CHAPTER 4

QUADRATIC EQUATIONS

4.1 INTRODUCTION

In chapter 2, we have learnt about polynomials and their zeros. We have also learnt about graphical representation of linear, quadratic and cubic polynomials. When a polynomial $f(x)$ is equated to zero, we get an equation which is known as a polynomial equation. If $f(x)$ is a linear polynomial, then $f(x) = 0$ is called a linear equation. For example, $3x - 2 = 0$, $4t + \frac{3}{5} = 0$ etc. are linear equations. In earlier classes, we have learnt about the method of solving a linear equation. If $f(x)$ is a quadratic polynomial i.e., $f(x) = ax^2 + bx + c,\ a \neq 0$. Then, $f(x) = 0$ i.e., $ax^2 + bx + c = 0, a \neq 0$ is called a quadratic equation. Such equations arise in many real life situations. In this chapter, we will learn about quadratic equations and various ways of finding their zeros or roots. In the end of the chapter, we will also discuss some applications of quadratic equations in daily life situations.

4.2 QUADRATIC EQUATION

QUADRATIC EQUATION *If $p(x)$ is a quadratic polynomial, then $p(x) = 0$ is called a quadratic equation.*

The general form of a quadratic equation is $ax^2 + bx + c = 0$, where $a, b, c \in R$ and $a \neq 0$.

ROOTS OF A QUADRATIC EQUATION *Let $p(x) = 0$ be a quadratic equation, then the zeros of the polynomial $p(x)$ are called the roots of the equation $p(x) = 0$.*

Thus, $x = \alpha$ is a roots of $p(x) = 0$ if and only if $p(\alpha) = 0$.

As we have seen in section 2.2 that a quadratic polynomial may or may not have real zeros. In case a quadratic polynomial has real zeros, it can have at most two zeros. It follows from this that a quadratic equation can have at most two real roots.

Finding the roots of a quadratic equation is known as solving the quadratic equation.

Various concepts discussed so far are illustrated by the following examples.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON DETERMINING WHETHER A GIVEW EQUATION IS QUADRATIC OR NOT**

EXAMPLE 1 *Which of the following are quadratic equations?*

(i) $x^2 - 6x + 4 = 0$

(ii) $2x^2 - 7x = 0$

(iii) $x + \frac{3}{x} = x^2$

(iv) $x^2 + \frac{1}{x^2} = 2$

(v) $x^2 + 2\sqrt{x} - 3 = 0$

(vi) $3x^2 - 4x + 2 = 2x^2 - 2x + 4$

SOLUTION (i) Let $p(x) = x^2 - 6x + 4$

Clearly, $p(x) = x^2 - 6x + 4$ is a quadratic polynomial. Therefore, $x^2 - 6x + 4 = 0$ is a quadratic equation.

(ii) Clearly, $2x^2 - 7x$ is a quadratic polynomial. So, the given equation is a quadratic equation.

(iii) We have,

$$x + \frac{3}{x} = x^2 \Rightarrow \frac{x^2 + 3}{x} = x^2 \Rightarrow x^2 + 3 = x^3 \Rightarrow x^3 - x^2 - 3 = 0$$

Clearly, $x^3 - x^2 - 3$, being a polynomial of degree 3, is not a quadratic polynomial. So, the given equation is not a quadratic equation.

(iv) We have,

$$x^2 + \frac{1}{x^2} = 2 \Rightarrow \frac{x^4 + 1}{x^2} = 2 \Rightarrow x^4 - 2x^2 + 1 = 0$$

Clearly, $x^4 - 2x^2 + 1$ is not a quadratic polynomial. So, the given equation is not a quadratic equation.

(v) Clearly, $x^2 + 2\sqrt{x} - 3$ is not a quadratic polynomial because it contains a term involving $x^{1/2}$, where 1/2 is not an integer. So, the given equation is not a quadratic equation.

(vi) We have,

$$3x^2 - 4x + 2 = 2x^2 - 2x + 4 \Rightarrow x^2 - 2x - 2 = 0$$

Clearly, $x^2 - 2x - 2$ is a quadratic polynomial. So, the given equation is a quadratic equation.

Type II ON DETERMINING WHETHER THE GIVEN VALUES ARE SOLUTIONS OF THE GIVEN EQUATION OR NOT

EXAMPLE 2 *In each of the following determine whether the given values are solution of the given equation or not:*

(i) $3x^2 - 2x - 1 = 0, x = 1$ (ii) $6x^2 - x - 2 = 0, x = -1/2, x = 2/3$

(iii) $x^2 - x + 1 = 0, x = 1, x = -1$ (iv) $x^2 + \sqrt{2}x - 4 = 0, x = \sqrt{2}, x = -2\sqrt{2}$

SOLUTION (i) Substituting $x = 1$ on the LHS of the given equation, we get

$$\text{LHS} = 3 \times 1^2 - 2 \times 1 - 1 = 0 = \text{RHS}$$

So, $x = 1$ is a solution of the given equation.

(ii) Substituting $x = -\frac{1}{2}$ in the LHS of the given equation, we get

$$\text{LHS} = 6 \times \left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) - 2 = \frac{6}{4} + \frac{1}{2} - 2 = 0 = \text{RHS}$$

So, $x = \frac{1}{2}$ is a solution of the given equation.

For $x = \frac{2}{3}$, we have

$$\text{LHS} = 6 \times \left(\frac{2}{3}\right)^2 - \frac{2}{3} - 2 = 0 = \text{RHS}$$

So, $x = \frac{2}{3}$ is also a solution of the given equation.

(iii) Substituting $x = 1$ on the LHS of the given equation, we get

$$\text{LHS} = 1^2 - 1 + 1 = 1 \neq \text{RHS}$$

So, $x = 1$ is not a solution of the given equation.
Similarly, $x = -1$ is not a solution of the given equation.

(iv) Substituting $x = \sqrt{2}$ on the LHS of the given equation, we get

$$\text{LHS} = (\sqrt{2})^2 + \sqrt{2} \times \sqrt{2} - 4 = 2 + 2 - 4 = 0 = \text{RHS}$$

So, $x = \sqrt{2}$ is solution of the given equation.

Substituting $x = -2\sqrt{2}$ on the LHS of the given equation, we get

$$\text{LHS} = (-2\sqrt{2})^2 + \sqrt{2} \times (-2\sqrt{2}) - 4 = 8 - 4 - 4 = 0 = \text{RHS}$$

So, $x = -2\sqrt{2}$ is also a solution of the given equation.

Type III ON DETERMINING AN UNKNOWN INVOLVED IN A QUADRATIC EQUATION WHEN ITS ROOT(S) IS (ARE) GIVEN

EXAMPLE 3 *In each of the following, determine the value of k for which the given value is a solution of the equation:*

(i) $kx^2 + 2x - 3 = 0, x = 2$ (ii) $3x^2 + 2kx - 3 = 0, x = -\frac{1}{2}$

(iii) $x^2 + 2ax - k = 0, x = -a$

SOLUTION (i) Since $x = 2$ is a root of the given equation. Therefore, it satisfies the equation

i.e. $k(2)^2 + 2 \times 2 - 3 = 0 \Rightarrow 4k + 4 - 3 = 0 \Rightarrow k = -1/4$

(ii) Since $x = -1/2$ is a root of the given equation. So, it satisfies the equation

i.e. $3\left(-\frac{1}{2}\right)^2 + 2k\left(-\frac{1}{2}\right) - 3 = 0 \Rightarrow \frac{3}{4} - k - 3 = 0 \Rightarrow k = -\frac{9}{4}$

(iii) Since $x = -a$ is a root of the equation $x^2 + 2ax - k = 0$

$\therefore$ $a^2 + 2a \times -a - k = 0 \Rightarrow k = -a^2$

EXAMPLE 4 *If one root of the quadratic equation* $2x^2 + kx - 6 = 0$ *is 2, find the value of k. Also, find the other root.* [CBSE 2002]

SOLUTION Since $x = 2$ is a root of the equation $2x^2 + kx - 6 = 0$.

$\therefore$ $2 \times 2^2 + 2k - 6 = 0 \Rightarrow 8 + 2k - 6 = 0 \Rightarrow 2k + 2 = 0 \Rightarrow k = -1$

Putting $k = -1$ in the equation $2x^2 + kx - 6 = 0$, we get

$$2x^2 - x - 6 = 0$$
$$\Rightarrow \quad 2x^2 - 4x + 3x - 6 = 0$$
$$\Rightarrow \quad 2x(x - 2) + 3(x - 2) = 0$$
$$\Rightarrow \quad (x - 2)(2x + 3) = 0$$
$$\Rightarrow \quad x - 2 = 0, 2x + 3 = 0 \Rightarrow x = 2, x = -3/2$$

Hence, the other root is $-3/2$.

LEVEL-2

EXAMPLE 5 *If $x = 2$ and $x = 3$ are roots of the equation $3x^2 - 2kx + 2m = 0$, find the value of k and m.*

SOLUTION It is given that $x = 2$ and $x = 3$ are roots of the equation $3x^2 - 2kx + 2m = 0$.

$\therefore \quad 3 \times 2^2 - 2k \times 2 + 2m = 0$ and $3 \times 3^2 - 2k \times 3 + 2m = 0$

$\Rightarrow \quad 12 - 4k + 2m = 0$ and $27 - 6k + 2m = 0$

$\Rightarrow \quad 12 = 4k - 2m$ and $27 = 6k - 2m$

Solving these two equations, we get $k = \frac{15}{2}$ and $m = 9$

EXERCISE 4.1

LEVEL-1

1. Which of the following are quadratic equations?

(i) $x^2 + 6x - 4 = 0$ (ii) $\sqrt{3}x^2 - 2x + \frac{1}{2} = 0$

(iii) $x^2 + \frac{1}{x^2} = 5$ (iv) $x - \frac{3}{x} = x^2$

(v) $2x^2 - \sqrt{3x} + 9 = 0$ (vi) $x^2 - 2x - \sqrt{x} - 5 = 0$

(vii) $3x^2 - 5x + 9 = x^2 - 7x + 3$ (viii) $x + \frac{1}{x} = 1$

(ix) $x^2 - 3x = 0$ (x) $\left(x + \frac{1}{x}\right)^2 = 3\left(x + \frac{1}{x}\right) + 4$

(xi) $(2x + 1)(3x + 2) = 6(x - 1)(x - 2)$ (xii) $x + \frac{1}{x} = x^2, x \neq 0$

(xiii) $16x^2 - 3 = (2x + 5)(5x - 3)$ (xiv) $(x + 2)^3 = x^3 - 4$

(xv) $x(x + 1) + 8 = (x + 2)(x - 2)$

2. In each of the following, determine whether the given values are solutions of the given equation or not:

(i) $x^2 - 3x + 2 = 0, x = 2, x = -1$ (ii) $x^2 + x + 1 = 0, x = 0, x = 1$

(iii) $x^2 - 3\sqrt{3}x + 6 = 0, x = \sqrt{3}, x = -2\sqrt{3}$ (iv) $x + \frac{1}{x} = \frac{13}{6}, x = \frac{5}{6}, x = \frac{4}{3}$

(v) $2x^2 - x + 9 = x^2 + 4x + 3, x = 2, x = 3$ (vi) $x^2 - \sqrt{2}x - 4 = 0, x = -\sqrt{2}, x = -2\sqrt{2}$

(vii) $a^2x^2 - 3abx + 2b^2 = 0, x = a/b, x = b/a$

3. In each of the following, find the value of k for which the given value is a solution of the given equation:

(i) $7x^2 + kx - 3 = 0, x = 2/3$ (ii) $x^2 - x(a + b) + k = 0, x = a$

(iii) $kx^2 + \sqrt{2}x - 4 = 0, x = \sqrt{2}$ (iv) $x^2 + 3ax + k = 0, x = -a$

4. Determine, if 3 is a root of the equation given below:

$$\sqrt{x^2 - 4x + 3} + \sqrt{x^2 - 9} = \sqrt{4x^2 - 14x + 16}$$

LEVEL-2

5. If $x = 2/3$ and $x = -3$ are the roots of the equation $ax^2 + 7x + b = 0$, find the values of a and b. [CBSE 2016]

ANSWERS

1. (i), (ii), (vii), (viii), (ix), (xiii), (xiv)
2. (i) $x = 2$ is a solution but $x = -1$ is not a solution
   (ii) $x = 0$ and $x = 1$ are not solutions
   (iii) $x = \sqrt{3}$ is a solution but $x = -2\sqrt{3}$ is not a solution
   (iv) $x = \frac{5}{6}$ and $x = \frac{4}{3}$ are not solutions.
   (v) $x = 2$ and $x = 3$ are solutions
   (vi) $x = -\sqrt{2}$ is a solution but $x = -2\sqrt{2}$ is not a solution
   (vii) $x = \frac{b}{a}$ is a solution but $x = \frac{a}{b}$ is not a solution
3. (i) $k = -\frac{1}{6}$ (ii) $k = ab$ (iii) $k = 1$ (iv) $k = 2a^2$
4. $x = 3$ is not a root of the given equation
5. $a = 3, b = -6$

4.3 FORMULATION OF QUADRATIC EQUATIONS

Quadratic equations arise in several situations in the world around us and in different fields of mathematics. Following examples will illustrate the formulation of quadratic equations.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *The product of two consecutive positive integers is 240. Formulate the quadratic equation whose roots are these integers.*

SOLUTION Let two consecutive positive integers be x and $x + 1$. Then, their product is $x(x + 1)$.

It is given that the product is 240.

$\therefore \quad x(x + 1) = 240 \Rightarrow x^2 + x - 240 = 0$

This is the required quadratic equation.

EXAMPLE 2 *The area of a rectangular plot is 528 m^2. The length of the plot (in metres) is one more than twice its breadth. Formulate the quadratic equation to determine the length and breadth of the plot.* [NCERT]

SOLUTION Let the breadth of the plot be x metres.

It is given that the length of the plot is one more than twice its breadth.

$\therefore \quad$ Length $= (2x + 1)$ metres

Now, Area of the plot $= 528 \text{ m}^2$

$\Rightarrow$ Length $\times$ Breadth $= 528 \text{ m}^2$

$\Rightarrow (2x + 1) \times x = 528 \Rightarrow 2x^2 + x - 528 = 0$

This is the required quadratic equation.

EXAMPLE 3 *A two digit number is such that the product of the digits is 12. When 36 is added to the number the digits interchange their places. Formulate the quadratic equation whose root(s) is (are) digit(s) of the number.*

SOLUTION Let the ten's digit of the number be x.

It is given that the product of digits is 12.

$$\therefore \quad \text{Unit's digit} = \frac{12}{x}$$

$$\therefore \quad \text{Number} = 10x + \frac{12}{x}$$

If 36 is added to the number the digits interchange their places.

$$\therefore \quad 10x + \frac{12}{x} + 36 = 10 \times \frac{12}{x} + x$$

$$\Rightarrow \quad 10x + \frac{12}{x} + 36 = \frac{120}{x} + x$$

$$\Rightarrow \quad 9x - \frac{108}{x} + 36 = 0$$

$$\Rightarrow \quad 9x^2 - 108 + 36x = 0$$

$$\Rightarrow \quad x^2 + 4x - 12 = 0 \qquad \text{[Dividing throughout by 9]}$$

Hence, required quadratic equation is $x^2 + 4x - 12 = 0$.

EXAMPLE 4 *Rohan's mother is 26 years older than him. The product of their ages 3 years from now will be 360. Formulate the quadratic equation to find their ages.* [NCERT]

SOLUTION Let Rohan's present age be x years. Then, his mother's age is $(x + 26)$ years.

Rohan's age after 3 years = $(x + 3)$ years

After 3 years the age of Rohan's mother $= (x + 26 + 3)$ years $= (x + 29)$ years.

It is given that after 3 years from now, the product of Rohan's and his mother's ages will be 360 years.

$$\therefore \quad (x + 3)(x + 29) = 360 \Rightarrow x^2 + 32x - 273 = 0$$

This is the required equation.

LEVEL-2

EXAMPLE 5 *A train travels a distance of 480 km at a uniform speed. If the speed had been 8 km/hr less, then it would have taken 3 hours more to cover the same distance. Formulate the quadratic equation in terms of the speed of the train.* [NCERT]

SOLUTION Let the speed of the train be x km/hr. Then,

Time taken to travel a distance of 480 km $= \frac{480}{x}$ hr

Also,

Time taken by the train to travel a distance of 480 km with the speed $(x-8)$ km/hr $= \frac{480}{x-8}$ hr

It is given that if the speed had been 8 km/hr less, then the train would have taken 3 hours more to cover the same distance

$$\therefore \quad \frac{480}{x-8} = \frac{480}{x} + 3$$

$\Rightarrow \quad \frac{480}{x-8} - \frac{480}{x} = 3$

$\Rightarrow \quad \frac{480\,(x - x + 8)}{x\,(x-8)} = 3$

$\Rightarrow \quad \frac{480 \times 8}{x\,(x-8)} = 3$

$\Rightarrow \quad 3x\,(x-8) = 480 \times 8$

$\Rightarrow \quad x\,(x-8) = 160 \times 8 \Rightarrow x^2 - 8x - 1280 = 0,$ which is the required equation.

EXAMPLE 6 *Sum of the areas of two squares is 468 m^2. If the difference of their perimeters is 24 m, formulate the quadratic equation to find the sides of the two squares.*

SOLUTION Let the length of each side of a square be x metres. Then, its perimeter is $4x$.

It is given that the difference of the perimeters of two squares is 24 m.

$\therefore \quad$ Perimeter of second square $= 24 + 4x$ metres

$\Rightarrow \quad$ Length of each side of second square $= \frac{24 + 4x}{4}$ metres $= (6 + x)$ metres

It is given that the sum of the areas of two squares is 468 m^2.

$\therefore \quad x^2 + (6 + x)^2 = 468$

$\Rightarrow \quad x^2 + (36 + 12x + x^2) = 468$

$\Rightarrow \quad 2x^2 + 12x - 432 = 0$

$\Rightarrow \quad x^2 + 6x - 216 = 0$

This is the required equation.

EXAMPLE 7 *Two water taps together can fill a tank in $9\frac{3}{8}$ hours. The larger takes 10 hours less than the smaller one to fill the tank separately. Formulate the quadratic equation to find the time in which each tap can separately fill the tank.* [CBSE 2016]

SOLUTION Suppose the smaller tap fills the tank in x hours. Then, the larger tap will fill the tank in $(x - 10)$ hours.

Since the smaller tap takes x hours to fill the tank.

$\therefore \quad$ Portion of the tank filled by the smaller tank in one hour $= \frac{1}{x}$

$\Rightarrow \quad$ Portion of the tank filled by the smaller tap in $9\frac{3}{8}$ hours i.e., in $\frac{75}{8}$ hours

$$= \frac{75}{8} \times \frac{1}{x} = \frac{75}{8x}$$

Similarly, we have,

Portion of the tank filled by the larger tap in $\frac{75}{8}$ hours $= \left(\frac{75}{8} \times \frac{1}{x-10}\right)$

$$= \frac{75}{8\,(x-10)}$$

It is given that the two taps fill the tank in $\frac{75}{8}$ hours.

$$\therefore \quad \frac{75}{8x}+\frac{75}{8(x-10)}=1$$

$$\Rightarrow \quad \frac{1}{x}+\frac{1}{x-10}=\frac{8}{75}$$

$$\Rightarrow \quad \frac{x-10+x}{x(x-10)}=\frac{8}{75}$$

$$\Rightarrow \quad \frac{2x-10}{x^2-10x}=\frac{8}{75}$$

$$\Rightarrow \quad (2x-10)\times 75 = 8(x^2-10x)$$

$$\Rightarrow \quad 150x-750=8x^2-80x \Rightarrow 8x^2-230x+750=0 \Rightarrow 4x^2-115x+375=0$$

This is the required quadratic equation.

EXERCISE 4.2

LEVEL-1

1. The product of two consecutive positive integers is 306. Form the quadratic equation to find the integers, if x denotes the smaller integer. [NCERT]
2. John and Jivanti together have 45 marbles. Both of them lost 5 marbles each, and the product of the number of marbles they now have is 128. Form the quadratic equation to find how many marbles they had to start with, if John had x marbles.
3. A cottage industry produces a certain number of toys in a day. The cost of production of each toy (in rupees) was found to be 55 minus the number of articles produced in a day. On a particular day, the total cost of production was ₹ 750. If x denotes the number of toys produced that day, form the quadratic equation fo find x. [NCERT]
4. The height of a right triangle is 7 cm less than its base. If the hypotenuse is 13 cm, form the quadratic equation to find the base of the triangle. [NCERT]

LEVEL-2

5. An express train takes 1 hour less than a passenger train to travel 132 km between Mysore and Bangalore. If the average speed of the express train is 11 km/hr more than that of the passenger train, form the quadratic equation to find the average speed of express train.
6. A train travels 360 km at a uniform speed. If the speed had been 5 km/hr more, it would have taken 1 hour less for the same journey. Form the quadratic equation to find the speed of the train.

ANSWERS

1. $x^2+x-306=0$ 2. $x^2-45x+324=0$ 3. $x^2-55x+750=0$

4. $x^2-7x-60=0$ 5. $x^2+11x-1452=0$ 6. $x^2+5x-1800=0$

4.4 SOLUTION OF A QUADRATIC EQUATION BY FACTORIZATION METHOD

In earlier class, we have learnt how to factorize quadratic and other simple polynomials. In this section, we will apply the method of factorization to solve simple quadratic equations.

Let the quadratic equation be $ax^2+bx+c=0, a \neq 0$.

Let the quadratic polynomial $ax^2 + bx + c$ be expressible as the product of two linear factors, say $(px + q)$ and $(rx + s)$, where p, q, r, s are real numbers such that $p \neq 0$ and $r \neq 0$. Then,

$$ax^2 + bx + c = 0$$

$$\Rightarrow \quad (px + q)(rx + s) = 0$$

$$\Rightarrow \quad px + q = 0 \text{ or, } rx + s = 0$$

Solving these linear equations, we get the possible roots of the given quadratic equation as

$$x = -\frac{p}{q} \text{ and } x = -\frac{s}{r}$$

Following examples will illustrate the above procedure for solving quadratic equations.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Solve the following quadratic equations by factorization:*

(i) $x^2 + 6x + 5 = 0$ (ii) $8x^2 - 22x - 21 = 0$ (iii) $9x^2 - 3x - 2 = 0$

SOLUTION (i) We have,

$$x^2 + 6x + 5 = 0$$

$$\Rightarrow \quad x^2 + 5x + x + 5 = 0$$

$$\Rightarrow \quad x(x + 5) + (x + 5) = 0$$

$$\Rightarrow \quad (x + 5)(x + 1) = 0 \Rightarrow x + 5 = 0 \text{ or, } x + 1 = 0 \Rightarrow x = -5 \text{ or, } x = -1$$

Thus, $x = -5$ and $x = -1$ are two roots of the equation $x^2 + 6x + 5 = 0$

(ii) We have,

$$8x^2 - 22x - 21 = 0$$

$$\Rightarrow \quad 8x^2 - 28x + 6x - 21 = 0$$

$$\Rightarrow \quad 4x(2x - 7) + 3(2x - 7) = 0$$

$$\Rightarrow \quad (2x - 7)(4x + 3) = 0 \Rightarrow 2x - 7 = 0 \text{ or, } 4x + 3 = 0 \Rightarrow x = \frac{7}{2} \text{ or, } x = -\frac{3}{4}$$

Thus, $x = \frac{7}{2}$ and $x = -\frac{3}{4}$ are two roots of the equation $8x^2 - 22x - 21 = 0$

(iii) We have,

$$9x^2 - 3x - 2 = 0$$

$$\Rightarrow \quad 9x^2 - 6x + 3x - 2 = 0$$

$$\Rightarrow \quad 3x(3x - 2) + (3x - 2) = 0$$

$$\Rightarrow \quad (3x - 2)(3x + 1) = 0$$

$$\Rightarrow \quad 3x - 2 = 0 \text{ or, } 3x + 1 = 0 \Rightarrow x = \frac{2}{3} \text{ or, } x = -\frac{1}{3}$$

Thus, $x = \frac{2}{3}$ and $x = -\frac{1}{3}$ are two roots of the equation $9x^2 - 3x - 2 = 0$

EXAMPLE 2 *Solve the following quadratic equations by factorization method:*

(i) $x^2 - 9 = 0$ (ii) $x^2 - 8x + 16 = 0$

SOLUTION (i) We have,

$$x^2 - 9 = 0$$

$$\Rightarrow \quad (x-3)(x+3) = 0$$

$$\Rightarrow \quad x - 3 = 0 \text{ or, } x + 3 = 0 \Rightarrow x = 3 \text{ or, } x = -3 \Rightarrow x = \pm 3$$

Thus, $x = 3$ and $x = -3$ are roots of the given equation.

(ii) We have,

$$x^2 - 8x + 16 = 0 \Rightarrow (x-4)^2 = 0 \Rightarrow x = 4, x = 4$$

Thus, both the roots of the given equation are equal and are equal to 4.

EXAMPLE 3 *Solve the following quadratic equations by factorization method:*

(i) $\frac{x}{x+1} + \frac{x+1}{x} = \frac{34}{15}, x \neq 0, x \neq -1$ (ii) $\frac{x+3}{x-2} - \frac{1-x}{x} = \frac{17}{4}$

(iii) $\frac{1}{x-2} + \frac{2}{x-1} = \frac{6}{x}$ [CBSE 2013]

SOLUTION (i) We have,

$$\frac{x}{x+1} + \frac{x+1}{x} = \frac{34}{15}$$

$$\Rightarrow \quad \frac{x^2 + (x+1)^2}{x(x+1)} = \frac{34}{15}$$

$$\Rightarrow \quad \frac{x^2 + x^2 + 2x + 1}{x^2 + x} = \frac{34}{15}$$

$$\Rightarrow \quad 34x^2 + 34x = 15x^2 + 15x^2 + 30x + 15$$

$$\Rightarrow \quad 4x^2 + 4x - 15 = 0$$

$$\Rightarrow \quad 4x^2 + 10x - 6x - 15 = 0$$

$$\Rightarrow \quad 2x(2x+5) - 3(2x+5) = 0$$

$$\Rightarrow \quad (2x-3)(2x+5) = 0 \Rightarrow 2x - 3 = 0 \text{ or, } 2x + 5 = 0 \Rightarrow x = \frac{3}{2} \text{ or, } x = -\frac{5}{2}.$$

(ii) We have,

$$\frac{x+3}{x-2} - \frac{1-x}{x} = \frac{17}{4}$$

$$\Rightarrow \quad \frac{x(x+3) - (1-x)(x-2)}{x(x-2)} = \frac{17}{4}$$

$$\Rightarrow \quad \frac{x^2 + 3x - (x - 2 - x^2 + 2x)}{x^2 - 2x} = \frac{17}{4}$$

$$\frac{2x^2 + 2}{x^2 - 2x} = \frac{17}{4}$$

$$\Rightarrow \quad 8x^2 + 8 = 17x^2 - 34x$$

$\Rightarrow \quad 9x^2 - 34x - 8 = 0$

$\Rightarrow \quad 9x^2 - 36x + 2x - 8 = 0$

$\Rightarrow \quad 9x(x-4) + 2(x-4) = 0$

$\Rightarrow \quad (x-4)(9x+2) = 0 \Rightarrow x - 4 = 0 \text{ or, } 9x + 2 = 0 \Rightarrow x = 4 \text{ or, } x = -\frac{2}{9}$

(iii) We have,

$$\frac{1}{x-2} + \frac{2}{x-1} = \frac{6}{x}$$

$$\Rightarrow \quad \frac{(x-1) + 2(x-2)}{(x-2)(x-1)} = \frac{6}{x}$$

$$\Rightarrow \quad \frac{3x-5}{x^2 - 3x + 2} = \frac{6}{x}$$

$\Rightarrow \quad 3x^2 - 5x = 6x^2 - 18x + 12$

$\Rightarrow \quad 3x^2 - 13x + 12 = 0$

$\Rightarrow \quad 3x^2 - 9x - 4x + 12 = 0$

$\Rightarrow \quad 3x(x-3) - 4(x-3) = 0$

$\Rightarrow \quad (x-3)(3x-4) = 0 \Rightarrow x - 3 = 0 \text{ or, } 3x - 4 = 0 \Rightarrow x = 3 \text{ or, } x = \frac{4}{3}$

EXAMPLE 4 *Solve the following quadratic equations by factorization method:*

(i) $\frac{4}{x} - 3 = \frac{5}{2x+3}\ x \neq 0, \frac{-3}{2}$ [CBSE 2006C, 2014]

(ii) $\frac{2x}{x-3} + \frac{1}{2x+3} + \frac{3x+9}{(x-3)(2x+3)} = 0, x \neq 3, \frac{-3}{2}$ [CBSE 2016]

SOLUTION (i) We have,

$$\frac{4}{x} - 3 = \frac{5}{2x+3}$$

$$\Rightarrow \quad \frac{4-3x}{x} = \frac{5}{2x+3}$$

$\Rightarrow \quad (4-3x)(2x+3) = 5x$

$\Rightarrow \quad 12 - x - 6x^2 = 5x$

$\Rightarrow \quad 6x^2 + 6x - 12 = 0$

$\Rightarrow \quad x^2 + x - 2 = 0$

$\Rightarrow \quad x^2 + 2x - x - 2 = 0$

$\Rightarrow \quad x(x+2) - (x+2) = 0$

$\Rightarrow \quad (x+2)(x-1) = 0 \Rightarrow x + 2 = 0 \text{ or, } x - 1 = 0 \Rightarrow x = -2 \text{ or, } x = 1$

(ii) Clearly, the given equation is valid if $x - 3 \neq 0$ and $2x + 3 \neq 0$ i.e. when $x \neq \frac{-3}{2}, 3$.

