CHAPTER 6

CO-ORDINATE GEOMETRY

6.1 INTRODUCTION

In class IX, we have seen that to locate the position of a point on a plane, we require a pair of mutually perpendicular lines which are known as the coordinate axes. The horizontal line is known as the x-axis and the vertical line is known as the y-axis. The intersection point of the coordinate axes is known as the origin. The distance of a point from the y-axis is called its x-coordinate, or abscissa and the distance from the x-axis is called its y-coordinate, or ordinate. We have seen that the coordinates of a point on the x-axis are of the form $(x, 0)$, and that of a point on y-axis are of the form $(0, y)$. We have also learnt about plotting of points in a plane when their coordinates are given. Also, we have seen that a linear equation in two variables, when represented graphically, gives a straight line.

In this chapter, we will see how we can find the distance between two points whose coordinates are given. We will also find the coordinates of the point which divides the line segment joining two given points in a given ratio. Finally, we will learn about the method of finding the area of a triangle in terms of the coordinates of its vertices.

6.2 RECAPITULATION

RECTANGULAR COORDINATE AXES Let $X'OX$ and $Y'OY$ be two mutually perpendicular lines through any point O in the plane of the paper. We call the point O, the origin. Now, choose a convenient unit of length and starting from the origin as zero, mark-off a number scale on the horizontal line $X'OX$, positive to the right of the origin O and negative to the left of origin O. Also, mark-off the same scale on the vertical line $Y'OY$, positive upwards and negative downwards of the origin O.

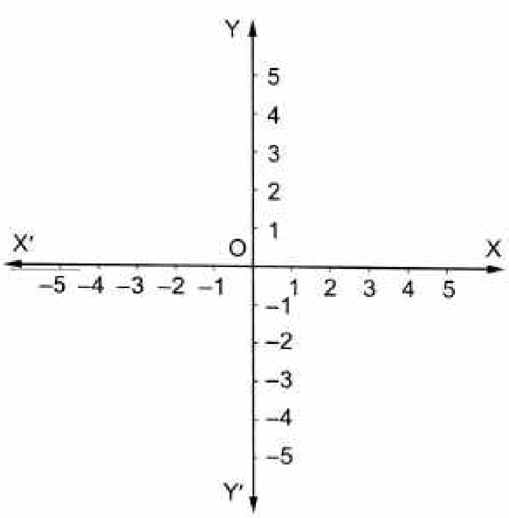


Fig. 6.1

The line $X'OX$ is called the x-axis or axis of x, the line $Y'OY$ is known as the y-axis or axis of y, and the two lines taken together are called the coordinate axes or the axes of coordinates.

CARTESIAN COORDINATES OF A POINT Let $X'OX$ and $Y'OY$ be the coordinate axes, and let P be any point in the plane. Draw perpendiculars PM and PN from P on x and y-axis respectively. The length of the directed line segment OM in the units of scale chosen is called the *x-coordinate* or *abscissa* of point P. Similarly, the length of the directed line segment ON on the same scale is called the *y-coordinate* or *ordinate* of point P. Let $OM = x$ and $ON = y$. Then the position of the point P in the plane with respect to the coordinate axes is represented by the ordered (x, y). The ordered pair (x, y) is called the coordinates of point P.

Thus, for a given point, the abscissa and ordinate are the distances of the given point from y-axis and x-axis respectively.

The above system of coordinating an ordered pair (x, y) with every point in a plane is called the *Rectangular Cartesian coordinate system*.

