3 - 2x^2$ cuts x-axis at two points.

(ii) We have, $f(x) = (x-0)(x-0)(x-2)$

So, 0 and 2 are two zeros of $f(x)$. The curve cuts x-axis at two points O (0, 0), P (2, 0) whose x-coordinates are the zeros of the polynomial $f(x)$.

ILLUSTRATION 3 *Draw the graph of the polynomial* $f(x) = x^3$.

SOLUTION Let $y = f(x)$ or, $y = x^3$.

The values of y for various values of x are given in the following table:

x	-4	-3	-2	-1	0	1	2	3	4
$y = x^3$	-64	-27	-8	-1	0	1	8	27	64

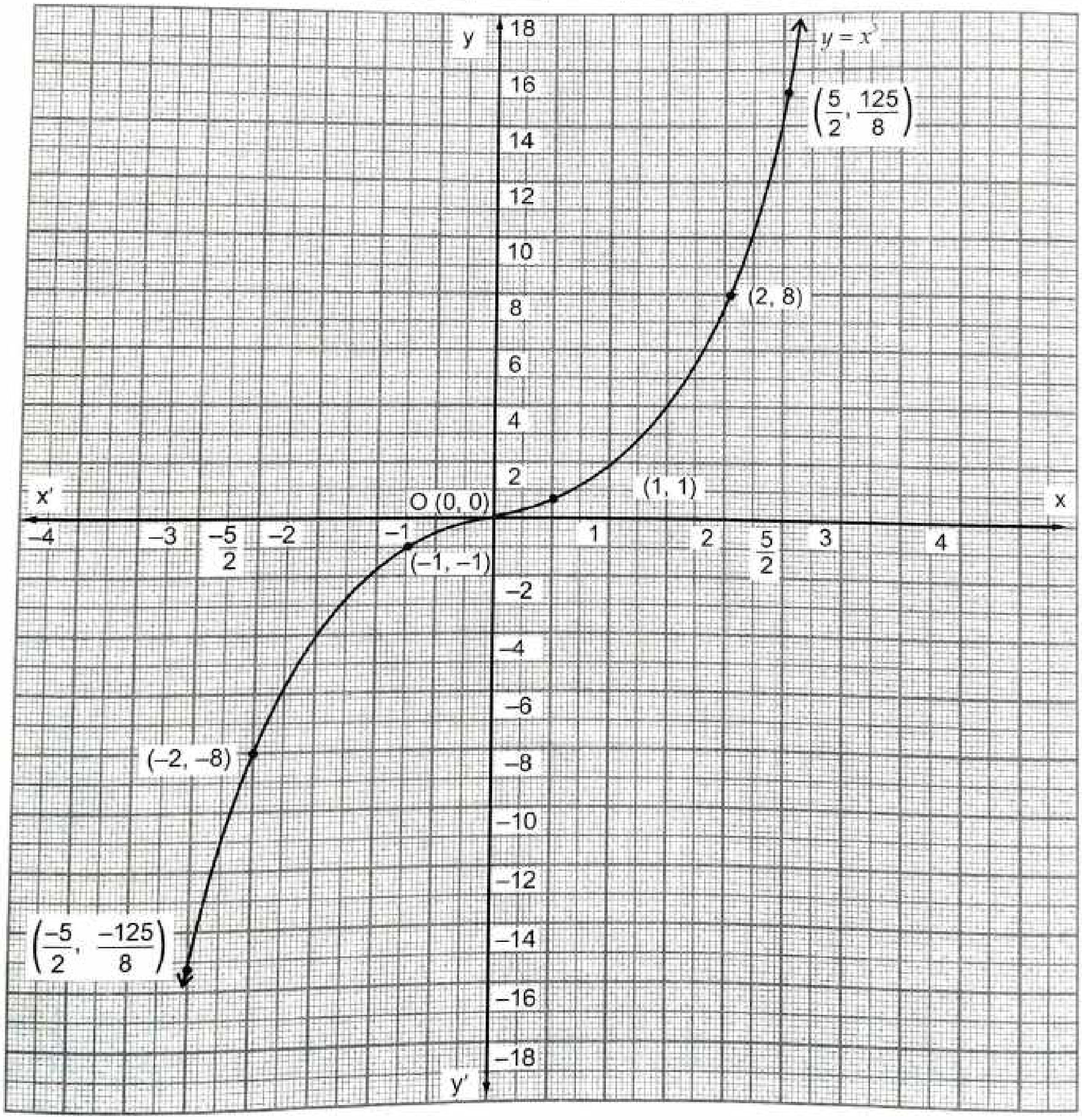


Fig. 2.14 Graph of $f(x) = x^3$.

Thus, the curve $y = x^3$ passes through the points $(-4, -64)$, $(-3, -27)$, $(-2, -8)$, $(-1, -1)$, $(0, 0)$, $(1, 1)$, $(2, 8)$, $(3, 27)$, $(4, 64)$ etc.

Plotting these points on a graph paper and drawing a free hand curve passing through these points, we obtain the graph of the given polynomial as shown in Fig. 2.14.

Observations: We make the following observation from the graph of the polynomial $f(x) = x^3$.

(i) The polynomial $f(x) = x^3 = (x - 0)(x - 0)(x - 0)$ has three identical factors. The curve $y = x^3$ cuts x-axis at three coincident points i.e. at exactly one point.

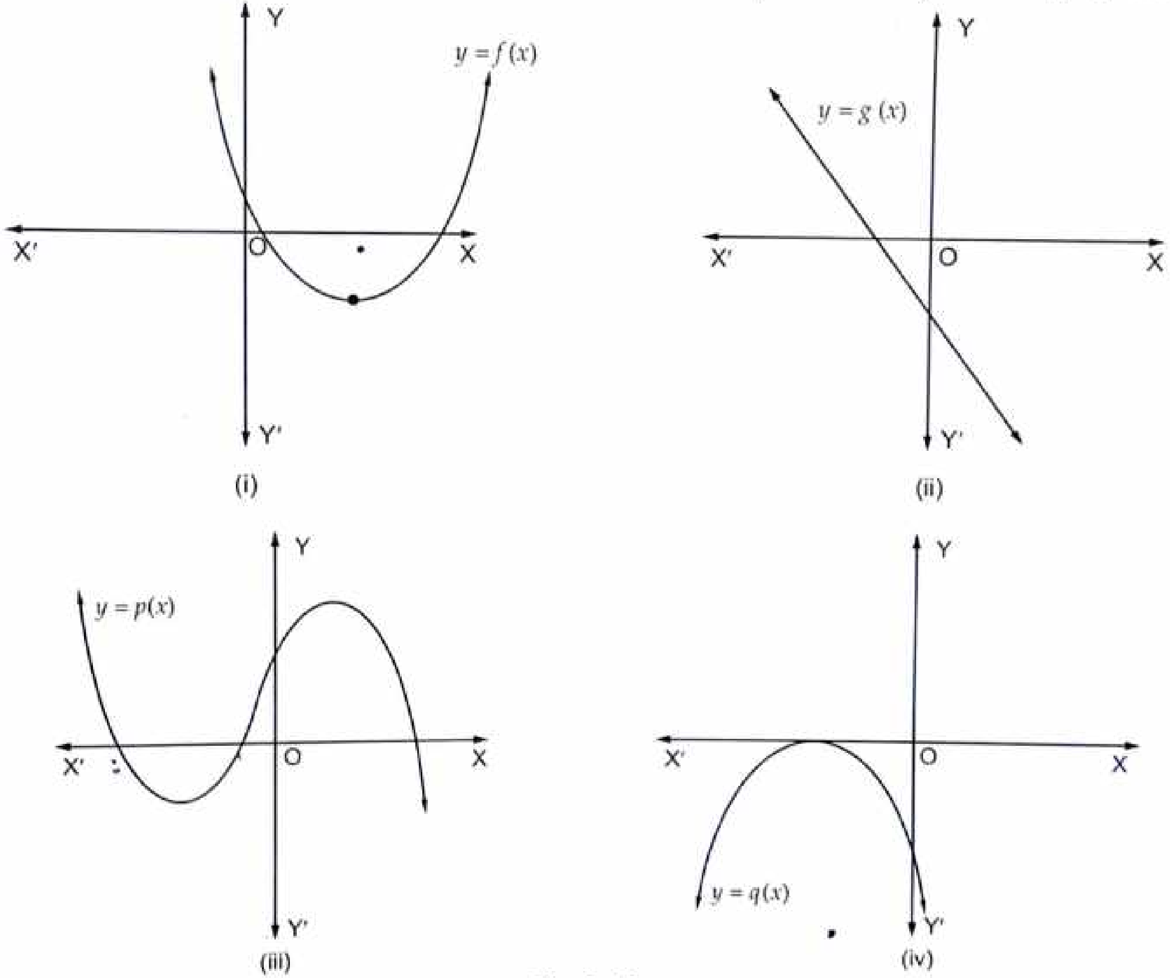
(ii) The polynomial $f(x) = x^3$ has exactly one zero equal to 0. The curve $y = x^3$ cuts x-axis at the point $O(0, 0)$ whose x-coordinate is equal to zero of the polynomial.

From the above discussion we infer that the graph of a linear polynomial crosses x-axis at one point only and the graph of a quadratic polynomial crosses x-axis at atmost two points. Also, the graph of a cubic polynomial crosses x-axis at atmost three points. In general, graph of an nth degree polynomial crosses the x-axis at atmost n points.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *If each one of the following graphs is the graph of a polynomial, then identify which one corresponds to a linear polynomial and which one corresponds to a quadratic polynomial?*



(i) (ii) (iii) (iv)

Fig. 2.15

SOLUTION (i) We observe that the graph $y = f(x)$ is a parabola opening upwards. Therefore, $f(x)$ is a quadratic polynomial in which coefficient of x^2 is positive.

(ii) We find that the graph $y = g(x)$ is a straight line. So, $g(x)$ is a linear polynomial.

(iii) Here, $p(x)$ is neither linear nor quadratic.

(iv) Here, $q(x)$ is a quadratic polynomial in which coefficient of x^2 is negative because the graph is a parabola opening downwards.

LEVEL-2

EXAMPLE 2 *The graphs of* $y = ax^2 + bx + c$ *are given in Fig. 2.16. Identify the signs of* a, b *and* c *in each of the following:*

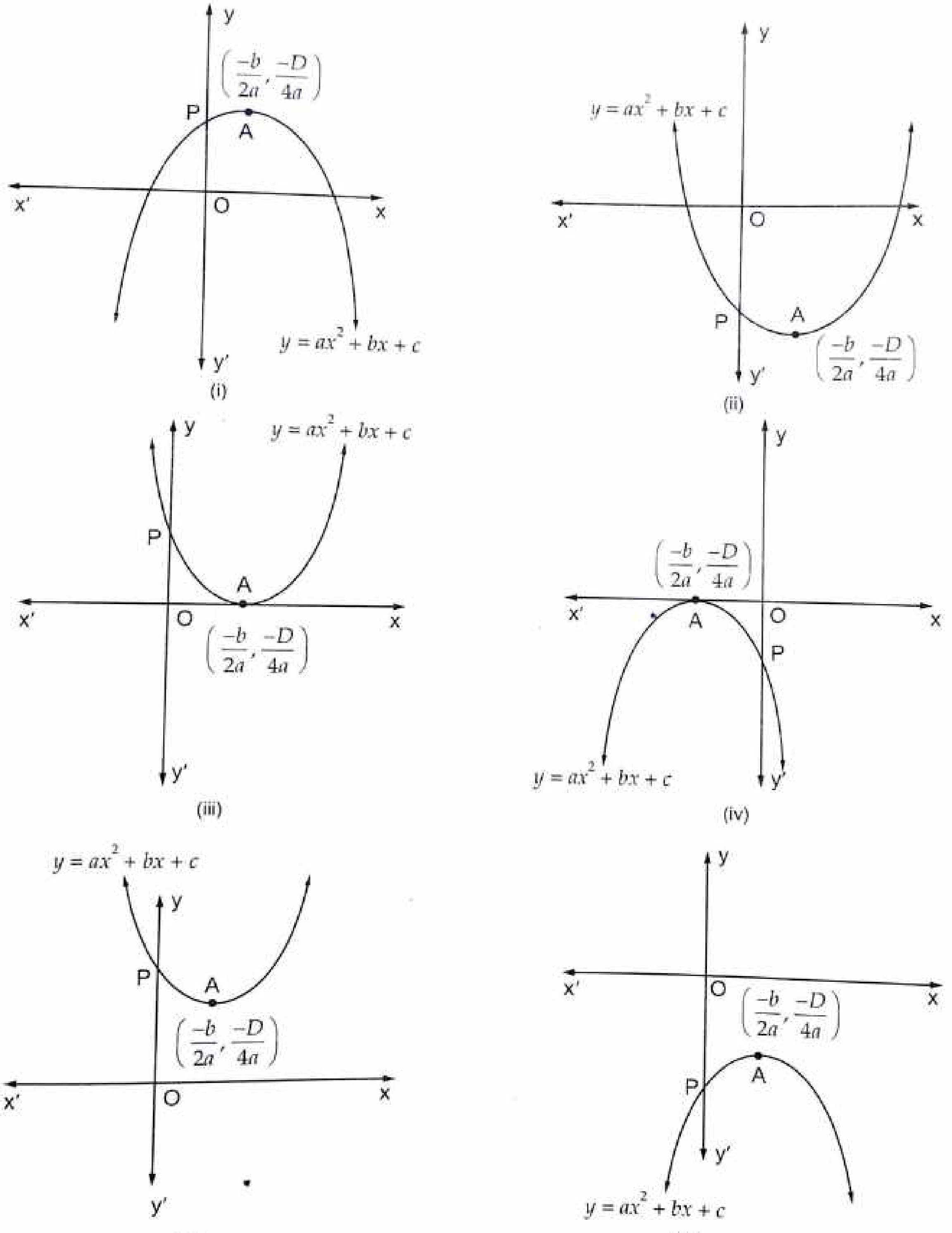


Fig. 2.16

SOLUTION (i) We observe that $y = ax^2 + bx + c$ represents a parabola opening downwards. Therefore, $a < 0$. We also observe that the vertex of the parabola is in first quadrant.

$$\therefore \qquad -\frac{b}{2a} > 0 \Rightarrow -b < 0 \Rightarrow b > 0 \qquad [\because a < 0]$$

Parabola $y = ax^2 + bx + c$ cuts y-axis at P. On y-axis, we have $x = 0$. Putting $x = 0$ in $y = ax^2 + bx + c$, we get $y = c$.

So, the coordinates of P are $(0, c)$. As P lies on the positive direction of y-axis. Therefore, $c > 0$. Hence, $a < 0$, $b > 0$ and $c > 0$.

(ii) We find that $y = ax^2 + bx + c$ represents a parabola opening upwards. Therefore, $a > 0$. The vertex of the parabola is in fourth quadrant.

$$\therefore \qquad \frac{-b}{2a} > 0 \Rightarrow -b > 0 \Rightarrow b < 0. \qquad [\because a > 0]$$

Parabola $y = ax^2 + bx + c$ cuts y-axis at P and on y-axis. We have $x = 0$. Therefore, on putting $x = 0$ in $y = ax^2 + bx + c$, we get $y = c$. So, the coordinates of P are $(0, c)$. As P lies on OY'. Therefore, $c < 0$. Hence, $a > 0$, $b < 0$ and $c < 0$.

(iii) Clearly, $y = ax^2 + bx + c$ represents a parabola opening upwards. Therefore, $a > 0$. The vertex of the parabola lies on OX.

$$\therefore \qquad \frac{-b}{2a} > 0 \Rightarrow -b > 0 \Rightarrow b < 0 \qquad [\because a > 0]$$

The parabola $y = ax^2 + bx + c$ cuts y-axis at P which lies on OY. Putting $x = 0$ in $y = ax^2 + bx + c$, we get $y = c$. So, the coordinates of P are $(0, c)$. Clearly, P lies on OY. Therefore, $c > 0$. Hence, $a > 0$, $b < 0$, and $c > 0$.

(iv) The parabola $y = ax^2 + bx + c$ opens downwards. Therefore, $a < 0$. The vertex $(-b/2a, \; -D/4a)$ of the parabola is on OX'.

$$\therefore \qquad \frac{-b}{2a} < 0 \Rightarrow b < 0 \qquad [\because a < 0]$$

Parabola $y = ax^2 + bx + c$ cuts y-axis at $P\,(0, c)$ which lies on OY'. Therefore, $c < 0$. Hence, $a < 0$, $b < 0$ and $c < 0$.

(v) We notice that the parabola $y = ax^2 + bx + c$ opens upwards. Therefore, $a > 0$. The vertex $(-b/2a, \; -D/4a)$ of the parabola lies in the first quadrant.

$$\therefore \qquad \frac{-b}{2a} > 0 \Rightarrow \frac{b}{2a} < 0 \Rightarrow b < 0 \qquad [\because a > 0]$$

As $P\,(0, c)$ lies on OY. Therefore, $c > 0$. Hence, $a > 0$, $b < 0$ and $c > 0$.

(vi) Clearly, $a < 0$. As $(-b/2a, \; -D/4a)$ lies in the fourth quadrant.

$$\therefore \qquad \frac{-b}{2a} > 0 \Rightarrow \frac{b}{2a} < 0 \Rightarrow b > 0 \qquad [\because a < 0]$$

As $P\,(0, c)$ lies on OY'. Therefore, $c < 0$. Hence, $a < 0$, $b > 0$ and $c < 0$.

2.4 GEOMETRICAL MEANING OF THE ZEROS OF A POLYNOMIAL

In the previous section, we have seen that the graph of a linear polynomial is a straight line and it cuts x-axis at exactly one point. The graph of a quadratic polynomial is a parabola which cuts x-axis at atmost two points. We have also seen that the graph of a cubic polynomial cuts x-axis at atmost 3 points. In general, the graph of an nth degree polynomial crosses x-axis at atmost n points. Also, x-coordinates of these points are the zeros of the polynomial. Thus, geometrically zeros of a polynomial are the x-coordinates of the points where its graph crosses or touches x-axis.

It follows from the above discussion that an nth degree polynomial can have at most n real zeros. That is the number of real zeros of a polynomial is less than or equal to the degree of the polynomial. In higher classes, we will study that the total number of zeros (real or imaginary) of an nth degree polynomial is exactly n.

2.5 RELATIONSHIP BETWEEN THE ZEROS AND COEFFICIENTS OF A POLYNOMIAL

In class IX, we have learnt about the factorization of polynomials. In the previous section, we have studied that a polynomial of degree n has exactly n zeros (real or imaginary). The zeros of a polynomial are closely connected to its coefficients. In this section, we will find out relationship between the zeros and coefficients of a polynomial.

Consider the quadratic polynomial $f(x) = 6x^2 - x - 2$. By the method of splitting the middle term, we obtain

$$f(x) = 6x^2 - x - 2 = 6x^2 - 4x + 3x - 2 = 2x(3x-2) + 1(3x-2) = (3x-2)(2x+1)$$

$$\therefore \quad f(x) = 0$$

$$\Rightarrow \quad (3x-2)(2x+1) = 0 \Rightarrow 3x - 2 = 0 \text{ or, } 2x + 1 = 0 \Rightarrow x = \frac{2}{3} \text{ or, } x = -\frac{1}{2}$$

Hence, the zeros of $6x^2 - x - 2$ are $\alpha = \frac{2}{3}$ and $\beta = -\frac{1}{2}$.