Now, $\quad \frac{2x}{x-3} + \frac{1}{2x+3} + \frac{3x+9}{(x-3)(2x+3)} = 0$

$\Rightarrow \quad 2x(2x+3)+(x-3)+3x+9=0$ [Multiplying throughout by $(x-3)(2x+3)$]

$\Rightarrow \quad 4x^2+6x+x-3+3x+9=0$

$\Rightarrow \quad 4x^2+10x+6=0$

$\Rightarrow \quad 2x^2+5x+3=0$ $\left[\text{Multiplying through } \frac{1}{2}\right]$

$\Rightarrow \quad 2x^2+2x+3x+3=0$

$\Rightarrow \quad 2x(x+1)+3(x+1)=0$

$\Rightarrow \quad (2x+3)(x+1)=0$

$\Rightarrow \quad x+1=0$

$\Rightarrow \quad x=-1$ $[\because 2x+3 \neq 0]$

Hence, $x=-1$ is the only solution of the given equation.

ALITER $\quad \frac{2x}{x-3}+\frac{1}{2x+3}+\frac{2}{x-3}-\frac{1}{2x+3}=0$

$\Rightarrow \quad \frac{2(x+1)}{x-3}=0$

$\Rightarrow \quad x+1=0 \Rightarrow x=-1$

LEVEL-2

EXAMPLE 5 *Solve the following quadratic equations by factorization method:*

(i) $x^2+2\sqrt{2}x-6=0$ (ii) $\sqrt{3}x^2+10x+7\sqrt{3}=0$

SOLUTION (i) We have,

$x^2+2\sqrt{2}x-6=0$

$\Rightarrow \quad x^2+3\sqrt{2}x-\sqrt{2}x-6=0$

$\Rightarrow \quad x(x+3\sqrt{2})-\sqrt{2}(x+3\sqrt{2})=0$

$\Rightarrow \quad (x+3\sqrt{2})(x-\sqrt{2})=0 \Rightarrow x+3\sqrt{2}=0$ or, $x-\sqrt{2}=0 \Rightarrow x=-3\sqrt{2}$ or, $x=\sqrt{2}$

Thus, $x=-3\sqrt{2}$ and $x=\sqrt{2}$ are two roots of the given equation.

(ii) $\quad \sqrt{3}x^2+10x+7\sqrt{3}=0$

$\Rightarrow \quad \sqrt{3}x^2+3x+7x+7\sqrt{3}=0$

$\Rightarrow \quad \sqrt{3}x(x+\sqrt{3})+7(x+\sqrt{3})=0$

$\Rightarrow \quad (x+\sqrt{3})(\sqrt{3}x+7)=0 \Rightarrow x+\sqrt{3}=0$ or, $\sqrt{3}x+7=0$

$\Rightarrow \quad x=-\sqrt{3}$ or, $x=-7/\sqrt{3}$

Thus, $x=-\sqrt{3}$ and, $x=-\frac{7}{\sqrt{3}}$ are two roots of the given equation.

REMARK *In order to solve the quadratic equations in the following examples we may use the following algorithm:*

ALGORITHM

STEP I *Factorize the constant term of the given quadratic equation.*

STEP II *Express the coefficient of middle term as the sum or difference of the factors obtained in step I. Clearly, the product of these two factors will be equal to the product of the coefficient of x^2 and constant term.*

STEP III *Split the middle term in two parts obtained in step II.*

STEP IV *Factorize the quadratic equation obtained in step III by grouping method.*

EXAMPLE 6 *Solve the following quadratic equations by factorization method:*

(i) $x^2 - 2ax + a^2 - b^2 = 0$ (ii) $x^2 - 4ax + 4a^2 - b^2 = 0$

(iii) $4x^2 - 4ax + (a^2 - b^2) = 0$ [CBSE 2012] (iv) $4x^2 - 4a^2x + (a^4 - b^4) = 0$ [CBSE 2004, 2015]

SOLUTION (i) We have,

$$x^2 - 2ax + a^2 - b^2 = 0$$

Here, Factors of constant term $(a^2 - b^2)$ are $(a - b)$ and $(a + b)$.

Also, Coefficient of the middle term $= -2a = -\{(a - b) + (a + b)\}$

$\therefore$ $x^2 - 2ax + a^2 - b^2 = 0$

$\Rightarrow$ $x^2 - \{(a - b) + (a + b) x + (a - b)(a + b) = 0\}$

$\Rightarrow$ $x^2 - (a - b)x - (a + b)x + (a - b)(a + b) = 0$

$\Rightarrow$ $\{x^2 - (a - b)x\} - \{(a + b)x - (a - b)(a + b)\} = 0$

$\Rightarrow$ $x\{x - (a - b)\} - (a + b)\{x - (a - b)\} = 0$

$\Rightarrow$ $\{x - (a - b)\}\{x - (a + b)\} = 0$

$\Rightarrow$ $x - (a - b) = 0$ or, $x - (a + b) = 0 \Rightarrow x = a - b$ or, $x = a + b$

(ii) We have,

$$x^2 - 4ax + (4a^2 - b^2) = 0$$

Here, Constant term $= (4a^2 - b^2) = (2a - b)(2a + b)$

and, Coefficient of middle term $= -4a$

Also, Coefficient of middle term $= -\{(2a - b) + (2a + b)\}$

$\therefore$ $x^2 - 4ax + (4a^2 - b^2) = 0$

$\Rightarrow$ $x^2 - \{(2a - b) + (2a + b)\}x + (2a - b)(2a + b) = 0$

$\Rightarrow$ $x^2 - (2a - b)x - (2a + b)x + (2a - b)(2a + b) = 0$

$\Rightarrow$ $\{x^2 - (2a - b)x\} - \{(2a + b)x - (2a - b)(2a + b)\} = 0$

$\Rightarrow$ $x\{x - (2a - b)\} - (2a + b)\{x - (2a - b)\} = 0$

$\Rightarrow$ $\{x - (2a - b)\}\{x - (2a + b)\} = 0$

$\Rightarrow$ $x - (2a - b) = 0$ or, $x - (2a + b) = 0 \Rightarrow x = 2a - b$ or, $x = 2a + b$

(iii) We have,

$$4x^2 - 4ax + (a^2 - b^2) = 0$$

Here, Constant term $= (a^2 - b^2) = (a - b)(a + b)$

and, Coefficient of middle term $= -4a$

Also, Coefficient of the middle term $-4a = -\{2(a+b) + 2(a-b)\}$

$\therefore \quad 4x^2 - 4ax + (a^2 - b^2) = 0$

$\Rightarrow \quad 4x^2 - \{2(a+b) + 2(a-b)\}x + (a+b)(a-b) = 0$

$\Rightarrow \quad 4x^2 - 2(a+b)x - 2(a-b)x + (a+b)(a-b) = 0$

$\Rightarrow \quad \{4x^2 - 2(a+b)x\} - \{2(a-b)x - (a+b)(a-b)\} = 0$

$\Rightarrow \quad 2x\{2x - (a+b)\} - (a-b)\{2x - (a+b)\} = 0$

$\Rightarrow \quad \{2x - (a+b)\}\{2x - (a-b)\} = 0$

$\Rightarrow \quad \{2x - (a+b)\} = 0$ or, $\{2x - (a-b)\} = 0$

$\Rightarrow \quad 2x = a + b$ or, $2x = a - b \Rightarrow x = \frac{a+b}{2}$ or, $x = \frac{a-b}{2}$

(iv) We have,

$4x^2 - 4a^2x + (a^4 - b^4) = 0$

Here, Constant term $= a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$

and, Coefficient of middle term $= -4a^2$

Also, Coefficient of the middle term $-4a^2 = -\{2(a^2 + b^2) + 2(a^2 - b^2)\}$

$\therefore \quad 4x^2 - 4a^2x + (a^4 - b^4) = 0$

$\Rightarrow \quad 4x^2 - \{2(a^2 + b^2) + 2(a^2 - b^2)\}x + (a^2 - b^2)(a^2 + b^2) = 0$

$\Rightarrow \quad 4x^2 - 2(a^2 + b)x - 2(a^2 - b^2)x + (a^2 - b^2)(a^2 + b^2) = 0$

$\Rightarrow \quad \{4x^2 - 2(a^2 + b^2)x\} - \{2(a^2 - b^2)x - (a^2 - b^2)(a^2 + b^2)\} = 0$

$\Rightarrow \quad 2x\{2x - (a^2 + b^2)\} - (a^2 - b^2)\{2x - (a^2 + b^2)\} = 0$

$\Rightarrow \quad \{2x - (a^2 + b^2)\}\{2x - (a^2 - b^2)\} = 0$

$\Rightarrow \quad 2x - (a^2 + b^2) = 0$ or, $2x - (a^2 - b^2) = 0 \Rightarrow x = \frac{a^2 + b^2}{2}$ or, $x = \frac{a^2 - b^2}{2}$

EXAMPLE 7 *Solve the following quadratic equations by factorization method:*

(i) $4x^2 - 2(a^2 + b^2)x + a^2b^2 = 0$ [CBSE 2004]

(ii) $9x^2 - 9(a+b)x + (2a^2 + 5ab + 2b^2) = 0$ [CBSE 2004, 2009, 2016]

SOLUTION (i) We have,

$4x^2 - 2(a^2 + b^2)x + a^2b^2 = 0$

Here, Constant term $= a^2b^2 = a^2 \times b^2$

and, Coefficient of middle term $= -2(a^2 + b^2)$

$\therefore \quad 4x^2 - 2(a^2 + b^2)x + a^2b^2 = 0$

$\Rightarrow \quad 4x^2 - 2a^2x - 2b^2x + a^2b^2 = 0$

$\Rightarrow \quad (4x^2 - 2a^2x) - (2b^2x - a^2b^2) = 0$

$\Rightarrow \quad 2x(2x - a^2) - b^2(2x - a^2) = 0$

$\Rightarrow \quad (2x - a^2)(2x - b^2) = 0$

$\Rightarrow \quad (2x - a^2) = 0$ or, $(2x - b^2) = 0 \Rightarrow x = \frac{a^2}{2}$ or, $x = \frac{b^2}{2}$

(ii) We have,

$$9x^2 - 9(a+b)x + (2a^2 + 5ab + 2b^2) = 0$$

Here, Constant term $= 2a^2 + 5ab + 2b^2$

$= 2a^2 + 4ab + ab + 2b^2$

$= 2a(a+2b) + b(a+2b) = (2a+b)(a+2b)$

and, Coefficient of middle term $= -9(a+b) = -3\{(2a+b) + (a+2b)\}$

$\therefore \quad 9x^2 - 9(a+b)x + (2a^2 + 5ab + 2b^2) = 0$

$\Rightarrow \quad 9x^2 - 3\{(2a+b) + (a+2b)\}x + (2a+b)(a+2b) = 0$

$\Rightarrow \quad 9x^2 - 3(2a+b)x - 3(a+2b)x + (2a+b)(a+2b) = 0$

$\Rightarrow \quad 3x\{3x - (2a+b)\} - (a+2b)\{3x - (2a+b)\} = 0$

$\Rightarrow \quad \{3x - (2a+b)\}\{3x - (a+2b)\} = 0$

$\Rightarrow \quad \{3x - (2a+b)\} = 0$ or, $\{3x - (a+2b)\} = 0 \Rightarrow x = \frac{2a+b}{3}$ or, $x = \frac{a+2b}{3}$

EXAMPLE 8 *Solve the following quadratic equations by factorization method:*

(i) $x^2 + \left(\frac{a}{a+b} + \frac{a+b}{a}\right)x + 1 = 0$ (ii) $x^2 + x - (a+1)(a+2) = 0$

(iii) $x^2 + 3x - (a^2 + a - 2) = 0$ (iv) $a^2b^2x^2 + b^2x - a^2x - 1 = 0$

[CBSE 2005]

SOLUTION (i) We have,

$$x^2 + \left(\frac{a}{a+b} + \frac{a+b}{a}\right)x + 1 = 0$$

$\Rightarrow \quad x^2 + \frac{ax}{a+b} + \frac{a+b}{a}x + \frac{a}{a+b} \times \frac{a+b}{a} = 0$

$\Rightarrow \quad x\left\{x + \frac{a}{a+b}\right\} + \frac{a+b}{a}\left\{x + \frac{a}{a+b}\right\} = 0$

$\Rightarrow \quad \left\{x + \frac{a}{a+b}\right\}\left\{x + \frac{a+b}{a}\right\} = 0$

$\Rightarrow \quad x + \frac{a}{a+b} = 0$ or, $x + \frac{a+b}{a} = 0 \Rightarrow x = -\frac{a}{a+b}$ or, $x = -\frac{a+b}{a}$

(ii) We have,

$$x^2 + x - (a+1)(a+2) = 0$$

$\Rightarrow$ $x^2 + x\{(a+2) - (a+1)\} - (a+1)(a+2) = 0$

$\Rightarrow$ $\{x^2 + x(a+2)\} - x(a+1) - (a+1)(a+2) = 0$

$\Rightarrow$ $x\{x + (a+2)\} - (a+1)\{x + (a+2)\} = 0$

$\Rightarrow$ $\{x + (a+2)\}\{x - (a+1)\} = 0$

$\Rightarrow$ $x + (a+2) = 0$ or, $x - (a+1) = 0 \Rightarrow x = -(a+2)$ or, $x = (a+1)$

(iii) We have,

$$x^2 + 3x - (a^2 + a - 2) = 0$$

$\Rightarrow$ $x^2 + 3x - (a+2)(a-1) = 0$

$\Rightarrow$ $x^2 + \{(a+2) - (a-1)\}x - (a+2)(a-1) = 0$

$\Rightarrow$ $\{x^2 + (a+2)x\} - (a-1)x - (a+2)(a-1) = 0$

$\Rightarrow$ $x\{x + (a+2)\} - (a-1)\{x + (a+2)\} = 0$

$\Rightarrow$ $\{x + (a+2)\}\{x - (a-1)\} = 0$

$\Rightarrow$ $x + (a+2) = 0$ or, $x - (a-1) = 0 \Rightarrow x = -(a+2)$ or, $x = a - 1$

(iv) We have,

$\Rightarrow$ $a^2b^2x^2 + b^2x - a^2x - 1 = 0$

$\Rightarrow$ $(a^2b^2x^2 + b^2x) - (a^2x + 1) = 0$

$\Rightarrow$ $(a^2x + 1)b^2x - (a^2x + 1) = 0$

$\Rightarrow$ $(a^2x + 1)(b^2x - 1) = 0$

$\Rightarrow$ $a^2x + 1 = 0, b^2x - 1 = 0$

$\Rightarrow$ $a^2x = -1, b^2x = 1 \Rightarrow x = -\dfrac{1}{a^2}, x = \dfrac{1}{b^2}$

EXAMPLE 9 *Solve the following quadratic equations by factorization method:*

$$\frac{1}{a+b+x} = \frac{1}{a} + \frac{1}{b} + \frac{1}{x}, a + b \neq 0$$

[CBSE 2005]

SOLUTION We have,

$$\frac{1}{a+b+x} - \frac{1}{x} = \frac{1}{a} + \frac{1}{b}$$

$\Rightarrow$ $\dfrac{x - (a+b+x)}{x(a+b+x)} = \dfrac{a+b}{ab}$

$\Rightarrow$ $\dfrac{-(a+b)}{x(a+b+x)} = \dfrac{a+b}{ab}$

$\Rightarrow$ $-ab(a+b) = (a+b)x(a+b+x)$

$\Rightarrow \quad (a+b)\{x(a+b+x)+ab\}=0$

$\Rightarrow \quad x(a+b+x)+ab=0 \qquad [\because a+b \neq 0]$

$\Rightarrow \quad x^2+ax+bx+ab=0$

$\Rightarrow \quad x(x+a)+b(x+a)=0$

$\Rightarrow \quad (x+a)(x+b)=0 \Rightarrow x+a=0$ or, $x+b=0 \Rightarrow x=-a$ or, $x=-b$

LEVEL-3

EXAMPLE 10 *Solve:*

$$x=\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-x}}}, x \neq 2$$

SOLUTION We have,

$$x=\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-x}}}$$

$$\Rightarrow \quad x=\cfrac{1}{2-\cfrac{1}{\cfrac{2(2-x)-1}{2-x}}}$$

$$\Rightarrow \quad x=\cfrac{1}{2-\cfrac{2-x}{4-2x-1}}$$

$$\Rightarrow \quad x=\cfrac{1}{2-\cfrac{2-x}{3-2x}}$$

$$\Rightarrow \quad x=\frac{3-2x}{2(3-2x)-(2-x)}$$

$$\Rightarrow \quad x=\frac{3-2x}{4-3x}$$

$\Rightarrow \quad x(4-3x)=(3-2x)$

$\Rightarrow \quad 4x-3x^2=3-2x$

$\Rightarrow \quad 3x^2-6x+3=0 \Rightarrow x^2-2x+1=0 \Rightarrow (x-1)^2=0 \Rightarrow x=1,1$

EXAMPLE 11 *Solve:*

(i) $x+\frac{1}{x}=25\frac{1}{25}$ (ii) $(x-3)(x-4)=\frac{34}{(33)^2}$

SOLUTION (i) We have,

$$x + \frac{1}{x} = 25\frac{1}{25}$$

$$\Rightarrow \quad x + \frac{1}{x} = 25 + \frac{1}{25}$$

$$\Rightarrow \quad \frac{x^2 + 1}{x} = \left(25 + \frac{1}{25}\right)$$

$$\Rightarrow \quad x^2 + 1 = \left(25 + \frac{1}{25}\right)x$$

$$\Rightarrow \quad x^2 - \left(25 + \frac{1}{25}\right)x + 1 = 0$$

$$\Rightarrow \quad x^2 - \left(25 + \frac{1}{25}\right)x + 25 \times \frac{1}{25} = 0$$

$$\Rightarrow \quad x^2 - 25x - \frac{1}{25}x + 25 \times \frac{1}{25} = 0$$

$$\Rightarrow \quad (x^2 - 25x) - \left(\frac{x}{25} - 25 \times \frac{1}{25}\right) = 0$$

$$\Rightarrow \quad x(x - 25) - \frac{1}{25}(x - 25) = 0$$

$$\Rightarrow \quad (x - 25)\left(x - \frac{1}{25}\right) = 0$$

$$\Rightarrow \quad x - 25 = 0 \text{ or, } x - \frac{1}{25} = 0 \Rightarrow x = 25 \text{ or, } x = \frac{1}{25}$$

ALITER We have,

$$x + \frac{1}{x} = 25\frac{1}{25}$$

$$\Rightarrow \quad x + \frac{1}{x} = 25 + \frac{1}{25} \Rightarrow x = 25, \text{ or, } x = \frac{1}{25}$$ [By comparing two sides]

(ii) We have,

$$(x - 3)(x - 4) = \frac{34}{33^2}$$

$$\Rightarrow \quad x^2 - 7x + 12 - \frac{34}{33^2} = 0$$

$$\Rightarrow \quad x^2 - 7x + \frac{13034}{33^2} = 0$$

$$\Rightarrow \quad x^2 - 7x + \frac{98}{33} \times \frac{133}{33} = 0$$

$$\Rightarrow \quad x^2 - \frac{231}{33}x + \frac{98}{33} \times \frac{133}{33} = 0$$

$\Rightarrow \quad x^2 - \left(\frac{98}{33} + \frac{133}{33}\right)x + \frac{98}{33} \times \frac{133}{33} = 0$

$\Rightarrow \quad x^2 - \frac{98}{33}x - \frac{133}{33}x + \frac{98}{33} \times \frac{133}{33} = 0$

$\Rightarrow \quad \left(x^2 - \frac{98}{33}x\right) - \left(\frac{133}{33}x - \frac{98}{33} \times \frac{133}{33}\right) = 0$

$\Rightarrow \quad x\left(x - \frac{98}{33}\right) - \frac{133}{33}\left(x - \frac{98}{33}\right) = 0$

$\Rightarrow \quad \left(x - \frac{98}{33}\right)\left(x - \frac{133}{33}\right) = 0 \Rightarrow x = \frac{98}{33} \text{ or, } x = \frac{133}{33}$

EXERCISE 4.3

LEVEL-1

Solve the following quadratic equations by factorization:

1. $(x-4)(x+2) = 0$
2. $(2x+3)(3x-7) = 0$
3. $3x^2 - 14x - 5 = 0$
4. $9x^2 - 3x - 2 = 0$
5. $\frac{1}{x-1} - \frac{1}{x+5} = \frac{6}{7}, x \neq 1, -5$ [CBSE 2010]
6. $6x^2 + 11x + 3 = 0$
7. $5x^2 - 3x - 2 = 0$
8. $48x^2 - 13x - 1 = 0$
9. $3x^2 = -11x - 10$
10. $25x(x+1) = -4$
11. $16x - \frac{10}{x} = 27$ [CBSE 2014]
12. $\frac{1}{x} - \frac{1}{x-2} = 3, \; x \neq 0, 2$ [NCERT, CBSE 2010]
13. $x - \frac{1}{x} = 3, \; x \neq 0$ [NCERT, CBSE 2010]
14. $\frac{1}{x+4} - \frac{1}{x-7} = \frac{11}{30}, \; x \neq 4, 7$ [NCERT]
15. $\frac{1}{x-3} + \frac{2}{x-2} = \frac{8}{x}, \; x \neq 0, 2, 3$ [CBSE 2013]
16. $a^2x^2 - 3abx + 2b^2 = 0$
17. $9x^2 - 6b^2x - (a^4 - b^4) = 0$ [CBSE 2015]
18. $4x^2 + 4bx - (a^2 - b^2) = 0$
19. $ax^2 + (4a^2 - 3b)x - 12ab = 0$
20. $2x^2 + ax - a^2 = 0$ [CBSE 2014]
21. $\frac{16}{x} - 1 = \frac{15}{x+1}, x \neq 0, -1$ [CBSE 2014]

22. $\frac{x+3}{x+2} = \frac{3x-7}{2x-3}$, $x \neq -2, \frac{3}{2}$ [CBSE 2017]

23. $\frac{2x}{x-4} + \frac{2x-5}{x-3} = \frac{25}{3}$, $x \neq 3, 4$

24. $\frac{x+3}{x-2} - \frac{1-x}{x} = \frac{17}{4}$, $x \neq 0, 2$ [CBSE 2017]

25. $\frac{x-3}{x+3} - \frac{x+3}{x-3} = \frac{48}{7}$, $x \neq 3, x \neq -3$ [CBSE 2015]

26. $\frac{1}{x-2} + \frac{2}{x-1} = \frac{6}{x}$, $x \neq 0$ [CBSE 2013]

27. $\frac{x+1}{x-1} - \frac{x-1}{x+1} = \frac{5}{6}$, $x \neq 1, -1$

28. $\frac{x-1}{2x+1} + \frac{2x+1}{x-1} = \frac{5}{2}$, $x \neq -\frac{1}{2}, 1$

29. $\frac{4}{x} - 3 = \frac{5}{2x+3}$, $x \neq 0, -\frac{3}{2}$ [CBSE 2013]

30. $\frac{x-4}{x-5} + \frac{x-6}{x-7} = \frac{10}{3}$; $x \neq 5, 7$ [CBSE 2014]

31. $\frac{x-2}{x-3} + \frac{x-4}{x-5} = \frac{10}{3}$; $x \neq 3, 5$ [CBSE 2014]

32. $\frac{5+x}{5-x} - \frac{5-x}{5+x} = 3\frac{3}{4}$; $x \neq 5, -5$ [CBSE 2014]

33. $\frac{3}{x+1} - \frac{1}{2} = \frac{2}{3x-1}$, $x \neq -1, \frac{1}{3}$ [CBSE 2014]

34. $\frac{3}{x+1} + \frac{4}{x-1} = \frac{29}{4x-1}$; $x \neq 1, -1, \frac{1}{4}$ [CBSE 2015]

35. $\frac{2}{x+1} + \frac{3}{2(x-2)} = \frac{23}{5x}$; $x \neq 0, -1, 2$ [CBSE 2015]

LEVEL-2

36. $x^2 - (\sqrt{3}+1)x + \sqrt{3} = 0$

37. $3\sqrt{5}x^2 + 25x - 10\sqrt{5} = 0$

38. $\sqrt{3}x^2 - 2\sqrt{2}x - 2\sqrt{3} = 0$ [CBSE 2015]

39. $4\sqrt{3}x^2 + 5x - 2\sqrt{3} = 0$ [CBSE 2014]

40. $\sqrt{2}x^2 - 3x - 2\sqrt{2} = 0$ [CBSE 2006C, 2013, 2016]

41. $x^2 - (\sqrt{2}+1)x + \sqrt{2} = 0$

42. $3x^2 - 2\sqrt{6}x + 2 = 0$ [NCERT, CBSE 2010, 2012]

43. $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$ [CBSE 2017]

44. $\frac{m}{n}x^2 + \frac{n}{m} = 1 - 2x$

45. $\frac{x-a}{x-b} + \frac{x-b}{x-a} = \frac{a}{b} + \frac{b}{a}$

46. $\frac{1}{(x-1)(x-2)}+\frac{1}{(x-2)(x-3)}+\frac{1}{(x-3)(x-4)}=\frac{1}{6}$

47. $\frac{a}{x-b}+\frac{b}{x-a}=2, x \neq a, b$ [CBSE 2016]

48. $\frac{x+1}{x-1}+\frac{x-2}{x+2}=4-\frac{2x+3}{x-2}; x \neq 1, -2, 2$ [CBSE 2016]

49. $\frac{a}{x-a}+\frac{b}{x-b}=\frac{2c}{x-c}$ [CBSE 2016]

50. $x^2+2ab=(2a+b)x$

51. $(a+b)^2x^2-4abx-(a-b)^2=0$

52. $a(x^2+1)-x(a^2+1)=0$

53. $x^2-x-a(a+1)=0$

54. $x^2+\left(a+\frac{1}{a}\right)x+1=0$

55. $abx^2+(b^2-ac)x-bc=0$ [CBSE 2005]

56. $a^2b^2x^2+b^2x-a^2x-1=0$ [CBSE 2005]

57. $\frac{x-1}{x-2}+\frac{x-3}{x-4}=3\frac{1}{3}, x \neq 2, 4$ [CBSE 2005]

58. $\frac{1}{2a+b+2x}=\frac{1}{2a}+\frac{1}{b}+\frac{1}{2x}$ [CBSE 2013]

59. $3\left(\frac{3x-1}{2x+3}\right)-2\left(\frac{2x+3}{3x-1}\right)=5; x \neq \frac{1}{3}, -\frac{3}{2}$ [CBSE 2014]

60. $3\left(\frac{7x+1}{5x-3}\right)-4\left(\frac{5x-3}{7x+1}\right)=11; x \neq \frac{3}{5}, -\frac{1}{7}$ [CBSE 2014]

LEVEL-3

61. $(x-5)(x-6)=\frac{25}{(24)^2}$

62. $7x+\frac{3}{x}=35\frac{3}{5}$

ANSWERS

1. $4, -2$	2. $-\frac{3}{2}, \frac{7}{3}$	3. $5, -\frac{1}{3}$	4. $\frac{2}{3}, \frac{-1}{3}$
5. $2, -6$	6. $-\frac{3}{2}, \frac{-1}{3}$	7. $\frac{-2}{5}, 1$	8. $\frac{-1}{16}, \frac{1}{3}$
9. $-\frac{5}{3}, -2$	10. $-\frac{4}{5}, -\frac{1}{5}$	11. $2, -\frac{5}{16}$	12. $\frac{3 \pm \sqrt{3}}{2}$
13. $\frac{3 \mp \sqrt{13}}{2}$	14. $1, 2$	15. $4, \frac{12}{5}$	16. $\frac{2b}{a}, \frac{b}{a}$
17. $\frac{a^2+b^2}{3}, \frac{b^2-a^2}{3}$	18. $\frac{a-b}{2}, -\frac{a+b}{2}$	19. $\frac{3b}{a}, -4a$	20. $\frac{a}{2}, -a$
21. ± 4	22. $-1, 5$	23. $6, \frac{40}{13}$	24. $4, \frac{-2}{9}$

25. $-4, \frac{9}{4}$
26. $3, \frac{4}{3}$
27. $5, \frac{-1}{5}$
28. -1
29. $-2, 1$
30. $8, \frac{11}{2}$
31. $6, \frac{7}{2}$
32. $3, -\frac{25}{3}$
33. $3, 1$
34. $4, -7$
35. $4, -\frac{23}{11}$
36. $\sqrt{3}, 1$
37. $-2\sqrt{5}, \frac{\sqrt{5}}{3}$
38. $\sqrt{6}, -\sqrt{\frac{2}{3}}$
39. $\frac{\sqrt{3}}{4}, \frac{-2}{\sqrt{3}}$
40. $-\frac{1}{\sqrt{2}}, 2\sqrt{2}$
41. $\sqrt{2}, 1$
42. $\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}$
43. $-\frac{5}{\sqrt{2}}, -\sqrt{2}$
44. $\frac{-n \pm \sqrt{mn}}{m}$
45. $0, a+b$
46. $-2, 7$
47. $\frac{a+b}{2}, a+b$
48. $-5, \frac{6}{5}$
49. $0, \frac{2ab-bc-ac}{a+b-2c}$
50. $2a, b$
51. $1, -\left(\frac{a-b}{a+b}\right)$
52. $a, \frac{1}{a}$
53. $-a, a+1$
54. $-a, -\frac{1}{a}$
55. $\frac{-b}{a}, \frac{c}{b}$
56. $-\frac{1}{a^2}, \frac{1}{b^2}$
57. $5, \frac{5}{2}$
58. $-a, \frac{-b}{2}$
59. $0, -7$
60. $0, 1$
61. $6\frac{1}{24}, 4\frac{23}{24}$
62. $5, \frac{3}{35}$

4.5 SOLUTION OF A QUADRATIC EQUATION BY COMPLETING THE SQUARE

In the previous section, we learnt about the factorization method to obtain the roots of a quadratic equation. In this section, we shall learn about the method of completing squares. We may use the following algorithm to obtain the roots of a quadratic equation by using the method of completing squares.