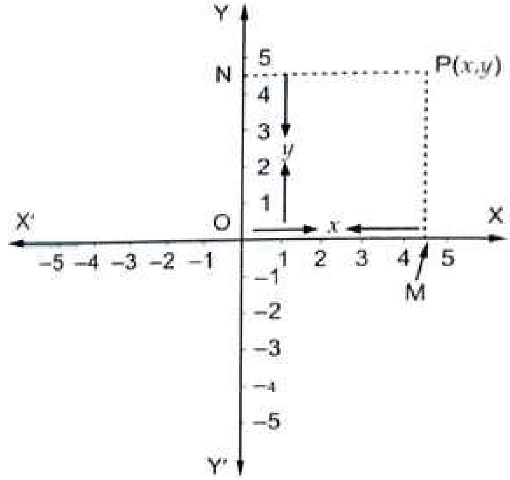


Fig. 6.2

It follows from the above discussion that corresponding to every point P in the Euclidean plane there is a unique ordered pair (x, y) of real numbers called its cartesian coordinates. Conversely, when we are given an ordered pair (x, y) and a Cartesian coordinate system, we can determine a point in the Euclidean plane having its coordinates (x, y). For this we mark-off a directed line segment $OM = x$ on the x-axis and another directed line segment $ON = y$ on y-*axis*. Now, draw perpendiculars at M and N to X and Y axes respectively. The point of intersection of these two perpendiculars determines point P in the Euclidean space having coordinates (x, y).

Thus, there is one-to-one correspondence between the set of all ordered pairs (x, y) of real numbers and the points in the Euclidean plane. The set of all ordered pairs (x, y) of real numbers is called the Cartesian plane and is denoted by R^2.

QUADRANTS Let $X'OX$ and $Y'OY$ be the coordinate axes. We observe that the two axes divide the Euclidean plane into four regions, called the quadrants. The regions XOY, $X'OY$, $X'OY'$ and $Y'OX$ are known as the first, the second, the third and the fourth quadrants respectively. The ray OX is taken as positive x-axis, OX' as negative x-axis, OY as positive y-axis and OY' as negative y-axis. In view of the above sign convention the four quadrants are characterised by the following signs of *abscissa* and *ordinate*.

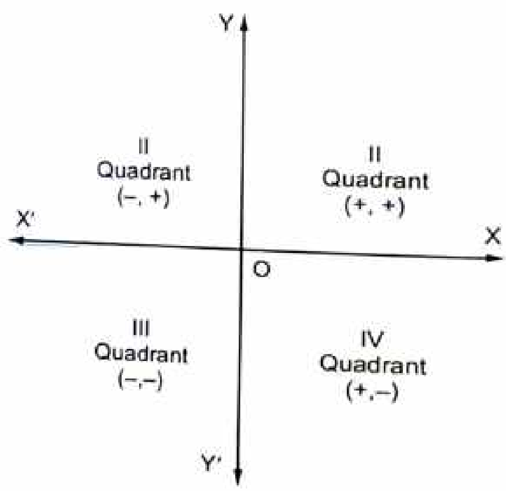


Fig. 6.3

I quadrant: $x > 0, y > 0$

II quadrant: $x < 0, y > 0$

III quadrant: $x < 0, y < 0$

IV quadrant: $x > 0, y < 0$

The coordinates of the origin are taken as (0, 0). The coordinates of any point on x-axis are of the form (x, 0) and the coordinates of any point on y-axis are of the form (0, y). Thus, if the abscissa of a point is zero, it would lie somewhere on the y-axis and if its ordinate is zero it would lie on x-axis.

It follows from the above discussion that by simply looking at the coordinates of a point we can tell in which quadrant it would lie as discussed in the following illustration.

REMARK 1 *If the coordinates of a point P are (x, y), we shall frequently refer to it as P(x, y).*

REMARK 2 *It is evident from the above discussion that:*

(i) *The abscissa of a point is its perpendicular distance from y-axis.*

(ii) *The ordinate of a point is its perpendicular distance from x-axis.*
(iii) *The abscissa of every point situated on the right side of y-axis is positive and the abscissa of every point situated on the left side of y-axis is negative.*
(iv) *The ordinate of every point situated above x-axis is positive and that of every point below x-axis is negative.*
(v) *The abscissa of every point on y-axis is zero.*
(vi) *The ordinate of every point on x-axis is zero.*
(vii) *Coordinates of the origin are O (0, 0).*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *The base AB of two equilateral triangles ABC and ABC′ with side 2a lies along the X-axis such that the mid-point of AB is at the origin as shown in Fig. 6.4 . Find the coordinates of the vertices C and C′ of the triangles.*

SOLUTION Since the mid-point of AB is at the origin O and $AB = 2a$.