We observe that

$$\text{Sum of its zeros } = \alpha + \beta = \frac{2}{3} - \frac{1}{2} = \frac{1}{6} = \frac{-(-1)}{6} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

Product of its zeros

Let us now consider a cubic polynomial $p(x)$ given by

$$p(x) = 6x^3 + 5x^2 - 12x + 4$$

$$\Rightarrow \quad p(x) = (x+2)(2x-1)(3x-2) \qquad \text{[By using factorization]}$$

$$\therefore \quad p(x) = 0$$

$$\Rightarrow \quad (x+2)(2x-1)(3x-2) = 0$$

$$\Rightarrow \quad x = -2, \frac{1}{2}, \frac{2}{3}$$

Hence, the zeros of $p(x) = 6x^3 + 5x^2 - 12x + 4$ are $\alpha = -2, \beta = \frac{1}{2}$ and $\gamma = \frac{2}{3}$.

Also, $p(x)$ is a cubic polynomial. So, $p(x)$ can have atmost three real zeros.

Now,

$$\text{Sum of the zeros} = \alpha + \beta + \gamma = -2 + \frac{1}{2} + \frac{2}{3} = -\frac{5}{6} = -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

$$\begin{aligned}\text{Sum of the products of zeros taken two at a time} &= \alpha\beta + \beta\gamma + \gamma\alpha \\ &= (-2) \times \frac{1}{2} + \frac{1}{2} \times \frac{2}{3} + \frac{2}{3} \times (-2) \\ &= -1 + \frac{1}{3} - \frac{4}{3} = -2 \\ &= \frac{-12}{6} = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}\end{aligned}$$

$$\text{Product of all zeros} = \alpha\beta\gamma = (-2) \times \frac{1}{2} \times \frac{2}{3} = -\frac{2}{3} = -\frac{4}{6} = -\frac{\text{Constant term}}{\text{Coefficient of } x^3}$$

Let us now find a formal relation between zeros and coefficients of a polynomial.

2.5.1 RELATIONSHIP BETWEEN THE ZEROS AND COEFFICIENTS OF A QUADRATIC POLYNOMIAL

Let α and β be the zeros of a quadratic polynomial $f(x) = ax^2 + bx + c$. By factor theorem $(x - \alpha)$ and $(x - \beta)$ are the factors of $f(x)$.

$\therefore \quad f(x) = k(x - \alpha)(x - \beta)$, where k is a constant

$\Rightarrow \quad ax^2 + bx + c = k\left\{x^2 - (\alpha + \beta)x + \alpha\beta\right\}$

$\Rightarrow \quad ax^2 + bx + c = kx^2 - k(\alpha + \beta)x + k\alpha\beta$

Comparing the coefficients of x^2, x and constant terms on both sides, we get

$$a = k,\ b = -k(\alpha + \beta) \text{ and } c = k\alpha\beta$$

$$\Rightarrow \quad \alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

$$\Rightarrow \quad \alpha + \beta = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} \text{ and, } \alpha\beta = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

Hence,

$$\text{Sum of the zeros} = -\frac{b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}, \text{Product of the zeros} = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

REMARK *If α and β are the zeros of a quadratic polynomial $f(x)$. Then, the polynomial $f(x)$ is given by*

$$f(x) = k\{x^2 - (\alpha + \beta)x + \alpha\beta\}$$

or, $\quad f(x) = k\left\{x^2 - (\text{Sum of the zeros})\,x + \text{Product of the zeros}\right\}$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I ON VERIFYING THE RELATIONSHIP BETWEEN THE ZEROS AND COEFFICIENTS

EXAMPLE 1 *Find the zeros of the quadratic polynomial* $x^2+7x+12$, *and verify the relation between the zeros and its coefficients.*

SOLUTION We have,

$$f(x)=x^2+7x+12=x^2+4x+3x+12=x(x+4)+3(x+4)=(x+4)(x+3)$$

The zeros of $f(x)$ are given by $f(x)=0$.

Now, $f(x)=0$

$\Rightarrow \quad x^2+7x+12=0$

$\Rightarrow \quad (x+4)(x+3)=0$

$\Rightarrow \quad x+4=0$ or, $x+3=0$

$\Rightarrow \quad x=-4$ or, $x=-3$.

Thus, the zeros of $f(x)=x^2+7x+12$ are $\alpha=-4$ and $\beta=-3$.

Now,

$$\text{Sum of the zeros}=\alpha+\beta=(-4)+(-3)=-7 \text{ and, } -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}=-\frac{7}{1}=-7$$

$$\therefore \quad \text{Sum of the zeros}=-\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

$$\text{Product of the zeros}=\alpha\beta=(-4)\times(-3)=12 \text{ and, } \frac{\text{Constant term}}{\text{Coefficient of } x^2}=\frac{12}{1}=12$$

$$\therefore \quad \text{Product of the zeros}=\frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

EXAMPLE 2 *Find the zeros of the quadratic polynomial* $f(x)=6x^2-3$, *and verify the relationship between the zeros and its coefficients:*

SOLUTION We have,

$f(x)=6x^2-3$

$\Rightarrow \quad f(x)=(\sqrt{6}x)^2-(\sqrt{3})^2$

$\Rightarrow \quad f(x)=(\sqrt{6}x-\sqrt{3})(\sqrt{6}x+\sqrt{3})$

The zeros of $f(x)$ are given by $f(x)=0$.

Now, $f(x)=0$

$\Rightarrow \quad (\sqrt{6}x-\sqrt{3})(\sqrt{6}x+\sqrt{3})=0$

$\Rightarrow \quad \sqrt{6}x-\sqrt{3}=0$ or, $\sqrt{6}x+\sqrt{3}=0$

$\Rightarrow \quad x=\frac{\sqrt{3}}{\sqrt{6}}$ or, $x=\frac{-\sqrt{3}}{\sqrt{6}}$

$$\Rightarrow \qquad x = \frac{1}{\sqrt{2}} \text{ or, } x = -\frac{1}{\sqrt{2}}$$

Hence, the zeros of $f(x) = 6x^2 - 3$ are: $\alpha = \frac{1}{\sqrt{2}}$ and $\beta = -\frac{1}{\sqrt{2}}$.

Now,

$$\text{Sum of the zeros} = \alpha + \beta = \frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right) = 0 \text{ and, } -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} = -\frac{0}{6} = 0$$

$$\therefore \qquad \text{Sum of the zeros} = -\frac{\text{Coefficient of } x}{\text{coefficient of } x^2}$$

Also,

$$\text{Product of the zeros} = \alpha\beta = \frac{1}{\sqrt{2}} \times \frac{-1}{\sqrt{2}} = \frac{-1}{2} \text{ and, } \frac{\text{Constant term}}{\text{Coefficient of } x^2} = \frac{-3}{6} = \frac{-1}{2}$$

$$\therefore \qquad \text{Product of the zeros} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

EXAMPLE 3 *Find the zeros of the polynomial* $f(u) = 4u^2 + 8u$, *and verify the relationship between the zeros and its coefficients.* **[NCERT]**

SOLUTION We have,

$$f(u) = 4u^2 + 8u$$

$$\Rightarrow \qquad f(u) = 4u(u+2)$$

The zeros of $f(u)$ are given by $f(u) = 0$.

$$\text{Now,} \qquad f(u) = 0$$

$$\Rightarrow \qquad 4u(u+2) = 0$$

$$\Rightarrow \qquad u = 0 \text{ or, } u + 2 = 0$$

$$\Rightarrow \qquad u = 0 \text{ or, } u = -2$$

Hence, the zeros of $f(u)$ are: $\alpha = 0$ and $\beta = -2$

Now,

$$\alpha + \beta = 0 + (-2) = -2 \text{ and } \alpha\beta = 0 \times -2 = 0$$

$$\text{Also,} \qquad -\frac{\text{Coefficient of } u}{\text{Coefficient of } u^2} = -\frac{8}{4} = -2 \text{ and, } \frac{\text{Constant term}}{\text{Coefficient of } u^2} = \frac{0}{4} = 0$$

$$\therefore \quad \text{Sum of the zeros} = -\frac{\text{Coefficient of } u}{\text{Coefficient of } u^2} \text{ and, Product of the zeros} = \frac{\text{Constant term}}{\text{Coefficient of } u^2}$$

EXAMPLE 4 *Find the zeros of the polynomial* $f(x) = 4\sqrt{3}x^2 + 5x - 2\sqrt{3}$, *and verify the relationship between the zeros and its coefficients.*

SOLUTION We have,

$$f(x) = 4\sqrt{3}x^2 + 5x - 2\sqrt{3}$$

$\Rightarrow \quad f(x) = 4\sqrt{3}\,x^2 + 8x - 3x - 2\sqrt{3}$ [Splitting the middle term]

$\Rightarrow \quad f(x) = 4x(\sqrt{3}x + 2) - \sqrt{3}(\sqrt{3}x + 2)$

$\Rightarrow \quad f(x) = (\sqrt{3}x + 2)(4x - \sqrt{3})$

The zeros of $f(x)$ are given by $f(x) = 0$.

Now, $\quad f(x) = 0$

$\Rightarrow \quad (\sqrt{3}x + 2)(4x - \sqrt{3}) = 0$

$\Rightarrow \quad \sqrt{3}x + 2 = 0$ or, $4x - \sqrt{3} = 0$

$\Rightarrow \quad x = -\dfrac{2}{\sqrt{3}}$ or, $x = \dfrac{\sqrt{3}}{4}$

Hence, the zeros of $f(x)$ are: $\alpha = -\dfrac{2}{\sqrt{3}}$ and $\beta = \dfrac{\sqrt{3}}{4}$

Now,

$$\alpha + \beta = -\frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{4} = \frac{-8+3}{4\sqrt{3}} = -\frac{5}{4\sqrt{3}} \text{ and, } \alpha\beta = \frac{-2}{\sqrt{3}} \times \frac{\sqrt{3}}{4} = -\frac{1}{2}$$

Also,

$$-\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} = -\frac{5}{4\sqrt{3}} \text{ and, } \frac{\text{Constant term}}{\text{Coefficient of } x^2} = \frac{-2\sqrt{3}}{4\sqrt{3}} = -\frac{1}{2}$$

Hence, Sum of the roots $= -\dfrac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$ and, Prodcut of the roots $= \dfrac{\text{Constant term}}{\text{Coefficient of } x^2}$

EXAMPLE 5 *Find the zeros of the quadratic polynomial $f(x) = abx^2 + (b^2 - ac)x - bc$, and verify the relationship between the zeros and its coefficients.*

SOLUTION We have,

$f(x) = abx^2 + (b^2 - ac)x - bc$

$\Rightarrow \quad f(x) = abx^2 + b^2x - acx - bc$

$\Rightarrow \quad f(x) = bx(ax + b) - c(ax + b)$

$\Rightarrow \quad f(x) = (ax + b)(bx - c)$

The zeros of $f(x)$ are given by $f(x) = 0$.

Now, $\quad f(x) = 0$

$\Rightarrow \quad (ax + b)(bx - c) = 0$

$\Rightarrow \quad ax + b = 0$ or, $bx - c = 0$

$\Rightarrow \quad x = -\dfrac{b}{a}$ or, $x = \dfrac{c}{b}$

Thus, the zeros of $f(x)$ are: $\alpha = -\dfrac{b}{a}$ and, $\beta = \dfrac{c}{b}$.

Now,

$$\alpha + \beta = -\frac{b}{a} + \frac{c}{b} = \frac{ac - b^2}{ab} \text{ and, } \alpha\beta = -\frac{b}{a} \times \frac{c}{b} = -\frac{c}{a}$$

Also,

$$-\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} = -\left(\frac{b^2 - ac}{ab}\right) = \frac{ac - b^2}{ab} \text{ and, } \frac{\text{Constant term}}{\text{Coefficient of } x^2} = -\frac{bc}{ab} = -\frac{c}{a}$$

Hence,

$$\text{Sum of the zeros} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} \text{ and, Product of the zeros} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

EXAMPLE 6 *Find the zeros of the polynomial* $x^2 + \frac{1}{6}x - 2$, *and verify the relation between the coefficients and zeros of the polynomial.* **[NCERT EXEMPLAR]**

SOLUTION Let $f(x) = x^2 + \frac{1}{6}x - 2$. Then,

$$f(x) = \frac{1}{6}(6x^2 + x - 12) = \frac{1}{6}(6x^2 + 9x - 8x - 12)$$

$$\Rightarrow \quad f(x) = \frac{1}{6}\{(6x^2 + 9x) - (8x + 12)\} = \frac{1}{6}\{3x(2x + 3) - 4(2x + 3)\} = \frac{1}{6}(2x + 3)(3x - 4)$$

The zeros of $f(x)$ are given by $f(x) = 0$.

Now, $f(x) = 0 \Rightarrow \frac{1}{6}(2x + 3)(3x - 4) = 0 \Rightarrow 2x + 3 = 0$ or, $3x - 4 = 0 \Rightarrow x = \frac{-3}{2}$ or, $x = \frac{4}{3}$

Hence, $\alpha = \frac{-3}{2}$ and $\beta = \frac{4}{3}$ are the zeros of the given polynomial.

Now,

$$\alpha + \beta = \left(-\frac{3}{2}\right) + \frac{4}{3} = -\frac{1}{6} \text{ and, } \alpha\beta = \left(\frac{-3}{2}\right)\left(\frac{4}{3}\right) = -2$$

The given polynomial is $f(x) = x^2 + \frac{1}{6}x - 2$.

$$\therefore \quad -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} = \left(\frac{-1/6}{1}\right) = \frac{-1}{6} \text{ and, } \frac{\text{Constant term}}{\text{Coefficient of } x^2} = \frac{-2}{1} = -2$$

Clearly,

$$\alpha + \beta = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} \text{ and, } \alpha\beta = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

Hence, the relation between the coefficients and zeros is verified.

Type II ON FINDING THE VALUES OF SYMMETRIC EXPRESSIONS INVOLVING ZEROS OF A QUADRATIC POLYNOMIAL

EXAMPLE 7 *If* α *and* β *are the zeros of the quadratic polynomial* $f(x) = x^2 - px + q$, *then find the values of (i)* $\alpha^2 + \beta^2$ *(ii)* $\frac{1}{\alpha} + \frac{1}{\beta}$

SOLUTION It is given that α and β are the zero of the polynomial $f(x) = x^2 - px + q$.

$\therefore \quad \alpha + \beta = -\left(\frac{-p}{1}\right) = p$ and, $\alpha\beta = \frac{q}{1} = q$

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = p^2 - 2q$ $\qquad [\because \alpha + \beta = p \text{ and } \alpha\beta = q]$

(ii) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{p}{q}$

Type III ON FINDING A QUADRATIC POLYNOMIAL WHEN THE SUM AND PRODUCT OF ITS ZEROES ARE GIVEN

EXAMPLE 8 *Find a quadratic polynomial each with the given numbers as the sum and product of its zeros respectively*

(i) $\frac{1}{4}, -1$ [NCERT] (ii) $\sqrt{2}, \frac{1}{3}$ [NCERT] (iii) $0, \sqrt{5}$ [NCERT]

SOLUTION We know that a quadratic polynomial when the sum and product of its zeros are given by

$f(x) = k\left\{x^2 - (\text{Sum of the zeros})\, x + \text{Product of the zeros}\right\}$, where k is a constant.

(i) We have, Sum $= \frac{1}{4}$ and, Product $= -1$. So, required quadratic polynomial $f(x)$ is given by

$$f(x) = k\left(x^2 - \frac{1}{4}x - 1\right)$$

(ii) We have, Sum $= \sqrt{2}$ and, Product $= \frac{1}{3}$. So, required quadratic polynomial $f(x)$ is given by

$$f(x) = k\left(x^2 - \sqrt{2}x + \frac{1}{3}\right)$$

(iii) We have, Sum $= 0$ and, Product $= \sqrt{5}$. So, required quadratic polynomial $f(x)$ is given by

$$f(x) = k(x^2 - 0x + \sqrt{5}) = k(x^2 + \sqrt{5})$$

EXAMPLE 9 *Find a quadratic polynomial, the sum and product of whose zeroes are* $\sqrt{2}$ *and* $-\frac{3}{2}$ *respectively. Also, find its zeroes.* [NCERT EXEMPLAR]

SOLUTION Let α, β be the zeros of required polynomial. It is given that $\alpha + \beta = \sqrt{2}$ and $\alpha\beta = -\frac{3}{2}$.

The quadratic polynomial is $f(x) = x^2 - (\alpha + \beta)x + \alpha\beta$ or, $f(x) = x^2 - \sqrt{2}\,x - \frac{3}{2}$

Now, $\quad f(x) = x^2 - \sqrt{2}x - \frac{3}{2}$

$\Rightarrow \quad f(x) = \frac{1}{2}(2x^2 - 2\sqrt{2}x - 3)$

$\Rightarrow \quad f(x) = \frac{1}{2}(2x^2 - 3\sqrt{2}x + \sqrt{2}x - 3)$

$$\Rightarrow \quad f(x) = \frac{1}{2}\{\sqrt{2}x(\sqrt{2}x - 3) + (\sqrt{2}x - 3)\}$$

$$\Rightarrow \quad f(x) = \frac{1}{2}(\sqrt{2}x - 3)(\sqrt{2}x + 1)$$

The zeroes of $f(x)$ are given by $f(x) = 0$.

Now, $\quad f(x) = 0$

$$\Rightarrow \quad \frac{1}{2}(\sqrt{2}x - 3)(\sqrt{2}x + 1) = 0 \Rightarrow \sqrt{2}x - 3 = 0 \text{ or, } \sqrt{2}x + 1 = 0 \Rightarrow x = \frac{3}{\sqrt{2}} \text{ or, } x = -\frac{1}{\sqrt{2}}$$

Hence, the zeroes of $f(x)$ are $\frac{3}{\sqrt{2}}$ and $-\frac{1}{\sqrt{2}}$.

EXAMPLE 10 *If α and β are the zeros of the quadratic polynomial* $f(x) = x^2 - x - 2$, *find a polynomial whose zeros are* $2\alpha + 1$ *and* $2\beta + 1$.