ALGORITHM

STEP I *Obtain the quadratic equation. Let the quadratic equation be* $ax^2 + bx + c = 0,\ a \neq 0$.

STEP II *Make the coefficient of* x^2 *unity by dividing throughout by it, if it is not unity. i.e., obtain*

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

STEP III *Shift the constant term* $\frac{c}{a}$ *on RHS to get* $x^2 + \frac{b}{a}x = -\frac{c}{a}$.

STEP IV *Add square of half of the coefficient of x i.e.* $\left(\frac{b}{2a}\right)^2$ *on both sides to obtain*

$$x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

STEP V *Write LHS as the perfect square of a binomial expression and simplify RHS to get*

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

STEP VI *Take square root of both sides to get* $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$

STEP VII *Obtain the values of x by shifting the constant term* $\frac{b}{2a}$ *on RHS.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Solve the quadratic equation* $9x^2 - 15x + 6 = 0$ *by the method of completing the square.* [NCERT]

SOLUTION We have, $9x^2 - 15x + 6 = 0$

$\Rightarrow \quad x^2 - \frac{15}{9}x + \frac{6}{9} = 0$ [Dividing throughout by 9]

$\Rightarrow \quad x^2 - \frac{5}{3}x + \frac{2}{3} = 0$

$\Rightarrow \quad x^2 - \frac{5}{3}x = -\frac{2}{3}$ [Shifting the constant term on RHS]

$\Rightarrow \quad x^2 - 2\left(\frac{5}{6}\right)x + \left(\frac{5}{6}\right)^2 = \left(\frac{5}{6}\right)^2 - \frac{2}{3}$ [Adding square of half of coefficient of x on both sides]

$\Rightarrow \quad \left(x - \frac{5}{6}\right)^2 = \frac{25}{36} - \frac{2}{3}$

$\Rightarrow \quad \left(x - \frac{5}{6}\right)^2 = \frac{25-24}{36}$

$\Rightarrow \quad \left(x - \frac{5}{6}\right)^2 = \frac{1}{36}$

$\Rightarrow \quad x - \frac{5}{6} = \pm\frac{1}{6}$ [Taking square root of both sides]

$\Rightarrow \quad x = \frac{5}{6} \pm \frac{1}{6}$

$\Rightarrow \quad x = \frac{5}{6} + \frac{1}{6} = 1$ or, $x = \frac{5}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3} \Rightarrow x = 1$ or, $x = 2/3$

Hence, the roots of the equation are 1 and 2/3.

EXAMPLE 2 *Solve the equation* $2x^2 - 5x + 3 = 0$ *by the method of completing square.* [NCERT]

SOLUTION We have, $2x^2 - 5x + 3 = 0$

$\Rightarrow \quad x^2 - \frac{5}{2}x + \frac{3}{2} = 0$ [Dividing throughout by 2]

$\Rightarrow \quad x^2 - \frac{5}{2}x = -\frac{3}{2}$ [Shifting the constant term on RHS]

$\Rightarrow \quad x^2 - 2\left(\frac{5}{4}\right)x + \left(\frac{5}{4}\right)^2 = \left(\frac{5}{4}\right)^2 - \frac{3}{2}$ $\left[\text{Adding}\left(\frac{1}{2}\text{Coeff. of } x\right)^2 \text{ on both sides}\right]$

$\Rightarrow \quad \left(x - \frac{5}{4}\right)^2 = \frac{25}{16} - \frac{3}{2}$

$\Rightarrow \quad \left(x - \frac{5}{4}\right)^2 = \frac{1}{16}$

$\Rightarrow \quad x - \frac{5}{4} = \pm\frac{1}{4} \Rightarrow x = \frac{5}{4} \pm \frac{1}{4}$

$\Rightarrow \quad x = \frac{5}{4} + \frac{1}{4} = \frac{6}{4}$ or, $x = \frac{5}{4} - \frac{1}{4} = \frac{4}{4} \Rightarrow x = \frac{3}{2}$ or, $x = 1$

Hence, the roots of the equation $2x^2 - 5x + 3 = 0$ are $\frac{3}{2}$ and 1.

EXAMPLE 3 *Find the roots of the equation* $5x^2 - 6x - 2 = 0$ *by the method of completing the square.* [NCERT]

SOLUTION We have, $5x^2 - 6x - 2 = 0$

$$\Rightarrow \quad x^2 - \frac{6}{5}x - \frac{2}{5} = 0 \quad \text{[Dividing throughout by 5]}$$

$$\Rightarrow \quad x^2 - \frac{6}{5}x = \frac{2}{5}$$

$$\Rightarrow \quad x^2 - 2\left(\frac{3}{5}\right)x + \left(\frac{3}{5}\right)^2 = \frac{2}{5} + \left(\frac{3}{5}\right)^2$$

$$\Rightarrow \quad \left(x - \frac{3}{5}\right)^2 = \frac{19}{25}$$

$$\Rightarrow \quad x - \frac{3}{5} = \pm\frac{\sqrt{19}}{5}$$

$$\Rightarrow \quad x = \frac{3}{5} \pm \frac{\sqrt{19}}{5} = \frac{3 \pm \sqrt{19}}{5}$$

$$\Rightarrow \quad x = \frac{3 + \sqrt{19}}{5} \text{ or, } x = \frac{3 - \sqrt{19}}{5}$$

Hence, the roots of the given equation are $\frac{3+\sqrt{19}}{5}$ and $\frac{3-\sqrt{19}}{5}$.

EXAMPLE 4 *By using the method of completing the square, show that the equation* $4x^2 + 3x + 5 = 0$ *has no real roots.* [NCERT]

SOLUTION We have,

$$4x^2 + 3x + 5 = 0$$

$$\Rightarrow \quad x^2 + \frac{3}{4}x + \frac{5}{4} = 0$$

$$\Rightarrow \quad x^2 + 2\left(\frac{3}{8}x\right) = -\frac{5}{4}$$

$$\Rightarrow \quad x^2 + 2\left(\frac{3}{8}\right)x + \left(\frac{3}{8}\right)^2 = \left(\frac{3}{8}\right)^2 - \frac{5}{4}$$

$$\Rightarrow \quad \left(x + \frac{3}{8}\right)^2 = -\frac{71}{64}$$

Clearly, RHS is negative. But, $\left(x + \frac{3}{8}\right)^2$ cannot be negative for any real value of x. Hence, the given equation has no real roots.

LEVEL-2

EXAMPLE 5 *Find the roots of the following equation* $4x^2 + 4bx - (a^2 - b^2) = 0$ *by the method of completing the square.*

SOLUTION We have,

$$4x^2 + 4bx - (a^2 - b^2) = 0$$

$$\Rightarrow \quad x^2 + bx - \left(\frac{a^2 - b^2}{4}\right) = 0$$

$$\Rightarrow \quad x^2 + 2\left(\frac{b}{2}\right)x = \frac{a^2 - b^2}{4}$$

$$\Rightarrow \quad x^2 + 2\left(\frac{b}{2}\right)x + \left(\frac{b}{2}\right)^2 = \frac{a^2 - b^2}{4} + \left(\frac{b}{2}\right)^2$$

$$\Rightarrow \quad \left(x + \frac{b}{2}\right)^2 = \frac{a^2}{4}$$

$$\Rightarrow \quad x + \frac{b}{2} = \pm\frac{a}{2}$$

$$\Rightarrow \quad x = \frac{-b}{2} \pm \frac{a}{2} \Rightarrow x = \frac{-b-a}{2}, \frac{-b+a}{2}$$

Hence, the roots are $-\left(\frac{a+b}{2}\right)$ and $\left(\frac{a-b}{2}\right)$

EXAMPLE 6 *Find the roots of the equation* $a^2x^2 - 3abx + 2b^2 = 0$ *by the method of completing the square.*

SOLUTION We have,

$$a^2x^2 - 3abx + 2b^2 = 0$$

$$\Rightarrow \quad x^2 - \frac{3b}{a}x + 2\frac{b^2}{a^2} = 0$$

$$\Rightarrow \quad x^2 - \frac{3b}{a}x = -\frac{2b^2}{a^2}$$

$$\Rightarrow \quad x^2 - 2\left(\frac{3b}{2a}\right)x + \left(\frac{3b}{2a}\right)^2 = -\frac{2b^2}{a^2} + \left(\frac{3b}{2a}\right)^2$$

$$\Rightarrow \quad \left(x - \frac{3b}{2a}\right)^2 = -\frac{2b^2}{a^2} + \frac{9b^2}{4a^2}$$

$$\Rightarrow \quad \left(x - \frac{3b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

$$\Rightarrow \quad \left(x - \frac{3b}{2a}\right) = \pm\frac{b}{2a}$$

$$\Rightarrow \quad x = \frac{3b}{2a} \pm \frac{b}{2a} \Rightarrow x = \frac{3b}{2a} + \frac{b}{2a} = \frac{2b}{a} \text{ or, } x = \frac{3b}{2a} - \frac{b}{2a} = \frac{b}{a}$$

Hence, the roots are $\frac{2b}{a}$ and $\frac{b}{a}$.

EXAMPLE 7 *Solve the equation* $x^2 - (\sqrt{3}+1)x + \sqrt{3} = 0$ *by the method of completing the square.*

SOLUTION We have,

$$x^2 - (\sqrt{3}+1)x + \sqrt{3} = 0$$

$$\Rightarrow \quad x^2 - (\sqrt{3}+1)x = -\sqrt{3}$$

$$\Rightarrow \quad \left(x - \frac{\sqrt{3}+1}{2}\right)^2 = \frac{-4\sqrt{3} + (\sqrt{3}+1)^2}{4}$$

$$\Rightarrow \quad \left(x-\frac{\sqrt{3}+1}{2}\right)^2=\left(\frac{\sqrt{3}-1}{2}\right)^2$$

$$\Rightarrow \quad x-\frac{\sqrt{3}+1}{2}=\pm\frac{\sqrt{3}-1}{2} \Rightarrow x=\frac{\sqrt{3}+1}{2}\pm\frac{\sqrt{3}-1}{2} \Rightarrow x=\sqrt{3},1$$

Hence, the roots are $\sqrt{3}$ and 1.

EXERCISE 4.4

LEVEL-1

Find the roots of the following quadratic equations (if they exist) by the method of completing the square.

1. $x^2-4\sqrt{2x}+6=0$
2. $2x^2-7x+3=0$
3. $3x^2+11x+10=0$
4. $2x^2+x-4=0$
5. $2x^2+x+4=0$

LEVEL-2

6. $4x^2+4\sqrt{3}x+3=0$
7. $\sqrt{2}x^2-3x-2\sqrt{2}=0$
8. $\sqrt{3}x^2+10x+7\sqrt{3}=0$
9. $x^2-(\sqrt{2}+1)\,x+\sqrt{2}=0$
10. $x^2-4ax+4a^2-b^2=0$

ANSWERS

1. $\sqrt{2}, 3\sqrt{2}$
2. $3, \frac{1}{2}$
3. $-\frac{5}{3}, -2$
4. $\frac{\sqrt{33}-1}{4}, \frac{-\sqrt{33}-1}{4}$
5. No real roots
6. $-\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}$
7. $-\frac{1}{\sqrt{2}}, 2\sqrt{2}$
8. $-\sqrt{3}, -\frac{7}{\sqrt{3}}$
9. $\sqrt{2}, 1$
10. $2a-b, 2a+b$

4.6 SOLUTION OF A QUADRATIC EQUATION BY USING THE QUADRATIC FOR FORMULA (SHREEDHARACHARYA'S RULE)

In the previous section, we have learnt about factorization method of solving quadratic equations. In some cases, it is not convenient to solve quadratic equations by factorization method. For example, consider the equation $x^2+4x+2=0$. In order to solve this equation by factorization method we will have to split the coefficient of the middle term 4 into two integers whose sum is 4 and product is 2. Clearly, this not possible in integers. Therefore, this equation cannot be solved by using factorization method. In this section, we shall discuss a method to solve such quadratic equations. The method which we will discuss below is popularly known as Shreedharacharya's formula as it was first given by an ancient Indian mathematician Shreedharacharya around 1025 A.D.

Consider the quadratic equation

$$ax^2+bx+c=0, a\neq 0$$

$$\Rightarrow \quad x^2+\frac{b}{a}x+\frac{c}{a}=0 \qquad \text{[Dividing throught out by } a\text{]} \quad \ldots \text{(i)}$$

$\Rightarrow \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$

$\Rightarrow \quad x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$ $\quad \left[\text{Adding} \left(\frac{b}{2a}\right)^2 \text{ i.e.} \left(\frac{1}{2}\text{Coeff. of } x\right)^2 \text{ on both sides}\right]$

$\Rightarrow \quad x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$

$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \left(\frac{b^2 - 4ac}{4a^2}\right)$

$\Rightarrow \quad x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$ $\quad \left[\text{Taking square root of both sides and assuming } b^2 - 4ac \geq 0\right]$

$\Rightarrow \quad x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow \quad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ or, $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow \quad x = \frac{-b + \sqrt{D}}{2a}$ or, $x = \frac{-b - \sqrt{D}}{2a}$, where $D = b^2 - 4ac$

Thus, if $D = b^2 - 4ac \geq 0$, then the quadratic equation $ax^2 + bx + c = 0$ has real roots α and β given by

$$\alpha = \frac{-b + \sqrt{D}}{2a} \text{ and } \beta = \frac{-b - \sqrt{D}}{2a}$$

DISCRIMINANT *If $ax^2 + bx + c = 0, a \neq 0$ is a quadratic equation, then the expression $b^2 - 4ac$ is known as its discriminant and is generally denoted by D.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I ON DETERMINING THE DISCRIMINANT OF A QUADRATIC EQUATION

EXAMPLE 1 *Write the discriminant of the following quadratic equations:*

(i) $x^2 - 4x + 2 = 0$

(ii) $3x^2 + 2x - 1 = 0$

(iii) $x^2 - 4x + a = 0$

(iv) $\sqrt{3}x^2 - 2\sqrt{2}x - 2\sqrt{3} = 0$

(v) $x^2 + x + 1 = 0$

(vi) $x^2 + px + 2q = 0$

SOLUTION (i) The given equation is $x^2 - 4x + 2 = 0$

Here, $\quad a = 1, b = -4$ and, $c = 2$

$\therefore \quad D = b^2 - 4ac = (-4)^2 - 4 \times 1 \times 2 = 16 - 8 = 8$

(ii) The given equation is $3x^2 + 2x - 1 = 0$

Here, $a = 3, b = 2$ and, $c = -1$

$\therefore$ $D = b^2 - 4ac = 2^2 - 4 \times 3 \times -1 = 4 + 12 = 16$

(iii) The given equation is $x^2 - 4x + a = 0$

Here, $a = 1, b = -4$ and, $c = a$.

$\therefore$ $D = b^2 - 4ac = (-4)^2 - 4 \times 1 \times a = 16 - 4a$

(iv) The given equation is $\sqrt{3}x^2 - 2\sqrt{2}x - 2\sqrt{3} = 0$

Here, $a = \sqrt{3}, b = -2\sqrt{2}$ and, $c = -2\sqrt{3}$

$\therefore$ $D = b^2 - 4ac = (-2\sqrt{2})^2 - 4 \times \sqrt{3} \times -2\sqrt{3} = 8 + 24 = 32$

(v) The given equation is $x^2 + x + 1 = 0$

Here, $a = 1, b = 1$ and, $c = 1$.

$\therefore$ $D = b^2 - 4ac = 1^2 - 4 \times 1 \times 1 = -3$

(vi) The given equation is $x^2 + px + 2q = 0$

Here, $a = 1, b = p$ and, $c = 2q$

$\therefore$ $D = b^2 - 4ac = p^2 - 4 \times 1 \times 2q = p^2 - 8q$

Type II ON SOLVING A QUADRATIC EQUATION HAVING REAL ROOTS BY USING QUADRATIC FORMULA

EXAMPLE 2 *In the following, determine whether the given quadratic equations have real roots and if so, find the roots*

(i) $9x^2 + 7x - 2 = 0$

(ii) $2x^2 + 5\sqrt{3}x + 6 = 0$

(iii) $3x^2 + 2\sqrt{5}x - 5 = 0$

(iv) $x^2 + 5x + 5 = 0$

(v) $6x^2 + x - 2 = 0$

(vi) $25x^2 + 20x + 7 = 0$

SOLUTION (i) The given equation is $9x^2 + 7x - 2 = 0$

Here, $a = 9, b = 7$ and $c = -2$

$\therefore$ $D = b^2 - 4ac = 7^2 - 4 \times 9 \times -2 = 49 + 72 = 121 > 0$

So, the given equation has real roots, given by

$$\alpha = \frac{-b + \sqrt{D}}{2a} = \frac{-7 + \sqrt{121}}{2 \times 9} = \frac{-7 + 11}{18} = \frac{4}{18} = \frac{2}{9}$$

and, $$\beta = \frac{-b - \sqrt{D}}{2a} = \frac{-7 - \sqrt{121}}{2 \times 9} = \frac{-7 - 11}{18} = -1$$

(ii) The given equation is $2x^2 + 5\sqrt{3}x + 6 = 0$

Here, $a = 2, b = 5\sqrt{3}$ and $c = 6$

$\therefore$ $D = b^2 - 4ac = 75 - 4 \times 2 \times 6 = 27 > 0$

So, the given equation has real roots given by

$$\alpha = \frac{-b+\sqrt{D}}{2a} = \frac{-5\sqrt{3}+\sqrt{27}}{2\times 2} = \frac{-5\sqrt{3}+3\sqrt{3}}{4} = \frac{-2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2}$$

and, $$\beta = \frac{-b-\sqrt{D}}{2a} = \frac{-5\sqrt{3}-\sqrt{27}}{2\times 2} = \frac{-5\sqrt{3}-3\sqrt{3}}{4} = -2\sqrt{3}$$

(iii) The given equation is $3x^2 + 2\sqrt{5}x - 5 = 0$

Here, $a = 3, b = 2\sqrt{5}$ and, $c = -5$

$\therefore$ $D = b^2 - 4ac = (2\sqrt{5})^2 - 4\times 3\times -5 = 20 + 60 = 80 > 0$

So, the given equation has real roots, given by

$$\alpha = \frac{-b+\sqrt{D}}{2a} = \frac{-2\sqrt{5}+\sqrt{80}}{2\times 3} = \frac{-2\sqrt{5}+4\sqrt{5}}{6} = \frac{2\sqrt{5}}{6} = \frac{\sqrt{5}}{3}$$

and, $$\beta = \frac{-b-\sqrt{D}}{2a} = \frac{-2\sqrt{5}-\sqrt{80}}{2\times 3} = \frac{-2\sqrt{5}-4\sqrt{5}}{6} = -\sqrt{5}$$

(iv) The given equation is $x^2 + 5x + 5 = 0$

Here, $a = 1, b = 5$ and, $c = 5$

$\therefore$ $D = b^2 - 4ac = 25 - 4\times 1\times 5 = 5 > 0$

So, the given equation has real roots, given by

$$\alpha = \frac{-b+\sqrt{D}}{2a} = \frac{-5+\sqrt{5}}{2} \text{ and } \beta = \frac{-b-\sqrt{D}}{2a} = \frac{-5-\sqrt{5}}{2}$$

(v) The given equation is $6x^2 + x - 2 = 0$

Here, $a = 6, b = 1$ and, $c = -2$

$\therefore$ $D = b^2 - 4ac = 1 - 4\times 6\times -2 = 49 > 0$

So, the given equation has real roots, given by

$$\alpha = \frac{-b+\sqrt{D}}{2a} = \frac{-1+\sqrt{49}}{2\times 6} = \frac{-1+7}{12} = \frac{6}{12} = \frac{1}{2}$$

and, $$\beta = \frac{-b-\sqrt{D}}{2a} = \frac{-1-\sqrt{49}}{2\times 6} = \frac{-1-7}{12} = \frac{-8}{12} = \frac{-2}{3}$$

(vi) The given equation is $25x^2 + 20x + 7 = 0$

Here, $a = 25, b = 20$ and, $c = 7$

$\therefore$ $D = b^2 - 4ac = (20)^2 - 4\times 25\times 7 = 400 - 700 = -300 < 0$

So, the given equation has no real roots.

EXAMPLE 3 *Solve for x:* $\frac{x-1}{x+2} + \frac{x-3}{x-4} = \frac{10}{3}, x \neq -2, 4$

SOLUTION We have,

$$\frac{x-1}{x+2} + \frac{x-3}{x-4} = \frac{10}{3}$$

$\Rightarrow \quad \dfrac{(x^2-5x+4)+(x^2-x-6)}{(x+2)(x-4)} = \dfrac{10}{3}$

$\Rightarrow \quad \dfrac{2x^2-6x-2}{x^2-2x-8} = \dfrac{10}{3}$

$\Rightarrow \quad 6x^2-18x-6 = 10x^2-20x-80$

$\Rightarrow \quad 4x^2-2x-74 = 0$

$\Rightarrow \quad 2x^2-x-37 = 0$

$\Rightarrow \quad x = \dfrac{1 \pm \sqrt{1+296}}{4} \Rightarrow x = \dfrac{1 \pm \sqrt{297}}{4}$

EXAMPLE 4 *Solve for x:* $\dfrac{1}{x+1} + \dfrac{2}{x+2} = \dfrac{4}{x+4}, x \neq 1, -2, -4.$

SOLUTION We have,

$\dfrac{1}{x+1} + \dfrac{2}{x+2} = \dfrac{4}{x+4}$

$\Rightarrow \quad \dfrac{1}{x+1} + \dfrac{2}{x+2} = \dfrac{1}{x+4} + \dfrac{3}{x+4}$

$\Rightarrow \quad \dfrac{1}{x+1} - \dfrac{1}{x+4} = \dfrac{3}{x+4} - \dfrac{2}{x+2}$

$\Rightarrow \quad \dfrac{x+4-x-1}{(x+1)(x+4)} = \dfrac{3x+6-2x-8}{(x+4)(x+2)}$

$\Rightarrow \quad \dfrac{3}{(x+1)(x+4)} = \dfrac{x-2}{(x+4)(x+2)}$

$\Rightarrow \quad \dfrac{3}{x+1} = \dfrac{x-2}{x+2}$ [Multiplying both sides by $(x+4)$]

$\Rightarrow \quad 3x+6 = x^2-x-2$

$\Rightarrow \quad x^2-4x-8 = 0 \Rightarrow x = \dfrac{4 \pm \sqrt{16+32}}{2} = \dfrac{4 \pm 4\sqrt{3}}{2} = 2 \pm 2\sqrt{3}$

LEVEL-2

EXAMPLE 5 *Using quadratic formula solve the following quadratic equations:*

(i) $p^2x^2 + (p^2-q^2)x - q^2 = 0, p \neq 0$ [CBSE 2004]

(ii) $9x^2 - 9(a+b)x + (2a^2+5ab+2b^2) = 0$ [CBSE 2004, 2009]

SOLUTION (i) We have, $p^2x^2 + (p^2-q^2)x - q^2 = 0$

Comparing this equation with $ax^2+bx+c=0$, we have

$a = p^2, b = p^2 - q^2$ and $c = -q^2$

$D = b^2 - 4ac = (p^2-q^2)^2 - 4 \times p^2 \times -q^2 = (p^2-q^2)^2 + 4p^2q^2 = (p^2+q^2)^2 > 0$

So, the given equation has real roots given by

$\alpha = -\dfrac{b-\sqrt{D}}{2a} = -\dfrac{(p^2-q^2)+(p^2+q^2)}{2p^2} = \dfrac{q^2}{p^2}$

and, $\beta = \dfrac{-b - \sqrt{D}}{2a} = \dfrac{-(p^2 - q^2) - (p^2 + q^2)}{2p^2} = -1$

ALITER We have,

$$p^2x^2 + (p^2 - q^2)x - q^2 = 0$$

$\Rightarrow$ $p^2x^2 + p^2x - q^2x - q^2 = 0$

$\Rightarrow$ $(p^2x^2 + p^2x) - (q^2x + q^2) = 0$

$\Rightarrow$ $p^2x(x+1) - q^2(x+1) = 0$

$\Rightarrow$ $(x+1)(p^2x - q^2) = 0$

$\Rightarrow$ $x + 1 = 0$ or, $p^2x - q^2 = 0 \Rightarrow x = -1$ or, $x = \dfrac{q^2}{p^2}$

(ii) We have,

$$9x^2 - 9(a+b)x + (2a^2 + 5ab + 2b^2) = 0$$

Comparing this equation with $Ax^2 + Bx + C = 0$, we have

$$A = 9,\ B = -9(a+b) \text{ and } C = 2a^2 + 5ab + 2b^2$$

$\therefore$ $D = B^2 - 4AC$

$\Rightarrow$ $D = 81(a+b)^2 - 36(2a^2 + 5ab + 2b^2)$

$\Rightarrow$ $D = 9a^2 + 9b^2 - 18ab$

$\Rightarrow$ $D = 9(a-b)^2 \geq 0$

So, the roots of the given equation are real and are given by

$$\alpha = \frac{-B + \sqrt{D}}{2A} = \frac{9(a+b) + 3(a-b)}{18} = \frac{12a + 6b}{18} = \frac{2a+b}{3}$$

and, $\beta = \dfrac{-B - \sqrt{D}}{2A} = \dfrac{9(a+b) - 3(a-b)}{18} = \dfrac{6a + 12b}{18} = \dfrac{a+2b}{3}$

ALITER See Ex 7 (i) on page 4.14.

EXAMPLE 6 *Using quadratic formula, solve the following equation for x:*

$$abx^2 + (b^2 - ac)x - bc = 0$$ [CBSE 2005]

SOLUTION We have,

$$abx^2 + (b^2 - ac)x - bc = 0$$

$\therefore$ $x = \dfrac{-(b^2 - ac) \pm \sqrt{(b^2 - ac)^2 - 4(ab)(-bc)}}{2ab}$

$\Rightarrow$ $x = \dfrac{-(b^2 - ac) \pm \sqrt{(b^2 - ac)^2 - 4ab^2c}}{2ab}$

$\Rightarrow$ $x = \dfrac{-(b^2 - ac) \pm \sqrt{b^4 - 2ab^2c + a^2c^2 + 4ab^2c}}{2ab}$

$$\Rightarrow \quad x = \frac{-(b^2 - ac) \pm \sqrt{(b^2 + ac)^2}}{2\,ab}$$

$$\Rightarrow \quad x = \frac{-(b^2 - ac) \pm (b^2 + ac)}{2\,ab}$$

$$\Rightarrow \quad x = \frac{-(b^2 - ac) + (b^2 + ac)}{2\,ab}, x = \frac{-(b^2 - ac) - (b^2 + ac)}{2\,ab}$$

$$\Rightarrow \quad x = \frac{2\,ac}{2\,ab}, x = \frac{-2b^2}{2\,ab} \Rightarrow x = \frac{c}{b}, x = \frac{-b}{a}$$

EXERCISE 4.5

LEVEL-1

1. Write the discriminant of the following quadratic equations:
   (i) $2x^2 - 5x + 3 = 0$
   (ii) $x^2 + 2x + 4 = 0$
   (iii) $(x - 1)(2x - 1) = 0$
   (iv) $x^2 - 2x + k = 0, k \in R$
   (v) $\sqrt{3}x^2 + 2\sqrt{2}x - 2\sqrt{3} = 0$
   (vi) $x^2 - x + 1 = 0$
2. In the following, determine whether the given quadratic equations have real roots and if so, find the roots:
   (i) $16x^2 = 24x + 1$
   (ii) $x^2 + x + 2 = 0$
   (iii) $\sqrt{3}x^2 + 10x - 8\sqrt{3} = 0$
   (iv) $3x^2 - 2x + 2 = 0$
   (v) $2x^2 - 2\sqrt{6}x + 3 = 0$
   (vi) $3a^2x^2 + 8abx + 4b^2 = 0, a \neq 0$
   (vii) $3x^2 + 2\sqrt{5}x - 5 = 0$
   (viii) $x^2 - 2x + 1 = 0$
   (ix) $2x^2 + 5\sqrt{3}x + 6 = 0$
   (x) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$ [CBSE 2013, NCERT]
   (xi) $2x^2 - 2\sqrt{2}x + 1 = 0$ [NCERT]
   (xii) $3x^2 - 5x + 2 = 0$ [NCERT]
3. Solve for x:
   (i) $\frac{x-1}{x-2} + \frac{x-3}{x-4} = 3\frac{1}{3}; x \neq 2, 4$ [CBSE 2005]
   (ii) $\frac{1}{x} + \frac{2}{2x-3} = \frac{1}{x-2}, \; x \neq 0, \frac{3}{2}, \; 2$ [CBSE 2016]
   (iii) $x + \frac{1}{x} = 3, \; x \neq 0$ [NCERT]
   (iv) $\frac{16}{x} - 1 = \frac{15}{x+1}, x \neq 0, -1$ [CBSE 2014]
   (v) $\frac{1}{x-3} - \frac{1}{x+5} = \frac{1}{6}, \; x \neq 3, -5$ [CBSE 2016]

ANSWERS

1. (i) 1 (ii) -12 (iii) 1 (iv) $4 - 4k$
   (v) 32 (vi) -3
2. (i) $\frac{3 \pm \sqrt{10}}{4}$ (ii) Not real (iii) $-4\sqrt{3}, \frac{2}{\sqrt{3}}$ (iv) Not real

(v) Real and equal, $\sqrt{\frac{3}{2}}$ (vi) $\frac{-2b}{a}, \frac{-2b}{3a}$ (vii) $\frac{\sqrt{5}}{3}, -\sqrt{5}$ (viii) 1

(ix) $-2\sqrt{3}, \frac{-\sqrt{3}}{2}$ (x) $-\sqrt{2}, -\frac{5}{\sqrt{2}}$ (xi) $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ (xii) $1, \frac{2}{3}$

3. (i) $5, \frac{5}{2}$ (ii) 1, 3 (iii) $\frac{3 \pm \sqrt{5}}{2}$ (iv) ± 4 (v) $-9, 7$

4.7 NATURE OF ROOTS

Let $ax^2 + bx + c = 0, a \neq 0$ be a quadratic equation. In the previous section, we have seen that the roots of this equation are given by

$$\alpha = \frac{-b + \sqrt{D}}{2a} \text{ and } \beta = \frac{-b - \sqrt{D}}{2a}, \text{ provided that } D = b^2 - 4ac \geq 0.$$

If $D = b^2 - 4ac > 0$, then α and β are real.