$\therefore \quad OA = OB = a.$

Thus, the coordinates of A and B are $(a, 0)$ and $(-a, 0)$ respectively.

Since triangles ABC and ABC' are equilateral. Therefore, their third vertices C and C′ lie on the perpendicular bisector of base AB. Clearly, $Y'OY$ is the perpendicular bisector of AB. Thus, C and C′ lie on Y-axis. Consequently, their x-coordinates are equal to zero.

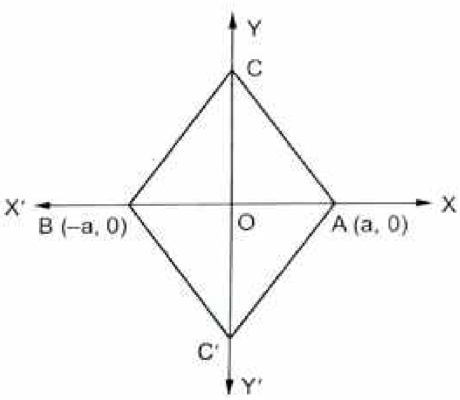


Fig. 6.4

In ΔAOC, we have

$$OA^2 + OC^2 = AC^2 \quad \text{[Using Pythagoras theorem]}$$
$$\Rightarrow \quad a^2 + OC^2 = (2a)^2 \quad [\because AB = AC = BC \text{ and } AB = 2a \therefore AC = 2a]$$
$$\Rightarrow \quad OC^2 = 4a^2 - a^2$$
$$\Rightarrow \quad OC^2 = 3a^2$$
$$\Rightarrow \quad OC = \sqrt{3}a$$

Similarly, by applying Pythagoras theorem in $\Delta\, AOC'$, we have

$$OC' = \sqrt{3}\, a$$

Thus, the coordinates of C and C′ are $(0, \sqrt{3}\, a)$ and $(0, -\sqrt{3}\, a)$ respectively.

LEVEL-2

EXAMPLE 2 *Find the coordinates of the vertices of an equilateral triangle of side 2a as shown in Fig. 6.5.*

SOLUTION Since OAB is an equilateral triangle of side $2a$. Therefore,

$$OA = AB = OB = 2a$$

Let BL perpendicular from B on OA. Then,

$$OL = LA = a$$

In $\Delta\, OLB$, we have

$$OB^2 = OL^2 + LB^2$$
$$\Rightarrow \quad (2a)^2 = a^2 + LB^2$$
$$\Rightarrow \quad LB^2 = 3a^2$$

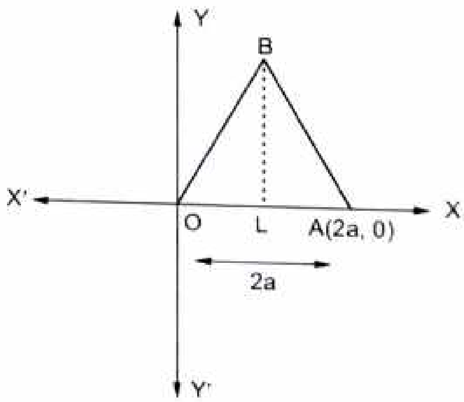


Fig. 6.5

$\Rightarrow \quad LB = \sqrt{3}\,a$

Clearly, coordinates of O are $(0, 0)$ and that of A are $(2a, 0)$. Since $OL = a$ and $LB = \sqrt{3}\,a$. So, the coordinates of B are $(a, \sqrt{3}\,a)$.