SOLUTION It is given that α and β are the zeros of the polynomial $f(x) = x^2 - x - 2$.

$$\therefore \quad \alpha + \beta = -\left(-\frac{1}{1}\right) = 1 \text{ and, } \alpha\beta = -\frac{2}{1} = -2$$

Let S and P denote respectively the sum and the product of zeros of the required polynomial. Then,

$$S = (2\alpha + 1) + (2\beta + 1) = 2(\alpha + \beta) + 2 = 2 \times 1 + 2 = 4 \qquad [\because \alpha + \beta = 1]$$

and, $\quad P = (2\alpha + 1)(2\beta + 1) = 4\alpha\beta + 2\alpha + 2\beta + 1 = 4\alpha\beta + 2(\alpha + \beta) + 1$

$$= 4 \times -2 + 2 \times 1 + 1 = -8 + 2 + 1 = -5 \qquad [\because \alpha + \beta = 1 \text{ and } \alpha\beta = -2]$$

Hence, required polynomial $g(x)$ is

$$g(x) = k\{x^2 - Sx + P\} = k(x^2 - 4x - 5), \text{ where } k \text{ is any non-zero constant.}$$

LEVEL-2

Type IV ON FINDING THE VALUES OF SYMMETRIC EXPRESSIONS INVOLVING ZEROES OF A QUADRATIC POLYNOMIAL

EXAMPLE 11 *If α and β are the zeros of the quadratic polynomial* $f(x) = ax^2 + bx + c$, *then evaluate:*

(i) $\alpha^2 + \beta^2$ (ii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (iii) $\alpha^3 + \beta^3$

(iv) $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$ (v) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$

SOLUTION It is given that α and β are the zeros of the quadratic polynomial $f(x) = ax^2 + bx + c$.

$$\therefore \quad \alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

(i) We know that

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$\therefore \quad \alpha^2 + \beta^2 = \left(\frac{-b}{a}\right)^2 - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}$$

(ii) $$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{\left(\frac{-b}{a}\right)^2-2\left(\frac{c}{a}\right)}{\frac{c}{a}}=\frac{b^2-2ac}{ac}$$

(iii) We know that $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

$$\therefore \quad \alpha^3+\beta^3=\left(\frac{-b}{a}\right)^3-3\frac{c}{a}\left(\frac{-b}{a}\right)=\frac{-b^3}{a^3}+\frac{3bc}{a^2}=\frac{-b^3+3abc}{a^3}=\frac{3abc-b^3}{a^3}$$

(iv) $$\frac{1}{\alpha^3}+\frac{1}{\beta^3}=\frac{\alpha^3+\beta^3}{(\alpha\beta)^3}=\frac{\frac{3abc-b^3}{a^3}}{\left(\frac{c}{a}\right)^3}=\frac{3abc-b^3}{c^3}$$ [Using (iii)]

(v) $$\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}=\frac{\alpha^3+\beta^3}{\alpha\beta}=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}=\frac{\left(-\frac{b}{a}\right)^3-3\left(\frac{c}{a}\right)\left(-\frac{b}{a}\right)}{\frac{c}{a}}=\frac{3abc-b^3}{a^2c}$$

EXAMPLE 12 *If α and β are the zeros of the quadratic polynomial $f(x)=ax^2+bx+c$, then evaluate:*

(i) $\alpha^4+\beta^4$ *(ii)* $\frac{\alpha^2}{\beta^2}+\frac{\beta^2}{\alpha^2}$

SOLUTION It is given that α and β are the zeros of the quadratic polynomial $f(x)=ax^2+bx+c$.

$$\therefore \quad \alpha+\beta=-\frac{b}{a} \text{ and, } \alpha\beta=\frac{c}{a}$$

(i) We have,

$$\alpha^4+\beta^4=(\alpha^2+\beta^2)^2-2\alpha^2\beta^2$$

$$\Rightarrow \quad \alpha^4+\beta^4=\left\{(\alpha+\beta)^2-2\alpha\beta\right\}^2-2(\alpha\beta)^2$$

$$\Rightarrow \quad \alpha^4+\beta^4=\left\{\left(-\frac{b}{a}\right)^2-2\frac{c}{a}\right\}^2-2\left(\frac{c}{a}\right)^2 \qquad \left[\because \alpha+\beta=-\frac{b}{a}, \alpha\beta=\frac{c}{a}\right]$$

$$\Rightarrow \quad \alpha^4+\beta^4=\left(\frac{b^2-2ac}{a^2}\right)^2-\frac{2c^2}{a^2}=\frac{(b^2-2ac)^2-2a^2c^2}{a^4}$$

(ii) We have,

$$\frac{\alpha^2}{\beta^2}+\frac{\beta^2}{\alpha^2}=\frac{\alpha^4+\beta^4}{\alpha^2\beta^2}=\frac{(b^2-2ac)^2-2a^2c^2}{a^4\times\left(\frac{c}{a}\right)^2}$$ [Using (i)]

$$=\frac{(b^2-2ac)^2-2a^2c^2}{a^2c^2}$$

Type V ON FINDING AN UNKNOWN WHEN A RELATION BETWEEN ZEROS AND COEFFICIENTS IS GIVEN

EXAMPLE 13 *If α and β are the zeros of the polynomial $f(x) = x^2 - 5x + k$ such that $\alpha - \beta = 1$, find the value of k.*

SOLUTION It is given that α and β are the zeros of the polynomial $f(x) = x^2 - 5x + k$.

$$\therefore \quad \alpha + \beta = -\left(\frac{-5}{1}\right) = 5 \text{ and, } \alpha\beta = \frac{k}{1} = k$$

Now, $\alpha - \beta = 1$ [Given]

$$\Rightarrow \quad (\alpha - \beta)^2 = 1$$

$$\Rightarrow \quad (\alpha + \beta)^2 - 4\alpha\beta = 1$$

$$\Rightarrow \quad 25 - 4k = 1 \Rightarrow 24 = 4k \Rightarrow k = 6$$

Hence, the value of k is 6.

EXAMPLE 14 *If α and β are the zeros of the quadratic polynomial $f(x) = kx^2 + 4x + 4$ such that $\alpha^2 + \beta^2 = 24$, find the values of k.*

SOLUTION It is given that α and β are the zeros of the quadratic polynomial $f(x) = kx^2 + 4x + 4$

$$\therefore \quad \alpha + \beta = -\frac{4}{k} \text{ and, } \alpha\beta = \frac{4}{k}$$

We have,

$$\alpha^2 + \beta^2 = 24$$

$$\Rightarrow \quad (\alpha + \beta)^2 - 2\alpha\beta = 24 \qquad [\because \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta]$$

$$\Rightarrow \quad \left(-\frac{4}{k}\right)^2 - 2 \times \frac{4}{k} = 24$$

$$\Rightarrow \quad \frac{16}{k^2} - \frac{8}{k} = 24$$

$$\Rightarrow \quad 16 - 8k = 24k^2$$

$$\Rightarrow \quad 3k^2 + k - 2 = 0$$

$$\Rightarrow \quad 3k^2 + 3k - 2k - 2 = 0$$

$$\Rightarrow \quad 3k(k + 1) - 2(k + 1) = 0$$

$$\Rightarrow \quad (k + 1)(3k - 2) = 0 \Rightarrow k + 1 = 0 \text{ or, } 3k - 2 = 0 \Rightarrow k = -1 \text{ or, } k = \frac{2}{3}$$

Hence, $k = -1$ or, $k = \dfrac{2}{3}$

EXAMPLE 15 *If α, β are the zeros of the polynomial $f(x) = 2x^2 + 5x + k$ satisfying the relation $\alpha^2 + \beta^2 + \alpha\beta = \frac{21}{4}$, then find the value of k for this to be possible.*

SOLUTION It is given that α and β are the zeros of the polynomial $f(x) = 2x^2 + 5x + k$.

$$\therefore \quad \alpha + \beta = -\frac{5}{2} \text{ and, } \alpha\beta = \frac{k}{2}$$

We have,

$$\alpha^2 + \beta^2 + \alpha\beta = \frac{21}{4}$$

$$\Rightarrow \quad (\alpha^2 + \beta^2 + 2\alpha\beta) - \alpha\beta = \frac{21}{4}$$

$$\Rightarrow \quad (\alpha + \beta)^2 - \alpha\beta = \frac{21}{4}$$

$$\Rightarrow \quad \frac{25}{4} - \frac{k}{2} = \frac{21}{4} \qquad \left[\because \alpha + \beta = -\frac{5}{2} \text{ and, } \alpha\beta = \frac{k}{2}\right]$$

$$\Rightarrow \quad -\frac{k}{2} = -1 \Rightarrow k = 2$$

EXAMPLE 16 *If sum of the squares of zeros of the quadratic polynomial* $f(x) = x^2 - 8x + k$ *is 40, find the value of k.*

SOLUTION Let α, β be the zeros of the polynomial $f(x) = x^2 - 8x + k$. Then,

$$\alpha + \beta = -\left(\frac{-8}{1}\right) = 8 \text{ and, } \alpha\beta = \frac{k}{1} = k$$

It is given that

$$\alpha^2 + \beta^2 = 40$$

$$\Rightarrow \quad (\alpha + \beta)^2 - 2\alpha\beta = 40$$

$$\Rightarrow \quad 8^2 - 2k = 40 \qquad [\because \alpha + \beta = 8 \text{ and } \alpha\beta = k]$$

$$\Rightarrow \quad 2k = 64 - 40 \Rightarrow 2k = 24 \Rightarrow k = 12$$

Type VI ON FINDING A QUADRATIC POLYNOMIAL WHEN THE SUM AND PRODUCT OF ITS ZEROS ARE GIVEN

EXAMPLE 17 *If* α *and* β *are the zeros of the quadratic polynomial* $f(x) = 2x^2 - 5x + 7$, *find a polynomial whose zeros are* $2\alpha + 3\beta$ *and* $3\alpha + 2\beta$.

SOLUTION It is given that α and β are the zeros of the quadratic polynomial $f(x) = 2x^2 - 5x + 7$.

$$\therefore \quad \alpha + \beta = -\left(-\frac{5}{2}\right) = \frac{5}{2} \text{ and, } \alpha\beta = \frac{7}{2}$$

Let S and P denote respectively the sum and product of zeros of the required polynomial. Then,

$$S = (2\alpha + 3\beta) + (3\alpha + 2\beta) = 5(\alpha + \beta) = 5 \times \frac{5}{2} = \frac{25}{2}$$

and, $$P = (2\alpha + 3\beta)(3\alpha + 2\beta) = 6(\alpha^2 + \beta^2) + 13\alpha\beta = 6\alpha^2 + 6\beta^2 + 12\alpha\beta + \alpha\beta$$

$$= 6(\alpha + \beta)^2 + \alpha\beta = 6 \times \left(\frac{5}{2}\right)^2 + \frac{7}{2} = \frac{75}{2} + \frac{7}{2} = 41$$

Hence, the required polynomial $g(x)$ is given by

$g(x) = k(x^2 - Sx + P)$ or, $g(x) = k\left(x^2 - \frac{25}{2}x + 41\right)$, where k is any non-zero real number.

EXAMPLE 18 *If α and β are the zeros of the quadratic polynomial* $f(x) = 3x^2 - 4x + 1$*, find a quadratic polynomial whose zeros are* $\frac{\alpha^2}{\beta}$ *and* $\frac{\beta^2}{\alpha}$.

SOLUTION It is given that α and β are the zeros of the polynomial $f(x) = 3x^2 - 4x + 1$.

$$\therefore \qquad \alpha + \beta = -\left(-\frac{4}{3}\right) = \frac{4}{3} \text{ and, } \alpha\beta = \frac{1}{3}$$

Let S and P denote respectively the sum and product of the zeros of the polynomial whose zeros are $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$. Then,

$$S = \frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3 + \beta^3}{\alpha\beta} = \frac{(\alpha + \beta)^3 - 3\alpha\beta\,(\alpha + \beta)}{\alpha\beta} = \frac{\left(\frac{4}{3}\right)^3 - 3 \times \frac{1}{3} \times \frac{4}{3}}{\frac{1}{3}} = \frac{28}{9}$$

and, $$P = \frac{\alpha^2}{\beta} \times \frac{\beta^2}{\alpha} = \alpha\beta = \frac{1}{3}$$

Hence, the required polynomial $g(x)$ is given by

$g(x) = k\,(x^2 - Sx + P)$ or, $g(x) = k\left(x^2 - \frac{28}{9}x + \frac{1}{3}\right)$, where k is any non-zero real number.

EXAMPLE 19 *Find a quadratic polynomial whose zeros are reciprocals of the zeros of the polynomial* $f(x) = ax^2 + bx + c, a \neq 0, c \neq 0$.

SOLUTION Let α and β be the zeros of the polynomial $f(x) = ax^2 + bx + c$. Then,

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

Let S and P denote respectively the sum and product of the zeros of a polynomial whose zeros are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$. Then,

$$S = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{-\frac{b}{a}}{\frac{c}{a}} = -\frac{b}{c} \text{ and } P = \frac{1}{\alpha} \times \frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{\frac{c}{a}} = \frac{a}{c}$$

Hence, the required polynomial $g(x)$ is given by

$$g(x) = k(x^2 - Sx + P) = k\left(x^2 + \frac{bx}{c} + \frac{a}{c}\right), \text{ where } k \text{ is any non-zero constant.}$$

EXERCISE 2.1

LEVEL-1

1. Find the zeros of each of the following quadratic polynomials and verify the relationship between the zeros and their coefficients:
   (i) $f(x) = x^2 - 2x - 8$ [NCERT] (ii) $g(s) = 4s^2 - 4s + 1$ [NCERT]
   (iii) $h(t) = t^2 - 15$ [NCERT] (iv) $f(x) = 6x^2 - 3 - 7x$ [NCERT]
   (v) $p(x) = x^2 + 2\sqrt{2}x - 6$ (vi) $q(x) = \sqrt{3}x^2 + 10x + 7\sqrt{3}$

(vii) $f(x) = x^2 - (\sqrt{3}+1)x + \sqrt{3}$ (viii) $g(x) = a(x^2+1) - x(a^2+1)$

(ix) $h(s) = 2s^2 - (1+2\sqrt{2})s + \sqrt{2}$ [NCERT EXEMPLAR]

(x) $f(v) = v^2 + 4\sqrt{3}v - 15$ [NCERT EXEMPLAR]

(xi) $p(y) = y^2 + \frac{3\sqrt{5}}{2}y - 5$ [NCERT EXEMPLAR]

(xii) $q(y) = 7y^2 - \frac{11}{3}y - \frac{2}{3}$ [NCERT EXEMPLAR]

2. For each of the following, find a quadratic polynomial whose sum and product respectively of the zeroes are as given. Also, find the zeroes of these polynomials by factorization.

(i) $-\frac{8}{3}, \frac{4}{3}$ (ii) $\frac{21}{8}, \frac{5}{16}$ (iii) $-2\sqrt{3}, -9$ (iv) $\frac{-3}{2\sqrt{5}}, -\frac{1}{2}$

[NCERT EXEMPLAR]

3. If α and β are the zeros of the quadratic polynomial $f(x) = x^2 - 5x + 4$, find the value of $\frac{1}{\alpha} + \frac{1}{\beta} - 2\alpha\beta$.

4. If α and β are the zeros of the quadratic polynomial $p(y) = 5y^2 - 7y + 1$, find the value of $\frac{1}{\alpha} + \frac{1}{\beta}$.

5. If α and β are the zeros of the quadratic polynomial $f(x) = x^2 - x - 4$, find the value of $\frac{1}{\alpha} + \frac{1}{\beta} - \alpha\beta$.

6. If α and β are the zeros of the quadratic polynomial $f(x) = x^2 + x - 2$, find the value of $\frac{1}{\alpha} - \frac{1}{\beta}$.

7. If one zero of the quadratic polynomial $f(x) = 4x^2 - 8kx - 9$ is negative of the other, find the value of k.

8. If the sum of the zeros of the quadratic polynomial $f(t) = kt^2 + 2t + 3k$ is equal to their product, find the value of k.

LEVEL-2

9. If α and β are the zeros of the quadratic polynomial $p(x) = 4x^2 - 5x - 1$, find the value of $\alpha^2\beta + \alpha\beta^2$.

10. If α and β are the zeros of the quadratic polynomial $f(t) = t^2 - 4t + 3$, find the value of $\alpha^4\beta^3 + \alpha^3\beta^4$.

11. If α and β are the zeros of the quadratic polynomial $f(x) = 6x^2 + x - 2$, find the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$.

12. If α and β are the zeros of the quadratic polynomial $p(s) = 3s^2 - 6s + 4$, find the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} + 2\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + 3\alpha\beta$.

13. If the squared difference of the zeros of the quadratic polynomial $f(x) = x^2 + px + 45$ is equal to 144, find the value of p.

14. If α and β are the zeros of the quadratic polynomial $f(x) = x^2 - px + q$, prove that
$$\frac{\alpha^2}{\beta^2} + \frac{\beta^2}{\alpha^2} = \frac{p^4}{q^2} - \frac{4p^2}{q} + 2.$$

15. If α and β are the zeros of the quadratic polynomial $f(x) = x^2 - p(x + 1) - c$, show that $(\alpha + 1)(\beta + 1) = 1 - c$.

16. If α and β are the zeros of a quadratic polynomial such that $\alpha + \beta = 24$ and $\alpha - \beta = 8$, find a quadratic polynomial having α and β as its zeros.

17. If α and β are the zeros of the quadratic polynomial $f(x) = x^2 - 1$, find a quadratic polynomial whose zeros are $\frac{2\alpha}{\beta}$ and $\frac{2\beta}{\alpha}$.

18. If α and β are the zeros of the quadratic polynomial $f(x) = x^2 - 3x - 2$, find a quadratic polynomial whose zeros are $\frac{1}{2\alpha + \beta}$ and $\frac{1}{2\beta + \alpha}$.

19. If α and β are the zeros of the polynomial $f(x) = x^2 + px + q$, form a polynomial whose zeros are $(\alpha + \beta)^2$ and $(\alpha - \beta)^2$.

20. If α and β are the zeros of the quadratic polynomial $f(x) = x^2 - 2x + 3$, find a polynomial whose roots are (i) $\alpha + 2, \beta + 2$ (ii) $\frac{\alpha - 1}{\alpha + 1}, \frac{\beta - 1}{\beta + 1}$.