Also,

$$\alpha - \beta = \left(\frac{-b + \sqrt{D}}{2a}\right) - \left(\frac{-b - \sqrt{D}}{2a}\right) = \frac{-b + \sqrt{D} + b + \sqrt{D}}{2a} = \frac{2\sqrt{D}}{2a} = \frac{\sqrt{D}}{a}$$

$$\Rightarrow \quad \alpha - \beta \neq 0 \Rightarrow \alpha \neq \beta$$

Thus, if $D = b^2 - 4ac > 0$ i.e. the discriminant of the equation is positive, then the equation has real and distinct roots α and β given by

$$\alpha = \frac{-b + \sqrt{D}}{2a} \text{ and } \beta = \frac{-b - \sqrt{D}}{2a}$$

If $D = b^2 - 4ac = 0$, then α and β are real.

Also, $\alpha = -\frac{b}{2a} = \beta$ [Putting $D = 0$ in the expression for α and β]

Thus, if $D = b^2 - 4ac = 0$ i.e. the discriminant of the equation is zero, then the equation has real and equal roots both equal to $-\frac{b}{2a}$.

Now, a natural question arises: what is the nature of the roots of the equation $ax^2 + bx + c = 0$ when its discriminant D is negative? To answer this question, let us go back to the equation

$$ax^2 + bx + c = 0$$

$$\Rightarrow \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad \text{[Dividing Throughout by } a]$$

$$\Rightarrow \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\Rightarrow \quad x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\Rightarrow \quad x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad \ldots \text{(i)}$$

If $D = b^2 - 4ac < 0$, then $\frac{b^2 - 4ac}{4a^2} < 0$

Therefore, LHS of equation (i) is positive (being perfect square of a real number) and its RHS is negative. So, there is no real value of x satisfying equation (i). Hence, there is no real root of the given quadratic equation in this case.

Thus, if $D = b^2 - 4ac < 0$, i.e. the discriminant of the quadratic equation is negative, then the equation has not real roots.

Let us now discuss problems on determining the nature of the roots of quadratic equations.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON DETERMINING THE NATURE OF ROOTS OF QUADRATIC EQUATION**

EXAMPLE 1 *Determine the nature of the roots of the following quadratic equations:*

(i) $2x^2 + x - 1 = 0$ (ii) $x^2 - 4x + 4 = 0$ (iii) $x^2 + x + 1 = 0$

(iv) $4x^2 - 4x + 1 = 0$ (v) $2x^2 + 5x + 5 = 0$

SOLUTION (i) The given quadratic equation is $2x^2 + x - 1 = 0$. Here, $a = 2, b = 1$ and $c = -1$.

$\therefore$ $D = b^2 - 4ac = 1^2 - 4 \times 2 \times -1 = 9$

We find that $D > 0$. So, roots of the given equation are real and distinct.

(ii) The given equation is $x^2 - 4x + 4 = 0$. Here, $a = 1, b = -4$ and, $c = 4$.

$\therefore$ $D = b^2 - 4ac = (-4)^2 - 4 \times 1 \times 4 = 0$

We find that $D = 0$. Therefore, roots of the given equation are real and equal.

(iii) The given equation is $x^2 + x + 1 = 0$. Here, $a = 1$, $b = 1$ and, $c = 1$

$\therefore$ $D = b^2 - 4ac = 1^2 - 4 \times 1 \times 1 = -3$

We find that $D < 0$. Therefore, roots of the given equation are not real.

(iv) The given equation is $4x^2 - 4x + 1 = 0$. Here, $a = 4, b = -4$ and, $c = 1$

$\therefore$ $D = b^2 - 4ac = (-4)^2 - 4 \times 4 \times 1 = 0$

We find that $D = 0$. Therefore, roots of the given equation are real and equal.

(v) The given equation is $2x^2 + 5x + 5 = 0$. Here, $a = 2, b = 5$ and, $c = 5$

$\therefore$ $D = b^2 - 4ac = 5^2 - 4 \times 2 \times 5 = 25 - 40 = -15$

We find that $D < 0$. Therefore, roots of the given equation are not real.

Type II **ON DETERMINING THE VALUES OF AN UNKNOWN INVOLVED IN THE QUADRATIC EQUATION WHEN THE NATURE OF ITS ROOTS IS GIVEN**

Results to Remember:

(i) $ax - b > 0 \Rightarrow x > \frac{b}{a}$ if, $a > 0$ and $x < \frac{b}{a}$ if, $a < 0$

(ii) $x^2 - a^2 > 0 \Rightarrow x < -a$ or, $x > a$ (iii) $x^2 - a^2 \geq 0 \Rightarrow x \leq -a$ or, $x \geq a$

(iv) $x^2 - a^2 < 0 \Rightarrow -a < x < a$ (v) $(x - a)(x - b) > 0, a < b \Rightarrow x < a$ or, $x > b$

(vi) $(x - a)(x - b) < 0, a < b \Rightarrow a < x < b$

EXAMPLE 2 *Find the values of k for which the given equation has real and equal roots:*

(i) $2x^2 - 10x + k = 0$ (ii) $9x^2 + 3kx + 4 = 0$

(iii) $12x^2 + 4kx + 3 = 0$ (iv) $2x^2 + 3x + k = 0$

(v) $2x^2 - kx + 1 = 0$ (vi) $kx^2 - 5x + k = 0$

(vii) $x^2 + k(4x + k - 1) + 2 = 0$ (viii) $x^2 - 2x(1 + 3k) + 7(3 + 2k) = 0$ [CBSE 2002 C]

(ix) $(k + 1)x^2 - 2(k - 1)x + 1 = 0$ [CBSE 2015]

SOLUTION (i) The given equation is $2x^2 - 10x + k = 0$. Here, $a = 2, b = -10$ and $c = k$

$$\therefore \quad D = b^2 - 4ac = (-10)^2 - 4 \times 2 \times k = 100 - 8k$$

The given equation will have real and equal roots, if

$$D = 0 \Rightarrow 100 - 8k = 0 \Rightarrow k = \frac{100}{8} = \frac{25}{2}$$

(ii) The given equation is $9x^2 + 3kx + 4 = 0$. Here, $a = 9, b = 3k$ and $c = 4$.

$$\therefore \quad D = b^2 - 4ac = (3k)^2 - 4 \times 9 \times 4 = 9k^2 - 144$$

The given equation will have real and equal roots, if

$$D = 0 \Rightarrow 9k^2 - 144 = 0 \Rightarrow k^2 = \frac{144}{9} \Rightarrow k^2 = 16 \Rightarrow k = \pm 4$$

(iii) The given equation is $12x^2 + 4kx + 3 = 0$. Here, $a = 12, b = 4k$ and, $c = 3$

$$\therefore \quad D = b^2 - 4ac = (4k)^2 - 4 \times 12 \times 3 = 16k^2 - 144$$

The given equation will have real and equal roots, if

$$D = 0 \Rightarrow 16k^2 - 144 = 0 \Rightarrow 16k^2 = 144 \Rightarrow k^2 = 9 \Rightarrow k = \pm 3$$

(iv) The given equation is $2x^2 + 3x + k = 0$. Here, $a = 2, b = 3$ and, $c = k$

$$\therefore \quad D = b^2 - 4ac = 9 - 4 \times 2 \times k = 9 - 8k$$

The given equation will have real and equal roots, if

$$D = 0 \Rightarrow 9 - 8k = 0 \Rightarrow k = \frac{9}{8}$$

(v) The given equation is $2x^2 - kx + 1 = 0$. Here, $a = 2, b = -k$ and, $c = 1$

$$\therefore \quad D = b^2 - 4ac = (-k)^2 - 4 \times 2 \times 1 = k^2 - 8$$

The given equation will have real and equal roots, if

$$D = 0 \Rightarrow k^2 - 8 = 0 \Rightarrow k^2 = 8 \Rightarrow k = \pm 2\sqrt{2}$$

(vi) The given equation is $kx^2 - 5x + k = 0$. Here, $a = k, b = -5$ and, $c = k$

$$\therefore \quad D = b^2 - 4ac = (-5)^2 - 4 \times k \times (k) = 25 - 4k^2$$

The given equation will have real and equal roots, if

$$D = 0 \Rightarrow 25 - 4k^2 = 0 \Rightarrow 25 = 4k^2 \Rightarrow k^2 = \frac{25}{4} \Rightarrow k = \pm \frac{5}{2}$$

(vii) The given equation is $x^2 + k(4x + k - 1) + 2 = 0$ or, $x^2 + 4kx + k(k - 1) + 2 = 0$.

Here, $a = 1, b = 4k, c = k(k-1) + 2 = 0$

$\therefore \quad D = (4k)^2 - 4 \times 1 \times \{k(k-1)+2\}$

$\Rightarrow \quad D = 16k^2 - 4k(k-1) - 8$

$\Rightarrow \quad D = 16k^2 - 4k^2 + 4k - 8$

$\Rightarrow \quad D = 12k^2 + 4k - 8$

$\Rightarrow \quad D = 4(3k^2 + k - 2)$

$\Rightarrow \quad D = 4(3k^2 + 3k - 2k - 2) = 4[3k(k+1) - 2(k+1)] = 4(3k-2)(k+1)$

The given equation will have equal roots, if

$$D = 0 \Rightarrow 4(3k-2)(k+1) = 0 \Rightarrow 3k - 2 = 0 \text{ or, } k+1 = 0 \Rightarrow k = \frac{2}{3} \text{ or, } k = -1$$

(viii) The given equation is $x^2 - 2x(1+3k) + 7(3+2k) = 0$. Here, $a = 1$, $b = -2(1+3k)$ and $c = 7(3+2k)$.

$\therefore \quad D = 4(1+3k)^2 - 4 \times \{7(3+2k)\}$

$\Rightarrow \quad D = 4(3k+1)^2 - 4 \times 1 \times 7(3+2k) = 4(9k^2 + 6k + 1 - 21 - 14k) = 4(9k^2 - 8k - 20)$

The given equation will have equal roots, if

$D = 0$

$\Rightarrow \quad 4(9k^2 - 8k - 20) = 0$

$\Rightarrow \quad 9k^2 - 8k - 20 = 0$

$\Rightarrow \quad 9k^2 - 18k + 10k - 20 = 0$

$\Rightarrow \quad (k-2)(9k+10) = 0 \Rightarrow k - 2 = 0 \text{ or, } 9k + 10 = 0 \Rightarrow k = 2 \text{ or, } k = -\frac{10}{9}$

(ix) The given equation is $(k+1)x^2 - 2(k-1)x + 1 = 0$. Here, $a = k+1$, $b = -2(k-1)$, $c = 1$. Let D be the discriminant of the given equation. Then,

$$D = b^2 - 4ac = 4(k-1)^2 - 4(k+1) = 4(k^2 - 3k)$$

The given equation will have real and equal roots, if

$$D = 0 \Rightarrow 4(k^2 - 3k) = 0 \Rightarrow k^2 - 3k = 0 \Rightarrow k(k-3) = 0 \Rightarrow k = 0, 3$$

EXAMPLE 3 *Find the values of k for which the following equation has equal roots:*

$$(k-12)x^2 + 2(k-12)x + 2 = 0$$ [CBSE 2013, 2017]

SOLUTION The quadratic equation is $(k-12)x^2 + 2(k-12)x + 2 = 0$. Here, $a = k - 12$, $b = 2(k-12)$ and $c = 2$.

$\therefore \quad D = b^2 - 4ac = 4(k-12)^2 - 4(k-12) \times 2$

$\Rightarrow \quad D = 4(k-12)\{(k-12) - 2\} = 4(k-12)(k-14)$

The given equation will have equal roots, if

$$D = 0 \Rightarrow 4(k-12)(k-14) = 0 \Rightarrow k - 12 = 0 \text{ or, } k - 14 = 0 \Rightarrow k = 12 \text{ or, } k = 14$$

EXAMPLE 4 *If -4 is a root of the quadratic equation $x^2 + px - 4 = 0$ and the quadratic equation $x^2 + px + k = 0$ has equal roots, find the value of k*

SOLUTION It is given that -4 is a root of the equation $x^2 + px - 4 = 0$.

$\therefore$ $(-4)^2 + p \times (-4) - 4 = 0$ [$\because$ A root always satisfies the equation]

$\Rightarrow$ $16 - 4p - 4 = 0 \Rightarrow 4p = 12 \Rightarrow p = 3$

The equation $x^2 + px + k = 0$ has equal roots. Therefore, its discriminant is zero.

i.e. $p^2 - 4k = 0$ [$\because a = 1, b = p$ and $c = k$]

$\Rightarrow$ $9 - 4k = 0$ [$\because p = 3$]

$\Rightarrow$ $k = 9/4$

EXAMPLE 5 *Show that the equation $x^2 + ax - 4 = 0$ has real and distinct roots for all real values of a.*

SOLUTION The given equation is $x^2 + ax - 4 = 0$. Let D be its discriminant. Then,

$$D = a^2 - 4 \times -4 = a^2 + 16$$

Clearly, $D = a^2 + 16 > 0$ for all $a \in R$.

Hence, the given equation has real and distinct roots.

EXAMPLE 6 *Find the value of k for which the given equation has equal roots. Also, find the roots.*

(i) $9x^2 - 24x + k = 0$ (ii) $2kx^2 - 40x + 25 = 0$

SOLUTION (i) The given equation is $9x^2 - 24x + k = 0$. Here, $a = 9, b = -24$ and $c = k$.

$\therefore$ $D = b^2 - 4ac = (-24)^2 - 4 \times 9 \times k = 576 - 36k$

The given equation will have real and equal roots, if

$$D = 0 \Rightarrow 576 - 36k = 0 \Rightarrow k = 16$$

Putting $k = 16$ in the given equation, we get

$$9x^2 - 24x + 16 = 0 \Rightarrow (3x - 4)^2 = 0 \Rightarrow 3x - 4 = 0 \Rightarrow x = 4/3.$$

Hence, both the roots of the given equation are equal to 4/3.

(ii) The given equation is $2kx^2 - 40x + 25 = 0$. Here, $a = 2k, b = -40$ and $c = 25$

$\therefore$ $D = b^2 - 4ac = (-40)^2 - 4 \times 2k \times 25 = 1600 - 200k$

The equation will have equal roots, if

$$D = 0 \Rightarrow 1600 - 200k = 0 \Rightarrow k = 8$$

Substituting $k = 8$ in the given equation, we get

$$16x^2 - 40x + 25 = 0 \Rightarrow (4x - 5)^2 = 0 \Rightarrow x = \frac{5}{4}$$

Hence, the roots of the given equation are each equal to 5/4.

EXAMPLE 7 *Find the value of k for which the quadratic equation $(k + 4)x^2 + (k + 1)x + 1 = 0$ has equal roots.* **[CBSE 2000 C, 2013]**

SOLUTION Here, $a = k + 4, b = k + 1$ and $c = 1$. Let D be the discriminatn of this equation. Then,

$$D = b^2 - 4ac = (k+1)^2 - 4(k+4) = k^2 - 2k - 15 = (k-5)(k+3)$$

If the roots of the given equation are equal, then

$$D = 0 \Rightarrow (k-5)(k+3) = 0 \Rightarrow k = 5, -3.$$

EXAMPLE 8 *If -5 is a root of the quadratic equation $2x^2 + px - 15 = 0$ and the quadratic equation $p(x^2 + x) + k = 0$ has equal roots, find the value of k.* [CBSE 2002, 2009]

SOLUTION It is given -5 is a root of the equation $2x^2 + px - 15 = 0$. Therefore, $x = -5$ satisfies it.

i.e. $$2(-5)^2 - 5p - 15 = 0 \Rightarrow 50 - 5p - 15 = 0 \Rightarrow 5p = 35 \Rightarrow p = 7$$

Putting $p = 7$ in $p(x^2 + x) + k = 0$, we get $7x^2 + 7x + k = 0$.

This equation will have equal roots, if its disriminant is zero.

i.e. $$49 - 4 \times 7 \times k = 0 \Rightarrow k = \frac{49}{28} \Rightarrow k = \frac{7}{4}$$

LEVEL-2

EXAMPLE 9 *Find the values of k for which the equation $x^2 - 4x + k = 0$ has distinct real roots.*

SOLUTION The given equation is $x^2 - 4x + k = 0$. Here, $a = 1, b = -4$ and $c = k$.

$\therefore$ $$D = (-4)^2 - 4 \times 1 \times k = 16 - 4k$$

The given equation will have real and distinct roots, if

$$D > 0 \Rightarrow 16 - 4k > 0 \Rightarrow 16 > 4k \Rightarrow 4k < 16 \Rightarrow k < \frac{16}{4} = 4$$

Hence, the given equation will have distinct roots, if $k < 4$.

EXAMPLE 10 *Determine the positive values of 'k' for which the equation $x^2 + kx + 64 = 0$ and $x^2 - 8x + k = 0$ will both have real roots.* [CBSE 2016]

SOLUTION Given equations are

$$x^2 + kx + 64 = 0 \quad \ldots (i)$$

and, $$x^2 - 8x + k = 0 \quad \ldots (ii)$$

Let D_1 and D_2 be the discriminants of equations (i) and (ii) respectively. Then,

$$D_1 = k^2 - 4 \times 64 = k^2 - 256 \text{ and, } D_2 = (-8)^2 - 4k = 64 - 4k$$

Both the equations will have real roots, if

$$D_1 \geq 0 \text{ and } D_2 \geq 0$$

$\Rightarrow$ $$k^2 - 256 \geq 0 \text{ and } 64 - 4k \geq 0$$

$\Rightarrow$ $$k^2 \geq 256 \text{ and } 64 \geq 4k$$

$\Rightarrow$ $$k \geq 16 \ (\because k > 0) \text{ and } k \leq 16$$

$\Rightarrow$ $$k = 16$$

Hence, both the equations will have real roots, when $k = 16$.

EXAMPLE 11 *Find the values of k for which the given equation has real roots:*

(i) $kx^2 - 6x - 2 = 0$ (ii) $9x^2 + 3kx + 4 = 0$ (iii) $5x^2 - kx + 1 = 0$

SOLUTION (i) We have, $kx^2 - 6x - 2 = 0$. Here, $a = k, b = -6$ and $c = -2$

$\therefore \quad D = b^2 - 4ac = (-6)^2 - 4 \times k \times -2 = 36 + 8k$

The given equation will have real roots, if

$$D \geq 0 \Rightarrow 36 + 8k \geq 0 \Rightarrow 8k \geq -36 \Rightarrow k \geq \frac{36}{8} \Rightarrow k \geq -\frac{9}{2}$$

(ii) The given equation is $9x^2 + 3kx + 4 = 0$. Here, $a = 9, b = 3k$ and $c = 4$

$\therefore \quad D = b^2 - 4ac = 9k^2 - 4 \times 9 \times 4 = 9k^2 - 144$

The given equation will have real roots, if

$D \geq 0$

$\Rightarrow \quad 9k^2 - 144 \geq 0$

$\Rightarrow \quad 9(k^2 - 16) \geq 0$

$\Rightarrow \quad k^2 - 16 \geq 0 \qquad [\because ab > 0 \text{ and } a > 0 \Rightarrow b > 0]$

$\Rightarrow \quad k \leq -4 \text{ or } k \geq 4 \qquad [\because x^2 - a^2 \geq 0 \Rightarrow x \leq -a \text{ or, } x \geq a]$

(iii) The given equation is $5x^2 - kx + 1 = 0$. Here, $a = 5, b = -k$ and $c = 1$.

$\therefore \quad D = b^2 - 4ac = (-k)^2 - 4 \times 5 \times 1 = k^2 - 20$

The given equation will have real roots, if

$D \geq 0$

$\Rightarrow \quad k^2 - 20 \geq 0$

$\Rightarrow \quad k \leq -\sqrt{20} \text{ or, } k \geq \sqrt{20} \qquad [\because x^2 - a^2 \geq 0 \Rightarrow x \leq -a \text{ or, } x \geq a]$

EXAMPLE 12 *If p, q, r are real and $p \neq q$, then show that the roots of the equation $(p - q)x^2 + 5(p + q)x - 2(p - q) = 0$ are real and unequal.*

SOLUTION The given equation is $(p - q)x^2 + 5(p + q)x - 2(p - q) = 0$.

Here, $\quad a = p - q, b = 5(p + q)$ and $c = -2(p - q)$

Let D be the discriminant of the given equation. Then,

$$D = b^2 - 4ac = 25(p + q)^2 - 4(p - q) \times -2(p - q) = 25(p + q)^2 + 8(p - q)^2$$

We find that: $25(p + q)^2 > 0$ and $8(p - q)^2 > 0 \qquad [\because p \neq q]$

$\therefore \quad D = 25(p + q)^2 + 8(p - q)^2 > 0$

Hence, roots of the given equation are real and unequal.

EXAMPLE 13 *Find the values of k for which the equation $x^2 + 5kx + 16 = 0$ has no real roots.*

SOLUTION The given equation is $x^2 + 5kx + 16 = 0$. Here, $a = 1, b = 5k$ and $c = 16$.

The discriminant D of this equation is given by

$$D = b^2 - 4ac = (5k)^2 - 4 \times 1 \times 16 = 25k^2 - 16$$

The given equation will have no real roots, if

$D < 0$

$\Rightarrow \quad 25k^2 - 64 < 0$

$\Rightarrow \quad 25\left(k^2 - \frac{64}{25}\right) < 0$

$\Rightarrow \quad k^2 - \frac{64}{25} < 0 \quad [\because ab < 0 \text{ and } a > 0 \Rightarrow b < 0]$

$\Rightarrow \quad -\frac{8}{5} < k < \frac{8}{5} \quad [\because x^2 - a^2 < 0 \Rightarrow -a < x < a]$

Type III **ON DETERMINING OR PROVING THE NATURE OF THE ROOTS**

EXAMPLE 14 *If p, q, r and s are real numbers such that $pr = 2(q+s)$, then show that at least one of the equations $x^2 + px + q = 0$ and $x^2 + rx + s = 0$ has real roots.*

SOLUTION We have,

$$x^2 + px + q = 0 \quad ...(i)$$

and, $$x^2 + rx + s = 0 \quad ...(ii)$$

Let D_1 and D_2 be the discriminants of equations (i) and (ii) respectively. Then,

$$D_1 = p^2 - 4q \text{ and } D_2 = r^2 - 4s$$

$\Rightarrow \quad D_1 + D_2 = p^2 - 4q + r^2 - 4s = (p^2 + r^2) - 4(q+s)$

$\Rightarrow \quad D_1 + D_2 = p^2 + r^2 - 4\left(\frac{pr}{2}\right) \quad \left[\because pr = 2(q+s) \therefore q + s = \frac{pr}{2}\right]$

$\Rightarrow \quad D_1 + D_2 = p^2 + r^2 - 2pr = (p-r)^2 \geq 0 \quad [\because (p-r)^2 \geq 0 \text{ for all real } p, r]$

$\Rightarrow$ At least one of D_1 and D_2 is greater than or equal to zero

$\Rightarrow$ At least one of the two equations has real roots.

EXAMPLE 15 *If the roots of the equation $x^2 + 2cx + ab = 0$ are real unequal, prove that the equation $x^2 - 2(a+b)x + a^2 + b^2 + 2c^2 = 0$ has no real roots.* **[CBSE 2016]**

SOLUTION The two equations are

$$x^2 + 2cx + ab = 0 \quad ...(i)$$

and, $$x^2 - 2(a+b)x + a^2 + b^2 + 2c^2 = 0 \quad ...(ii)$$

Let D_1 and D_2 be the discriminants of equations (i) and (ii) respectively. Then,

$$D_1 = (2c)^2 - 4 \times 1 \times ab = 4c^2 - 4ab = 4(c^2 - ab)$$

and, $$D_2 = [-2(a+b)]^2 - 4 \times 1 \times (a^2 + b^2 + 2c^2)$$

$\Rightarrow \quad D_2 = 4(a+b)^2 - 4(a^2 + b^2 + 2c^2)$

$\Rightarrow \quad D_2 = 4\{a^2 + b^2 + 2ab - a^2 - b^2 - 2c^2\}$

$\Rightarrow \quad D_2 = 4(2ab - 2c^2) = -8(c^2 - ab)$

It is given that the roots of equation (i) are real and unequal. Therefore,

$$D_1 > 0$$

$\Rightarrow \quad 4(c^2 - ab) > 0$

$\Rightarrow \quad c^2 - ab > 0$

$\Rightarrow \quad -8(c^2 - ab) < 0$

$\Rightarrow \quad D_2 < 0$

$\Rightarrow$ Roots of equations (ii) are not real.

EXAMPLE 16 *Prove that the equation* $x^2(a^2 + b^2) + 2x(ac + bd) + (c^2 + d^2) = 0$ *has no real root, if* $ad \neq bc$.

SOLUTION Let D be the discriminant of the equation $(a^2+b^2)x^2+2x(ac+bd)+(c^2+d^2)=0$. Then,

$$D = 4(ac + bd)^2 - 4(a^2 + b^2)(c^2 + d^2)$$

$$\Rightarrow \quad D = 4[(ac + bd)^2 - (a^2 + b^2)(c^2 + d^2)]$$

$$\Rightarrow \quad D = 4[a^2c^2 + b^2d^2 + 2ac.bd - a^2c^2 - a^2d^2 - b^2c^2 - b^2d^2]$$

$$\Rightarrow \quad D = 4[2ac.bd - a^2d^2 - b^2c^2] = -4[a^2d^2 + b^2c^2 - 2ad.bc] = -4(ad - bc)^2$$

It is given that $ad \neq bc$.

$\therefore \quad ad - bc \neq 0 \Rightarrow (ad - bc)^2 > 0 \Rightarrow -4(ad - bc)^2 < 0 \Rightarrow D < 0$

Hence, the given equation has no real roots.

EXERCISE 4.6

LEVEL-1

1. Determine the nature of the roots of the following quadratic equations:
   (i) $2x^2 - 3x + 5 = 0$ [NCERT]
   (ii) $2x^2 - 6x + 3 = 0$ [NCERT]
   (iii) $\frac{3}{5}x^2 - \frac{2}{3}x + 1 = 0$
   (iv) $3x^2 - 4\sqrt{3}x + 4 = 0$ [NCERT]
   (v) $3x^2 - 2\sqrt{6}x + 2 = 0$
2. Find the values of k for which the roots are real and equal in each of the following equations:
   (i) $kx^2 + 4x + 1 = 0$
   (ii) $kx^2 - 2\sqrt{5}x + 4 = 0$
   (iii) $3x^2 - 5x + 2k = 0$
   (iv) $4x^2 + kx + 9 = 0$
   (v) $2kx^2 - 40x + 25 = 0$
   (vi) $9x^2 - 24x + k = 0$
   (vii) $4x^2 - 3kx + 1 = 0$
   (viii) $x^2 - 2(5 + 2k)x + 3(7 + 10k) = 0$
   (ix) $(3k + 1)x^2 + 2(k + 1)x + k = 0$
   (x) $kx^2 + kx + 1 = -4x^2 - x$
   (xi) $(k + 1)x^2 + 2(k + 3)x + (k + 8) = 0$
   (xii) $x^2 - 2kx + 7k - 12 = 0$
   (xiii) $(k + 1)x^2 - 2(3k + 1)x + 8k + 1 = 0$
   (xiv) $5x^2 - 4x + 2 + k(4x^2 - 2x - 1) = 0$
   (xv) $(4 - k)x^2 + (2k + 4)x + (8k + 1) = 0$
   (xvi) $(2k + 1)x^2 + 2(k + 3)x + (k + 5) = 0$
   (xvii) $4x^2 - 2(k + 1)x + (k + 4) = 0$
   (xviii) $4x^2 - 2(k + 1)x + (k + 1) = 0$ [CBSE 2017]

3. In the following, determine the set of values of k for which the given quadratic equation has real roots:
   (i) $2x^2 + 3x + k = 0$ (ii) $2x^2 + x + k = 0$ (iii) $2x^2 - 5x - k = 0$
   (iv) $kx^2 + 6x + 1 = 0$ (v) $3x^2 + 2x + k = 0$
4. Find the values of k *for which the following equations have real and equal roots:*
   (i) $x^2 - 2(k+1)x + k^2 = 0$ [CBSE 2001 C, 2013]
   (ii) $k^2x^2 - 2(2k-1)x + 4 = 0$ [CBSE 2001 C]
   (iii) $(k+1)x^2 - 2(k-1)x + 1 = 0$ [CBSE 2002 C]
   (iv) $x^2 + k(2x + k - 1) + 2 = 0$ [CBSE 2017]
5. Find the values of k for which the following equations have real roots
   (i) $2x^2 + kx + 3 = 0$ [NCERT] (ii) $kx(x-2) + 6 = 0$ [NCERT, 2013]
   (iii) $x^2 - 4kx + k = 0$ [CBSE 2012] (iv) $kx(x - 2\sqrt{5}) + 10 = 0$ [CBSE 2013]
   (v) $kx(x-3) + 9 = 0$ [CBSE 2014] (vi) $4x^2 + kx + 3 = 0$ [CBSE 2014]
6. Find the values of k for which the given quadratic equation has real and distinct roots:
   (i) $kx^2 + 2x + 1 = 0$ (ii) $kx^2 + 6x + 1 = 0$
7. For what value of k, $(4-k)x^2 + (2k+4)x + (8k+1) = 0$, is a perfect square.
8. Find the least positive value of k for which the equation $x^2 + kx + 4 = 0$ has real roots.
9. Find the values of k for which the quadratic equation $(3k+1)x^2 + 2(k+1)x + 1 = 0$ has equal roots. Also, find the roots. [CBSE 2014]
10. Find the values of p for which the equadratic equation $(2p+1)x^2 - (7p+2)x + (7p-3) = 0$ has equal roots. Also, find these roots. [CBSE 2014]
11. If -5 is a root of the quadratic equation $2x^2 + px - 15 = 0$ and the quadratic equation $p(x^2 + x) + k = 0$ has equal roots, find the value of k. [CBSE 2014]
12. If 2 is a root of the quadratic equation $3x^2 + px - 8 = 0$ and the quadratic equation $4x^2 - 2px + k = 0$ has equal roots, find the value of k. [CBSE 2014]
13. If 1 is a root of the quadratic equation $3x^2 + ax - 2 = 0$ and the quadratic equation $a(x^2 + 6x) - b = 0$ has equal roots, find the value of b.
14. Find the value of p for which the quadratic equation $(p+1)x^2 - 6(p+1)x + 3(p+q) = 0$, $p \neq -1$ has equal roots. Hence, find the roots of the equation. [CBSE 2015]
15. Determine the nature of the roots of the following quadratic equations:
    (i) $(x-2a)(x-2b) = 4ab$ (ii) $9a^2b^2x^2 - 24abcdx + 16c^2d^2 = 0, a \neq 0, b \neq 0$
    (iii) $2(a^2+b^2)x^2 + 2(a+b)x + 1 = 0$ (iv) $(b+c)x^2 - (a+b+c)x + a = 0$
16. Determine the set of values of k for which the following quadratic equations have real roots:
    (i) $x^2 - kx + 9 = 0$ (ii) $2x^2 + kx + 2 = 0$
    (iii) $4x^2 - 3kx + 1 = 0$ (iv) $2x^2 + kx - 4 = 0$

LEVEL-2

If the roots of the equation $(b-c)x^2 + (c-a)x + (a-b) = 0$ are equal, then prove that $2b = a + c$. [CBSE 2002 C]

18. If the roots of the equation $(a^2+b^2)x^2-2(ac+bd)x+(c^2+d^2)=0$ are equal, prove that $\frac{a}{b}=\frac{c}{d}$. [CBSE 2017]

19. If the roots of the equations $ax^2+2bx+c=0$ and $bx^2-2\sqrt{ac}x+b=0$ are simultaneously real, then prove that $b^2=ac$.

20. If p, q are real and $p \neq q$, then show that the roots of the equation $(p-q)x^2+5\ (p+q)x-2(p-q)=0$ are real and unequal.

21. If the roots of the equation $(c^2-ab)x^2-2(a^2-bc)x+b^2-ac=0$ are equal, prove that either $a=0$ or $a^3+b^3+c^3=3abc$.

22. Show that the equation $2(a^2+b^2)x^2+2(a+b)x+1=0$ has no real roots, when $a \neq b$.

23. Prove that both the roots of the equation $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$ are real but they are equal only when $a=b=c$.

24. If a, b, c are real numbers such that $ac \neq 0$, then show that at least one of the equations $ax^2+bx+c=0$ and $-ax^2+bx+c=0$ has real roots.

25. If the equation $(1+m^2)x^2+2mcx+(c^2-a^2)=0$ has equal roots, prove that $c^2=a^2(1+m^2)$. [CBSE 2017]

ANSWERS

1. (i) Not real (ii) Real and distinct (iii) Not real (iv) Real and equal (v) Real and equal

2. (i) $k=4$ (ii) $k=\frac{5}{4}$ (iii) $k=\frac{25}{24}$ (iv) $k=\pm 12$ (v) $k=8$ (vi) $k=16$ (vii) $k=\pm\frac{4}{3}$ (viii) $k=2, \frac{1}{2}$ (ix) $k=\frac{-1}{2}, 1$ (x) $k=5, -3$ (xi) $k=\frac{1}{3}$ (xii) $k=4, 3$ (xiii) $k=0, 3$ (xiv) $k=-\frac{6}{5}, 1$ (xv) $k=0, 3$ (xvi) $k=\frac{-5\pm\sqrt{41}}{2}$ (xvii) $k=-3, -7$ (xviii) $k=-1, 3$

3. (i) $k \leq \frac{9}{8}$ (ii) $k \leq \frac{1}{8}$ (iii) $k \geq -\frac{25}{8}$ (iv) $k \leq 9$

4. (i) $k=\frac{-1}{2}$ (ii) $k=\frac{1}{4}$ (iii) $k=0, 3$ (iv) $k=2$

5. (i) $k=\pm 2\sqrt{6}$ (ii) $k=6$ (iii) $k=0, \frac{1}{4}$ (iv) $k=2$ (v) $k=4$ (vi) $k=\pm 4\sqrt{3}$

6. (i) $k<1$ (ii) $k<9$

7. $k=0, 3$

8. $k=4$

9. $k=0, 1; x=-1, -\frac{1}{2}$

10. $p=4, -\frac{4}{7}; x=\frac{5}{3}, 7$

11. $k=2$

12. 1

13. -9

14. $p=3; x=3$

15. (i) Real and distinct (ii) Real and equal (iii) Not real (iv) Real and unequal

16. (i) $k \leq -6$ or $k \geq 6$ (ii) $k \leq -4$ or, $k \geq 4$ (iii) $k \leq -\frac{4}{3}$ or, $k \geq \frac{4}{3}$ (iv) $k \in R$

HINTS TO SELECTED PROBLEMS

17. $D = 0 \Rightarrow (c-a)^2 - 4(b-c)(a-b) = 0$

$\Rightarrow \quad c^2 + a^2 + 4b^2 - 2ac - 4ab + 4ac - 4bc = 0 \Rightarrow (c+a-2b)^2 = 0 \Rightarrow c + a - 2b = 0$

ALITER Clearly, $x = 1$ satisfies the given equation. Since it has equal roots. So, both roots are equal to 1.

$\therefore \quad$ Product of the roots $= 1 \Rightarrow \dfrac{a-b}{b-c} = 1 \Rightarrow a - b = b - c \Rightarrow 2b = a + c$

19. Let D_1 and D_2 be the discriminants of the two equations. Then,

$D_1 \geq 0$ and $D_2 \geq b^2 \Rightarrow 4b^2 - 4ac \geq 0$ and $4ac - 4b^2 \geq 0 \Rightarrow b^2 \geq ac$ and $ac \geq b^2 \Rightarrow b^2 = ac.$

21. We have, $D = 4a(a^3 + b^3 + c^3 - 3abc)$. For the roots to be equal, we must have

$D = 0 \Rightarrow 4a(a^3 + b^3 + c^3 - 3abc) = 0 \Rightarrow a = 0$ or, $a^3 + b^3 + c^3 = 3abc$

23. The given equation is $3x^2 - 2x(a+b+c) + (ab+bc+ca) = 0$. Let D be its discriminant. Then,

$D = 4(a+b+c)^2 - 12(ab+bc+ca)$

$\Rightarrow \quad D = 4[(a+b+c)^2 - 3(ab+bc+ca)]$

$\Rightarrow \quad D = 4[a^2 + b^2 + c^2 - ab - bc - ca]$

$\Rightarrow \quad D = 2[2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca] = 2[(a-b)^2 + (b-c)^2 + (c-a)^2$

Clearly $D \geq 0$. If $D = 0$, then,

$(a-b)^2 + (b-c)^2 + (c-a)^2 = 0 \Rightarrow a - b = 0, b - c = 0, c - a = 0.$

4.8 SOLUTIONS OF PROBLEMS INVOLVING QUADRATIC EQUATIONS

In this section, we will discuss some simple problems on practical applications of quadratic equation. In this type of problems we first formulate a quadratic equation whose solution is a solution of the given problem. Sometimes it may happen that, out of the roots of the quadratic equation only one has a meaning for the problem. Any root of the quadratic equation, which does not satisfy the condition of the problem will be rejected.

In order to solve this type of problems, we may use the following algorithm.

ALGORITHM

STEP I *Translate the word problem into symbolic language and formulate the quadratic equation.*

STEP II *Solve the quadratic equation formed in Step I.*

STEP III *Translate the solution into verbal language and reject the solution which does not have a meaning for the problem.*

4.8.1 APPLICATIONS OF QUADRATIC EQUATIONS FOR SOLVING PROBLEMS ON NUMBERS

In this section, we will discuss problems on numbers. Following examples will illustrate the same.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *The sum of the squares of two consecutive natural numbers is 313. Find the numbers.*

SOLUTION Let two consecutive natural numbers be x and $x + 1$. Then,

$$x^2 + (x+1)^2 = 313 \quad \text{[Given]}$$

$\Rightarrow \quad 2x^2 + 2x + 1 = 313$

$\Rightarrow \quad 2x^2 + 2x - 312 = 0$

$\Rightarrow \quad x^2 + x - 156 = 0$

$\Rightarrow \quad x^2 + 13x - 12x - 156 = 0$

$\Rightarrow \quad x(x+13) - 12(x+13) = 0$

$\Rightarrow \quad (x+13)(x-12) = 0$

$\Rightarrow \quad x + 13 = 0$ or, $x - 12 = 0 \Rightarrow x = 12$ or, $x = -13$

Since x, being a natural number, cannot be negative. Therefore, $x = 12$.
Hence, the two consecutive natural numbers are 12 and 13.

EXAMPLE 2 *The sum of two numbers is 15. If the sum of their reciprocals is* $\frac{3}{10}$, *find the numbers.* [CBSE 2000, 2005]

SOLUTION Let the required numbers be x and $15 - x$. Then,

$$\frac{1}{x} + \frac{1}{15-x} = \frac{3}{10}$$

$\Rightarrow \quad \frac{15 - x + x}{x(15-x)} = \frac{3}{10}$

$\Rightarrow \quad \frac{15}{x(15-x)} = \frac{3}{10}$

$\Rightarrow \quad 150 = 3x(15 - x)$

$\Rightarrow \quad 150 = 45x - 3x^2$

$\Rightarrow \quad x^2 - 15x + 50 = 0$

$\Rightarrow \quad x^2 - 10x - 5x + 50 = 0$

$\Rightarrow \quad x(x-10) - 5(x-10) = 0$

$\Rightarrow \quad (x-10)(x-5) = 0$

$\Rightarrow \quad x - 10 = 0$ or, $x - 5 = 0 \Rightarrow x = 10$ or, $x = 5$

Hence, the two numbers are 10 and 5.

EXAMPLE 3 *The sum of a number and its reciprocal is* $2\frac{1}{30}$. *Find the number.*

SOLUTION Let the required number be x. Then,

$$x + \frac{1}{x} = 2\frac{1}{30}$$

$\Rightarrow \quad \frac{x^2 + 1}{x} = \frac{61}{30}$

$\Rightarrow \quad 30x^2 + 30 = 61x$

$\Rightarrow \quad 30x^2 - 61x + 30 = 0$

$\Rightarrow \quad 30x^2 - 36x - 25x + 30 = 0$

$\Rightarrow \quad 6x(5x - 6) - 5(5x - 6) = 0$

$\Rightarrow \quad (6x - 5)(5x - 6) = 0$

$\Rightarrow \quad 6x - 5 = 0$ or, $5x - 6 = 0 \Rightarrow x = \frac{5}{6}$ or, $x = \frac{6}{5}$

Hence, the required number is 5/6 or, 6/5

EXAMPLE 4 *Divide 16 into two parts such that twice the square of the larger part exceeds the square of the smaller part by 164.*

SOLUTION Let the larger part be x. Then, the smaller part $= 16 - x$.
By hypothesis, we have

$2x^2 = (16 - x)^2 + 164$

$\Rightarrow \quad 2x^2 - (16 - x)^2 - 164 = 0$

$\Rightarrow \quad x^2 + 32x - 420 = 0$

$\Rightarrow \quad (x + 42)(x - 10) = 0$

$\Rightarrow \quad x = -42$ or, $x = 10$

$\Rightarrow \quad x = 10$ $\quad [\because x > 0 \therefore x = -42$ is not possible$]$

Hence, the required parts are 10 and 6.

EXAMPLE 5 *The sum of the squares of two positive integers is 208. If the square of the larger number is 18 times the smaller number, find the numbers.*

SOLUTION Let the smaller number be x. Then,

Square of larger number $= 18x$

Also, Square of the smaller number $= x^2$

It is given that the sum of the square of the integers is 208.

$\therefore \quad x^2 + 18x = 208$

$\Rightarrow \quad x^2 + 18x - 208 = 0$

$\Rightarrow \quad x^2 + 26x - 8x - 208 = 0$

$\Rightarrow \quad (x + 26)(x - 8) = 0 \Rightarrow x = 8, x = -26$

But, the numbers are positive. Therefore, $x = 8$

$\therefore$ Square of the larger number $= 18x = 18 \times 8 = 144$

$\Rightarrow$ Larger number $= \sqrt{144} = 12$

Hence, the numbers are 8 and 12.

EXAMPLE 6 *The difference of the squares of two numbers is 45. The square of the smaller number is 4 times the larger number. Determine the numbers.*

SOLUTION Let the larger number be x. Then,

Square of the smaller number $= 4x$

Also, Square of the larger number $= x^2$

It is given that the difference of the squares of the numbers is 45.

$\therefore \quad x^2 - 4x = 45$

$\Rightarrow \quad x^2 - 4x - 45 = 0$

$\Rightarrow \quad x^2 - 9x + 5x - 45 = 0$

$\Rightarrow \quad x(x-9) + 5(x-9) = 0$

$\Rightarrow \quad (x-9)(x+5) = 0$

$\Rightarrow \quad x - 9 = 0$ or, $x + 5 = 0 \Rightarrow x = 9, -5$

CASE I *When* $x = 9$: In this case, we have
Square of the smaller number $= 4x = 36$

$\therefore$ Smaller number $= \pm 6$.

Thus, the numbers are 9, 6 or 9, − 6

CASE II *When* $x = -5$: In this case, we have
Square of the smaller number $= 4x = -20$. But, square of a number is always positive. Therefore, $x = -5$ is not possible.

Hence, the numbers are 9, 6 or 9, − 6.

EXAMPLE 7 *A two digit number is such that the product of its digits is 18. When 63 is subtracted from the number, the digits interchange their places. Find the number.* [CBSE 2006C]

SOLUTION Let the tens digit be x. Then, the units digits $= \frac{18}{x}$.

$\therefore \quad$ Number $= 10x + \frac{18}{x}$

and, Number obtained by interchanging the digits $= 10 \times \frac{18}{x} + x$

$\therefore \quad \left(10x + \frac{18}{x}\right) - \left(10 \times \frac{18}{x} + x\right) = 63$

$\Rightarrow \quad 10x + \frac{18}{x} - \frac{180}{x} - x = 63$

$\Rightarrow \quad 9x - \frac{162}{x} - 63 = 0$

$\Rightarrow \quad 9x^2 - 63x - 162 = 0$

$\Rightarrow \quad x^2 - 7x - 18 = 0$

$\Rightarrow \quad (x-9)(x+2) = 0 \Rightarrow x = 9$ or, $x = -2$

But, a digit can never be negative. So, $x = 9$. Hence, the required number $= 10 \times 9 + \frac{18}{9} = 92$.

EXAMPLE 8 *A two digit number is such that the product of the digits is 14. When 45 is added to the number, then the digits are reversed. Find the number.*

SOLUTION Let the tens digit be x. Then, units digit $= \frac{14}{x}$.

$\therefore \quad$ Number $= 10x + \frac{14}{x}$

and, Number formed by reversing the digits $= 10 \times \frac{14}{x} + x$

$\therefore \quad 10x + \frac{14}{x} + 45 = 10 \times \frac{14}{x} + x$ [Given]

$\Rightarrow \quad 10x + \frac{14}{x} + 45 = \frac{140}{x} + x$

$\Rightarrow \quad 9x - \frac{126}{x} + 45 = 0$

$\Rightarrow \quad 9x^2 + 45x - 126 = 0$

$\Rightarrow \quad x^2 + 5x - 14 = 0$

$\Rightarrow \quad x^2 + 7x - 2x - 14 = 0$

$\Rightarrow \quad (x+7)(x-2) = 0 \Rightarrow x = -7 \text{ or, } x = 2 \Rightarrow x = 2$

Hence, the required number $= 10 \times 2 + \frac{14}{2} = 27$.

EXAMPLE 9 *Find two consecutive odd positive integers, sum of whose squares is 290.* [NCERT, CBSE 2014]

SOLUTION Let x be an odd positive integer. Then, an odd positive integer just greater than x is $x+2$. It is given that

$x^2 + (x+2)^2 = 290$

$\Rightarrow \quad 2x^2 + 4x + 4 = 290$

$\Rightarrow \quad 2x^2 + 4x - 286 = 0$

$\Rightarrow \quad x^2 + 2x - 143 = 0$

$\Rightarrow \quad x^2 + 13x - 11x - 143 = 0$

$\Rightarrow \quad x(x+13) - 11(x+13) = 0$

$\Rightarrow \quad (x+13)(x-11) = 0$

$\Rightarrow \quad x - 11 = 0$ $[\because x > 0 \therefore x + 13 \neq 0]$

$\Rightarrow \quad x = 11$

Hence, required integers are 11 and 13.

LEVEL-2

EXAMPLE 10 *If the sum of first n even natural numbers is 420, find the value of n.*

SOLUTION We have,

$2 + 4 + 6 + 8 + \dots \text{ to } n \text{ terms} = 420$

$\Rightarrow \quad \frac{n}{2}[2 \times 2 + (n-1) \times 2] = 420$

$\Rightarrow \quad n(2 + n - 1) = 420$

$\Rightarrow \quad n(n+1) = 420$

$\Rightarrow \quad n^2 + n - 420 = 0$

$\Rightarrow \quad n^2 + 21n - 20n - 420 = 0$

$\Rightarrow \quad n(n+21) - 20(n+21) = 0$

$\Rightarrow \quad (n+21)(n-20) = 0$

$\Rightarrow \quad n = 20, -21 \Rightarrow n = 20$ $[\because n \text{ is a natural number} \therefore n > 0]$

EXAMPLE 11 *The denominator of a fraction is one more than twice the numerator. If the sum of the fraction and its reciprocal is* $2\frac{16}{21}$, *find the fraction.* [CBSE 2016]

SOLUTION Let the numerator of the fraction be x. Then,

Denominator $= 2x + 1$ [Given]

$\therefore$ Fraction $= \frac{x}{2x+1} \Rightarrow$ Reciprocal of the fraction $= \frac{2x+1}{x}$

It is given that the sum of the fraction and its reciprocal is $2\frac{16}{21}$.

$$\therefore \quad \frac{x}{2x+1} + \frac{2x+1}{x} = 2\frac{16}{21}$$

$$\Rightarrow \quad \frac{x^2 + (2x+1)^2}{x(2x+1)} = \frac{58}{21}$$

$$\Rightarrow \quad \frac{5x^2 + 4x + 1}{2x^2 + x} = \frac{58}{21}$$

$$\Rightarrow \quad 21(5x^2 + 4x + 1) = 58(2x^2 + x)$$

$$\Rightarrow \quad 105x^2 + 84x + 21 = 116x^2 + 58x$$

$$\Rightarrow \quad 11x^2 - 26x - 21 = 0$$

$$\Rightarrow \quad 11x^2 - 33x + 7x - 21 = 0$$

$$\Rightarrow \quad 11x(x-3) + 7(x-3) = 0$$

$$\Rightarrow \quad (11x+7)(x-3) = 0$$

$$\Rightarrow \quad x = 3, -\frac{7}{11} \Rightarrow x = 3$$ [$\because x$ is a natural number $\therefore x > 0$]

Hence, fraction $= \frac{x}{2x+1} = \frac{3}{7}$

EXAMPLE 12 *A two-digit number is four times the sum and three times the product of its digits. Find the number.* [CBSE 2016]

SOLUTION Let the digits at tens and units place of the number be x and y respectively. Then,

Number $= 10x + y$

It is given that

Number $= 4 \times$ Sum of the digits. Also, Number $= 3 \times$ Product of digits

$\Rightarrow \quad 10x + y = 4(x + y)$ and $10x + y = 3xy$

$\Rightarrow \quad 6x - 3y = 0$ and $10x + y = 3xy$

$\Rightarrow \quad y = 2x$ and $10x + y = 3xy$

$\Rightarrow \quad 10x + 2x = 3x \times 2x$ [On eliminating y]

$\Rightarrow \quad 6x^2 - 12x = 0$

$\Rightarrow \quad 6x(x-2) = 0 \Rightarrow x = 0$ or, $x = 2$

Since the given number is a two-digit number. So, its tens digit cannot be zero.

$\therefore \quad x = 2 \Rightarrow y = 2 \times 2 = 4$ [$\because y = 2x$]

Hence, required number $= 10x + y = 10 \times 2 + 4 = 24$.

EXAMPLE 13 *If the sum of n successive odd natural numbers starting from 3 is 48, find the value of n.*

SOLUTION We have,

$$3 + 5 + 7 + 9 + \dots \text{ to } n \text{ terms} = 48$$

$$\Rightarrow \quad \frac{n}{2}\{2 \times 3 + (n-1) \times 2\} = 48 \quad \left[\text{Using}: S_n = \frac{n}{2}\{2a + (n-1)d\} \text{ where } a = 3 \text{ and } d = 2\right]$$

$$\Rightarrow \quad n(3 + n - 1) = 48$$

$$\Rightarrow \quad n^2 + 2n - 48 = 0$$

$$\Rightarrow \quad n^2 + 8n - 6n - 48 = 0$$

$$\Rightarrow \quad n(n+8) - 6(n+8) = 0$$

$$\Rightarrow \quad (n+8)(n-6) = 0$$

$$\Rightarrow \quad n = -8 \text{ or, } n = 6 \Rightarrow n = 6 \quad [\because n > 0]$$

LEVEL-3

EXAMPLE 14 *One-fourth of a herd of camels was seen in the forest. Twice the square root of the herd had gone to mountains and the remaining 15 camels were seen on the bank of a river. Find the total number of camels.*

SOLUTION Let the total number of camels be x. Then,

Number of camels seen in the forest $= \frac{x}{4}$

Number of camels gone to mountains $= 2\sqrt{x}$

Number of camels on the bank of river = 15

Total number of camels $= \frac{x}{4} + 2\sqrt{x} + 15$

By hypothesis, we have

$$\frac{x}{4} + 2\sqrt{x} + 15 = x$$

$$\Rightarrow \quad 3x - 8\sqrt{x} - 60 = 0$$

$$\Rightarrow \quad 3y^2 - 8y - 60 = 0, \text{ where } x = y^2$$

$$\Rightarrow \quad 3y^2 - 18y + 10y - 60 = 0$$

$$\Rightarrow \quad 3y(y-6) + 10(y-6) = 0$$

$$\Rightarrow \quad (3y+10)(y-6) = 0$$

$$\Rightarrow \quad y = 6 \text{ or, } y = -\frac{10}{3}$$

Now, $y = -\frac{10}{3} \Rightarrow x = \left(-\frac{10}{3}\right)^2 = \frac{100}{9}$ $\quad [\because x = y^2]$

But, the number of camels cannot be a fraction.

$\therefore \quad y = 6 \Rightarrow x = 6^2 = 36$ $\quad [\because x = y^2]$

Hence, the number of camels = 36.

EXAMPLE 15 *O Girl! Out of a group of swans, $\frac{7}{2}$ times the square root of the number are playing on the shore of a tank. The two remaining ones are playing, with amorous fight, in the water. What is the total number of swans?*

SOLUTION Let the total number of swans be x. Then,

$$\text{Number of swans playing on the shore of the tank} = \frac{7}{2}\sqrt{x}$$

It is given that there are two remaining swans.

$\therefore \quad x = \frac{7}{2}\sqrt{x} + 2$

$\Rightarrow \quad x - \frac{7}{2}\sqrt{x} - 2 = 0$

$\Rightarrow \quad y^2 - \frac{7}{2}y - 2 = 0$, where $y^2 = x$

$\Rightarrow \quad 2y^2 - 7y - 4 = 0$

$\Rightarrow \quad 2y^2 - 8y + y - 4 = 0$

$\Rightarrow \quad 2y(y-4) + (y-4) = 0$

$\Rightarrow \quad (y-4)(2y+1) = 0$

$\Rightarrow \quad y = 4$ or, $y = -\frac{1}{2}$

$\Rightarrow \quad y = 4 \qquad \left[\because y = -\frac{1}{2} \text{ is not possible}\right]$

$\Rightarrow \quad x = y^2 = 4^2 = 16$

Hence, the total number of swans is 16.

EXERCISE 4.7

LEVEL-1

1. Find two consecutive numbers whose squares have the sum 85. [CBSE 2000]
2. Divide 29 into two parts so that the sum of the squares of the parts is 425.
3. Two squares have sides x cm and $(x+4)$ cm. The sum of their areas is 656 cm^2. Find the sides of the squares.
4. The sum of two numbers is 48 and their product is 432. Find the numbers.
5. If an integer is added to its square, the sum is 90. Find the integer with the help of quadratic equation.
6. Find the whole number which when decreased by 20 is equal to 69 times the reciprocal of the number.
7. Find two consecutive natural numbers whose product is 20.
8. The sum of the squares of two consecutive odd positive integers is 394. Find them. [CBSE 2009, 2017]
9. The sum of two numbers is 8 and 15 times the sum of their reciprocals is also 8. Find the numbers.
10. The sum of a number and its positive square root is 6/25. Find the number.
11. The sum of a number and its square is 63/4, find the numbers.

12. There are three consecutive integers such that the square of the first increased by the product of the other two gives 154. What are the integers?
13. The product of two successive integral multiples of 5 is 300. Determine the multiples.
14. The sum of the squares of two numbers is 233 and one of the numbers is 3 less than twice the other number. Find the numbers.
15. Find the consecutive even integers whose squares have the sum 340.
16. The difference of two numbers is 4. If the difference of their reciprocals is $\frac{4}{21}$, find the numbers. [CBSE 2008]
17. Find two natural numbers which differ by 3 and whose squares have the sum 117.
18. The sum of the squares of three consecutive natural numbers is 149. Find the numbers.
19. The sum of two numbers is 16. The sum of their reciprocals is 1/3. Find the numbers. [CBSE 2005]
20. Determine two consecutive multiples of 3 whose product is 270.
21. The sum of a number and its reciprocal is 17/4. Find the number.
22. A two-digit number is such that the product of its digits is 8. When 18 is subtracted from the number, the digits interchange their places. Find the number.
23. A two-digit number is such that the product of the digits is 12. When 36 is added to the number the digits interchange their places. Determine the number.
24. A two-digit number is such that the product of the digits is 16. When 54 is subtracted from the number, the digits are interchanged. Find the number.
25. Two numbers differ by 3 and their product is 504. Find the numbers. [CBSE 2002 C]
26. Two numbers differ by 4 and their product is 192. Find the numbers. [CBSE 2000 C]
27. A two digit number is 4 times the sum of its digits and twice the product of its digits. Find the number.
28. The difference of the squares of two positive integers is 180. The square of the smaller number is 8 times the larger, find the numbers. [CBSE 2014]
29. The sum of two numbers is 18. The sum of their reciprocals is 1/4. Find the numbers. [CBSE 2005]
30. The sum of two numbers a and b is 15, and the sum of their reciprocals $\frac{1}{a}$ and $\frac{1}{b}$ is 3/10. Find the numbers a and b. [CBSE 2005]
31. The sum of two numbers is 9. The sum of their reciprocals is 1/2. Find the numbers. [CBSE 2012]
32. Three consecutive positive integers are such that the sum of the square of the first and the product of other two is 46, find the integers. [CBSE 2010]
33. The difference of squares of two numbers is 88. If the larger number is 5 less than twice the smaller number, then find the two numbers. [CBSE 2010]
34. The difference of squares of two numbers is 180. The square of the smaller number is 8 times the larger number. Find two numbers. [NCERT]
35. Find two consecutive odd positive integers, sum of whose squares is 970. [CBSE 2014]
36. The difference of two natural numbers is 3 and the difference of their reciprocals is $\frac{3}{28}$. Find the numbers. [CBSE 2014]
37. The sum of the squares of two consecutive odd numbers is 394. Find the numbers. [CBSE 2014]

38. The sum of the squares of two consecutive multiples of 7 is 637. Find the multiples. [CBSE 2014]

39. The sum of the squares of two consecutive even numbers is 340. Find the numbers. [CBSE 2014]

40. The numerator of a fraction is 3 less than the denominator. If 2 is added to both the numerator and the denominator, then the sum of the new fraction and the original fraction is $\frac{29}{20}$. Find the original fraction. [CBSE 2015]

41. Find a natural number whose square diminished by 84 is equal to thrice of 8 more than the given number. . [NCERT EXEMPLAR]

42. A natural number when increased by 84 equals 160 times its reciprocal. Find the number. [NCERT EXEMPLAR]

ANSWERS

1. 6, 7 or − 6, − 7   2. 13, 16   3. 16 cm, 20 cm   4. 36, 12
5. − 10, 9   6. 23   7. 4, 5   8. 13, 15
9. 3, 5   10. $\frac{1}{25}$ 11. $\frac{7}{2}$ or $-\frac{9}{2}$ 12. 8, 9, 10
13. 15, 20 or − 20, − 15   14. 8, 13   15. 12, 14
16. 7, 3 or − 3, − 7   17. 6, 9   18. 6, 7, 8   19. 4, 12
20. 15, 18   21. 4 or $\frac{1}{4}$ 22. 42   23. 26
24. 82   25. 21, 24, or − 24, − 21   26. 12, 16, or − 16, − 12
27. 36   28. 8, 12   29. 12, 6
30. $a = 5, b = 10$ or $a = 10, b = 5$ 31. 3, 6   32. 4, 5, 6
33. 13, 9   34. 18, 12; 18, −12   35. 21, 23   36. 7, 4
37. 13, 15   38. 14, 21   39. 12, 14   40. $\frac{7}{10}$
41. 12   42. 8

HINTS TO SELECTED PROBLEMS

1. Let the natural numbers be x and $x + 1$. Then, by hypothesis, we have
$$x^2 + (x+1)^2 = 85.$$

2. Let the two parts be x and $29 - x$. Then, by hypothesis, we have
$$x^2 + (29 - x)^2 = 425.$$

4. Let the numbers be x and $48 - x$. Then, by using the given condition, we have
$$x(48 - x) = 432.$$

5. We have, $x + x^2 = 90 \Rightarrow x^2 + x - 90 = 0 \Rightarrow (x + 10)(x - 9) = 0$

6. Let the whole number be x. It is given that
$$(x - 20) = 60\left(\frac{1}{x}\right) \Rightarrow x^2 - 20x - 69 = 0 \Rightarrow (x - 23)(x + 3) = 0 \Rightarrow x = 23, -3.$$

7. Let the numbers be x and $x + 1$. It is given that
$$x(x + 1) = 20 \Rightarrow x^2 + x - 20 = 0$$

8. Let the consecutive odd positive integers be $2x - 1$ and $2x + 1$. Then,
$$(2x - 1)^2 + (2x + 1)^2 = 394 \Rightarrow 8x^2 + 2 = 394 \Rightarrow 4x^2 = 392 \Rightarrow x = 6$$

9. Let the numbers be x and $8 - x$. It is given that

$$15\left(\frac{1}{x}+\frac{1}{8-x}\right)=8 \Rightarrow 15=x(8-x) \Rightarrow x^2-8x+15=0$$

10. Let the number be x. It is given that

$$x+\sqrt{x}=\frac{6}{25} \Rightarrow y^2+y=6/25, \text{ where } x=y^2 \Rightarrow 25y^2+25y-6=0.$$

11. Let the number be x. It is given that: $x+x^2=\frac{63}{4}$.
12. Let the integers be x, $x+1$ and $x+2$. Then, $x^2+(x+1)(x+2)=154$.
13. Let the successive multiples of 5 be $5x$ and $5(x+1)$. Then,
$5x.5(x+1)=300 \Rightarrow x^2+x=12 \Rightarrow x^2+x-12=0$
14. Let one number be x. Then, other number $=2x-3$. It is given that: $x^2+(2x-3)^2=233$.
15. Let the consecutive even integers be $2x$ and $2x+2$. Then, by hypothesis
$(2x)^2+(2x+2)^2=340 \Rightarrow 8x^2+8x-336=0 \Rightarrow x^2+x-42=0$.
17. Let the numbers be x and $x-3$. Then, $x^2+(x-3)^2=117$.
18. Let the numbers be x, $x+1$ and $x+2$. Then, $x^2+(x+1)^2+(x+2)^2=149$.
19. Let the two parts be x and $57-x$. Then, $x(57-x)=782$.
20. Let the required numbers be $3x$ and $3x+3$. Then, $(3x)(3x+3)=270$

4.8.2 APPLICATIONS OF QUADRATIC EQUATIONS FOR SOLVING PROBLEMS ON TIME AND DISTANCE

For solving problems on time and distance, we use the following formulae:

$$\text{Speed}=\frac{\text{Distance}}{\text{Time}}, \text{ Time}=\frac{\text{Distance}}{\text{Speed}}$$

Following example will illustrate the same.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *A train travels a distance of 300 km at constant speed. If the speed of the train is increased by 5 km an hour, the journey would have taken 2 hours less. Find the original speed of the train.*

SOLUTION Let x km/hr be the constant speed of the train. Then,

Time taken to cover 300 km $=\frac{300}{x}$ hrs.