21. If α and β are the zeros of the quadratic polynomial $f(x) = ax^2 + bx + c$, then evaluate:

(i) $\alpha - \beta$ (ii) $\frac{1}{\alpha} - \frac{1}{\beta}$ (iii) $\frac{1}{\alpha} + \frac{1}{\beta} - 2\alpha\beta$

(iv) $\alpha^2\beta + \alpha\beta^2$ (v) $\alpha^4 + \beta^4$ (vi) $\frac{1}{a\alpha + b} + \frac{1}{a\beta + b}$

(vii) $\frac{\beta}{a\alpha + b} + \frac{\alpha}{a\beta + b}$ (viii) $a\left(\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}\right) + b\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)$

ANSWERS

1. (i) $4, -2$ (ii) $\frac{1}{2}, \frac{1}{2}$ (iii) $\sqrt{15}, -\sqrt{15}$ (iv) $\frac{3}{2}, -\frac{1}{3}$

(v) $-3\sqrt{2}, \sqrt{2}$ (vi) $-\sqrt{3}, -\frac{7}{\sqrt{3}}$ (vii) $\sqrt{3}, 1$ (viii) $a, \frac{1}{a}$

(ix) $\sqrt{2}, \frac{1}{2}$ (x) $\sqrt{3}, -5\sqrt{3}$ (xi) $-2\sqrt{5}, \frac{\sqrt{5}}{2}$ (xii) $-\frac{1}{7}, \frac{2}{3}$

2. (i) $f(x) = k\left(x^2 + \frac{8}{3}x + \frac{4}{3}\right)$ (ii) $f(x) = k\left(x^2 - \frac{21}{8}x + \frac{5}{16}\right)$

(iii) $f(x) = k\left(x^2 + 2\sqrt{3}x - 9\right)$ (iv) $f(x) = k\left(x^2 + \frac{3}{2\sqrt{5}}x - \frac{1}{2}\right)$, where k is any constant

3. $-\frac{27}{4}$ **4.** 7 **5.** $\frac{15}{4}$ **6.** $-\frac{3}{2}$

7. 0 **8.** $-\frac{2}{3}$ **9.** $\frac{-5}{16}$ **10.** 108

11. $-\frac{25}{12}$ **12.** 8 **13.** ± 18

16. $f(x) = k(x^2 - 24x + 128)$ **17.** $f(x) = k(x^2 + 4x + 4)$

18. $f(x) = k\left(x^2 - \frac{9}{16}x + \frac{1}{16}\right)$ **19.** $f(x) = k\{x^2 - 2(p^2 - 2q)x + p^2(p^2 - 4q)\}$

20. (i) $f(x) = k(x^2 - 6x + 11)$ (ii) $f(x) = k\left\{x^2 - \frac{2}{3}x + \frac{1}{3}\right\}$

21. (i) $\frac{\sqrt{b^2 - 4ac}}{a}$ (ii) $\frac{\sqrt{b^2 - 4ac}}{c}$ (iii) $-\left(\frac{b}{c} + \frac{2c}{a}\right)$ (iv) $-\frac{bc}{a^2}$

(v) $\frac{(b^2 - 2ac)^2 - 2a^2c^2}{a^4}$ (vi) $\frac{b}{ac}$ (vii) $\frac{-2}{a}$ (viii) b

2.5.2 RELATIONSHIP BETWEEN ZEROS AND COEFFICIENTS OF A CUBIC POLYNOMIAL

Let α, β, γ be the zeros of a cubic polynomial $f(x) = ax^3 + bx^2 + cx + d, a \neq 0$. Then, by factor theorem, $x - \alpha, x - \beta$ and $x - \gamma$ are factors of $f(x)$. Also, $f(x)$, being a cubic polynomial, cannot have more than three linear factors.

$$\therefore \quad f(x) = k(x - \alpha)(x - \beta)(x - \gamma)$$

$$\Rightarrow \quad ax^3 + bx^2 + cx + d = k(x - \alpha)(x - \beta)(x - \gamma)$$

$$\Rightarrow \quad ax^3 + bx^2 + cx + d = k\{x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma\}$$

$$\Rightarrow \quad ax^3 + bx^2 + cx + d = kx^3 - k(\alpha + \beta + \gamma)x^2 + k(\alpha\beta + \beta\gamma + \gamma\alpha)x - k\alpha\beta\gamma$$

Comparing the coefficients of x^3, x^2, x and constant terms on both sides, we get

$$a = k, b = -k(\alpha + \beta + \gamma), c = k(\alpha\beta + \beta\gamma + \gamma\alpha) \text{ and } d = -k(\alpha\beta\gamma)$$

$$\Rightarrow \quad \alpha + \beta + \gamma = -\frac{b}{a}, \ \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} \text{ and, } \alpha\beta\gamma = -\frac{d}{a}$$

$$\Rightarrow \quad \text{Sum of the zeros} = -\frac{b}{a} = -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

$$\text{Sum of the products of the zeros taken two at a time} = \frac{c}{a} = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$$

$$\text{Product of the zeros} = -\frac{d}{a} = -\frac{\text{Constant term}}{\text{Coefficient of } x^3}$$

REMARK 1 *It follows from the above discussion that a cubic polynomial having α, β and γ as its zeros is given by*

$$f(x) = k(x - \alpha)(x - \beta)(x - \gamma)$$

or, $f(x) = k\{x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma\}$, where k is any non-zero real number.

REMARK 2 *If* $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ *is a polynomial of degree 4 having* α, β, γ *and* δ *as its zeros, then*

$$\alpha + \beta + \gamma + \delta = -\frac{b}{a} = -\frac{\text{Coefficient of } x^3}{\text{Coefficient of } x^4}$$

$$\alpha\beta + \beta\gamma + \gamma\delta + \alpha\delta + \alpha\gamma + \beta\delta = \frac{c}{a} = \frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^4}$$

or,
$$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = \frac{c}{a} = \frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^4}$$

$$\alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \alpha\gamma\delta = -\frac{d}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^4}$$

or,
$$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -\frac{d}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^4}$$

$$\alpha\beta\gamma\delta = \frac{e}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^4}$$

OR

$$\text{Sum of the zeros} = -\frac{b}{a} = -\frac{\text{Coefficient of } x^3}{\text{Coefficient of } x^4}$$

$$\text{Sum of the products of the zeros taken two at a time} = \frac{c}{a} = \frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^4}$$

$$\text{Sum of the products of the zeros taken three at a time} = -\frac{d}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^4}$$

$$\text{Product of the zeros} = \frac{e}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^4}$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I ON VERIFYING THE RELATIONSHIP BETWEEN THE ZEROS AND COEFFICIENTS OF A POLYNOMIAL

EXAMPLE 1 *Verify that* $3, -1$ *and* $-\frac{1}{3}$ *are the zeros of the cubic polynomial* $p(x) = 3x^3 - 5x^2 - 11x - 3$ *and then verify the relationship between the zeros and its coefficients.* [NCERT]

SOLUTION Given polynomial is $p(x) = 3x^3 - 5x^2 - 11x - 3$

$\therefore$
$$p(3) = 3 \times 3^3 - 5 \times 3^2 - 11 \times 3 - 3 = 81 - 45 - 33 - 3 = 0$$

$$p(-1) = 3 \times (-1)^3 - 5 \times (-1)^2 - 11 \times (-1) - 3 = -3 - 5 + 11 - 3 = 0$$

and,
$$p\left(-\frac{1}{3}\right) = 3 \times \left(-\frac{1}{3}\right)^3 - 5 \times \left(-\frac{1}{3}\right)^2 - 11 \times \left(-\frac{1}{3}\right) - 3 = -\frac{1}{9} - \frac{5}{9} + \frac{11}{3} - 3 = 0$$

So, $3, -1$ and $-\frac{1}{3}$ are the zeros of polynomial $p(x)$.

Let $\alpha = 3, \beta = -1$ and $\gamma = -\frac{1}{3}$. Then,

$$\alpha + \beta + \gamma = 3 - 1 - \frac{1}{3} = \frac{5}{3} \text{ and, } -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3} = -\left(\frac{-5}{3}\right) = \frac{5}{3}$$

$$\therefore \quad \alpha + \beta + \gamma = -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = 3 \times (-1) + (-1) \times \left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right) \times 3 = -3 + \frac{1}{3} - 1 = -\frac{11}{3}$$

and, $$\frac{\text{Coefficient of } x}{\text{Coefficient of } x^3} = -\frac{11}{3}$$

$$\therefore \quad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$$

$$\alpha\beta\gamma = 3 \times (-1) \times \left(-\frac{1}{3}\right) = 1 \text{ and, } -\frac{\text{Constant term}}{\text{Coefficient of } x^3} = -\left(\frac{-3}{3}\right) = 1$$

$$\therefore \quad \alpha\beta\gamma = -\frac{\text{Constant term}}{\text{Coefficient of } x^3}$$

Type II ON FINDING A CUBIC POLYNOMIAL WHEN SUM, SUM OF THE PRODUCTS OF ITS ZEROS TAKEN TWO AT A TIME, AND PRODUCT OF ITS ZEROS ARE GIVEN

EXAMPLE 2 *Find a cubic polynomial with the sum, sum of the products of its zeros taken two at a time, and product of its zeros as 2, – 7, – 14 respectively.* [NCERT]

SOLUTION If α, β and γ are the zeros of a cubic polynomial $f(x)$, then

$$f(x) = k\,\{x^3 - (\alpha + \beta + \gamma)\, x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)\, x - \alpha\beta\gamma\},$$

where k is any non-zero real number.

Here, $\alpha + \beta + \gamma = 2, \alpha\beta + \beta\gamma + \gamma\alpha = -7$ and $\alpha\beta\gamma = -14$

$\therefore$ $f(x) = k\,(x^3 - 2x^2 - 7x + 14)$, where k is any non-zero real number.

Type III ON FINDING THE ZEROS OF A CUBIC POLYNOMIAL

EXAMPLE 3 *If two zeros of the polynomial* $f(x) = x^3 - 4x^2 - 3x + 12$ *are* $\sqrt{3}$ *and* $-\sqrt{3}$, *then find its third zero.* [CBSE 2010]

SOLUTION Let $\alpha = \sqrt{3}, \beta = -\sqrt{3}$ be the given zeros and γ be the third zero of $f(x)$. Then,

$$\alpha + \beta + \gamma = -\left(\frac{-4}{1}\right) \qquad \left[\text{Using } \alpha + \beta + \gamma = -\frac{\text{Coeff. of } x^2}{\text{Coeff. of } x^3}\right]$$

$$\Rightarrow \quad \sqrt{3} - \sqrt{3} + \gamma = 4$$

$$\Rightarrow \quad \gamma = 4$$

Hence, third zero is 4.

EXAMPLE 4 *Find the zeros of the polynomial* $f(x) = x^3 - 5x^2 - 16x + 80$, *if its two zeros are equal in magnitude but opposite in sign.*

SOLUTION Let α, β, γ be the zeros of polynomial $f(x)$ such that $\alpha + \beta = 0$. Then,

$$\text{Sum of the zeros} = -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

$$\Rightarrow \quad \alpha + \beta + \gamma = -\left(-\frac{5}{1}\right)$$

$$\Rightarrow \quad 0 + \gamma = 5 \qquad [\because \alpha + \beta = 0]$$

$$\Rightarrow \quad \gamma = 5$$

$$\text{Product of the zeros} = -\frac{\text{Constant term}}{\text{Coefficient of } x^3}$$

$$\Rightarrow \quad \alpha\beta\gamma = -\frac{80}{1}$$

$$\Rightarrow \quad 5\,\alpha\beta = -80 \qquad [\because \gamma = 5]$$

$$\Rightarrow \quad \alpha\beta = -16$$

$$\Rightarrow \quad -\alpha^2 = -16 \qquad [\because \alpha + \beta = 0 \therefore \beta = -\alpha]$$

$$\Rightarrow \quad \alpha = \pm 4$$

CASE I *When* $\alpha = 4$: In this case,

$$\alpha + \beta = 0 \Rightarrow 4 + \beta = 0 \Rightarrow \beta = -4$$

So, the zeros are $\alpha = 4, \beta = -4$ and $\gamma = 5$

CASE II *When* $\alpha = -4$: In this case,

$$\alpha + \beta = 0 \Rightarrow -4 + \beta = 0 \Rightarrow \beta = 4$$

So, the zeros are $\alpha = -4, \beta = 4$ and $\gamma = 5$

Hence, in either case the zeros are 4, – 4 and 5.

EXAMPLE 5 *If the zeros of the polynomial* $f(x) = x^3 - 3x^2 + x + 1$ *are* $a - b$, a, $a + b$, *find* a *and* b. **[NCERT]**

SOLUTION It is given that $a - b$, a and $a + b$ are the zeros of $f(x)$.

$$\text{Now, Sum of the zeros} = -\frac{\text{Coeff. of } x^2}{\text{Coeff. of } x^3}$$

$$\Rightarrow \quad a - b + a + a + b = -\frac{-3}{1} \Rightarrow 3a = 3 \Rightarrow a = 1$$

$$\text{and,} \quad \text{Product of zeros} = -\frac{\text{Constant term}}{\text{Coeff. of } x^3}$$

$$\Rightarrow \quad (a - b)\,a(a + b) = -\frac{1}{1}$$

$$\Rightarrow \quad a(a^2 - b^2) = -1$$

$$\Rightarrow \quad 1 - b^2 = -1 \qquad [\because a = 1]$$

$$\Rightarrow \quad b^2 = 2 \Rightarrow b = \pm\sqrt{2}$$

LEVEL-2

Type IV ON FINDING THE RELATIONSHIP BETWEEN THE COEFFICIENTS OF A POLYNOMIAL WHEN ITS ZEROS SATISFY CERTAIN RELATIONSHIP

EXAMPLE 6 *Given than the zeroes of the cubic polynomial* $f(x) = x^3 - 6x^2 + 3x + 10$ *are of the form* $a, a+b, a+2b$ *for some real numbers a and b, find the values of a and b as well as the zeros of the given polynomial.* [NCERT EXEMPLAR]

SOLUTION It is given that $a, a+b$, and $a+2b$ are zeros of $f(x) = x^3 - 6x^2 + 3x + 10$.

$$\therefore \quad \text{Sum of the zeros} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

$$\Rightarrow \quad a + (a+b) + (a+2b) = -\left(\frac{-6}{1}\right) = 6 \qquad \text{...(i)}$$

$$\Rightarrow \quad 3a + 3b = 6 \Rightarrow a + b = 2 \Rightarrow b = 2 - a \qquad \text{...(ii)}$$

and, $$\text{Product of the zeros} = -\frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

$$\Rightarrow \quad a(a+b)(a+2b) = \frac{-10}{1}$$

$$\Rightarrow \quad a(a+b)(a+2b) = -10 \qquad \text{...(iii)}$$

$$\Rightarrow \quad a \times 2 \times (a + 4 - 2a) = -10 \qquad [\text{From (ii) } b = 2 - a]$$

$$\Rightarrow \quad 2a(4 - a) = -10$$

$$\Rightarrow \quad a(4 - a) = -5$$

$$\Rightarrow \quad 4a - a^2 = -5 \Rightarrow a^2 - 4a - 5 = 0 \Rightarrow (a-5)(a+1) = 0 \Rightarrow a = 5, -1$$

CASE I *When* $a = 5$: In this case,

$$b = 2 - a \Rightarrow b = 2 - 5 = -3$$

So, the roots are $a = 5$, $a + b = 2$ and $a + 2b = 5 - 6 = -1$

CASE II *When* $a = -1$: In this case,

$$b = 2 - a \Rightarrow b = 2 + 1 = 3$$

So, the roots are $a = -1$, $a + b = -1 + 3 = 2$ and $a + 2b = -1 + 6 = 5$

Hence, either $a = -1$ and $b = 3$ or, $a = 5$ and $b = -3$

In either case, the zeros of the polynomial are 5, 2 and –1.

ALITER *It is given that the roots are* a, $a+b$, $a+2b$ *which are in A.P.*

Let $a = \alpha - \beta$, $a + b = \alpha$ and $a + 2b = \alpha + \beta$.

Now, proceed as in Example 10 on page 2.43.

EXAMPLE 7 *Find the condition which must be satisfied by the coefficients of the polynomial* $f(x) = x^3 - px^2 + qx - r$ *when the sum of its two zeros is zero.*

SOLUTION Let α, β and γ be the zeros of the polynomial $f(x)$ such that $\alpha + \beta = 0$.

Now, $$\text{Sum of the zeros} = -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

$$\alpha + \beta + \gamma = -\left(\frac{-p}{1}\right)$$

$\Rightarrow \quad \alpha + \beta + \gamma = p$

$\Rightarrow \quad 0 + \gamma = p \qquad [\because \alpha + \beta = 0]$

$\Rightarrow \quad \gamma = p$

Since γ is a zero of the polynomial $f(x)$. Therefore,

$f(\gamma) = 0$

$\Rightarrow \quad \gamma^3 - p\gamma^2 + q\gamma - r = 0$

$\Rightarrow \quad p^3 - p^3 + qp - r = 0 \qquad [\because \gamma = p]$

$\Rightarrow \quad pq = r$, which is the required condition.

EXAMPLE 8 *Find the condition that the zeros of the polynomial* $f(x) = x^3 - px^2 + qx - r$ *may be in arithmetic progression.*

SOLUTION Let $a - d$, a and $a + d$ be the zeros of the polynomial $f(x)$. Then,

$$\text{Sum of the zeros} = -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

$$\Rightarrow \quad (a - d) + a + (a + d) = -\frac{(-p)}{1}$$

$$\Rightarrow \quad 3a = p \Rightarrow a = \frac{p}{3}$$

Since a is a zero of the polynomial $f(x)$. Therefore,

$f(a) = 0$

$\Rightarrow \quad a^3 - pa^2 + qa - r = 0$

$$\Rightarrow \quad \left(\frac{p}{3}\right)^3 - p\left(\frac{p}{3}\right)^2 + q\left(\frac{p}{3}\right) - r = 0 \qquad \left[\because a = \frac{p}{3}\right]$$

$\Rightarrow \quad p^3 - 3p^3 + 9pq - 27r = 0$

$\Rightarrow \quad 2p^3 - 9pq + 27r = 0$, which is the required condition.

EXAMPLE 9 *Find the zeros of the polynomial* $f(x) = x^3 - 5x^2 - 2x + 24$, *if it is given that the product of its two zeros is 12.*

SOLUTION Let α, β, γ be the zeros of polynomial $f(x)$ such that $\alpha\beta = 12$. Then,

$$\alpha + \beta + \gamma = -\left(-\frac{5}{1}\right) = 5 \qquad \text{...(i)}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{-2}{1} = -2 \qquad \text{...(ii)}$$

and, $$\alpha\beta\gamma = -\frac{24}{1} = -24 \qquad \text{...(iii)}$$

Putting $\alpha\beta = 12$ in $\alpha\beta\gamma = -24$, we get

$$12\gamma = -24 \Rightarrow \gamma = -\frac{24}{12} = -2$$

Putting $\gamma = -2$ in (i), we get

$$\alpha + \beta - 2 = 5$$

$$\Rightarrow \quad \alpha + \beta = 7$$

Now, $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$$\Rightarrow \quad (\alpha - \beta)^2 = 7^2 - 4 \times 12 \qquad [\because \alpha + \beta = 7 \text{ and } \alpha\beta = 12]$$

$$\Rightarrow \quad (\alpha - \beta)^2 = 1$$

$$\Rightarrow \quad \alpha - \beta = \pm 1$$

Thus, we have

$$\alpha + \beta = 7 \text{ and } \alpha - \beta = 1 \text{ or, } \alpha + \beta = 7 \text{ and } \alpha - \beta = -1$$

CASE I *When* $\alpha + \beta = 7$ *and* $\alpha - \beta = 1$

Solving $\alpha + \beta = 7$ and $\alpha - \beta = 1$, we get $\alpha = 4$ and $\beta = 3$

CASE II *When* $\alpha + \beta = 7$ *and* $\alpha - \beta = -1$

Solving $\alpha + \beta = 7$ and $\alpha - \beta = -1$, we get $\alpha = 3$ and $\beta = 4$.

Hence, the zeros of the given polynomial are 3, 4 and – 2.