Time taken to cover 300 km when the speed is increased by 5 km/hr $=\frac{300}{x+5}$ hours.

It is given that the time to cover 300 km is reduced by 2 hours.

$$\therefore \quad \frac{300}{x}-\frac{300}{x+5}=2$$

$$\Rightarrow \quad \frac{300(x+5)-300x}{x(x+5)}=2$$

$$\Rightarrow \quad \frac{300x+1500-300x}{x^2+5x}=2$$

$$\Rightarrow \quad 2x^2+10x=1500$$

$$\Rightarrow \quad x^2+5x-750=0$$

$$\Rightarrow \quad x^2+30x-25x-750=0$$

$$\Rightarrow \quad x(x+30)-25(x+30)=0$$

$\Rightarrow \quad (x+30)(x-25)=0 \Rightarrow x=25$ or, $x=-30$.

But, x cannot be negative. Therefore, $x = 25$.

Hence, the original speed of the train is 25 km/hr.

EXAMPLE 2 *The speed of a boat in still water is 15 km/hr. It can go 30 km upstream and return downstream to the original point in 4 hours 30 minutes. Find the speed of the stream.* [CBSE 2017]

SOLUTION Let the speed of the stream be x km/hr. Then,

Speed downstream = $(15 + x)$ km/hr.

$\therefore$ Speed upstream = $(15 - x)$ km/hr.

Time taken by the boat to go 30 km upstream $= \frac{30}{15-x}$ hours.

Time taken by the boat to return 30 km downstream $= \frac{30}{15+x}$ hours.

It is given that the boat returns to the same point in 4 hours 30 minutes

$$\therefore \quad \frac{30}{15-x}+\frac{30}{15+x}=\frac{9}{2}$$

$$\Rightarrow \quad \frac{30(15+x)+30(15-x)}{(15+x)(15-x)}=\frac{9}{2}$$

$$\Rightarrow \quad \frac{450+30x+450-30x}{225-x^2}=\frac{9}{2}$$

$$\Rightarrow \quad \frac{900}{225-x^2}=\frac{9}{2}$$

$$\Rightarrow \quad 9(225-x^2)=1800$$

$$\Rightarrow \quad 225-x^2=200 \Rightarrow x^2=25 \Rightarrow x=\pm 5$$

But, the speed of the stream can never be negative.

Hence, the speed of the stream is 5 km/hr.

EXAMPLE 3 *A fast train takes 3 hours less than a slow train for a journey of 600 km. If the speed of the slow train is 10 km/hr less than that of the fast train, find the speeds of the two trains.*

SOLUTION Let the speed of the slow train be x km/hr. Then, speed of the fast train is $(x + 10)$ km/hr.

Time taken by the slow train to cover 600 km $= \frac{600}{x}$ hrs

Time taken by the fast train to cover 600 km $= \frac{600}{x+10}$ hrs

$$\therefore \quad \frac{600}{x}-\frac{600}{x+10}=3$$

$$\Rightarrow \quad \frac{600(x+10)-600x}{x(x+10)}=3$$

$$\Rightarrow \quad \frac{6000}{x^2+10x}=3$$

$$\Rightarrow \quad 3(x^2+10x)=6000$$

$$\Rightarrow \quad x^2+10x-2000=0$$

$$\Rightarrow \quad x^2+50x-40x-2000=0$$

$$\Rightarrow \quad x(x+50)-40(x+50)=0$$

$\Rightarrow \quad (x+50)(x-40)=0$

$\Rightarrow \quad x=-50$ or $x=40 \Rightarrow x=40$ [$\because$ x cannot be negative]

Hence, the speeds of two trains are 40 km/hr and 50 km/hr.

EXAMPLE 4 *A plane left 30 minutes later than the schedule time and in order to reach its destination 1500 km away in time it has to increase its speed by 250 km/hr from its usual speed. Find its usual speed.* [CBSE 2016]

SOLUTION Let the usual speed of the plane be x km/hr. Then,

Time taken to cover 1500 km with the usual speed $=\frac{1500}{x}$ hrs

Time taken to cover 1500 km with the speed of $(x+250)$ km/hr $=\frac{1500}{x+250}$

$$\therefore \quad \frac{1500}{x}=\frac{1500}{x+250}+\frac{1}{2}$$

$$\Rightarrow \quad \frac{1500}{x}-\frac{1500}{x+250}=\frac{1}{2}$$

$$\Rightarrow \quad \frac{1500x+1500\times 250-1500x}{x(x+250)}=\frac{1}{2}$$

$$\Rightarrow \quad \frac{1500\times 250}{x^2+250x}=\frac{1}{2}$$

$$\Rightarrow \quad 750000=x^2+250x$$

$$\Rightarrow \quad x^2+250x-750000=0$$

$$\Rightarrow \quad x^2+1000x-750x-750000=0$$

$$\Rightarrow \quad x(x+1000)-750(x+1000)=0$$

$$\Rightarrow \quad (x+1000)(x-750)=0$$

$\Rightarrow \quad x=-1000$ or, $x=750 \Rightarrow x=750$ [$\because$ speed cannot be negative]

Hence, the usual speed of the plane is 750 km/hr.

EXAMPLE 5 *In a flight of 600 km, a aircraft was slowed down due to bad weather. Its average speed for the trip was reduced by 200 km/hr and the time of flight increased by 30 minutes. Find the duration of flight.*

SOLUTION Let the original speed of the aircraft be x km/hr. Then,

New speed $=(x-200)$ km/hr.

Duration of flight at original speed $=\left(\frac{600}{x}\right)$ hr

Duration of flight at reduced speed $=\left(\frac{600}{x-200}\right)$ hr

$$\therefore \quad \frac{600}{x-200}-\frac{600}{x}=\frac{1}{2}$$

$$\Rightarrow \quad \frac{600x-600(x-200)}{x(x-200)}=\frac{1}{2}$$

$$\Rightarrow \quad \frac{120000}{x^2-200x}=\frac{1}{2}$$

$$\Rightarrow \quad x^2-200x-240000=0$$

$\Rightarrow \quad x^2 - 600x + 400x - 240000 = 0$

$\Rightarrow \quad x(x - 600) + 400(x - 600) = 0$

$\Rightarrow \quad (x - 600)(x + 400) = 0$

$\Rightarrow \quad x - 600 = 0 \text{ or, } x + 400 = 0$

$\Rightarrow \quad x = 600 \text{ or, } x = -400 \Rightarrow x = 600$ [$\because x$ cannot be negative]

So, the original speed of the aircraft was 600 km/hr.

Hence, Duration of flight $= \left(\frac{600}{x}\right)\text{hr} = \left(\frac{600}{600}\right)\text{hr} = 1 \text{ hr}$

LEVEL-2

EXAMPLE 6 *Swati can row her boat at a speed of 5 km/hr in still water. If it takes her 1 hour more to row the boat 5.25 km upstream than to return downstream, find the speed of the stream.*

SOLUTION Let the speed of the stream be x km/hr.

$\therefore$ Speed of the boat upstream $= (5 - x)$ km/hr.

Speed of the boat downstream $= (5 + x)$ km/hr.

Time taken for going 5.25 km upstream $= \frac{5.25}{5 - x}$ hours.

Time taken for going 5.25 km downstream $= \frac{5.25}{5 + x}$ hours.

Obviously, time taken for going 5.25 km upstream is more than the time taken for going 5.25 km. downstream.

It is given that the time taken for going 5.25 km. upstream is 1 hour more than the time taken for going 5.25 downstream.

$\therefore \quad \frac{5.25}{5 - x} - \frac{5.25}{5 + x} = 1$

$\Rightarrow \quad 5.25\left\{\frac{1}{5 - x} - \frac{1}{5 + x}\right\} = 1$

$\Rightarrow \quad \frac{21}{4}\left\{\frac{5 + x - 5 + x}{(5 - x)(5 + x)}\right\} = 1$

$\Rightarrow \quad \frac{21}{4} \times \frac{2x}{25 - x^2} = 1$

$\Rightarrow \quad \frac{21}{2} \times \frac{x}{25 - x^2} = 1$

$\Rightarrow \quad 21x = 50 - 2x^2$

$\Rightarrow \quad 2x^2 + 21x - 50 = 0$

$\Rightarrow \quad 2x^2 + 25x - 4x - 50 = 0$

$\Rightarrow \quad x(2x + 25) - 2(2x + 25) = 0$

$\Rightarrow \quad (2x + 25)(x - 2) = 0$

$\Rightarrow \quad x - 2 = 0, 2x + 25 = 0 \Rightarrow x = 2$ $\left[\because x \neq -\frac{25}{2} \text{ as } x > 0\right]$

Hence, the speed of the stream is 2 km/hr.

EXAMPLE 7 *Two trains leave a railway station at the same time. The first train travels due west and the second train due north. The first train travels 5 km/hr faster than the second train. If after two hours, they are 50 km apart, find the average speed of each train.*

SOLUTION Let the speed of the second train be x km/hr. Then, the speed of the first train is $(x+5)$ km/hr.

Let O be the position of the railway station from which the two trains leave.

Distance travelled by the first train in 2 hours $= OA = \text{Speed} \times \text{Time}$
$= 2(x+5)$ km

Distance travelled by the second train in 2 hours $= OB = \text{Speed} \times \text{Time} = 2x$ km

By using pythagoras theorem, we have,

$AB^2 = OA^2 + OB^2$

$\Rightarrow \quad 50^2 = \{2(x+5)\}^2 + \{2x\}^2$

$\Rightarrow \quad 2500 = 4(x+5)^2 + 4x^2$

$\Rightarrow \quad 8x^2 + 40x - 2400 = 0$

$\Rightarrow \quad x^2 + 5x - 300 = 0$

$\Rightarrow \quad x^2 + 20x - 15x - 300 = 0$

$\Rightarrow \quad x(x+20) - 15(x+20) = 0$

$\Rightarrow \quad (x+20)(x-15) = 0$

$\Rightarrow \quad x = -20$ or, $x = 15$

$\Rightarrow \quad x = 15$ [$\because x$ cannot be negative]

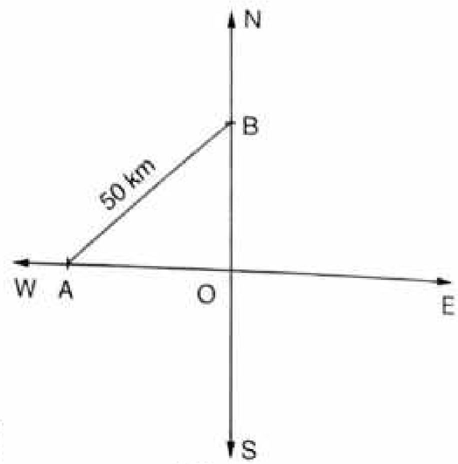

Fig. 4.1

Hence, the speed of the second train is 15 km/hr and, the speed of the first train is 20 km/hr.

EXERCISE 4.8

LEVEL-1

1. The speed of a boat in still water is 8 km/hr It can go 15 km upstream and 22 km downstream in 5 hours. Find the speed of the stream.
2. A train, travelling at a uniform speed for 360 km, would have taken 48 minutes less to travel the same distance if its speed were 5 km/hr more. Find the original speed of the train. **[NCERT EXEMPLAR]**
3. A fast train takes one hour less than a slow train for a journey of 200 km. If the speed of the slow train is 10 km/hr less than that of the fast train, find the speed of the two trains.
4. A passenger train takes one hour less for a journey of 150 km if its speed is increased by 5 km/hr from its usual speed. Find the usual speed of the train.
5. The time taken by a person to cover 150 km was 2.5 hrs more than the time taken in the return journey. If he returned at a speed of 10 km/hr more than the speed of going, what was the speed per hour in each direction?
6. A plane left 40 minutes late due to bad weather and in order to reach its destination, 1600 km away in time, it had to increase its speed by 400 km/hr from its usual speed. Find the usual speed of the plane. **[CBSE 2018]**
7. An aeroplane takes 1 hour less for a journey of 1200 km if its speed is increased by 100 km/hr from its usual speed. Find its usual speed.
8. A train travels at a certain average speed for a distance 63 km and then travels a distance of 72 km at an average speed of 6 km/hr more than the original speed. If it takes 3 hours to complete total journey, what is its original asverage speed? **[NCERT EXEMPLAR, CBSE 2018]**

9. A train covers a distance of 90 km at a uniform speed. Had the speed been 15 km/hour more, it would have taken 30 minutes less for the journey. Find the original speed of the train. [CBSE 2006C]

10. A train travels 360 km at a uniform speed. If the speed had been 5 km/hr more, it would have taken 1 hour less for the same journey. Find the speed of the train. [NCERT]

11. An express train takes 1 hour less than a passenger train to travel 132 km between Mysore and Bangalore (without taking into consideration the time they stop at intermediate stations). If the average speed of the express train is 11 km/hr more than that of the passenger train, find the average speeds of the two trains. [NCERT]

12. An aeroplane left 50 minutes later than its scheduled time, and in order to reach the destination, 1250 km away, in time, it had to increase its speed by 250 km/hr from its usual speed. Find its usual speed. [CBSE 2010]

13. While boarding an aeroplane, a passenger got hurt. The pilot showing promptness and concern, made arrangements to hospitalise the injured and so the plane started late by 30 minutes to reach the destination, 1500 km away in time, the pilot increased the speed by 100 km/hr. Find the original speed/hour of the plane. [CBSE 2013]

14. A motor boat whose speed in still water is 18 km/hr takes 1 hour more to go 24 km up stream that to return down stream to the same spot. Find the speed of the stream. [CBSE 2014, 2018]

15. A car moves a distance of 2592 km with uniform speed. The number of hours taken for the journey is one-half the number representing the speed, in km/hour. Find the time taken to cover the distance. [CBSE 2017]

ANSWERS

1. 3 km/hr 2. 45 km/hr 3. 50 km/hr, 40 km/hr
4. 25 km/hr 5. 20 km/hr, 30 km/hr 6. 800 km/hr
7. 300 km/hr 8. 42 km/hr 9. 45 km/hr 10. 40 km/hr
11. Speed of the passenger train = 33 km/hr, Speed of the express train = 44 km/hr
12. 500 km/hr 13. 500 km/hr 14. 6 km/hr 15. 36 hours

HINTS TO SELECTED PROBLEMS

2. Let the usual speed of the train be x km/hr. Then,

$$\frac{360}{x} - \frac{360}{x+10} = 3 \Rightarrow 1200 = x(x+10) \Rightarrow x^2 + 10x - 1200 = 0$$

3. Let the speed of the fast train be x km/hr. Then, speed of slow train = $(x-10)$ km/hr. Since, fast train takes one hour less than a slow train to cover 200 km.

$$\therefore \quad \frac{200}{x-10} - \frac{200}{x} = 1$$

4. Let the usual speed be x km./hr. Then, $\frac{150}{x} - \frac{150}{x+5} = 1$.

10. Let the speed in the upward journey be x km/hr. Then, the speed in the return journey = $(x+5)$ km/hr.

$$\therefore \quad \frac{360}{x} - \frac{360}{x+5} = 1$$

4.8.3 APPLICATIONS OF QUADRATIC EQUATIONS FOR SOLVING PROBLEMS ON AGES

The following illustrations will illustrate the problems on ages.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *One year ago, a man was 8 times as old as his son. Now his age is equal to the square of his son's age. Find their present ages.*

SOLUTION Suppose, one year ago, son's age be x years.
Then, man's age one year ago = $8x$ years.

$\therefore$ Present age of son = $(x + 1)$ years and, Present age of man = $(8x + 1)$ years.

$\therefore$ $8x + 1 = (x + 1)^2$ [Given]

$\Rightarrow$ $x^2 - 6x = 0$

$\Rightarrow$ $x(x - 6) = 0$

$\Rightarrow$ $x = 0$ or, $x = 6$

$\Rightarrow$ $x = 6$ [$\because$ Son's age cannot be 0]

So, Present age of son = $(x + 1)$ years = 7 years.

and, Present age of man = $(8x + 1)$ years = 49 years.

EXAMPLE 2 *The product of Ramu's age (in years) five years ago with his age (in years) 9 years later is 15. Find Ramu's present age.*

SOLUTION Let Ramu's present age be x years. Then,

His age 5 years ago = $(x - 5)$ years.

His age 9 years later = $(x + 9)$ years.

It is given that the product of these ages is 15.

$\therefore$ $(x - 5)(x + 9) = 15$

$\Rightarrow$ $x^2 + 4x - 60 = 0$

$\Rightarrow$ $x^2 + 10x - 6x - 60 = 0$

$\Rightarrow$ $(x + 10)(x - 6) = 0$

$\Rightarrow$ $x = 6$ or, $x = -10$

But, $x \neq -10$. So, $x = 6$.

Hence, Ramu's present age is 6 years.

EXAMPLE 3 *The sum of ages of a father and his son is 45 years. Five years ago, the product of their ages (in years) was 124. Determine their present ages.*

SOLUTION Let the present age of father be x years. Then,

Son's present age = $(45 - x)$ yeras.

Five years ago:

Father's age = $(x - 5)$ years

Son's age = $(45 - x - 5)$ years = $(40 - x)$ years.

It is given that five years ago, the product of their ages was 124.

$\therefore$ $(x - 5)(40 - x) = 124$

$\Rightarrow$ $40x - x^2 - 200 + 5x = 124$

$\Rightarrow$ $x^2 - 45x + 324 = 0$

$\Rightarrow$ $x^2 - 36x - 9x + 324 = 0$

$\therefore$ $x(x - 36) - 9(x - 36) = 0$

$(x - 9)(x - 36) = 0 \Rightarrow x = 9$, or, $x = 36$

When $x = 36$, we have

Father's present age = 36 years

Son's present age = 9 years

When $x = 9$, we have

Father's present age = 9 years

Son's present age = 36 years

Clearly, this is not possible.

Hence, Father's present age = 36 years and Son's present age = 9 years.

LEVEL-2

EXAMPLE 4 *Seven years ago Varun's age was five times the square of Swati's age. Three years hence Swati's age will be two fifth of Varun's age. Find their present ages.* [CBSE 2006C]

SOLUTION Seven years ago, let Swati's age be x years. Then, seven years ago Varun's age was $5x^2$ years.

$\therefore$ Swati's present age = $(x + 7)$ years, Varun's present age = $(5x^2 + 7)$ years

Three years hence, we have

Swati's age = $(x + 7 + 3)$ years = $(x + 10)$ years

Varun's age = $(5x^2 + 7 + 3)$ years = $(5x^2 + 10)$ years

It is given that three years hence Swati's age will be $\frac{2}{5}$ of Varun's age.

$\therefore \quad x + 10 = \frac{2}{5}(5x^2 + 10)$

$\Rightarrow \quad x + 10 = 2x^2 + 4$

$\Rightarrow \quad 2x^2 - x - 6 = 0$

$\Rightarrow \quad 2x^2 - 4x + 3x - 6 = 0$

$\Rightarrow \quad 2x(x - 2) + 3(x - 2) = 0$

$\Rightarrow \quad (2x + 3)(x - 2) = 0$

$\Rightarrow \quad x - 2 = 0 \qquad [\because 2x + 3 \neq 0 \text{ as } x > 0]$

$\Rightarrow \quad x = 2$

Hence, Swati's present age = $(2 + 7)$ years = 9 years

Varun's present age = $(5 \times 2^2 + 7)$ years = 27 years

EXERCISE 4.9

LEVEL-1

1. Ashu is x years old while his mother Mrs Veena is x^2 years old. Five years hence Mrs Veena will be three times old as Ashu. Find their present ages.
2. The sum of the ages of a man and his son is 45 years. Five years ago, the product of their ages was four times the man's age at the time. Find their present ages.
3. The product of Shikha's age five years ago and her age 8 years later is 30, her age at both times being given in years. Find her present age.
4. The product of Ramu's age (in years) five years ago and his age (in years) nine years later is 15. Determine Ramu's present age.
5. Is the following situation possible? If so, determine their present ages. The sum of the ages of two friends is 20 years. Four years ago, the product of their ages in years was 48. [NCERT]
6. A girl is twice as old as her sister. Four years hence, the product of their ages (in years) will be 160. Find their present ages. [CBSE 2010]

7. The sum of the reciprocals of Rehman's ages (in years) 3 years ago and 5 years from now is 1/3. Find his present age. [NCERT]

LEVEL-2

8. If Zeba were younger by 5 years than what she really is, then the square of her age (in years) would have been 11 more than 5 times her actual age. What is her age now? [NCERT EXEMPLAR]

9. At present Asha's age (in years) is 2 more than the square of her daughter Nisha's age. When Nisha grows to her mother's present age, Asha's age would be one year less than 10 times the present age of Nisha. Find the present ages of both Asha and Nisha. [NCERT EXEMPLAR]

ANSWERS

1. 5 years, 25 years	2. 36 years, 9 years	3. 7 years	4. 6 years
5. No.	6. 6 years, 12 years	7. 7 years	8. 14 years
9. Asha: 27 years, Nisha: 5 years			

HINTS TO SELECTED PROBLEMS

2. Let the present age of the man be x years. Then, present age of his son is $(45-x)$ years. Five years ago, man's age $= (x-5)$ years. Son's age $= (45-x-5)$ years.
$\therefore \quad (x-5)(45-x-5) = 4(x-5)$
3. Let the present age be x years. Then, $(x-5)(x+8) = 30$.
5. Let the present ages of two friends be x years and $(20-x)$ years respectively. According to the given condition, we have
$$(x-4)(20-x-4) = 48 \Rightarrow (x-4)(16-x) = 48 \Rightarrow x^2 - 20x + 112 = 0$$
Let D be the discriminant of this quadratic. Then,
$$D = 400 - 448 = -48 < 0$$
So, above equation does not have real roots. Hence, the given situation is not possible.
7. Let his present age be x years. Then,
$$\frac{1}{x-3} + \frac{1}{x+5} = \frac{1}{3} \Rightarrow x^2 - 4x - 21 = 0 \Rightarrow x = 7$$

4.8.4 APPLICATIONS OF QUADRATIC EQUATIONS FOR SOLVING PROBLEMS IN GEOMETRY

The following examples will illustrate the above application.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *The hypotenuse of right-angled triangle is 6 metres more than twice the shortest side. If the third side is 2 metres less than the hypotenuse, find the sides of the triangle.*

SOLUTION Let the length of the shortest side be x metres. Then,

Hypotenuse $= (2x+6)$ metres

And, The third side $= (2x+6-2)$ metres $= (2x+4)$ metres

By, Pythagoras theorem, we have

$$(2x+6)^2 = x^2 + (2x+4)^2$$

$$\Rightarrow \quad x^2 - 8x - 20 = 0$$

$$\Rightarrow \quad x^2 - 10x - 2x - 20 = 0$$

$\Rightarrow$ $(x-10)(x+2)=0$

$\Rightarrow$ $x=10$ or, $x=-2 \Rightarrow x=10$ [Since side of a triangle is never negative]

$\therefore$ Length of the shortest side = 10 metres.

Length of the hypotenuse = $(2x+6)$ metres = 26 metres

Length of the third side = $(2x+4)$ metres = 24 metres

Hence, the sides of the triangle are 10 m, 26 m and 24 m.

EXAMPLE 2 *The hypotenuse of a grassy land in the shape of a right triangle is 1 metre more than twice the shortest side. If the third side is 7 metres more than the shortest side, find the sides of the grassy land.*

SOLUTION Let the length of the shortest side be x metres. Then, by hypothesis

Hypotenuse = $(2x+1)$ metres, Third side = $(x+7)$ metres.

By Pythagoras theorem, we have

$(\text{Hypotenuse})^2$ = Sum of the square of the remaining two sides

$$\Rightarrow \quad (2x+1)^2 = x^2+(x+7)^2$$

$$\Rightarrow \quad 4x^2+4x+1 = 2x^2+14x+49$$

$$\Rightarrow \quad 2x^2-10x-48=0$$

$$\Rightarrow \quad x^2-5x-24=0$$

$$\Rightarrow \quad x^2-8x+3x-24=0$$

$$\Rightarrow \quad x(x-8)+3(x-8)=0$$

$$\Rightarrow \quad (x-8)(x+3)=0$$

$$\Rightarrow \quad x=8,-3$$

$\Rightarrow$ $x=8$ [$\because x=-3$ is not possible]

Hence, the lengths of the sides of the grassy land are 8 metres, 17 metres and 15 metres.

LEVEL-2

EXAMPLE 3 *The hypotenuse of a right triangle is $3\sqrt{5}$ cm. If the smaller side is tripled and the larger side is doubled, the new hypotenuse will be 15 cm. Find the length of each side.*

SOLUTION Let the smaller side of the right triangle be x cm and the larger side by y cm. Then,

$x^2+y^2=(3\sqrt{5})^2$ [Using Pythagoras Theorem]

$\Rightarrow$ $x^2+y^2=45$ ...(i)

If the smaller side is tripled and the larger side be doubled, the new hypotenuse is 15 cm.

$\therefore$ $(3x)^2+(2y)^2=15^2 \Rightarrow 9x^2+4y^2=225$ ...(ii)

From equation (i), we get $y^2=45-x^2$

Putting $y^2=45-x^2$ in equation (ii), we get

$$9x^2+4(45-x^2)=225$$

$$\Rightarrow \quad 5x^2 + 180 = 225 \Rightarrow 5x^2 = 45 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$$

But, length of a side cannot be negative. Therefore, $x = 3$.

Putting $x = 3$ in (i), we get

$$9 + y^2 = 45 \Rightarrow y^2 = 36 \Rightarrow y = 6$$

Hence, the length of the smaller side is 3 cm and the length of the larger side is 6 cm.

EXAMPLE 4 *Vikram wishes to fit three rods together in the shape of a right triangle. The hypotenuse is to be 2 cm longer than the base and 4 cm longer than the altitude. What should be the lengths of the rods?*

SOLUTION Let the length of the hypotenuse be x cm. Then,

Base $= (x-2)$ cm and, Altitude $= (x-4)$ cm.

By Pythagoras theorem, we have

$$(\text{Base})^2 + (\text{Altitude})^2 = (\text{Hypotenuse})^2$$

$$\Rightarrow \quad (x-2)^2 + (x-4)^2 = x^2$$

$$\Rightarrow \quad x^2 - 4x + 4 + x^2 - 8x + 16 = x^2$$

$$\Rightarrow \quad x^2 - 12x + 20 = 0$$

$$\Rightarrow \quad x^2 - 10x - 2x + 20 = 0$$

$$\Rightarrow \quad x(x-10) - 2(x-10) = 0$$

$$\Rightarrow \quad (x-10)(x-2) = 0$$

$$\Rightarrow \quad x = 2, \text{ or, } x = 10 \Rightarrow x = 10 \qquad [\text{For } x = 2, \text{ Base} = 0 \text{ cm which is not possible}]$$

Hence, the length of the rods are 8 cm 6 cm and 10 cm.