EXAMPLE 10 *Find the zeros of the polynomial* $f(x) = x^3 - 12x^2 + 39x - 28$, *if it is given that the zeros are in A.P.*

SOLUTION Let $\alpha = a - d$, $\beta = a$ and $\gamma = a + d$ be the zeros of the polynomial

$$f(x) = x^3 - 12x^2 + 39x - 28.$$

$$\therefore \quad \alpha + \beta + \gamma = -\left(\frac{-12}{1}\right) = 12 \text{ and, } \alpha\beta\gamma = -\left(\frac{-28}{1}\right) = 28$$

$$\Rightarrow \quad (a - d) + a + (a + d) = 12 \text{ and } (a - d)\, a\, (a + d) = 28$$

$$\Rightarrow \quad 3a = 12 \text{ and } a(a^2 - d^2) = 28$$

$$\Rightarrow \quad a = 4 \text{ and } 4(16 - d^2) = 28$$

$$\Rightarrow \quad a = 4 \text{ and } 16 - d^2 = 7$$

$$\Rightarrow \quad a = 4 \text{ and } d^2 = 9$$

$$\Rightarrow \quad a = 4 \text{ and } d = \pm 3$$

CASE I *When* $a = 4$ *and* $d = 3$: In this case,

$$\alpha = a - d = 4 - 3 = 1, \beta = a = 4 \text{ and } \gamma = a + d = 7$$

CASE II *When* $a = 4$ *and* $d = -3$: In this case,

$$\alpha = a - d = 4 + 3 = 7, \beta = a = 4 \text{ and } \gamma = a + d = 4 - 3 = 1$$

Hence, in either case the zeros of the given polynomial are 1, 4 and 7.

EXERCISE 2.2

LEVEL-1

1. Verify that the numbers given along side of the cubic polynomials below are their zeros. Also, verify the relationship between the zeros and coefficients in each case: [NCERT]

   (i) $f(x) = 2x^3 + x^2 - 5x + 2\,;\ \frac{1}{2}, 1, -2$

   (ii) $g(x) = x^3 - 4x^2 + 5x - 2\,;\ 2, 1, 1$

2. Find a cubic polynomial with the sum, sum of the product of its zeros taken two at a time, and product of its zeros as 3, – 1 and – 3 respectively.

LEVEL-2

3. If the zeros of the polynomial $f(x) = 2x^3 - 15x^2 + 37x - 30$ are in A.P., find them.
4. Find the condition that the zeros of the polynomial $f(x) = x^3 + 3px^2 + 3qx + r$ may be in A.P.
5. If the zeros of the polynomial $f(x) = ax^3 + 3bx^2 + 3cx + d$ are in A.P., prove that $2b^3 - 3abc + a^2d = 0$.
6. If the zeros of the polynomial $f(x) = x^3 - 12x^2 + 39x + k$ are in A.P., find the value of k.

ANSWERS

2. $f(x) = k\,(x^3 - 3x^2 - x + 3)$, k is any non-zero real number. 3. $2, 3, \frac{5}{2}$

4. $2p^2 - 3pq + r = 0$ 6. $k = -28$

2.6 DIVISION ALGORITHM FOR POLYNOMIALS

In earlier classes, we have studied division of integers. We have seen that on division of an integer (called dividend) by a non-zero integer (called divisor), we obtain the quotient and the remainder which is either zero or less than the divisor. Also, dividend, divisor, quotient and the remainder always satisfy the following relation.

$$\text{Dividend} = \text{Quotient} \times \text{Divisor} + \text{Remainder}$$

This is known as Euclid's division lemma which we have studied in chapter 1.

In earlier classes, we have studied about division of polynomials. In this section, we shall show that the division of polynomials also follows the similar rule which is known as the division algorithm for polynomials. We will also discuss problems on finding zeros of cubic and biquadratic polynomials when some of its zeros are given.

Let us first refresh the method of dividing one polynomial by another polynomial through following illustrations.

ILLUSTRATION 1 *Divide the polynomial* $f(x) = 14x^3 - 5x^2 + 9x - 1$ *by the polynomial* $g(x) = 2x - 1$. *Also, find the quotient and remainder.*

SOLUTION Using long division method, we obtain

$$\begin{array}{rl}
2x-1\,\big) & \overline{14x^3-5x^2+9x-1}\ \big(7x^2+x+5\\
& \underline{\underset{-}{14x^3}\ \underset{+}{-7x^2}}\\
& \qquad 2x^2+9x-1\\
& \qquad \underline{\underset{-}{2x^2}\ \underset{+}{-x}}\\
& \qquad\qquad 10x-1\\
& \qquad\qquad \underline{\underset{-}{10x}\ \underset{+}{-5}}\\
& \qquad\qquad\qquad 4
\end{array}$$

Clearly, quotient $q(x) = 7x^2 + x + 5$ and remainder $r(x) = 4$

Now,

$$\begin{aligned}
q(x)\,g(x) + r(x) &= (7x^2 + x + 5)(2x - 1) + 4\\
&= 14x^3 + 2x^2 + 10x - 7x^2 - x - 5 + 4\\
&= 14x^3 - 5x^2 + 9x - 1\\
&= f(x)
\end{aligned}$$

i.e. $f(x) = g(x)\,q(x) + r(x)$ or, Dividend = Quotient × Divisor+ Remainder

ILLUSTRATION 2 *Divide the polynomial* $f(x) = 6x^3 + 11x^2 - 39x - 65$ *by the polynomial* $g(x) = x^2 - 1 + x$. *Also, find the quotient and remainder.*

SOLUTION Using long division method, we obtain

$$\begin{array}{rl}
x^2+x-1\,\big) & \overline{6x^3+11x^2-39x-65}\ \big(6x+5\\
& 6x^3+6x^2-6x\\
& \underline{\ -\quad\ -\qquad +\quad}\\
& \qquad 5x^2-33x-65\\
& \qquad 5x^2+5x-5\\
& \qquad \underline{\ -\quad\ -\quad\ +\ }\\
& \qquad\qquad -38x-60
\end{array}$$

Clearly, quotient $q(x) = 6x + 5$ and remainder $r(x) = -38x - 60$.

Also, $$\begin{aligned} g(x)\,q(x) + r(x) &= (x^2 + x - 1)(6x + 5) + (-38x - 60)\\ &= 6x^3 + 6x^2 - 6x + 5x^2 + 5x - 5 - 38x - 60\end{aligned}$$

i.e. $f(x) = g(x)\,q(x) + r(x) = 6x^3 + 11x^2 - 39x - 65 = f(x)$

or, Dividend = Quotient × Divisor + Remainder

ILLUSTRATION 3 *Divide the polynomial* $u(x) = 9x^4 - 4x^2 + 4$ *by the polynomial* $v(x) = 3x^2 + x - 1$. *Also, find the quotient and remainder.*

SOLUTION Using long division method, we obtain

$$
\begin{array}{rl}
3x^2+x-1\,\big) & 9x^4+0x^3-4x^2+0x+4\,\big(\,3x^2-x \\
& 9x^4+3x^3-3x^2 \\
& -\qquad -\qquad + \\
\hline
& -3x^3-x^2+0x+4 \\
& -3x^3-x^2+x \\
& +\qquad +\qquad - \\
\hline
& -x+4
\end{array}
$$

Cleraly, quotient $q(x)=3x^2-x$ and remainder $r(x)=-x+4$.

Also,

$$v(x)\,q(x)+r(x)=(3x^2+x-1)(3x^2-x)+(-x+4)$$
$$=9x^4+3x^3-3x^2-3x^3-x^2+x-x+4$$

i.e. $u(x)=v(x)\,q(x)+r(x)=9x^4+0x^3-4x^2+0x+4=u(x)$

or, Dividend = Quotient × Divisor + Remainder

ILLUSTRATION 4 *Divide the polynomial* $f(x)=30x^4+11x^3-82x^2-12x+48$ *by* $3x^2+2x-4$. *Also, find the quotient and remainder.*

SOLUTION Using long division method, we obtain

$$
\begin{array}{rl}
3x^2+2x-4\,\big) & 30x^4+11x^3-82x^2-12x+48\,\big(\,10x^2-3x-12 \\
& 30x^4+20x^3-40x^2 \\
& -\qquad -\qquad + \\
\hline
& -9x^3-42x^2-12x+48 \\
& -9x^3-6x^2+12x \\
& +\qquad +\qquad - \\
\hline
& -36x^2-24x+48 \\
& -36x^2-24x+48 \\
& +\qquad +\qquad - \\
\hline
& 0
\end{array}
$$

Clearly, quotient $q(x)=10x^2-3x-12$ and remainder $r(x)=0$.

Also,

$$g(x)\,q(x)+r(x)=(3x^2+2x-4)(10x^2-3x-12)+0$$
$$=30x^4-9x^3-36x^2+20x^3-6x^2-24x-40x^2+12x+48+0$$

i.e. $f(x)=g(x)\,q(x)+r(x)=30x^4+11x^3-82x^2-12x+48=f(x)$

or, Dividend = Quotient × Divisor + Remainder

In the above illustrations, we observe that the division process is stoped when either the remainder is zero or its degree is less than the degree of divisor. Also, divident, divisor, quotient and remainder satisfy the relation

$$\text{Dividend} = \text{Quotient} \times \text{Divisor} + \text{Remainder}$$

This is an algorithm similar to Euclid's division algorithm for integers and is known as the division algorithm for polynomials as defined below.

DIVISION ALGORITHM *If $f(x)$ and $g(x)$ are any two polynomials with $g(x) \neq 0$, then we can always find polynomials $q(x)$ and $r(x)$ such that*

$$f(x) = q(x)\, g(x) + r(x), \text{ where } r(x) = 0 \text{ or degree } r(x) < \text{degree } g(x).$$

REMARK *If $r(x) = 0$, then polynomial $g(x)$ is a factor of polynomial $f(x)$.*

Following examples will illustrate various applications of division algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I ON VERIFYING THE DIVISION ALGORITHM FOR POLYNOMIALS

EXAMPLE 1 *Divide the polynomial $f(x) = 3x^2 - x^3 - 3x + 5$ by the polynomial $g(x) = x - 1 - x^2$ and verify the division algorithm.*

SOLUTION Writing the given polynomials in standard form, we get

$$f(x) = -x^3 + 3x^2 - 3x + 5 \text{ and } g(x) = -x^2 + x - 1$$

Using long division method, we obtain

$$\begin{array}{rl}
-x^2+x-1\,\big) & -x^3+3x^2-3x+5\;\big(\,x-2 \\
& -x^3+\;\;x^2-\;x \\
& \;\;+\qquad-\qquad+ \\
\hline
& \qquad 2x^2-2x+5 \\
& \qquad 2x^2-2x+2 \\
& \qquad\;\;-\qquad+\qquad- \\
\hline
& \qquad\qquad\qquad 3
\end{array}$$

$\therefore$ Quotient $q(x) = x - 2$ and, Remainder $r(x) = 3$

Now,

$$\begin{aligned}
\text{Quotient} \times \text{Divisor} + \text{Remainder} &= (x-2)(-x^2+x-1)+3 \\
&= -x^3 + x^2 - x + 2x^2 - 2x + 2 + 3 \\
&= -x^3 + 3x^2 - 3x + 5 = \text{Dividend}
\end{aligned}$$

Hence, the division algorithm is verified.

Type II ON FINDING THE REMAINING ZEROS OF A POLYNOMIAL WHEN SOME OF ITS ZEROS ARE GIVEN

EXAMPLE 2 *Find all the zeros of the polynomial $f(x) = 2x^4 - 3x^3 - 3x^2 + 6x - 2$, if two of its zeros are $\sqrt{2}$ and $-\sqrt{2}$.* [NCERT]

SOLUTION We know that, if $x = \alpha$ is a zero of a polynomial, then $x - \alpha$ is a factor of $f(x)$. Since $\sqrt{2}$ and $-\sqrt{2}$ are zeros of $f(x)$. Therefore, $(x - \sqrt{2})(x + \sqrt{2}) = x^2 - 2$ is a factor of $f(x)$. Let us now divide $f(x) = 2x^4 - 3x^3 - 3x^2 + 6x - 2$ by $g(x) = x^2 - 2$ to find the other zeros of $f(x)$.

Using long division method, we obtain

$$
\begin{array}{rrrrrrl}
x^2-2\,\big) & 2x^4 & -3x^3 & -3x^2 & +6x & -2 & \big(\,2x^2-3x+1 \\
 & 2x^4 & & -4x^2 & & & \\
 & - & & + & & & \\ \cline{2-6}
 & & -3x^3 & +x^2 & +6x & -2 & \\
 & & -3x^3 & & +6x & & \\
 & & + & & - & & \\ \cline{2-6}
 & & & x^2 & & -2 & \\
 & & & x^2 & & -2 & \\
 & & & - & & + & \\ \cline{4-6}
 & & & & & 0 &
\end{array}
$$

Thus, Quotient = $2x^2-3x+1$ and Remainder = 0.

By division algorithm, we have

$$f(x) = \text{Quotient} \times \text{Divisor} + \text{Remainder}$$

$$\Rightarrow \quad f(x) = (x^2-2)(2x^2-3x+1)+0$$

$$\Rightarrow \quad f(x) = (x-\sqrt{2})(x+\sqrt{2})(2x^2-2x-x+1)$$

$$\Rightarrow \quad f(x) = (x-\sqrt{2})(x+\sqrt{2})\{2x(x-1)-(x-1)\}$$

$$\Rightarrow \quad f(x) = (x-\sqrt{2})(x+\sqrt{2})(x-1)(2x-1)$$

On equating factors $x-\sqrt{2}, x+\sqrt{2}, x-1$ and $2x-1$ to zero, we get $x = \sqrt{2}, -\sqrt{2}, 1, \frac{1}{2}$.

Hence, the zeros of the given polynomial are $\sqrt{2}, -\sqrt{2}, 1$ and $\frac{1}{2}$.

EXAMPLE 3 *Obtain all the zeros of the polynomial* $f(x) = 3x^4+6x^3-2x^2-10x-5$, *if two of its zeros are* $\sqrt{\frac{5}{3}}$ *and* $-\sqrt{\frac{5}{3}}$. **[NCERT]**

SOLUTION It is given that $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$ are two zeros of $f(x)$. Therefore, $\left(x-\sqrt{\frac{5}{3}}\right)\left(x+\sqrt{\frac{5}{3}}\right)$ is a factor of $f(x)$. But, $\left(x-\sqrt{\frac{5}{3}}\right)\left(x+\sqrt{\frac{5}{3}}\right) = \left(x^2-\frac{5}{3}\right) = \frac{1}{3}(3x^2-5)$.

Therefore, $3x^2-5$ is a factor of $f(x)$. Let us now divide $f(x)$ by $3x^2-5$.

Using long division method, we obtain

$$
\begin{array}{rl}
3x^2-5\,\big) & 3x^4+6x^3-2x^2-10x-5\,\big(\,x^2+2x+1 \\
 & 3x^4+0x^3-5x^2 \\
 & -\quad\;\; -\quad\;\; + \\
\hline
 & \quad 6x^3+3x^2-10x-5 \\
 & \quad 6x^3+0x^2-10x \\
 & +\quad\;\; -\quad\;\; + \\
\hline
 & \qquad 3x^2-5 \\
 & \qquad 3x^2-5 \\
 & \qquad -\quad\; + \\
\hline
 & \qquad\quad 0
\end{array}
$$

Clearly, Quotient = x^2+2x+1 and Remainder = 0.
By division algorithm, we obtain

$$f(x)=(3x^2-5)(x^2+2x+1)+0$$

$$\Rightarrow \qquad f(x)=(\sqrt{3}x+\sqrt{5})(\sqrt{3}x-\sqrt{5})(x+1)^2$$

Thus, the factors of $f(x)$ are $\sqrt{3}x+\sqrt{5}, \sqrt{3}x-\sqrt{5}, x+1$ and $x+1$. Equating each factor to zero, we obtain $x=-\sqrt{\frac{5}{2}}, \sqrt{\frac{5}{2}}, -1, -1$. Hence, the zeros of $f(x)$ are $-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}, -1$ and -1.

EXAMPLE 4 *If two zeros of the polynomial* $f(x)=x^4-6x^3-26x^2+138x-35$ *are* $2\pm\sqrt{3}$, *find other zeros.* **[NCERT]**

SOLUTION It is given that $2+\sqrt{3}$ and $2-\sqrt{3}$ are two zeros of $f(x)$. Therefore, $\left(x-(2+\sqrt{3})\right)$ and $\left(x-(2-\sqrt{3})\right)$ are factors of $f(x)$.

But, $\{x-(2\sqrt{3})\}\{x-(2-\sqrt{3})\}=(x-2-\sqrt{3})(x-2+\sqrt{3})=(x-2)^2-(\sqrt{3})^2=x^2-4x+1$. Therefore, x^2-4x+1 is a factor of $f(x)$. Let us now divide $f(x)$ by x^2-4x+1.

Using long division method, we obtain

$$
\begin{array}{rl}
x^2-4x+1\,\big) & x^4-6x^3-26x^2+138x-35\,\big(\,x^2-2x-35 \\
 & x^4-4x^3+x^2 \\
 & -\quad\; +\quad\;\; - \\
\hline
 & -2x^3-27x^2+138x \\
 & -2x^3+8x^2-2x \\
 & +\qquad -\qquad + \\
\hline
 & \qquad -35x^2+140x-35 \\
 & \qquad -35x^2+140x-35 \\
 & \qquad +\qquad -\qquad + \\
\hline
 & \qquad\qquad 0
\end{array}
$$

Thus, Quotient $q(x) = x^2 - 2x - 35$ and Remainder $= 0$.

By division algorithm, we obtain

$$f(x) = (x^2 - 4x + 1)(x^2 - 2x - 35)$$

Hence, other two zeros of $f(x)$ are the zeros of the polynomial $x^2 - 2x - 35$.

Now,

$$x^2 - 2x - 35 = x^2 - 7x + 5x - 35 = (x - 7)(x + 5)$$

On equating each factor to zero, we obtain $x = 7, -5$.

Hence, other two zeros of $f(x)$ are 7 and -5.

LEVEL-2

Type III ON FINDING THE QUOTIENT AND REMAINDER USING DIVISION ALGORITHM

EXAMPLE 5 *Apply the division algorithm to find the quotient and remainder on dividing $f(x)$ by $g(x)$ as given below:*

(i) $f(x) = x^3 - 6x^2 + 11x - 6,\ g(x) = x + 2$

(ii) $f(x) = x^3 - 3x^2 + 5x - 3,\ g(x) = x^2 - 2$ [NCERT]

(iii) $f(x) = x^4 - 3x^2 + 4x + 5,\ g(x) = x^2 + 1 - x$ [NCERT]

(iv) $f(x) = x^4 - 5x + 6,\ g(x) = 2 - x^2$ [NCERT]

SOLUTION (i) We have,

$$f(x) = x^3 - 6x^2 + 11x - 6 \text{ and } g(x) = x + 2$$

We find that degree $f(x) = 3$ and degree $g(x) = 1$. Therefore, quotient $q(x)$ is of degree $3 - 1 = 2$ and the remainder $r(x)$ is of degree zero. Let $q(x) = ax^2 + bx + c$ and $r(x) = k$.

By using division algorithm, we obtain

$$f(x) = q(x) \times g(x) + r(x)$$

$$\Rightarrow \quad x^3 - 6x^2 + 11x - 6 = (ax^2 + bx + c)(x + 2) + k$$

$$\Rightarrow \quad x^3 - 6x^2 + 11x - 6 = ax^3 + (2a + b)x^2 + (2b + c)x + 2c + k$$

Equating the coefficients of like powers of x on both sides, we get

$1 = a$ [On equating the coefficients of x^3]

$-6 = 2a + b$ [On equating the coefficients of x^2]

$11 = 2b + c$ [On equating the coefficients of x]

and, $-6 = 2c + k$ [On equating the constant terms]

Solving these equations, we get

$$a = 1, b = -8, c = 27 \text{ and } k = -60$$

$\therefore$ Quotient $q(x) = ax^2 + bx + c = x^2 - 8x + 27$ and, Remainder $r(x) = k = -60$.

(ii) We have,

$$f(x) = x^3 - 3x^2 + 5x - 3 \text{ and } g(x) = x^2 - 2.$$

We find that degree $(f(x)) = 3$ and degree $(g(x)) = 2$. Therefore, quotient $q(x)$ is of degree 1 and the remainder $r(x)$ is of degree less than 2. Let $q(x) = ax + b$ and $r(x) = cx + d$.

Using division algorithm, we have

$$f(x) = g(x) \times q(x) + r(x)$$

$$\Rightarrow \quad x^3 - 3x^2 + 5x - 3 = (x^2 - 2)(ax + b) + (cx + d)$$

$$\Rightarrow \quad x^3 - 3x^2 + 5x - 3 = ax^3 + bx^2 + (c - 2a)x - 2b + d$$

On equating the coefficients of various powers of x on both sides, we get

$1 = a$ [On equating the coefficients of x^3]

$-3 = b$ [On equating the coefficients of x^2]

$5 = c - 2a$ [On equating the coefficients of x]

$-3 = -2b + d$ [On equating the constant terms]

Solving these equations, we get : $a = 1, b = -3, c = 7$ and $d = -9$

$\therefore$ Quotient $q(x) = ax + b = x - 3$ and Remainder $r(x) = 7x - 9$.

(iii) We have,

$$f(x) = x^4 - 3x^2 + 4x + 5 \text{ and } g(x) = x^2 - x + 1$$

We find that degree $(f(x)) = 4$ and degree $(g(x)) = 2$. Therefore, quotient $q(x)$ is of degree $2 (= 4 - 2)$ and remainder $r(x)$ is of degree less than 2 $(= \text{degree } (g(x)))$. So, let $q(x) = ax^2 + bx + c$ and $r(x) = px + q$.

Using division algorithm, we have

$$f(x) = g(x) \times q(x) + r(x)$$

$$\Rightarrow \quad x^4 + 0x^3 - 3x^2 + 4x + 5 = (x^2 - x + 1)(ax^2 + bx + c) + px + q$$

$$\Rightarrow \quad x^4 + 0x^3 - 3x^2 + 4x + 5 = ax^4 + (b - a)x^3 + (c - b + a)x^2 + (b - c + p)x + c + q$$

On equating the coefficients of various powers of x on both sides, we get

$a = 1$ [On equating the coefficients of x^2]

$b - a = 0$ [On equating the coefficients of x^3]

$c - b + a = -3$ [On equating the coefficients of x^2]

$b - c + p = 4$ [On equating the coefficient of x]

and, $c + q = 5$ [On equating the constant terms]

Solving these equations, we get

$$a = 1, b = 1, c = -3, p = 0 \text{ and } q = 8$$

$\therefore$ Quotient $q(x) = x^2 + x - 3$ and Remainder $r(x) = 8$

(iv) We have,

$$f(x) = x^4 + 0x^3 + 0x^2 - 5x + 6 \text{ and } g(x) = -x^2 + 2$$

We find that degree $(f(x)) = 4$ and degree $g(x) = 2$. Therefore, quotient $q(x)$ and remainder $r(x)$ are of degree 2 and less than 2 respectively.

Let $q(x) = ax^2 + bx + c$ and $r(x) = px + q$

By division algorithm, we have

$$f(x) = g(x) \times q(x) + r(x)$$

$$\Rightarrow \quad x^4 + 0x^3 + 0x^2 - 5x + 6 = (-x^2 + 2)(ax^2 + bx + c) + px + q$$

$$\Rightarrow \quad x^4 + 0x^3 + 0x^2 - 5x + 6 = -ax^4 - bx^3 + (2a - c)x^2 + (2b + p)x + 2c + q$$

Equating the coefficients of various powers of x, we get

$-a = 1$ [On equating the coefficients of x^4]

$-b = 0$ [On equating the coefficients of x^3]

$2a - c = 0$ [On equating the coefficients of x^2]

$2b + p = -5$ [On equating the coefficient of x]

and, $2c + q = 6$ [On equating the constant terms]

Solving these equations, we get

$$a = -1, b = 0, c = -2, p = -5 \text{ and } q = 10$$

$\therefore$ Quotient $q(x) = -x^2 - 2$ and Remainder $r(x) = -5x + 10$.

Type IV ON CHECKING WHETHER A GIVEN POLYNOMIAL IS A FACTOR OF THE OTHER POLYNOMIAL BY APPLYING THE DIVISION ALGORITHM

EXAMPLE 6 *By applying division algorithm prove that the polynomial* $g(x) = x^2 + 3x + 1$ *is a factor of the polynomial* $f(x) = 3x^4 + 5x^3 - 7x^2 + 2x + 2$.

SOLUTION We find that degree $(f(x)) = 4$ and degree $(g(x)) = 2$. Therefore, quotient $q(x)$ is of degree 2 $(= 4 - 2)$ and the remainder $r(x)$ is of degree 1 or less. Let $q(x) = ax^2 + bx + c$ and $r(x) = px + q$.

Using division algorithm, we have

$$f(x) = g(x) \times q(x) + r(x)$$

$$\Rightarrow \quad 3x^4 + 5x^3 - 7x^2 + 2x + 2 = (ax^2 + bx + c)(x^2 + 3x + 1) + (px + q)$$

$$\Rightarrow \quad 3x^4 + 5x^3 - 7x^2 + 2x + 2 = ax^4 + (3a + b)x^3 + (a + 3b + c)x^2 + (b + 3c + p)x + c + q$$

Equating coefficients of various powers of x, we get

$a = 3$ [On equating the coefficients of x^4]

$3a + b = 5$ [On equating the coefficients of x^3]

$a + c + 3b = -7$ [On equating the coefficients of x^2]

$b + 3c + p = 2$ [On equating the coefficient of x]

and, $c + q = 2$ [On equating the constant terms]

Solving these equations, we get

$$a = 3, b = -4, c = 2, p = 0 \text{ and } q = 0$$

$\therefore$ Quotient $q(x) = 3x^2 - 4x + 2$ and, Remainder $r(x) = 0x + 0 = 0$

Clearly, $r(x) = 0$. Hence, $g(x)$ is a factor of $f(x)$.

Type V MISCELLANEOUS APPLICATIONS OF DIVISION ALGORITHM

EXAMPLE 7 *On dividing the polynomial* $f(x) = x^3 - 3x^2 + x + 2$ *by a polynomial* $g(x)$, *the quotient* $q(x)$ *and remainder* $r(x)$ *where* $q(x) = x - 2$ *and* $r(x) = -2x + 4$ *respectively. Find the polynomial* $g(x)$. [NCERT]

SOLUTION By division algorithm, we obtain

$$f(x) = g(x) \times q(x) + r(x)$$

$$\Rightarrow \quad g(x) \times q(x) = f(x) - r(x)$$

$$\Rightarrow \quad g(x)(x - 2) = x^3 - 3x^2 + x + 2 - (-2x + 4)$$

$\Rightarrow \qquad g(x)(x-2) = x^3 - 3x^2 + 3x - 2$

Thus, $g(x)$ is a factor of $x^3 - 3x^2 + 3x - 2$ other than the factor $(x-2)$. So, to get $g(x)$, we divide $x^3 - 3x^2 + 3x - 2$ by $(x-2)$ as follows:

$$\begin{array}{r|l}
x-2 & \overline{x^3 - 3x^2 + 3x - 2} \; \big(x^2 - x + 1 \\
 & x^3 - 2x^2 \\
 & \underline{- \quad +} \\
 & \quad -x^2 + 3x - 2 \\
 & \quad -x^2 + 2x \\
 & \quad \underline{+ \qquad -} \\
 & \qquad\quad x - 2 \\
 & \qquad\quad x - 2 \\
 & \qquad\quad \underline{- \quad +} \\
 & \qquad\qquad 0
\end{array}$$

Hence, $g(x) = x^2 - x + 1$.

EXAMPLE 8 *What must be subtracted from* $8x^4 + 14x^3 - 2x^2 + 7x - 8$ *so that the resulting polynomial is exactly divisible by* $4x^2 + 3x - 2$.

SOLUTION We know that

$$\text{Dividend} = \text{Quotient} \times \text{Divisor} + \text{Remainder}$$

$$\Rightarrow \quad \text{Dividend} - \text{Remainder} = \text{Quotient} \times \text{Divisor}$$

Clearly, RHS of the above result is divisible by the divisor. Therefore, LHS is also divisible by the divisor. Thus, if we subtract remainder from the dividend, then it will be exactly divisible by the divisor.

Let us now divide $8x^4 + 14x^3 - 2x^2 + 7x - 8$ by $4x^2 + 3x - 2$ long division method.

$$\begin{array}{r|l}
4x^2 + 3x - 2 & \overline{8x^4 + 14x^3 - 2x^2 + 7x - 8} \; \big(2x^2 + 2x - 1 \\
 & 8x^4 + 6x^3 - 4x^2 \\
 & \underline{- \quad - \quad +} \\
 & \qquad 8x^3 + 2x^2 + 7x - 8 \\
 & \qquad 8x^3 + 6x^2 - 4x \\
 & \qquad \underline{- \quad - \quad +} \\
 & \qquad\qquad -4x^2 + 11x - 8 \\
 & \qquad\qquad -4x^2 - 3x + 2 \\
 & \qquad\qquad \underline{+ \quad + \quad -} \\
 & \qquad\qquad\qquad 14x - 10
\end{array}$$

$\therefore \qquad$ Quotient $= 2x^2 + 2x - 1$ and Remainder $= 14x - 10$

Thus, if we subtract the remainder $14x - 10$ from $8x^4 + 14x^3 - 2x^2 + 7x - 8$, it will be exactly divisible by $4x^2 + 3x - 2$.

EXAMPLE 9 *Find the values of a and b so that* $x^4 + x^3 + 8x^2 + ax + b$ *is divisible by* $x^2 + 1$.

SOLUTION If $x^4 + x^3 + 8x^2 + ax + b$ is exactly divisible by $x^2 + 1$, then the remainder should be zero. Let us now divide $x^4 + x^3 + 8x^2 + ax + b$ by $x^2 + 1$ by long division method.

$$
\begin{array}{rl}
x^2+1 \big) & x^4+x^3+8x^2+ax+b \;\big(\, x^2+x+7 \\
 & x^4 \qquad\quad + x^2 \\
 & - \qquad\quad\;\; - \\
\hline
 & \quad x^3+7x^2+ax+b \\
 & \quad x^3 \qquad\quad + x \\
 & \quad - \qquad\quad\;\; - \\
\hline
 & \qquad 7x^2+x(a-1)+b \\
 & \qquad 7x^2 \qquad\qquad +7 \\
 & \qquad - \qquad\qquad\;\; - \\
\hline
 & \qquad\qquad x(a-1)+b-7
\end{array}
$$

$\therefore$ Quotient $= x^2+x+7$ and, Remainder $= x(a-1)+(b-7)$

Now,

Remainder $= 0$

$\Rightarrow$ $x(a-1)+(b-7)=0$

$\Rightarrow$ $x(a-1)+(b-7)=0x+0$

$\Rightarrow$ $a-1=0$ and $b-7=0$ [On equating the coefficients of like powers of x]

$\Rightarrow$ $a=1$ and $b=7$

EXAMPLE 10 *What must be added to* $f(x)=4x^4+2x^3-2x^2+x-1$ *so that the resulting polynomial is divisible by* $g(x)=x^2+2x-3$?

SOLUTION By division algorithm, we have

$f(x)=g(x)\times q(x)+r(x)$

$\Rightarrow$ $f(x)-r(x)=g(x)\times q(x)$

$\Rightarrow$ $f(x)+\{-r(x)\}=g(x)\times q(x)$

Clearly, RHS is divisible by $g(x)$. Therefore, LHS is also divisible by $g(x)$. Thus, if we add $-r(x)$ to $f(x)$, then the resulting polynomial is divisible by $g(x)$. Let us now find the remainder when $f(x)$ is divided by $g(x)$. Using long division method, we obtain

$$
\begin{array}{rl}
x^2+2x-3 \big) & 4x^4+2x^3-2x^2+x-1 \;\big(\, 4x^2-6x+22 \\
 & 4x^4+8x^3-12x^2 \\
 & - \quad - \quad\;\; + \\
\hline
 & \quad -6x^3+10x^2+x-1 \\
 & \quad -6x^3-12x^2+18x \\
 & \quad + \qquad + \qquad - \\
\hline
 & \qquad\quad 22x^2-17x-1 \\
 & \qquad\quad 22x^2+44x-66 \\
 & \qquad\quad - \qquad - \qquad + \\
\hline
 & \qquad\qquad\quad -61x+65
\end{array}
$$

$\therefore$ $r(x)=-61x+65$

Hence, we should add $-r(x)=61x-65$ to $f(x)$ so that the resulting polynomial is divisible by $g(x)$.

EXAMPLE 11 *If the polynomial* $f(x) = x^4 - 6x^3 + 16x^2 - 25x + 10$ *is divided by another polynomial* $x^2 - 2x + k$, *the remainder comes out to be* $x + a$, *find k and a.* [NCERT]

SOLUTION By division algorithm, we have

$$\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}$$

$$\Rightarrow \quad \text{Dividend} - \text{Remainder} = \text{Divisor} \times \text{Quotient}$$

$\Rightarrow$ (Dividend – Remainder) is always divisible by the divisor.

It is given that $f(x) = x^4 - 6x^3 + 16x^2 - 25x + 10$ when divided by $x^2 - 2x + k$ leaves $x + a$ as remainder. Therefore,

$$f(x) - (x + a) = x^4 - 6x^3 + 16x^2 - 26x + 10 - a \text{ is exactly divisible by } x^2 - 2x + k.$$

Let us now divide $x^4 - 6x^3 + 16x^2 - 26x + 10 - a$ by $x^2 - 2x + k$. Using long division method:

$$\begin{array}{r|l}
x^2 - 2x + k & x^4 - 6x^3 + 16x^2 - 26x + 10 - a \;\big(\, x^2 - 4x + (8 - k) \\
 & x^4 - 2x^3 + kx^2 \\
 & - \quad + \quad - \\ \hline
 & -4x^3 + (16 - k)\,x^2 - 26x + 10 - a \\
 & -4x^3 + 8x^2 \qquad - 4kx \\
 & + \quad - \qquad + \\ \hline
 & (8 - k)x^2 - (26 - 4k)\,x + 10 - a \\
 & (8 - k)x^2 - (16 - 2k)x + (8k - k^2) \\
 & - \qquad + \qquad - \\ \hline
 & (-10 + 2k)x + (10 - a - 8k + k^2)
\end{array}$$

For $f(x) - (x + a) = x^4 - 6x^3 + 16x^2 - 26x + 10 - a$ to be exactly divisible by $x^2 - 2x + k$, we must have

Remainder = 0

$\Rightarrow \quad (-10 + 2k)x + (10 - a - 8k + k^2) = 0$ for all x

$\Rightarrow \quad -10 + 2k = 0, 10 - a - 8k + k^2 = 0$ [On equating the coefficients of like powers of x]

$\Rightarrow \quad k = 5, 10 - a - 40 + 25 = 0$

$\Rightarrow \quad k = 5$ and $a = -5$.

EXAMPLE 12 *If the polynomial* $6x^4 + 8x^3 + 17x^2 + 21x + 7$ *is divided by another polynomial* $3x^2 + 4x + 1$, *the remainder comes out to be* $ax + b$, *find a and b.* [CBSE 2009]

SOLUTION Let us divide the polynomial $f(x) = 6x^4 + 8x^3 + 17x^2 + 21x + 7$ by the polynomial $g(x) = 3x^2 + 4x + 1$ to find the remainder by long division method as shown below:

$$\begin{array}{r|l}
3x^2 + 4x + 1 & 6x^4 + 8x^3 + 17x^2 + 21x + 7 \;\big(\, 2x^2 + 5 \\
 & 6x^4 + 8x^3 + 2x^2 \\
 & - \quad - \quad - \\ \hline
 & 15x^2 + 21x + 7 \\
 & 15x^2 + 20x + 5 \\
 & - \quad - \quad - \\ \hline
 & x + 2
\end{array}$$

Clearly, remainder $= x + 2$. It is given that the remainder is $ax + b$.

$\therefore \quad ax + b = x + 2 \Rightarrow a = 1, b = 2.$ [On compairing the coefficients of like powers of x]

EXAMPLE 13 *Find k so that $x^2 + 2x + k$ is a factor of $2x^4 + x^3 - 14x^2 + 5x + 6$. Also, find all the zeroes of the two polynomials.* **[NCERT EXEMPLAR]**

SOLUTION It is given that $x^2 + 2x + k$ is a factor of the polynomial $f(x) = 2x^4 + x^3 - 14x^2 + 5x + 6$ when divided by $x^2 + 2x + k$, the remainder is zero.

Let us now divide $f(x) = 2x^4 + x^3 - 14x^2 + 5x + 6$ by $x^2 + 2x + k$ using by long division method.

$$
\begin{array}{r|l}
x^2 + 2x + k & 2x^4 + x^3 - 14x^2 + 5x + 6 \quad \left(2x^2 - 3x - 2(k+4)\right. \\
 & 2x^4 + 4x^3 + 2kx^2 \\
 & - \quad - \quad - \\
\hline
 & -3x^3 - 2x^2(k+7) + 5x + 6 \\
 & -3x^3 - 6x^2 \qquad - 3kx \\
 & + \quad + \quad + \\
\hline
 & -2x^2(k+4) + x(5+3k) + 6 \\
 & -2x^2(k+4) - 4x(k+4) - 2x(k+4) \\
 & + \quad + \quad + \\
\hline
 & x(7k+21) + (2k^2 + 8k + 6)
\end{array}
$$

Thus, Remainder $= x(7k + 21) + (2k^2 + 8k + 6)$ and Quotient $= 2x^2 - 3x - 2(k + 4)$.

$\therefore \qquad$ Remainder $= 0$.

$\Rightarrow \qquad x(7k + 21) + 2(k^2 + 4k + 3) = 0$ for all x.

$\Rightarrow \qquad 7k + 21 = 0$ and $k^2 + 4k + 3 = 0$

$\Rightarrow \qquad 7(k + 3) = 0$ and $(k + 1)(k + 3) = 0$

$\Rightarrow \qquad k + 3 = 0 \Rightarrow k = -3$

Substituting the value of k in $x^2 + 2x + k$, we obtain: $x^2 + 2x - 3 = (x + 3)(x - 1)$ as the divisor. Clearly, its zeros are -3 and 1. Consequently, two zeros of $f(x)$ are -3 and 1.