EXERCISE 4.10

LEVEL-1

1. The hypotenuse of a right triangle is 25 cm. The difference between the lengths of the other two sides of the triangle is 5 cm. Find the lengths of these sides.
2. The diagonal of a rectangular field is 60 metres more than the shorter side. If the longer side is 30 metres more than the shorter side, find the sides of the field.

LEVEL-2

3. The hypotenuse of a right triangle is $3\sqrt{10}$ cm. If the smaller leg is tripled and the longer leg doubled, new hypotenuse will be $9\sqrt{5}$ cm. How long are the legs of the triangle?
4. A pole has to be erected at a point on the boundary of a circular park of diameter 13 metres in such a way that the difference of its distances from two diametrically opposite fixed gates A and B on the boundary is 7 metres. Is it the possible to do so? If yes, at what distances from the two gates should the pole be erected? [NCERT]

ANSWERS

1. 15 cm, 20 cm 2. 120 m, 90 m 3. 3 cm, 9 cm
4. At a distance of 5 metres from the gate B

HINTS TO SELECTED PROBLEMS

4. Let P be the required location of the pole such that its distance from gate B is x metres. i.e. $BP = x$ metres. Therefore, $AP = x + 7$. Applying Pythagoras theorem in right triangle APB, we obtain

$$AP^2 + PB^2 = AB^2$$

$$\Rightarrow (x+7)^2 + x^2 = 13^2$$

$$\Rightarrow 2x^2 + 14x - 120 = 0 \Rightarrow x^2 + 7x - 60 = 0 \Rightarrow (x+12)(x-5) = 0 \Rightarrow x = 5$$

4.8.5 APPLICATIONS OF QUADRATIC EQUATIONS FOR SOLVING PROBLEMS ON MENSURATION

Following examples will illustrate the above applications.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *The area of a right angled triangle is 600 cm². If the base of the triangle exceeds the altitude by 10 cm, find the dimensions of the triangle.*

SOLUTION Let the altitude BC of right-angled triangle ABC be x cm. Then,

$$\text{Base} = BC = (x + 10) \text{ cm}.$$

$$\therefore \quad \text{Area} = \frac{1}{2}(\text{Base} \times \text{Height})$$

$$\Rightarrow \quad \text{Area} = \frac{1}{2}(x+10)\,x \text{ cm}^2$$

$$\Rightarrow \quad \frac{1}{2}x(x+10) = 600 \qquad [\because \text{Area} = 600 \text{ cm}^2]$$

$$\Rightarrow \quad x^2 + 10x = 1200$$

$$\Rightarrow \quad x^2 + 10x - 1200 = 0$$

$$\Rightarrow \quad x^2 + 40x - 30x - 1200 = 0$$

$$\Rightarrow \quad x(x+40) - 30(x+40) = 0$$

$$\Rightarrow \quad (x+40)(x-30) = 0$$

$$\Rightarrow \quad x = 30, -40 \Rightarrow x = 30 \qquad [\because x > 0]$$

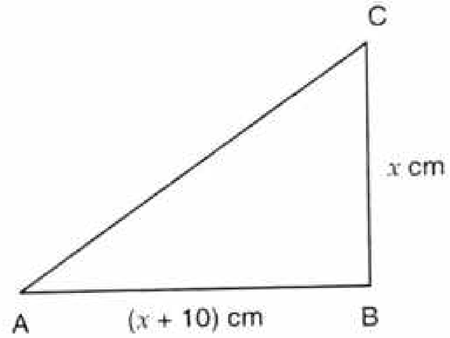

Fig. 4.2

Hence, Base = (30 + 10) cm = 40 cm and, Altitude = 30 cm.

EXAMPLE 2 *The perimeter of a rectangular field is 82 cm and its area is 400 m². Find the breadth of the rectangle.*

SOLUTION Let the breadth of the rectangle be x metres. Then,

$$\text{Perimeter} = 82 \text{ m}$$

$$\Rightarrow \quad 2(\text{Length} + \text{Breadth}) = 82$$

$$\Rightarrow \quad \text{Length} + x = 41$$

$$\Rightarrow \quad \text{Length} = 41 - x \text{ metres}$$

$$\text{Now,} \quad \text{Area} = 400 \text{ m}^2$$

$\Rightarrow$ Length × Breadth = 400

$\Rightarrow$ $(41 - x)\, x = 400$

$\Rightarrow$ $41x - x^2 = 400$

$\Rightarrow$ $x^2 - 41x + 400 = 0$

$\Rightarrow$ $x^2 - 25x - 16x + 400 = 0$

$\Rightarrow$ $x(x - 25) - 16(x - 25) = 0$

$\Rightarrow$ $(x - 25)(x - 16) = 0 \Rightarrow x = 25$ or, $x = 16$

Hence, breadth = 25 m or, 16 m.

EXAMPLE 3 *The length of the sides forming right angle of a right angled triangle are 5x cm and (3x – 1) cm. If the area of the triangle is 60 cm², find its hypotenuse.*

SOLUTION Let ABC be a right angled triangle with right angle at B.

Let $AB = 5x$ and $BC = 3x - 1$. Then,

$$\text{Area} = \Delta ABC = \frac{1}{2}(\text{Base} \times \text{Height})$$

$\Rightarrow$ $60 = \frac{1}{2}(AB \times BC)$

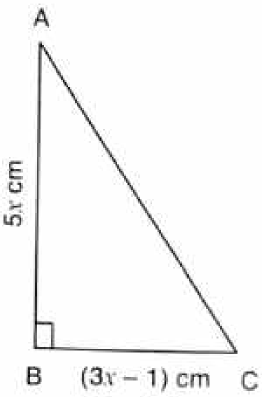

Fig. 4.3

$\Rightarrow$ $60 = \frac{1}{2} \times 5x(3x - 1)$

$\Rightarrow$ $120 = 5x(3x - 1)$

$\Rightarrow$ $24 = x(3x - 1)$

$\Rightarrow$ $3x^2 - x - 24 = 0$

$\Rightarrow$ $3x^2 - 9x + 8x - 24 = 0$

$\Rightarrow$ $3x(x - 3) + 8(x - 3) = 0$

$\Rightarrow$ $(x - 3)(3x + 8) = 0$

$\Rightarrow$ $x - 3 = 0$ or $3x + 8 = 0$

$\Rightarrow$ $x = 3$ or, $x = -\frac{8}{3} \Rightarrow x = 3$ [$\because x \neq -8/3$]

$\therefore$ $AB = 5x = 5 \times 3 = 15$ cm and $BC = (3x - 1) = (3 \times 3 - 1) = 8$ cm.

Now, $AC^2 = AB^2 + BC^2 \Rightarrow AC^2 = (15)^2 + (8)^2 \Rightarrow AC^2 = 289 \Rightarrow AC = 17$ cm.

Hence, Hypotenues = 17 cm.

EXAMPLE 4 *The length of a rectangle exceeds its width by 8 cm and the area of the rectangle is 240 sq. cm. Find the dimensions of the rectangle.*

SOLUTION Let the breadth of the given rectangle be x cm. Then, length = $(x + 8)$ cm.

Now, Area = 240 cm²

$\Rightarrow$ length × breadth = 240

$\Rightarrow$ $(x + 8)\, x = 240$

$\Rightarrow$ $x^2 + 8x - 240 = 0$

$\Rightarrow$ $x^2 + 20x - 12x - 240 = 0$

$\Rightarrow \quad x(x+20)-12(x+20)=0$

$\Rightarrow \quad (x+20)(x-12)=0$

$\Rightarrow \quad x=12 \text{ or, } x=-20$

But, x cannot be negative. So, $x = 12$.
Hence, length $= x + 8 = 12 + 8 = 20$ cm and breadth $= 12$ cm.

EXAMPLE 5 *The side of a square exceeds the side of the another square by 4 cm and the sum of the areas of the two squares is 400 sq. cm. Find the dimensions of the squares.*

SOLUTION Let S_1 and S_2 be two squares. Let the side of the square S_2 be x cm in length. Then, the side of square S_1 is $(x + 4)$ cm.

$\therefore \quad \text{Area of square } S_1 = (x+4)^2$ $\quad [\because \text{Area} = (\text{side})^2]$

and, $\quad \text{Area of square } S_2 = x^2$

It is given that

$\text{Area of square } S_1 + \text{Area of square } S_2 = 400 \text{ cm}^2$

$\Rightarrow \quad (x+4)^2 + x^2 = 400$

$\Rightarrow \quad (x^2+8x+16)+x = 400$

$\Rightarrow \quad 2x^2+8x-384=0$

$\Rightarrow \quad x^2+4x-192=0$

$\Rightarrow \quad x^2+16x-12x-192=0$

$\Rightarrow \quad x(x+16)-12(x+16)=0$

$\Rightarrow \quad (x+16)(x-12)=0$

$\Rightarrow \quad x=12 \text{ or, } x=-16$

As the length of the side of a square cannot be negative. Therefore, $x = 12$.

$\therefore \quad \text{Side of square } S_1 = x+4 = 12+4 = 16 \text{ cm}$ and, Side of square $S_2 = 12$ cm.

LEVEL-2

EXAMPLE 6 *If twice the area of a smaller square is subtracted from the area of a larger square, the result is 14 cm². However, if twice the area of the larger square is added to three times the area of the smaller square, the result is 203 cm². Determine the sides of the square.*

SOLUTION Let the lengths of each side of the smaller square be x cm and that of the larger square be y cm. Then,

Area of the smaller square $= x^2$ cm², Area of the larger square $= y^2$ cm²

It is given that

$$y^2 - 2x^2 = 14 \quad \ldots \text{(i)}$$

and, $$2y^2 + 3x^2 = 203 \quad \ldots \text{(ii)}$$

From (i), we have

$$y^2 = 14 + 2x^2$$

Substituting this value of y^2 in (ii), we get

$$2(14+2x^2)+3x^2 = 203$$

$\Rightarrow \quad 28+4x^2+3x^2 = 203$

$\Rightarrow \quad 7x^2 = 203-28$

$\Rightarrow \quad 7x^2 = 175 \Rightarrow x^2 = 25 \Rightarrow x = 5 \text{ cm}.$

Putting $x = 5$ in (i), we get

$$y^2 - 2 \times 5^2 = 14 \Rightarrow y^2 = 64 \Rightarrow y = 8$$

Hence, the lengths of the sides of the square are 5 cm and 8 cm respectively.

EXAMPLE 7 *A farmer wishes to grow a 100 m² rectangular vegetable garden. Since he has with the only 30 m barbed wire, he fences three sides of the rectangular garden letting compound wall of his house act as the fourth side-fence. Find the dimensions of his garden.*

SOLUTION Let the length of one side be x metres and other side by y metres. Then,

$$x + y + x = 30 \Rightarrow y = 30 - 2x \quad \ldots(i)$$

$\therefore$ Area of the vegetable garden = 100 m²

$\Rightarrow xy = 100$

$\Rightarrow x(30 - 2x) = 100$

$\Rightarrow 30x - 2x^2 = 100$

$\Rightarrow 15x - x^2 = 50$

$\Rightarrow x^2 - 15x + 50 = 0$

$\Rightarrow x^2 - 10x - 5x + 50 = 0$

$\Rightarrow (x - 10)(x - 5) = 0$

$\Rightarrow x = 5, 10$

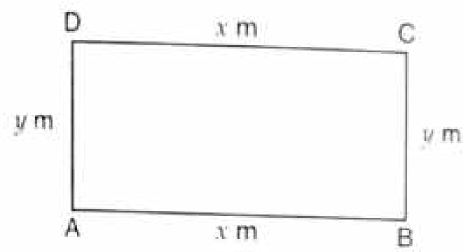

Fig. 4.4

When $x = 5$, we have

$$y = 30 - 2 \times 5 = 20 \quad \text{[Using (i)]}$$

When $x = 10$, we have

$$y = 30 - 2 \times 10 = 10 \quad \text{[Using (i)]}$$

Hence, the dimensions of the vegetable garden are: $5\,\text{m} \times 20\,\text{m}$ or, $10\,\text{m} \times 10\,\text{m}$

EXAMPLE 8 *The area of an isosceles triangle is 60 cm² and the length of each one of its equal sides is 13 cm. Find its base.* [CBSE 2015]

SOLUTION Let ABC be the given isosceles triangle in which $AB = AC = 13$ cm. Draw AD perpendicular from A on BC. Let $BC = 2x$ cm. Then, $BD = DC = x$ cm.

In ΔABD, we have

$$AB^2 = AD^2 + BD^2 \quad \text{[By Pythagoras Theorem]}$$

$\Rightarrow 13^2 = AD^2 + x^2$

$\Rightarrow AD = \sqrt{13^2 - x^2} = \sqrt{169 - x^2}$

Now, Area = 60 cm²

$\Rightarrow \frac{1}{2}(BC \times AD) = 60$

$\Rightarrow \frac{1}{2}\left\{(2x \times \sqrt{169 - x^2}\right\} = 60$

$\Rightarrow x\sqrt{169 - x^2} = 60$

$\Rightarrow x^2(169 - x^2) = 3600$

$\Rightarrow x^4 - 169x^2 + 3600 = 0$

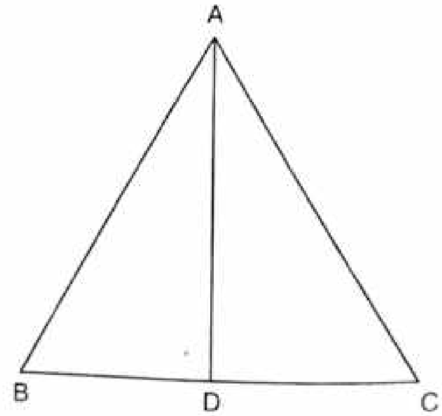

Fig. 4.5

$\Rightarrow \quad (x^2 - 144)(x^2 - 25) = 0$

$\Rightarrow \quad x^2 = 144$ or, $x^2 = 25 \Rightarrow x = 12$ or, $x = 5$

Hence, Base $= 2x = 24$ cm or, 10 cm.

EXAMPLE 9 *The perimeter of a right triangle is 60 cm. Its hypotenues is 25 cm. Find the area of the triangle.* [CBSE 2016]

SOLUTION Let ABC be the given right angled triangle such that base $= BC = x$ cm and hypotenuse $AC = 25$ cm.

Now, Perimeter $= 60$ cm

$\Rightarrow \quad AB + BC + AC = 60$

$\Rightarrow \quad AB + x + 25 = 60$

$\Rightarrow \quad AB = 35 - x$

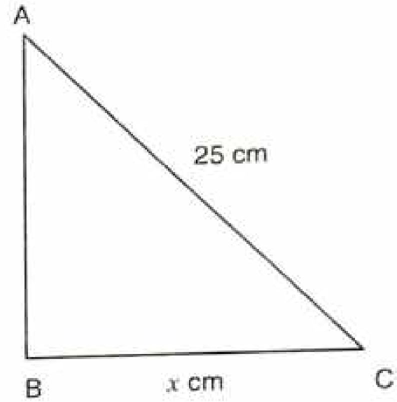

Fig. 4.6

By Pythagoras theorem, we have

$AB^2 + BC^2 = AC^2$

$\Rightarrow \quad (35 - x)^2 + x^2 = 25^2$

$\Rightarrow \quad 2x^2 - 70x + 600 = 0$

$\Rightarrow \quad x^2 - 35x + 300 = 0$

$\Rightarrow \quad x^2 - 20x - 15x + 300 = 0$

$\Rightarrow \quad (x - 20)(x - 15) = 0 \Rightarrow x = 20$ or, $x = 15$

If $x = 20$, then $AB = 35 - x = 15$ and $BC = x = 20$.

$\therefore \quad \text{Area} = \frac{1}{2}(BC \times AB) = \frac{1}{2}(20 \times 15) = 150 \text{ cm}^2$

If $x = 15$, then $AB = 35 - x = 20$ and $BC = x = 15$

$\therefore \quad \text{Area} = \frac{1}{2}(BC \times AB) = \frac{1}{2}(15 \times 20) = 150 \text{ cm}^2$

Hence, Area $= 150$ cm^2.

EXAMPLE 10 *There is a square field whose side is 44 m. A square flower bed is prepared in its centre leaving a gravel path all round the flower bed. The total cost of laying the flower bed and gravelling the path at ₹ 2.75 and ₹ 1.50 per square metre, respectively, is ₹ 4904. Find the width of the gravel path.*

SOLUTION Let the width of the gravel path be x metres. Then,

Each side of the square flower bed is $(44 - 2x)$ metres.

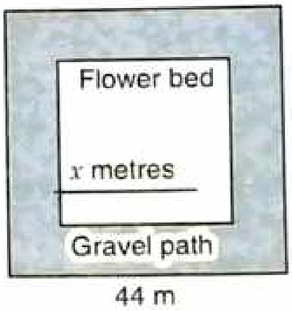

Fig. 4.7

Now, Area of the square field $= 44 \times 44 = 1936 \text{ m}^2$

Area of the flower bed $= (44 - 2x)^2 \text{ m}^2$

$\therefore$ Area of the gravel path

$=$ Area of the field $-$ Area of the flower bed

$= 1936 - (44 - 2x)^2$

$= 1936 - (1936 - 176x + 4x^2)$

$= (176x - 4x^2) \text{ m}^2$

Cost of laying the flower bed = (Area of the flower bed) (Rate per sq. m)

$$= (44-2x)^2 \times \frac{275}{100} = \frac{11}{4}(44-2x)^2 = 11(22-x)^2$$

Cost of gravelling the path = (Area of the path) × (Rate per sq. m)

$$= (176x - 4x^2)\frac{150}{100} = 6(44x - x^2)$$

It is given that the total cost of laying the flower bed and gravelling the path is ₹ 4904.

$\therefore \quad 11(22-x)^2 + 6(44x - x^2) = 4904$

$\Rightarrow \quad 11(484 - 44x + x^2) + (264x - 6x^2) = 4904$

$\Rightarrow \quad 5x^2 - 220x + 5324 = 4908$

$\Rightarrow \quad 5x^2 - 220x + 420 = 0$

$\Rightarrow \quad x^2 - 44x + 84 = 0$

$\Rightarrow \quad x^2 - 42x - 2x + 84 = 0$

$\Rightarrow \quad x(x-42) - 2(x-42) = 0$

$\Rightarrow \quad (x-2)(x-42) = 0$

$\Rightarrow \quad x = 2$ or, $x = 42$

But, $x \neq 42$, as the side of the square is 44 m. Therefore, $x = 2$.

Hence, the width of the gravel path is 2 metres.

EXAMPLE 11 *A chess board contains 64 equal squares and the area of each square is 6.25 cm². A border round the board is 2 cm wide. Find the length of the side of the chess board.*

SOLUTION Let the length of the side of the chess board be x cm. Then,

Area of 64 squares = $(x-4)^2$

$\therefore \quad (x-4)^2 = 64 \times 6.25$

$\Rightarrow \quad x^2 - 8x + 16 = 400$

$\Rightarrow \quad x^2 - 8x - 384 = 0$

$\Rightarrow \quad x^2 - 24x + 16x - 384 = 0$

$\Rightarrow \quad (x-24)(x+16) = 0$

$\Rightarrow \quad x = 24$ cm.

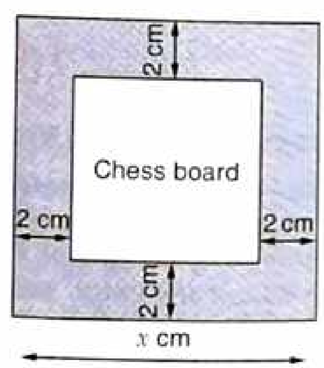

Fig. 4.8

EXERCISE 4.11

LEVEL-1

1. The perimeter of a rectangular field is 82 m and its area is 400 m². Find the breadth of the rectangle.
2. The length of a hall is 5 m more than its breadth. If the area of the floor of the hall is 84 m², what are the length and breadth of the hall?
3. Two squares have sides x cm and $(x+4)$ cm. The sum of their areas is 656 cm². Find the sides of the squares.

4. The area of a right angled triangle is 165 m^2. Determine its base and altitude if the latter exceeds the former by 7m.
5. Is it possible to design a rectangular mango grove whose length is twice its breadth and the area is 800 m^2? If so, find its length and breadth. [NCERT]
6. Is it possible to design a rectangular park of perimeter 80 m and area 400 m^2? If so, find its length and breadth. [NCERT]
7. Sum of the areas of two squares is 640 m^2. If the difference of their perimeters is 64 m, find the sides of the two squares. [NCERT, CBSE 2008]
8. Sum of the areas of two squares is 400 cm^2. If the difference of their perimeters is 16 cm, find the sides of two squares. [CBSE 2013]
9. The area of a rectangular plot is 528 m^2. The length of the plot (in metres) is one metre more then twice its breadth. Find the length and the breadth of the plot. [CBSE 2014]
10. In the centre of a rectangular lawn of dimensions $50 \text{ m} \times 40 \text{ m}$, a rectangular pond has to be constructed so that the area of the grass surrounding the pond would be 1184 m^2. Find the length and breadth of the pond. [NCERT EXEMPLAR]

ANSWERS

1. 16 m
2. Breadth = 7 m, Length = 12 m
3. 16 cm, 20 cm
4. Base = 15 m, Altitude = 22 m
5. Yes, 40 m, 20 m
6. Yes. 20m, 20 m
7. 24 m, 8 m
8. 16 cm, 12 cm
9. 33 m, 16 m
10. Length: 34 m, Breadth: 24 m.

HINTS TO SELECTED PROBLEMS

1. Let the breadth be x metres. Then,

$$2\,(\text{length} + \text{breadth}) = 82 \Rightarrow \text{length} = 41 - x \text{ metres}.$$

$$\therefore \quad \text{Area} = 400 \text{ m}^2 \Rightarrow x\,(41 - x) = 400 \Rightarrow x^2 - 41x + 400 = 0$$

4.8.6 APPLICATIONS OF QUADRATIC EQUATIONS FOR SOLVING PROBLEMS ON TIME AND WORK

Following examples will illustrate the above applications.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *A takes 6 days less than the time taken by B to finish a piece of work. If both A and B together can finish it in 4 days, find the time taken by B to finish the work.* [CBSE 2017]

SOLUTION Suppose B alone takes x days to finish the work. Then, A alone can finish it in $(x - 6)$ days.

Now, $\quad (A\text{'s one day's work}) + (B\text{'s one day's work}) = \dfrac{1}{x} + \dfrac{1}{x-6}$

and, $\quad (A + B)\text{'s one day's work} = \dfrac{1}{4}$

$$\therefore \quad \frac{1}{x} + \frac{1}{x-6} = \frac{1}{4}$$

$$\Rightarrow \quad \frac{x - 6 + x}{x\,(x-6)} = \frac{1}{4}$$

$$\Rightarrow \quad \frac{2x - 6}{x^2 - 6x} = \frac{1}{4}$$

$$\Rightarrow \quad 8x - 24 = x^2 - 6x$$

$\Rightarrow \quad x^2 - 14x + 24 = 0$

$\Rightarrow \quad x^2 - 12x - 2x + 24 = 0 \Rightarrow (x-12)(x-2) = 0 \Rightarrow x = 12$ or, $x = 2$

But, x cannot be less than 6. So, $x = 12$. Hence, B alone can finish the work in 12 days.

LEVEL-2

EXAMPLE 2 *A swimming pool is filled with three pipes with uniform flow. The first two pipes operating simultaneously, fill the pool in the same time during which the pool is filled by the third pipe alone. The second pipe fills the pool five hours faster than the first pipe and four hours slower than the third pipe. Find the time required by each pipe to fill the pool separately.*

SOLUTION Let V be the volume of the pool and x the number of hours required by the second pipe alone to fill the pool. Then, the first pipe takes $(x+5)$ hours, while the third pipe takes $(x-4)$ hours to fill the pool. So, the parts of the pool filled by the first, second and third pipes in one hour are respectively

$$\frac{V}{x+5}, \frac{V}{x} \text{ and } \frac{V}{x-4}$$

Let the time taken by the first and second pipes to fill the pool simultaneously be t hours. Then, the third pipe also takes the same time to fill the pool.

$$\therefore \quad \left(\frac{V}{x+5} + \frac{V}{x}\right)t = \text{Volume of the pool.}$$

Also, $$\frac{V}{x-4}t = \text{Volume of the pool.}$$

$$\Rightarrow \quad \left(\frac{V}{x+5} + \frac{V}{x}\right)t = \frac{V}{x-4}t$$

$$\Rightarrow \quad \frac{1}{x+5} + \frac{1}{x} = \frac{1}{x-4}$$

$$\Rightarrow \quad (2x+5)(x-4) = x^2 + 5x$$

$$\Rightarrow \quad x^2 - 8x - 20 = 0$$

$$\Rightarrow \quad x^2 - 10x + 2x - 20 = 0 \Rightarrow (x-10)(x+2) = 0 \Rightarrow x = 10 \text{ or, } x = -2$$

But, x cannot be negative. So, $x = 10$.

Hence, the timings required by first, second and third pipes to fill the pool individually are 15 hours, 10 hours and 6 hours respectively.

EXAMPLE 3 *Two pipes running together can fill a cistern in $3\frac{1}{13}$ minutes. If one pipe takes 3 minutes more than the other to fill it, find the time in which each pipe would fill the cistern.*

SOLUTION Suppose the faster pipe takes x minutes to fill the cistern. Therefore, the slower pipe will take $(x+3)$ minutes to fill the cistern.

Since the faster pipe takes x minutes to fill the cistern.

$\therefore$ Portion of the cistern filled by the faster pipe in one minute $= \frac{1}{x}$

$\Rightarrow$ Portion of the cistern filled by the faster pipe in $\frac{40}{13}$ minutes $= \frac{1}{x} \times \frac{40}{13} = \frac{40}{13x}$

Similarly,

Portion of the cistern filled by the slower pipe in $\frac{40}{13}$ minutes

$$= \frac{1}{x+3} \times \frac{40}{13} = \frac{40}{13(x+3)}$$

It is given that the cistern is filled in $\frac{40}{13}$ minutes.

$$\therefore \quad \frac{40}{13x} + \frac{40}{13(x+3)} = 1$$

$$\Rightarrow \quad \frac{1}{x} + \frac{1}{x+3} = \frac{13}{40}$$

$$\Rightarrow \quad \frac{x+3+x}{x(x+3)} = \frac{13}{40}$$

$$\Rightarrow \quad 40(2x+3) = 13x(x+3)$$

$$\Rightarrow \quad 80x + 120 = 13x^2 + 39x$$

$$\Rightarrow \quad 13x^2 - 41x - 120 = 0$$

$$\Rightarrow \quad 13x^2 - 65x + 24x - 120 = 0$$

$$\Rightarrow \quad 13x(x-5) + 24(x-5) = 0$$

$$\Rightarrow \quad (x-5)(13x+24) = 0$$

$$\Rightarrow \quad x - 5 = 0 \text{ or, } 13x + 24 = 0$$

$$\Rightarrow \quad x = 5 \text{ or, } x = \frac{-24}{13} \Rightarrow x = 5 \qquad [\because x > 0]$$

Hence, the faster pipe fills the cistern in 5 minutes and the slower pipe takes 8 minutes to fill the cistern.

EXERCISE 4.12

LEVEL-1

1. *A* takes 10 days less than the time taken by *B* to finish a piece of work. If both *A* and *B* together can finish the work in 12 days, find the time taken by *B* to finish the work.
2. If two pipes function simultaneously, a reservoir will be filled in 12 hours. One pipe fills the reservoir 10 hours faster than the other. How many hours will the second pipe take to fill the reservoir?
3. Two water taps together can fill a tank in $9\frac{3}{8}$ hours. The tap of larger diameter takes 10 hours less than the smaller one to fill the tank separately. Find the time in which each tap can separately fill the tank. [NCERT]
4. Two pipes running together can fill a tank in $11\frac{1}{9}$ minutes. If one pipe takes 5 minutes more than the other to fill the tank separately, find the time in which each pipe would fill the tank separately. [CBSE 2010]

5. To fill a swimming pool two pipes are used. If the pipe of larger diameter used for 4 hours and the pipe of smaller diameter for 9 hours, only half of the pool can be filled. Find, how long it would take for each pipe to fill the pool separately, if the pipe of smaller diameter takes 10 hours more than the pipe of larger diameter to fill the pool? [CBSE 2015]

ANSWERS

1. 30 days 2. 30 hours 3. 15 hours, 25 hours
5. Larger diameter pipe fills in 20 hours, Smaller diameter pipe fills in 30 hours.

HINT TO SELECTED PROBLEM

1. Suppose B alone takes x days to finish the work. Then, A alone can finish it in $(x-10)$ days.

   Now,

   A's one day's work + B's one day's work = $(A + B)$'s one day's work

   $\Rightarrow \quad \frac{1}{x} + \frac{1}{x-10} = \frac{1}{12}$

4.8.7 MISCELLANEOUS APPLICATIONS OF QUADRATIC EQUATIONS

Following examples will illustrate the above applications.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *A person on tour has ₹360 for his expenses. If he extends his tour for 4 days, he has to cut down his daily expenses by ₹3. Find the original duration of the tour.*

SOLUTION Let the original duration of the tour be x days.