For $k = -3$, we obtain

Quotient $= 2x^2 - 3x - 2 = 2x^2 - 4x + x - 2 = 2x(x - 2) + 1(x - 2) = (x - 2)(2x + 1)$

and, Divisor $= x^2 + 2x - 3 = x^2 + 3x - x - 3 = x(x + 3) - 1(x + 3) = (x - 1)(x + 3)$

$\therefore \qquad f(x) =$ (Quotient) × (Divisor)

$\Rightarrow \qquad f(x) = 2x^4 + x^3 - 14x^2 + 5x + 6 = (x - 2)(2x + 1)(x - 1)(x + 3)$

Hence, zeros of $f(x)$ are $2, -1/2, 1$ and -3.

EXAMPLE 14 *If the remainder on division of $x^3 + 2x^2 + kx + 3$ by $x - 3$ is 21, find the quotient and the value of k. Hence, find the zeroes of the cubic polynomial $x^3 + 2x^2 + kx - 18$.*

SOLUTION Let $f(x) = x^3 + 2x^2 + kx + 3$. It is given that $f(x)$ when divided by $x - 3$ gives 21 as remainder.

$$
\begin{aligned}
\therefore\quad & f(3) = 21\\
\Rightarrow\quad & 3^3 + 2\times 3^2 + 3k + 3 = 21\\
\Rightarrow\quad & 27 + 18 + 3k + 3 = 21\\
\Rightarrow\quad & 3k + 48 = 21\\
\Rightarrow\quad & 3k = -27 \Rightarrow k = -9
\end{aligned}
$$

Hence, the given polynomial is $f(x) = x^3 + 2x^2 - 9x + 3$.

Let us now divide $f(x)$ by $(x-3)$ to find the quotient. Using long division method, obtain

$$
\begin{array}{rl}
x-3\,\big) & x^3 + 2x^2 - 9x + 3 \;\big(\, x^2 + 5x + 6\\
 & x^3 - 3x^2\\
 & \underline{-\quad +\qquad\qquad}\\
 & \qquad 5x^2 - 9x + 3\\
 & \qquad 5x^2 - 15x\\
 & \qquad \underline{-\qquad +\qquad\quad}\\
 & \qquad\qquad 6x + 3\\
 & \qquad\qquad 6x - 18\\
 & \qquad\qquad \underline{-\quad +\quad}\\
 & \qquad\qquad\qquad 21
\end{array}
$$

So, Quotient $= x^2 + 5x + 6$

Thus, when $f(x)$ is divided by $x-3$ the quotient and the remainder are $x^2 + 5x + 6$ and 21 respectively. Therefore, using division algorithm, we obtain

$$
\begin{aligned}
& f(x) = (x^2 + 5x + 6)(x-3) + 21\\
\Rightarrow\quad & x^3 + 2x^2 - 9x + 3 - 21 = (x+2)(x+3)(x-3)\\
\Rightarrow\quad & x^3 + 2x^2 - 9x - 18 = (x+2)(x+3)(x-3)
\end{aligned}
$$

Hence, the zeros of $x^3 + 2x^2 - 9x - 18$ *i.e.* $x^3 + 2x^2 + kx - 18$ are $-2, -3$ and 3.

EXAMPLE 15 *For which values of a and b are the zeros of* $q(x) = x^3 + 2x^2 + a$ *also the zeros of the polynomial* $p(x) = x^5 - x^4 - 4x^3 + 3x^2 + 3x + b$? *Which zeros of* $p(x)$ *are not the zeros of* $q(x)$?

SOLUTION If zeros of $q(x)$ are also the zeroes of $p(x)$, then $p(x)$ is divisible by $q(x)$. In otherwords, when $p(x)$ is divided by $q(x)$, the remainder is zero. Let us now divide $p(x)$ by $q(x)$ to obtain the remainder.

$$
\begin{array}{rl}
x^3 + 2x^2 + a\,\big) & x^5 - x^4 - 4x^3 + 3x^2 + 3x + b \;\big(\, x^2 - 3x + 2\\
 & x^5 + 2x^4 + ax^2\\
 & \underline{-\quad -\quad -\qquad\qquad\qquad\qquad}\\
 & \quad -3x^4 - 4x^3 + (3-a)x^2 + 3x + b\\
 & \quad -3x^4 - 6x^3 - 3ax\\
 & \quad \underline{+\qquad +\qquad +\qquad\qquad\qquad\quad}\\
 & \qquad\quad 2x^3 + (3-a)x^2 + 3x(1+a) + b\\
 & \qquad\quad 2x^3 + 4x^2 \qquad\quad + 2a\\
 & \qquad\quad \underline{-\qquad -\qquad\qquad -\qquad\qquad}\\
 & \qquad\qquad (-1-a)x^2 + 3x(1+a) + b - 2a
\end{array}
$$

If $p(x)$ is divisible by $q(x)$, then remainder must be zero.

Now, Remainder = 0

$\Rightarrow \quad (-1-a)x^2 + 3x(1+a) + b - 2a = 0$ for all x.

$\Rightarrow \quad -1-a = 0,\ 3(1+a) = 0$ and $b - 2a = 0$ $\quad \left[\begin{array}{l}\text{On equating the coefficients}\\ \text{of like powers of } x\end{array}\right]$

$\Rightarrow \quad a = -1$ and $b - 2a = 0$

$\Rightarrow \quad a = -1$ and $b = -2$

Substituting $a = -1$ in $q(x) = x^3 + 2x^2 + a$, we obtain $q(x) = x^3 + 2x^2 - 1$.

Now, $\quad x^2 - 3x + 2 = (x-1)(x-2)$

So, zeros of $x^2 - 3x + 2$ are 1 and 2. We find that $q(1) = 1 + 2 - 1 = 2 \neq 0$ and $q(2) = 8 + 8 - 1 = 15 \neq 0$. So, the zeros of the quotient $x^2 - 3x + 2$ are not the zeros of $q(x)$. Hence, 1 and 2 are zeros of $p(x)$ which are not zeros of $q(x)$.

EXERCISE 2.3

LEVEL-1

1. Apply division algorithm to find the quotient $q(x)$ and remainder $r(x)$ on dividing $f(x)$ by $g(x)$ in each of the following:
   (i) $f(x) = x^3 - 6x^2 + 11x - 6,\ g(x) = x^2 + x + 1$
   (ii) $f(x) = 10x^4 + 17x^3 - 62x^2 + 30x - 3,\ g(x) = 2x^2 + 7x + 1$
   (iii) $f(x) = 4x^3 + 8x + 8x^2 + 7,\ g(x) = 2x^2 - x + 1$
   (iv) $f(x) = 15x^3 - 20x^2 + 13x - 12,\ g(x) = 2 - 2x + x^2$
2. Check whether the first polynomial is a factor of the second polynomial by applying the division algorithm:
   (i) $g(t) = t^2 - 3,\ f(t) = 2t^4 + 3t^3 - 2t^2 - 9t - 12$ [NCERT]
   (ii) $g(x) = x^3 - 3x + 1,\ f(x) = x^5 - 4x^3 + x^2 + 3x + 1$ [NCERT]
   (iii) $g(x) = 2x^2 - x + 3,\ f(x) = 6x^5 - x^4 + 4x^3 - 5x^2 - x - 15$
3. Obtain all zeros of the polynomial $f(x) = 2x^4 + x^3 - 14x^2 - 19x - 6$, if two of its zeros are -2 and -1.
4. Obtain all zeros of $f(x) = x^3 + 13x^2 + 32x + 20$, if one of its zeros is -2.
5. Obtain all zeros of the polynomial $f(x) = x^4 - 3x^3 - x^2 + 9x - 6$, if two of its zeros are $-\sqrt{3}$ and $\sqrt{3}$.
6. Find all zeros of the polynomial $f(x) = 2x^4 - 2x^3 - 7x^2 + 3x + 6$, if its two zeros are $-\sqrt{\frac{3}{2}}$ and $\sqrt{\frac{3}{2}}$.
7. Find all the zeros of the polynomial $x^4 + x^3 - 34x^2 - 4x + 120$, if two of its zeros are 2 and -2. [CBSE 2008]
8. Find all zeros of the polynomial $2x^4 + 7x^3 - 19x^2 - 14x + 30$, if two of its zeros are $\sqrt{2}$ and $-\sqrt{2}$. [CBSE 2008]

9. Find all the zeros of the polynomial $2x^3 + x^2 - 6x - 3$, if two of its zeros are $-\sqrt{3}$ and $\sqrt{3}$. [CBSE 2009]

10. Find all the zeros of the polynomial $x^3 + 3x^2 - 2x - 6$, if two of its zeros are $-\sqrt{2}$ and $\sqrt{2}$. [CBSE 2009]

11. Find all zeros of the polynomial $2x^4 - 9x^3 + 5x^2 + 3x - 1$, if two of its zeros are $2 + \sqrt{3}$ and $2 - \sqrt{3}$. [CBSE 2018]

LEVEL-2

12. What must be added to the polynomial $f(x) = x^4 + 2x^3 - 2x^2 + x - 1$ so that the resulting polynomial is exactly divisible by $x^2 + 2x - 3$?

13. What must be subtracted from the polynomial $f(x) = x^4 + 2x^3 - 13x^2 - 12x + 21$ so that the resulting polynomial is exactly divisible by $x^2 - 4x + 3$?

14. Given that $\sqrt{2}$ is a zero of the cumbic polynomial $6x^3 + \sqrt{2}x^2 - 10x - 4\sqrt{2}$, find its other two zeroes. [NCERT EXEMPLAR]

15. Given that $x - \sqrt{5}$ is a factor of the cubic polynomial $x^3 - 3\sqrt{5}x^2 + 13x - 3\sqrt{5}$, find all the zeroes of the polynomial. [NCERT EXEMPLAR]

ANSWERS

3. $-\frac{1}{2}, 3, -2, -1$
4. $-10, -1, -2$
5. $-\sqrt{3}, \sqrt{3}, 1, 2$
6. $2, -1, \sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}$
7. $2, -2, 5, -6$
8. $\sqrt{2}, -\sqrt{2}, -5, \frac{3}{2}$
9. $-\sqrt{3}, \sqrt{3}, -\frac{1}{2}$
10. $-\sqrt{2}, \sqrt{2}, -3$
11. $1, -\frac{1}{2}, 2 + \sqrt{3}, 2 - \sqrt{3}$
12. $x - 2$
13. $2x - 3$
14. $\frac{-\sqrt{2}}{2}, \frac{-2\sqrt{2}}{3}$
15. $\sqrt{5}, \sqrt{5} + \sqrt{2}, \sqrt{5} - \sqrt{2}$

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per the exact requirement of the questions:

1. Define a polynomial with real coefficients.
2. Define degree of a polynomial.
3. Write the standard form of a linear polynomial with real coefficients.
4. Write the standard form of a quadratic polynomial with real coefficients.
5. Write the standard form of a cubic polynomial with real coefficients.
6. Define value of a polynomial at a point.
7. Define zero of a polynomial.
8. The sum and product of the zeros of a quadratic polynomial are $-\frac{1}{2}$ and -3 respectively. What is the quadratic polynomial.
9. Write the family of quadratic polynomials having $-\frac{1}{4}$ and 1 as its zeros.
10. If the product of zeros of the quadratic polynomial $f(x) = x^2 - 4x + k$ is 3, find the value of k.
11. If the sum of the zeros of the quadratic polynomial $f(x) = kx^2 - 3x + 5$ is 1, write the value of k.

12. In Fig. 2.17, the graph of a polynomial $p(x)$ is given. Find the zeros of the polynomial.

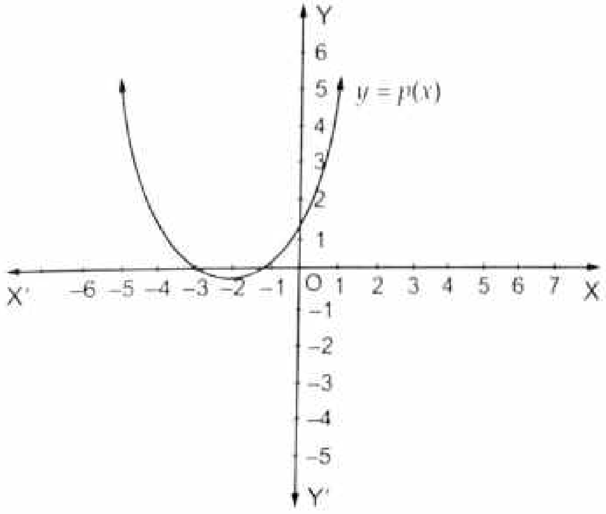


Fig. 2.17

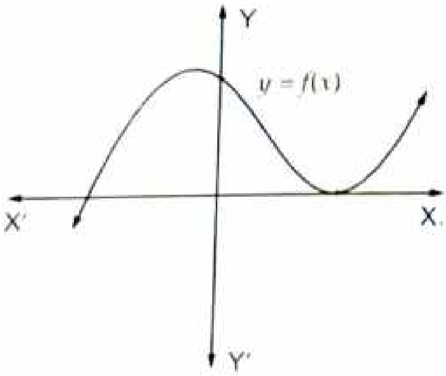


Fig. 2.18

13. The graph of a polynomial $y = f(x)$, shown in Fig. 2.18. Find the number of real zeros of $f(x)$.

14. The graph of the polynomial $f(x) = ax^2 + bx + c$ is as shown below (Fig. 2.19). Write the signs of 'a' and $b^2 - 4ac$.

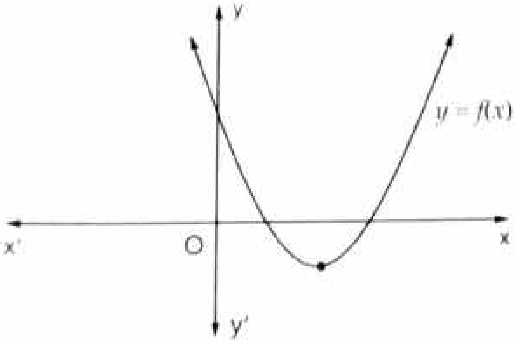


Fig. 2.19

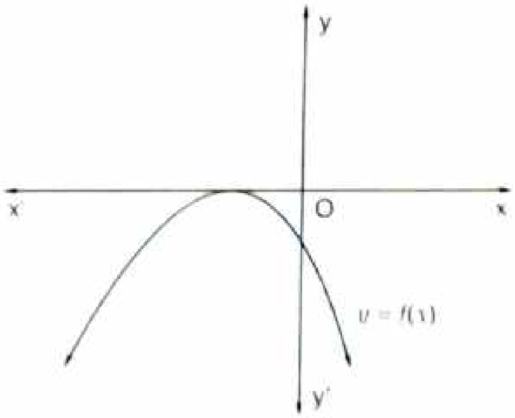


Fig. 2.20

15. The graph of the polynomial $f(x) = ax^2 + bx + c$ is as shown in Fig. 2.20. Write the value of $b^2 - 4ac$ and the number of real zeros of $f(x)$.

16. In Q. No. 14, write the sign of c.

17. In Q. No. 15, write the sign of c.

18. The graph of a polynomial $f(x)$ is as shown in Fig. 2.21. Write the number of real zeros of $f(x)$.

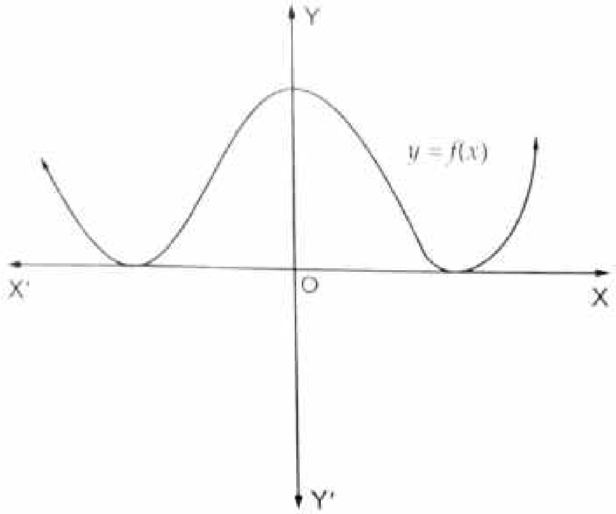


Fig. 2.21

19. If $x = 1$ is a zero of the polynomial $f(x) = x^3 - 2x^2 + 4x + k$, write the value of k.
20. State division algorithm for polynomials.
21. Give an example of polynomials $f(x), g(x), q(x)$ and $r(x)$ satisfying $f(x) = g(x)\, q(x) + r(x)$, where degree $r(x) = 0$.
22. Write a quadratic polynomial, sum of whose zeros is $2\sqrt{3}$ and their product is 2.
23. If fourth degree polynomial is divided by a quadratic polynomial, write the degree of the remainder.
24. If $f(x) = x^3 + x^2 - ax + b$ is divisible by $x^2 - x$ write the values of a and b.
25. If $a - b$, a and $a + b$ are zeros of the polynomial $f(x) = 2x^3 - 6x^2 + 5x - 7$, write the value of a.
26. Write the coefficients of the polynomial $p(z) = z^5 - 2z^2 + 4$.
27. Write the zeros of the polynomial $x^2 - x - 6$. **[CBSE 2008]**
28. If $(x + a)$ is a factor of $2x^2 + 2ax + 5x + 10$, find a. **[CBSE 2008]**
29. For what value of k, -4 is a zero of the polynomial $x^2 - x - (2k + 2)$? **[CBSE 2009]**
30. If 1 is a zero of the polynomial $p(x) = ax^2 - 3(a - 1)x - 1$, then find the value of a.
31. If α, β are the zeros of a polynomial such that $\alpha + \beta = -6$ and $\alpha\beta = -4$, then write the polynomial. **[CBSE 2010]**
32. If α, β are the zeros of the polynomial $2y^2 + 7y + 5$, write the value of $\alpha + \beta + \alpha\beta$. **[CBSE 2010]**
33. For what value of k, is 3 a zero of the polynomial $2x^2 + x + k$? **[CBSE 2010]**
34. For what value of k, is -3 a zero of the polynomial $x^2 + 11x + k$? **[CBSE 2010]**
35. For what value of k, is -2 a zero of the polynomial $3x^2 + 4x + 2k$? **[CBSE 2010]**
36. If a quadratic polynomial $f(x)$ is factorizable into linear distinct factors, then what is the total number of real and distinct zeros of $f(x)$?

37. If a quadratic polynomial $f(x)$ is a square of a linear polynomial, then its two zeroes are coincident. (True/False)

38. If a quadratic polynomial $f(x)$ is not factorizable into linear factors, then it has no real zero. (True/False)

39. If $f(x)$ is a polynomial such that $f(a)\,f(b) < 0$, then what is the number of zeros lying between a and b?

40. If graph of quadratic polynomial $ax^2 + bx + c$ cuts positive direction of y-axis, then what is the sign of c?

41. If the graph of quadratic polynomial $ax^2 + bx + c$ cuts negative direction of y-axis, then what is the sign of c?

ANSWERS

3. $f(x) = ax + b, a \neq 0$ 4. $f(x) = ax^2 + bx + c, a \neq 0$ 5. $f(x) = ax^3 + bx^2 + cx + d, a \neq 0$

8. $f(x) = k\left(x^2 + \frac{x}{2} - 3\right)$, where k is any non-zero real number.

9. $f(x) = k\left(x^2 - \frac{3}{4}x - \frac{1}{4}\right)$, where k is any non-zero real number. 10. $k = 3$ 11. 3

12. -3 and -1 13. 3 14. $a > 0, b^2 - 4ac > 0$ 15. $b^2 - 4ac = 0$, Two

16. $c > 0$ 17. $c < 0$ 18. 4 19. $k = -3$

21. $f(x) = x^3 + x^2 + x + 1$, $g(x) = x + 2$, $q(x) = x^2 - x + 3$, $r(x) = -5$

22. $f(x) = x^2 - 2\sqrt{3}x + 2$ 23. Less than or equal to 1 24. $a = 2, b = 0$

25. 1 26. 1, 0, 0, −2, 0, 4 27. 3, −2 28. 2 29. 9 30. 1

31. $f(x) = x^2 + 6x - 4$ 32. −1 33. −21 34. 24 35. −2 36. 2

37. True 38. True 39. At least one 40. Positive 41. Negative

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If α, β are the zeros of the polynomial $f(x) = x^2 + x + 1$, then $\frac{1}{\alpha} + \frac{1}{\beta} =$

(a) 1 (b) −1 (c) 0 (d) None of these

2. If α, β are the zeros of the polynomial $p(x) = 4x^2 + 3x + 7$, then $\frac{1}{\alpha} + \frac{1}{\beta}$ is equal to

(a) $\frac{7}{3}$ (b) $-\frac{7}{3}$ (c) $\frac{3}{7}$ (d) $-\frac{3}{7}$

3. If one zero of the polynomial $f(x) = (k^2 + 4)x^2 + 13x + 4k$ is reciprocal of the other, then $k =$

(a) 2 (b) − 2 (c) 1 (d) −1

4. If the sum of the zeros of the polynomial $f(x) = 2x^3 - 3kx^2 + 4x - 5$ is 6, then the value of k is

(a) 2 (b) 4 (c) -2 (d) -4

5. If α and β are the zeros of the polynomial $f(x) = x^2 + px + q$, then a polynomial having $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ is its zeros is

(a) $x^2 + qx + p$ (b) $x^2 - px + q$ (c) $qx^2 + px + 1$ (d) $px^2 + qx + 1$

6. If α, β are the zeros of polynomial $f(x) = x^2 - p(x+1) - c$, then $(\alpha + 1)(\beta + 1) =$

(a) $c - 1$ (b) $1 - c$ (c) c (d) $1 + c$

7. If α, β are the zeros of the polynomial $f(x) = x^2 - p(x+1) - c$ such that $(\alpha + 1)(\beta + 1) = 0$, then $c =$

(a) 1 (b) 0 (c) -1 (d) 2

8. If $f(x) = ax^2 + bx + c$ has no real zeros and $a + b + c < 0$, then

(a) $c = 0$ (b) $c > 0$ (c) $c < 0$ (d) None of these

9. If the diagram in Fig. 2.22 shows the graph of the polynomial $f(x) = ax^2 + bx + c$, then

(a) $a > 0, b < 0$ and $c > 0$ (b) $a < 0, b < 0$ and $c < 0$

(c) $a < 0, b > 0$ and $c > 0$ (d) $a < 0, b > 0$ and $c < 0$

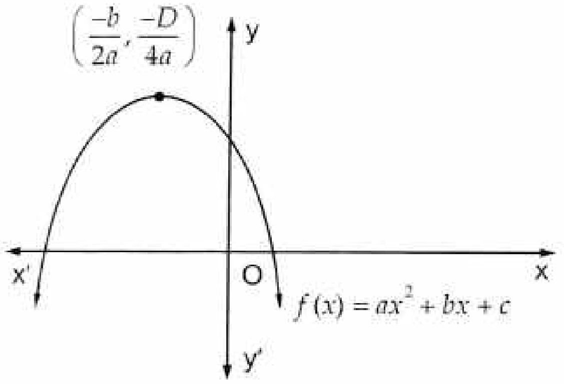


Fig. 2.22

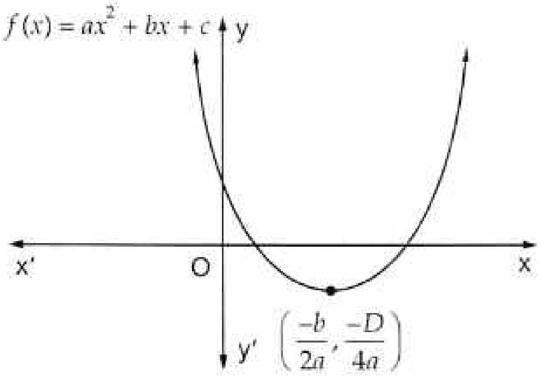


Fig. 2.23

10. Figure 2.23 shows the graph of the polynomial $f(x) = ax^2 + bx + c$ for which

(a) $a < 0, b > 0$ and $c > 0$ (b) $a < 0, b < 0$ and $c > 0$

(c) $a < 0, b < 0$ and $c < 0$ (d) $a > 0, b > 0$ and $c < 0$

11. If the product of zeros of the polynomial $f(x) = ax^3 - 6x^2 + 11x - 6$ is 4, then $a =$

(a) $\frac{3}{2}$ (b) $-\frac{3}{2}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

12. If zeros of the polynomial $f(x) = x^3 - 3px^2 + qx - r$ are in A.P., then

(a) $2p^3 = pq - r$ (b) $2p^3 = pq + r$ (c) $p^3 = pq - r$ (d) None of these

13. If the product of two zeros of the polynomial $f(x) = 2x^3 + 6x^2 - 4x + 9$ is 3, then its third zero is

(a) $\frac{3}{2}$ (b) $-\frac{3}{2}$ (c) $\frac{9}{2}$ (d) $-\frac{9}{2}$

14. If the polynomial $f(x) = ax^3 + bx - c$ is divisible by the polynomial $g(x) = x^2 + bx + c$, then $ab =$

(a) 1 (b) $\frac{1}{c}$ (c) -1 (d) $-\frac{1}{c}$

15. In Q. No. 14, $c =$

(a) b (b) $2b$ (c) $2b^2$ (d) $-2b$

16. If one root of the polynomial $f(x) = 5x^2 + 13x + k$ is reciprocal of the other, then the value of k is

(a) 0 (b) 5 (c) $\frac{1}{6}$ (d) 6

17. If α, β, γ are the zeros of the polynomial $f(x) = ax^3 + bx^2 + cx + d$, then $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} =$

(a) $-\frac{b}{d}$ (b) $\frac{c}{d}$ (c) $-\frac{c}{d}$ (d) $-\frac{c}{a}$

18. If α, β, γ are the zeros of the polynomial $f(x) = ax^3 + bx^2 + cx + d$, then $\alpha^2 + \beta^2 + \gamma^2 =$

(a) $\frac{b^2 - ac}{a^2}$ (b) $\frac{b^2 - 2ac}{a}$ (c) $\frac{b^2 + 2ac}{b^2}$ (d) $\frac{b^2 - 2ac}{a^2}$

19. If α, β, γ are the zeros of the polynomial $f(x) = x^3 - px^2 + qx - r$, then $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} =$

(a) $\frac{r}{p}$ (b) $\frac{p}{r}$ (c) $-\frac{p}{r}$ (d) $-\frac{r}{p}$

20. If α, β are the zeros of the polynomial $f(x) = ax^2 + bx + c$, then $\frac{1}{\alpha^2} + \frac{1}{\beta^2} =$

(a) $\frac{b^2 - 2ac}{a^2}$ (b) $\frac{b^2 - 2ac}{c^2}$ (c) $\frac{b^2 + 2ac}{a^2}$ (d) $\frac{b^2 + 2ac}{c^2}$

21. If two of the zeros of the cubic polynomial $ax^3 + bx^2 + cx + d$ are each equal to zero, then the third zero is

(a) $\frac{-d}{a}$ (b) $\frac{c}{a}$ (c) $\frac{-b}{a}$ (d) $\frac{b}{a}$

22. If two zeros of $x^3 + x^2 - 5x - 5$ are $\sqrt{5}$ and $-\sqrt{5}$, then its third zero is

(a) 1 (b) -1 (c) 2 (d) -2

23. The product of the zeros of $x^3 + 4x^2 + x - 6$ is

(a) -4 (b) 4 (c) 6 (d) -6

24. What should be added to the polynomial x^2-5x+4, so that 3 is the zero of the resulting polynomial?

(a) 1 (b) 2 (c) 4 (d) 5

25. What should be subtracted to the polynomial $x^2-16x+30$, so that 15 is the zero of the resulting polynomial?

(a) 30 (b) 14 (c) 15 (d) 16

26. A quadratic polynomial, the sum of whose zeroes is 0 and one zero is 3, is

(a) x^2-9 (b) x^2+9 (c) x^2+3 (d) x^2-3

27. If two zeroes of the polynomial x^3+x^2-9x-9 are 3 and –3, then its third zero is

(a) –1 (b) 1 (c) –9 (d) 9

28. If $\sqrt{5}$ and $-\sqrt{5}$ are two zeroes of the polynomial $x^3+3x^2-5x-15$, then its third zero is

(a) 3 (b) –3 (c) 5 (d) –5

29. If $x+2$ is a factor of $x^2+ax+2b$ and $a+b=4$, then

(a) $a=1, b=3$ (b) $a=3, b=1$ (c) $a=-1, b=5$ (d) $a=5, b=-1$

30. The polynomial which when divided by $-x^2+x-1$ gives a quotient $x-2$ and remainder 3, is

(a) x^3-3x^2+3x-5 (b) $-x^3-3x^2-3x-5$

(c) $-x^3+3x^2-3x+5$ (d) x^3-3x^2-3x+5

31. The number of polynomials having zeroes –2 and 5 is

(a) 1 (b) 2 (c) 3 (d) more than 3.

32. If one of the zeroes of the quadratic polynomial $(k-1)x^2+kx+1$ is –3, then the value of k is

(a) $\frac{4}{3}$ (b) $-\frac{4}{3}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

33. The zeroes of the quadratic polynomial $x^2+99x+127$ are

(a) both positive (b) both negative

(c)both equal (d) one positive and one negative

34. If the zeroes of the quadratic polynomial $x^2+(a+1)x+b$ are 2 and –3, then

(a) $a=-7, b=-1$ (b) $a=5, b=-1$ (c) $a=2, b=-6$ (d) $a=0, b=-6$

35. Given that one of the zeroes of the cubic polynomial ax^3+bx^2+cx+d is zero, the product of the other two zeroes is

(a) $-\frac{c}{a}$ (b) $\frac{c}{a}$ (c) 0 (d) $-\frac{b}{a}$

36. The zeroes of the quadratic polynomial $x^2+ax+a, a \neq 0$,

(a) cannot both be positive (b) cannot both be negative

(c)are always unequal (d) are always equal

37. If one of the zeros of the cubic polynomial x^3+ax^2+bx+c is –1, then the product of other two zeros is

(a) $b-a+1$ (b) $b-a-1$ (c) $a-b+1$ (d) $a-b-1$

38. Given that two of the zeros of the cubic polynomial $ax^3 + bx^2 + cx + d$ are 0, the third zero is

(a) $-\frac{b}{a}$ (b) $\frac{b}{a}$ (c) $\frac{c}{a}$ (d) $-\frac{d}{a}$

39. If one zero of the quadratic polynomial $x^2 + 3x + k$ is 2, then the value of k is

(a) 10 (b) -10 (c) 5 (d) -5

40. If the zeros of the quadratic polynomial $ax^2 + bx + c, c \neq 0$ are equal, then

(a) c and a have opposite signs (b) c and b have opposite signs

(c) c and a have the same sign (d) c and b have the same sign

41. If one of the zeros of a quadratic polynomial of the form $x^2 + ax + b$ is the negative of the other, then it

(a) has no linear term and constant term is negative.

(b) has no linear term and the constant term is positive.

(c) can have a linear term but the constant term is negative.

(d) can have a linear term but the constant term is positive.

42. Which of the following is not the graph of a quadratic polynomial?

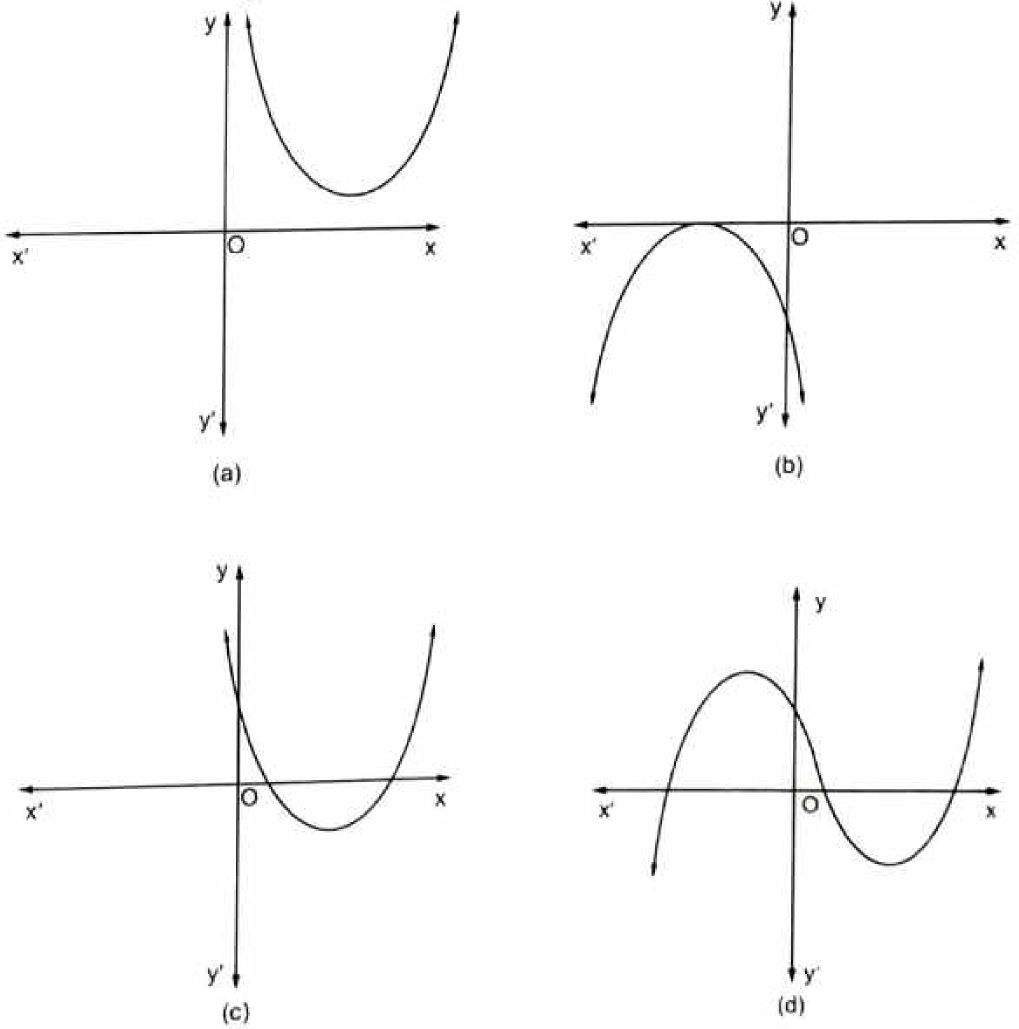


Fig. 2.24

ANSWERS

1. (b)	2. (d)	3. (a)	4. (b)	5. (c)	6. (b)	7. (a)
8. (c)	9. (a)	10. (b)	11. (a)	12. (a)	13. (b)	14. (a)
15. (b)	16. (b)	17. (c)	18. (d)	19. (b)	20. (b)	21. (c)
22. (b)	23. (c)	24. (b)	25. (c)	26. (a)	27. (a)	28. (b)
29. (b)	30. (c)	31. (d)	32. (a)	33. (b)	34. (d)	35. (b)
36. (a)	37. (a)	38. (a)	39. (b)	40. (c)	41. (a)	42. (d)

SUMMARY

1. Let x be a variable, n be a positive integer and $a_0, a_1, a_2, \ldots, a_n$ be constants (real numbers). Then, $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, is called a polynomial in variable x.
2. The exponent of the highest degree term in a polynomial is known as its degree.

   A polynomial of degree 0 is called a constant polynomial.

   A polynomial of degree 1, 2 or 3 is called a linear polynomial, a quadratic polynomial or a cubic polynomial respectively.

   Following are the forms of various degree polynomials.

Degree	Name of the polynomial	Form of the polynomial
0	Constant polynomial	$f(x) = a$, a is a constant
1	Linear polynomial	$f(x) = ax + b, a \neq 0$
2	Quadratic polynomial	$f(x) = ax^2 + bx + c, a \neq 0$
3	Cubic polynomial	$f(x) = ax^3 + bx^2 + cx + d, a \neq 0$
4	Biquadratic polynomial	$f(x) = ax^4 + bx^3 + cx^2 + dx + e, \ a \neq 0$

3. If $f(x)$ is a polynomial and α is any real number, then the real number obtained by replacing x by α in $f(x)$ is known as the value of $f(x)$ at $x = \alpha$ and is denoted by $f(\alpha)$.
4. A real number α is a zero of a polynomial $f(x)$, if $f(\alpha) = 0$.
5. A polynomial of degree n can have at most n real zeros.
6. Geometrically the zeros of a polynomial $f(x)$ are the x-coordinates of the points where the graph $y = f(x)$ intersects x-axis.
7. If α and β are the zeros of a quadratic polynomial $f(x) = ax^2 + bx + c$, then

$$\alpha + \beta = -\frac{b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}, \quad \alpha\beta = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

8. If α, β, γ are the zeros of a cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$, then

$$\alpha + \beta + \gamma = -\frac{b}{a} = -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}, \quad \alpha\beta\gamma = -\frac{d}{a} = -\frac{\text{Constant term}}{\text{Coefficient of } x^3}$$

9. If $\alpha, \beta, \gamma, \delta$ are the zeros of a biquadratic polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, then

$$\alpha + \beta + \gamma + \delta = -\frac{b}{a} = -\frac{\text{Coefficient of } x^3}{\text{Coefficient of } x^4}$$

$$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = \frac{c}{a} = \frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^4}$$

$$(\alpha + \beta)\gamma\delta + \alpha\beta(\gamma + \delta) = -\frac{d}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^4}, \quad \alpha\beta\gamma\delta = \frac{e}{a} = \frac{\text{Constant terms}}{\text{Coefficient of } x^4}$$

10. If $f(x)$ is a polynomial and $g(x)$ is a non-zero polynomial, then there exist two polynomials $q(x)$ and $r(x)$ such that $f(x) = g(x) \times q(x) + r(x)$, where $r(x) = 0$ or degree $r(x) <$ degree $g(x)$. This is known as the division algorithm.

NOTE: Formative assessment also includes lab activities, projects, assignments (Home work), oral and visual testings.