Total expenditure on tour = ₹ 360

$\therefore$ Expenditure per day = ₹ $\frac{360}{x}$

Duration of the extended tour = $(x+4)$ days

$\therefore$ Expenditure per day according to new schedule = ₹ $\frac{360}{x+4}$

It is given that the daily expenses are cut down by ₹ 3.

$$\therefore \quad \frac{360}{x} - \frac{360}{x+4} = 3$$

$$\Rightarrow \quad \frac{360(x+4) - 360x}{x(x+4)} = 3$$

$$\Rightarrow \quad \frac{360x + 1440 - 360x}{x(x+4)} = 3$$

$$\Rightarrow \quad \frac{1440}{x^2 + 4x} = 3$$

$$\Rightarrow \quad x^2 + 4x = 480$$

$$\Rightarrow \quad x^2 + 4x - 480 = 0$$

$$\Rightarrow \quad x^2 + 24x - 20x - 480 = 0$$

$$\Rightarrow \quad x(x+24) - 20(x+24) = 0$$

$$\Rightarrow \quad (x-20)(x+24) = 0 \Rightarrow x - 20 = 0 \text{ or, } x + 24 = 0 \Rightarrow x = 20 \text{ or, } x = -24$$

But, the number of days cannot be negative. So, $x = 20$.
Hence, the original duration of the tour was of 20 days.

EXAMPLE 2 *A piece of cloth costs ₹200. If the piece was 5 m longer and each metre of cloth costs ₹2 less the cost of the piece would have remained unchanged. How long is the piece and what is the original rate per metre?*

SOLUTION Let the length of the piece be x metres. Then, rate per metre = ₹ $\frac{200}{x}$

New length = $(x + 5)$ metres.

Since the cost remains same.

$\therefore$ New rate per metre = ₹ $\frac{200}{x+5}$

It is given that

$$\frac{200}{x} - \frac{200}{x+5} = 2$$

$$\Rightarrow \quad \frac{200(x+5) - 200x}{x(x+5)} = 2$$

$$\Rightarrow \quad \frac{1000}{x(x+5)} = 2$$

$$\Rightarrow \quad x^2 + 5x = 500$$

$$\Rightarrow \quad x^2 + 5x - 500 = 0$$

$$\Rightarrow \quad x^2 + 25x - 20x - 500 = 0$$

$$\Rightarrow \quad (x+25)(x-20) = 0$$

$$\Rightarrow \quad x + 25 = 0 \text{ or, } x - 20 = 0$$

$$\Rightarrow \quad x + 25 = 0 \text{ or, } x - 20 = 0 \Rightarrow x = 20 \text{ or, } x = -25$$

But, x cannot be negative. So, $x = 20$.

Rate per metre = ₹ $\frac{200}{x}$ = ₹ $\frac{200}{20}$ = ₹ 10

Hence, the length of the piece of cloth is 20 metres and rate = ₹ 10 per metre.

EXAMPLE 3 *₹6500 were divided equally among a certain number of persons. Had there been 15 more persons, each would have got ₹30 less. Find the original number of persons.*

SOLUTION Let the original number of persons be x. Then,

Share of each person = ₹ $\frac{6500}{x}$

If the number of persons is increased by 15. Then,

New share of each person = ₹ $\frac{6500}{x+15}$

Since each person gets ₹ 30 less, if number of persons is increased by 15.

$$\therefore \quad \frac{6500}{x} - \frac{6500}{x+15} = 30$$

$$\Rightarrow \quad \frac{6500(x+15)-6500x}{x(x+15)}=30$$

$$\Rightarrow \quad \frac{6500\times 15}{x(x+15)}=30$$

$$\Rightarrow \quad \frac{3250}{x(x+15)}=1$$

$$\Rightarrow \quad x^2+15x-3250=0$$

$$\Rightarrow \quad x^2+65x-50x-3250=0$$

$$\Rightarrow \quad x(x+65)-50(x+65)=0$$

$$\Rightarrow \quad (x+65)(x-50)=0$$

$$\Rightarrow \quad x-50=0 \text{ or, } x+65=0 \Rightarrow x=50 \text{ or, } x=-65$$

Since the number of persons cannot be negative. Therefore, $x = 50$.
Hence, the original number of persons is 50.

EXAMPLE 4 *A shopkeeper buys a number of books for ₹80. If he had bought 4 more books for the same amount, each book would have cost ₹1 less. How many books did he buy?*

[CBSE 2012]

SOLUTION Let the number of books bought be x. Then,

Cost of x books = ₹ 80 ⇒ Cost of one book = ₹ $\frac{80}{x}$

If the number of books bought is $x+4$, then

Cost of one book = ₹ $\frac{80}{x+4}$

It is given that the cost of one book is reduced by one rupee.

$$\therefore \quad \frac{80}{x}-\frac{80}{x+4}=1$$

$$\Rightarrow \quad 80\left(\frac{1}{x}-\frac{1}{x+4}\right)=1$$

$$\Rightarrow \quad 80\left\{\frac{x+4-x}{x(x+4)}\right\}=1$$

$$\Rightarrow \quad \frac{320}{x^2+4x}=1$$

$$\Rightarrow \quad x^2+4x=320$$

$$\Rightarrow \quad x^2+4x-320=0$$

$$\Rightarrow \quad x^2+20x-16x-320=0$$

$$\Rightarrow \quad x(x+20)-16(x+20)=0$$

$$\Rightarrow \quad (x+20)(x-16)=0$$

$$\Rightarrow \quad x=-20 \text{ or, } x=16 \Rightarrow x=16 \qquad [\because x \text{ cannot be negative}]$$

Hence, the number of books is 16.

EXAMPLE 5 *If the price of a book is reduced by ₹5, a person can buy 5 more books for ₹300. Find the original list price of the book.*

SOLUTION Let the original list price of the book be ₹ x.

$$\therefore \quad \text{Number of books bought for ₹ } 300 = \frac{300}{x}$$

$$\text{Reduced list price of the book} = ₹\,(x-5)$$

$$\therefore \quad \text{Number of books bought for ₹ } 300 = \frac{300}{x-5}$$

It is given that

$$\frac{300}{x-5} - \frac{300}{x} = 5$$

$$\Rightarrow \quad \frac{300x - 300x + 1500}{x(x-5)} = 5$$

$$\Rightarrow \quad \frac{1500}{x^2 - 5x} = 5$$

$$\Rightarrow \quad x^2 - 5x = 300$$

$$\Rightarrow \quad x^2 - 5x - 300 = 0$$

$$\Rightarrow \quad x^2 - 20x + 15x - 300 = 0$$

$$\Rightarrow \quad (x-20)(x+15) = 0$$

$$\Rightarrow \quad x - 20 = 0 \text{ or, } x + 15 = 0$$

$$\Rightarrow \quad x = 20, x = -15 \Rightarrow x = 20 \qquad [\because x = -15 \text{ is not possible}]$$

Hence, the list price of the book = ₹ 20

LEVEL-2

EXAMPLE 6 *A factory kept increasing its output by the same percentage every year. Find the percentage if it is known that the output is doubled in the last two years.*

SOLUTION Let P be the initial production (2 years ago), and let the increase in product every year be x%. Then,

$$\text{Product at the end of first year} = P + \frac{Px}{100} = P\left(1 + \frac{x}{100}\right)$$

Product at the end of the second year

$$= P\left(1 + \frac{x}{100}\right) + \frac{x}{100}\left\{P\left(1 + \frac{x}{100}\right)\right\}$$

$$= P\left(1 + \frac{x}{100}\right)\left(1 + \frac{x}{100}\right) = P\left(1 + \frac{x}{100}\right)^2$$

Since product is doubled in last two years

$$\therefore \quad P\left(1 + \frac{x}{100}\right)^2 = 2P$$

$$\Rightarrow \quad \left(1+\frac{x}{100}\right)^2 = 2$$

$$\Rightarrow \quad (100+x)^2 = 2\times 100^2$$

$$\Rightarrow \quad x^2+200x-10000=0$$

$$\Rightarrow \quad x = \frac{-200 \pm \sqrt{(200)^2+40000}}{2} = -100 \pm 100\sqrt{2} = 100(-1 \pm \sqrt{2})$$

$$\Rightarrow \quad x = 100(-1+\sqrt{2})$$

[$\because x$ cannot be negative]

EXAMPLE 7 *A dealer sells a toy for ₹24 and gains as much per cent as the cost price of the toy. Find the cost price of the toy.*

SOLUTION Let the cost price of the toy be ₹ x. Then,

$$\text{Gain} = x\% \Rightarrow \text{Gain} = ₹\left(x \times \frac{x}{100}\right) = ₹\,\frac{x^2}{100}$$

$$\therefore \quad \text{S.P.} = \text{C.P.} + \text{Gain} = x + \frac{x^2}{100}$$

But, S.P. = ₹ 24.

$$\therefore \quad x + \frac{x^2}{100} = 24$$

[Given]

$$\Rightarrow \quad 100x + x^2 = 2400$$

$$\Rightarrow \quad x^2+100x-2400=0$$

$$\Rightarrow \quad x^2+120x-20x-2400=0$$

$$\Rightarrow \quad x(x+120)-20(x+120)=0$$

$$\Rightarrow \quad (x+120)(x-20)=0$$

$$\Rightarrow \quad x = 20, -120 \Rightarrow x = 20$$

Hence, the cost price of the toy is ₹ 20. [$\because x > 0$]

EXAMPLE 8 *A peacock is sitting on the top of a pillar, which is 9 m high. From a point 27 m away from the bottom of the pillar, a snake is coming to its hole at the base of the pillar. Seeing the snake the peacock pounces on it. If their speeds are equal, at what distance from the whole is the snake caught?*

[CBSE 2008]

SOLUTION Let PQ be the pole and the peacock is sitting at the top P of the pole. Let the hole be at Q. Initially, the snake is at S when the peacock notices the snake such that $QS = 27$ m.

Suppose v m/sec is the common speed of both the snake and the peacock and the peacock catches the snake after t seconds at point T. Clearly, distance travelled by the snake in t seconds is same as the distance flown by peacock.

$$\therefore \quad PT = ST = x \quad \text{(say)}$$

Thus, in right triangle PQT, we have

$$QT = 27 - x,\ PT = x \text{ and } PQ = 9$$

Using Pythagoras theorem, we have

$$PT^2 = PQ^2 + QT^2$$

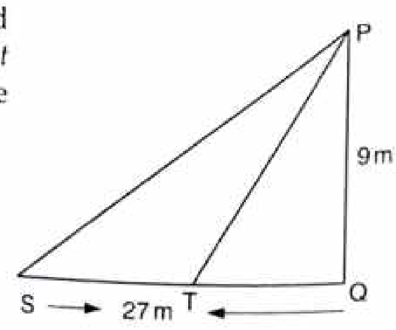

Fig. 4.9

$\Rightarrow \quad x^2 = 9^2 + (27 - x)^2$

$\Rightarrow \quad x^2 = 81 + 729 - 54x + x^2$

$\Rightarrow \quad 0 = 810 - 54x$

$\Rightarrow \quad 54x = 810$

$\Rightarrow \quad x = 15$

$\therefore \quad QT = SQ - ST = (27 - 15)\text{ m} = 12\text{ m}$

Hence, the snake is caught at a distance of 12 m from the hole.

EXAMPLE 9 *The angry Arjun carried some arrows for fighting with Bheeshm. With half the arrows, he cut down the arrows thrown by Bheeshm on him and with six other arrows he killed the rath driver of Bheeshm. With one arrow each he knocked down respectively the rath, flag and the bow of Bheeshm. Finally, with one more than four times the square root of arrrows he laid Bheeshm unconscious on an arrow bed. Find the total number of arrows Arjun had.*

SOLUTION Suppose Arjun had x arrows.

Number of arrows used to cut arrows of Bheeshm = $x/2$

Number of arrows used to kill the rath driver = 6

Number of other arrows used = 3

Remaining arrows $= 4\sqrt{x} + 1$

By hypothesis, we have

$\therefore \quad \frac{x}{2} + 6 + 3 + 4\sqrt{x} + 1 = x$

$\Rightarrow \quad x + 20 + 8\sqrt{x} = 2x$

$\Rightarrow \quad x = 20 + 8\sqrt{x}$

Putting $x = y^2$, the above equation becomes

$y^2 = 20 + 8y$

$\Rightarrow \quad y^2 - 8y - 20 = 0$

$\Rightarrow \quad y^2 - 10y + 2y - 20 = 0$

$\Rightarrow \quad (y - 10)(y + 2) = 0$

$\Rightarrow \quad y = 10 \text{ or, } y = -2$

$\Rightarrow \quad y = 10$ [$\because$ y cannot be negative]

$\Rightarrow \quad x = y^2 \Rightarrow x = 100$

Hence, the number of arrows which Arjun had = 100.

EXAMPLE 10 *One-fourth of a herd of camels was seen in the forest. Twice the square root of the herd had gone to mountains and the remaining 15 camels were seen on the bank of a river. Find the total number of camels.*

SOLUTION Let the total number of camels be x. Then,

Number of camels seen in the forest $= \frac{x}{4}$

Number of camels gone to mountains $= 2\sqrt{x}$

Number of camels on the bank of river = 15

Total number of camels $= \frac{x}{4} + 2\sqrt{x} + 15$

By hypothesis, we have

$\therefore \quad \frac{x}{4} + 2\sqrt{x} + 15 = x$

$\Rightarrow \quad 3x - 8\sqrt{x} - 60 = 0$

$\Rightarrow \quad 3y^2 - 8y - 60 = 0$, where $x = y^2$

$\Rightarrow \quad 3y^2 - 18y + 10y - 60 = 0$

$\Rightarrow \quad 3y(y-6) + 10(y-6) = 0$

$\Rightarrow \quad (3y+10)(y-6) = 9$

$\Rightarrow \quad y = 6$ or, $y = -\frac{10}{3}$

Now, $\quad y = -\frac{10}{3} \Rightarrow x = \left(-\frac{10}{3}\right)^2 = \frac{100}{9}$ $\quad [\because x = y^2]$

But, the number of camels cannot be a fraction.

$\therefore \quad y = 6 \Rightarrow x = 6^2 = 36$ $\quad [\because x = y^2]$

Hence, the number of camels = 36.

EXERCISE 4.13

LEVEL-1

1. A piece of cloth costs ₹ 35. If the piece were 4 m longer and each metre costs ₹ one less, the cost would remain unchanged. How long is the piece?
2. Some students planned a picnic. The budget for food was ₹ 480. But eight of these failed to go and thus the cost of food for each member increased by ₹ 10. How many students attended the picnic?
3. A dealer sells an article for ₹ 24 and gains as much percent as the cost price of the article. Find the cost price of the article.
4. Out of a group of swans, 7/2 times the square root of the total number are playing on the share of a pond. The two remaining ones are swinging in water. Find the total number of swans.
5. If the list price of a toy is reduced by ₹ 2, a person can buy 2 toys more for ₹ 360. Find the original price of the toy. [CBSE 2002 C]
6. ₹ 9000 were divided equally among a certain number of persons. Had there been 20 more persons, each would have got ₹ 160 less. Find the original number of persons.
7. Some students planned a picnic. The budget for food was ₹ 500. But, 5 of them failed to go and thus the cost of food for each member increased by ₹ 5. How many students attended the picnic?
8. A pole has to be erected at a point on the boundary of a circular park of diameter 13 metres in such a way that the difference of its distances from two diametrically opposite fixed gates A and B on the boundary is 7 metres. Is it the possible to do so? If yes, at what distances from the two gates should the pole be erected? [CBSE 2016]
9. In a class test, the sum of the marks obtained by P in Mathematics and science is 28. Had he got 3 marks more in Mathematics and 4 marks less in Science. The product of his marks, would have been 180. Find his marks in the two subjects. [CBSE 2008]

10. In a class test, the sum of Shefali's marks in Mathematics and English is 30. Had she got 2 marks more in Mathematics and 3 marks less in English, the product of her marks would have been 210. Find her marks in two subjects. [CBSE 2014, NCERT]

LEVEL-2

11. A cottage industry produces a certain number of pottery articles in a day. It was observed on a particular day that the cost of production of each article (in rupees) was 3 more than twice the number of articles produced on that day. If the total cost of production on that day was ₹ 90, find the number of articles produced and the cost of each article. [NCERT]

12. At t minutes past 2 pm the time needed by the minutes hand and a clock to show 3 pm was found to be 3 minutes less than $\frac{t^2}{4}$ minutes. Find t.

[NCERT EXEMPLAR]

ANSWERS

1. 10 m 2. 16 3. ₹ 20 4. 16 5. ₹ 20 6. 25 7. 20

8. At a distance of 5 metres from the gate B
9. Marks in Mathematics = 12, Marks in Science = 16

or

Marks in Mathematics = 9, Marks in Science = 19

10. Marks in Mathematics = 12, Marks in English = 18

or

Marks in Mathematics = 13, Marks in English = 17

11. Number of articles = 6, Cost of each article = 15 12. 14

HINTS TO SELECTED PROBLEMS

2. Suppose x students planned a picnic. Then, share of each student ₹ $\frac{480}{x}$

Share of each student when 8 students failed to go is ₹ $\frac{480}{x-8}$

$\therefore \quad \frac{480}{x-8} - \frac{480}{x} = 10.$

3. Let the cost price be ₹ x. Then, gain percent = x

$\therefore \quad$ Selling Price = ₹ $\left(x + \frac{x}{100} \times x\right)$ = ₹ $\left(\frac{x^2 + 100x}{100}\right)$

$\Rightarrow \quad \frac{x^2 + 100x}{100} = 24 \Rightarrow x^2 + 100x - 2400 = 0.$

4. Let the number of swans be x. Then,

$\frac{7}{2}\sqrt{x} + 2 = x \Rightarrow 2x - 7\sqrt{x} - 4 = 0 \Rightarrow 2y^2 - 7y - 4 = 0$, where $x = y^2$.

8. Let P be the required location of the pole such that its distance from gate B is x metres. i.e. $BP = x$ metres.

$\therefore \quad AP = x + 7$

In right triangle APB, we have

$$AP^2 + PB^2 = AB^2$$

$$\Rightarrow (x+7)^2 + x^2 = 13^2$$

$$\Rightarrow 2x^2 + 14x - 120 = 0 \Rightarrow x^2 + 7x - 60 = 0 \Rightarrow (x+12)(x-5) = 0 \Rightarrow x = 5$$

12. Time needed by the minutes hand to show 3 pm = $(60 - t)$ minutes.

$$\therefore \quad 60 - t = \frac{t^2}{4} - 3 \Rightarrow t^2 + 4t - 252 = 0$$

VERY SHORT ANSWER TYPE QUESTION (VSAQs)

Answer each of the following questions either in one word or one sentence or as per requirement of the question:

1. Write the value of k for which the quadratic equation $x^2 - kx + 4 = 0$ has equal roots.
2. What is the nature of roots of the quadratic equation $4x^2 - 12x - 9 = 0$?
3. If $1 + \sqrt{2}$ is a root of a quadratic equation with rational coefficients, write its other root.
4. Write the number of real roots of the equation $x^2 + 3|x| + 2 = 0$.
5. Write the sum of real roots of the equation $x^2 + |x| - 6 = 0$.
6. Write the set of values of 'a' for which the equation $x^2 + ax - 1 = 0$ has real roots.
7. Is there any real value of 'a' for which the equation $x^2 + 2x + (a^2 + 1) = 0$ has real roots?
8. Write the value of λ for which $x^2 + 4x + \lambda$ is a perfect square.
9. Write the condition to be satisfied for which equations $ax^2 + 2bx + c = 0$ and $bx^2 - 2\sqrt{ac}\,x + b = 0$ have equal roots.
10. Write the set of values of k for which the quadratic equation has $2x^2 + kx - 8 = 0$ has real roots.
11. Write a quadratic polynomial, sum of whose zeros is $2\sqrt{3}$ and their product is 2.
12. Show that $x = -3$ is a solution of $x^2 + 6x + 9 = 0$. **[CBSE 2008]**
13. Show that $x = -2$ is a solution of $3x^2 + 13x + 14 = 0$. **[CBSE 2008]**
14. Find the discriminant of the quadratic equation $3\sqrt{3}\,x^2 + 10x + \sqrt{3} = 0$ **[CBSE 2009]**
15. If $x = \frac{-1}{2}$, is a solution of the quadratic equation $3x^2 + 2kx - 3 = 0$, find the value of k. **[CBSE 2015]**
16. If $x = 3$ is one root of the quadratic equation $x^2 - 2kx - 6 = 0$, then find the value of k. **[CBSE 2018]**

ANSWERS

1. $k = \pm 4$ 2. Real and distinct 3. $1 - \sqrt{2}$ 4. No real root
5. 0 6. All real values 7. No 8. $\lambda = 4$
9. $b^2 = ac$ 10. All real values
11. $f(x) = k(x^2 - 2\sqrt{3}x + 2)$, where k is any real number 14. 64 15. $\frac{9}{4}$ 16. $k = \frac{1}{2}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If the equation $x^2 + 4x + k = 0$ has real and distinct roots, then

(a) $k < 4$ (b) $k > 4$ (c) $k \geq 4$ (d) $k \leq 4$

2. If the equation $x^2 - ax + 1 = 0$ has two distinct roots, then
(a) $|a| = 2$ (b) $|a| < 2$ (c) $|a| > 2$ (d) None of these

3. If the equation $9x^2 + 6kx + 4 = 0$ has equal roots, then the roots are both equal to
(a) $\pm\frac{2}{3}$ (b) $\pm\frac{3}{2}$ (c) 0 (d) ± 3

4. If $ax^2 + bx + c = 0$ has equal roots, then $c =$
(a) $\frac{-b}{2a}$ (b) $\frac{b}{2a}$ (c) $\frac{-b^2}{4a}$ (d) $\frac{b^2}{4a}$

5. If the equation $ax^2 + 2x + a = 0$ has two distinct roots, if
(a) $a = \pm 1$ (b) $a = 0$ (c) $a = 0, 1$ (d) $a = -1, 0$

6. The positive value of k for which the equation $x^2 + kx + 64 = 0$ and $x^2 - 8x + k = 0$ will both have real roots, is
(a) 4 (b) 8 (c) 12 (d) 16

7. The value of $\sqrt{6 + \sqrt{6 + \sqrt{6 +}}} \ldots$ is
(a) 4 (b) 3 (c) -2 (d) 3.5

8. If 2 is a root of the equation $x^2 + bx + 12 = 0$ and the equation $x^2 + bx + q = 0$ has equal roots, then $q =$
(a) 8 (b) -8 (c) 16 (d) -16

9. If the equation $(a^2 + b^2)x^2 - 2(ac + bd)x + c^2 + d^2 = 0$ has equal roots, then
(a) $ab = cd$ (b) $ad = bc$ (c) $ad = \sqrt{bc}$ (d) $ab = \sqrt{cd}$

10. If the roots of the equation $(a^2 + b^2)x^2 - 2b(a + c)x + (b^2 + c^2) = 0$ are equal, then
(a) $2b = a + c$ (b) $b^2 = ac$ (c) $b = \frac{2ac}{a + c}$ (d) $b = ac$

11. If the equation $x^2 - bx + 1 = 0$ does not possess real roots, then
(a) $-3 < b < 3$ (b) $-2 < b < 2$ (c) $b > 2$ (d) $b < -2$

12. If $x = 1$ is a common root of the equations $ax^2 + ax + 3 = 0$ and $x^2 + x + b = 0$, then $ab =$
(a) 3 (b) 3.5 (c) 6 (d) -3

13. If p and q are the roots of the equation $x^2 - px + q = 0$, then
(a) $p = 1, q = -2$ (b) $b = 0, q = 1$ (c) $p = -2, q = 0$ (d) $p = -2, q = 1$

14. If a and b can take values 1, 2, 3, 4. Then the number of the equations of the form $ax^2 + bx + 1 = 0$ having real roots is
(a) 10 (b) 7 (c) 6 (d) 12

15. The number of quadratic equations having real roots and which do not change by squaring their roots is
(a) 4 (b) 3 (c) 2 (d) 1

16. If $(a^2 + b^2)x^2 + 2(ab + bd)x + c^2 + d^2 = 0$ has no real roots, then
    (a) $ad = bc$ (b) $ab = cd$ (c) $ac = bd$ (d) $ad \neq bc$
17. If the sum of the roots of the equation $x^2 - x = \lambda(2x - 1)$ is zero, then $\lambda =$
    (a) -2 (b) 2 (c) $-\frac{1}{2}$ (d) $\frac{1}{2}$
18. If $x = 1$ is a common root of $ax^2 + ax + 2 = 0$ and $x^2 + x + b = 0$ then, $ab =$
    (a) 1 (b) 2 (c) 4 (d) 3
19. The value of c for which the equation $ax^2 + 2bx + c = 0$ has equal roots is
    (a) $\frac{b^2}{a}$ (b) $\frac{b^2}{4a}$ (c) $\frac{a^2}{b}$ (d) $\frac{a^2}{4b}$
20. If $x^2 + k(4x + k - 1) + 2 = 0$ has equal roots, then $k =$
    (a) $-\frac{2}{3}, 1$ (b) $\frac{2}{3}, -1$ (c) $\frac{3}{2}, \frac{1}{3}$ (d) $-\frac{3}{2}, -\frac{1}{3}$
21. If the sum and product of the roots of the equation $kx^2 + 6x + 4k = 0$ are equal, then $k =$
    (a) $-\frac{3}{2}$ (b) $\frac{3}{2}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$
22. If $\sin\alpha$ and $\cos\alpha$ are the roots of the equation $ax^2 + bx + c = 0$, then $b^2 =$
    (a) $a^2 - 2ac$ (b) $a^2 + 2ac$ (c) $a^2 - ac$ (d) $a^2 + ac$
23. If 2 is a root of the equation $x^2 + ax + 12 = 0$ and the quadratic equation $x^2 + ax + q = 0$ has equal roots, then $q =$
    (a) 12 (b) 8 (c) 20 (d) 16
24. If the sum of the roots of the equation $x^2 - (k + 6)x + 2(2k - 1) = 0$ is equal to half of their product, then $k =$
    (a) 6 (b) 7 (c) 1 (d) 5
25. If a and b are roots of the equation $x^2 + ax + b = 0$, then $a + b =$
    (a) 1 (b) 2 (c) -2 (d) -1
26. A quadratic equation whose one root is 2 and the sum of whose roots is zero, is
    (a) $x^2 + 4 = 0$ (b) $x^2 - 4 = 0$ (c) $4x^2 - 1 = 0$ (d) $x^2 - 2 = 0$
27. If one root of the equation $ax^2 + bx + c = 0$ is three times the other, then $b^2 : ac =$
    (a) $3:1$ (b) $3:16$ (c) $16:3$ (d) $16:1$
28. If one root of the equation $2x^2 + kx + 4 = 0$ is 2, then the other root is
    (a) 6 (b) -6 (c) -1 (d) 1
29. If one root of the equation $x^2 + ax + 3 = 0$ is 1, then its other root is
    (a) 3 (b) -3 (c) 2 (d) -2

30. If one root of the equation $4x^2 - 2x + (\lambda - 4) = 0$ be the reciprocal of the other, then $\lambda =$

(a) 8 (b) −8 (c) 4 (d) −4

31. If $y = 1$ is a common root of the equations $ay^2 + ay + 3 = 0$ and $y^2 + y + b = 0$, then ab equals

(a) 3 (b) −7/2 (c) 6 (d) − 3

[CBSE 2012]

32. The values of k for which the quadratic equation $16x^2 + 4kx + 9 = 0$ has real and equal roots are

(a) $6, -\frac{1}{6}$ (b) 36, − 36 (c) 6, − 6 (d) $\frac{3}{4}, -\frac{3}{4}$

[CBSE 2014]

ANSWERS

1. (a)	**2.** (c)	**3.** (a)	**4.** (d)	**5.** (a)	**6.** (d)
7. (b)	**8.** (c)	**9.** (d)	**10.** (b)	**11.** (b)	**12.** (a)
13. (a)	**14.** (a)	**15.** (c)	**16.** (d)	**17.** (c)	**18.** (b)
19. (a)	**20.** (b)	**21.** (a)	**22.** (b)	**23.** (d)	**24.** (b)
25. (d)	**26.** (b)	**27.** (c)	**28.** (d)	**29.** (a)	**30.** (a)
31. (a)	**32.** (c)				

SUMMARY

1. A polynomial of degree 2 is called a quadratic polynomial. The general form of a quadratic polynomial is $ax^2 + bx + c$, where a, b, c are real numbers such that $a \neq 0$ and x is a real variable.
2. If $p(x) = ax^2 + bx = + c, a \neq 0$ is a quadratic polynomial and α is a real number, then $p(\alpha) = a\alpha^2 + b\alpha + c$ is known as the value of the quadratic polynomial $p(x)$.
3. A real number α is said to be a zero of the quadratic polynomial $p(x) = ax^2 + bx + c$, if $p(\alpha) = 0.0$
4. If $p(x) = ax^2 + bx + c$ is a quadratic polynomial, then $p(x) = 0$ i.e., $ax^2 + bx + c = 0$, $a \neq 0$ is called a quadratic equation.
5. A real number α is said to be a root of the quadratic equation $ax^2 + bx + c = 0$, if $a\alpha^2 + b\alpha + c = 0$.

   In other words, α is a root of $ax^2 + bx + c = 0$ if and only if α is a zero of the polynomial $p(x) = ax^2 + bx + c$.
6. If $ax^2 + bx + c, a \neq 0$ is factorizable into a product of two linear factors, then the roots of the quadratic equation $ax^2 + bx + c = 0$ can be found by equating each factor to zero.
7. The roots of a quadratic equation can also be found by using the method of completing the square.

8. The roots of the quadratic equation $ax^2 + bx + c = 0, a \neq 0$ can be found by using the quadratic formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, provided that $b^2 - 4ac \geq 0$.

9. Nature of the roots of quadratic equation $ax^2 + bx + c = 0, a \neq 0$ depends upon the value of $D = b^2 - 4ac$, which is known as the discriminate of the quadratic equation.

10. The quadratic equation $ax^2 + bx + c = 0, a \neq 0$ has:

    (i) two distinct real roots, if $D = b^2 - 4ac > 0$

    (ii) two equal roots i.e. coincident real roots if $D = b^2 - 4ac = 0$

    (iii) no real roots, if $D = b^2 - 4ac < 0$.