EXERCISE 6.1

LEVEL-1

1. On which axis do the following points lie?
   (i) $P(5, 0)$ (ii) $Q(0 -2)$ (iii) $R(-4, 0)$ (iv) $S(0, 5)$
2. Let $ABCD$ be a square of side $2a$. Find the coordinates of the vertices of this square when
   (i) A coincides with the origin and AB and AD are along OX and OY respectively.
   (ii) The centre of the square is at the origin and coordinate axes are parallel to the sides AB and AD respectively.

LEVEL-2

3. The base PQ of two equilateral triangles PQR and PQR' with side $2a$ lies along y-axis such that the mid-point of PQ is at the origin. Find the coordinates of the vertices R and R' of the triangles.

ANSWERS

1. P on x-axis, Q on y-axis, R on x-axis, S on y-axis
2. (i) $A(0,0)$, $B(2a,0)$, $C(2a,2a)$, $D(0,2a)$ (ii) $A(a,a)$, $B(-a,a)$, $C(-a,-a)$, $D(a,-a)$
3. $R(\sqrt{3}a, 0)$, $R'(-\sqrt{3}\,a, 0)$

6.3 DISTANCE BETWEEN TWO POINTS

The distance between any two points in the plane is the length of the line segment joining them.

THEOREM *The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by*

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

i.e., $$PQ = \sqrt{(\text{Difference of abscissae})^2 + (\text{Difference of ordinates})^2}$$

PROOF Let $X'OX$ and $Y'OY$ be the coordinate axes. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two given points in the plane. Draw PL and QM perpendicular from P and Q on x-axis. From P draw PN perpendicular to QM. Then,

$$OL = x_1, OM = x_2, PL = y_1 \text{ and } QM = y_2$$

$\therefore \quad PN = LM = OM - OL = x_2 - x_1$

and, $\quad QN = QM - NM = QM - PL = y_2 - y_1$

Clearly, $\Delta\, PNQ$ is a right triangle right angled at N. Therefore, by Pythagoras theorem, we have

$$PQ^2 = PN^2 + QN^2$$

$\Rightarrow \quad PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$\Rightarrow \quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

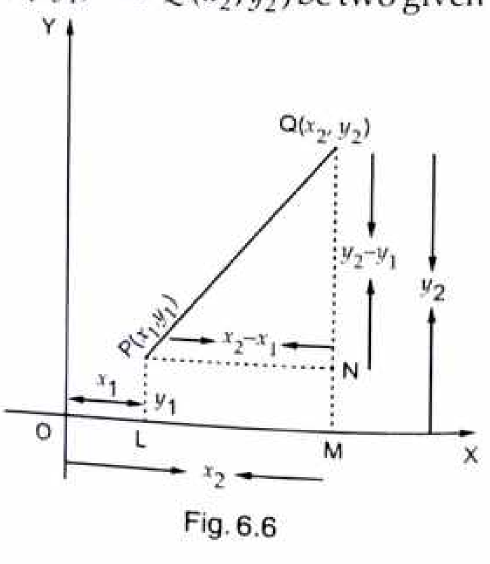


Fig. 6.6

Hence, distance btween any two points is given by

$$\sqrt{(\text{Diff. of abscissae})^2 + (\text{Diff. of ordinates})^2}$$

Q.E.D

NOTE *If O is the origin and P (x, y) is any point, then from the above formula, we have*

$$OP = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

SOME USEFUL POINTS

(I) *In order to prove that a given figure is a*

(i) *square, prove that the four sides are equal and the diagonals are also equal.*
(ii) *rhombus, prove that the four sides are equal.*
(iii) *rectangle, prove that opposite sides are equal and the diagonals are also equal.*
(iv) *parallelogram, prove that the opposite sides are equal.*
(v) *parallelogram but not a rectangle, prove that its opposite sides are equal but the diagonals are not equal.*
(vi) *rhombus but not a square, prove that its all sides are equal but the diagonals are not equal.*

(II) *For three points to be collinear, prove that the sum of the distances between two pairs of points is equal to the third pair of points.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the distance between the points*

(i) $P(-6, 7)$ *and* $Q(-1, -5)$ (ii) $R(a+b, a-b)$ *and* $S(a-b, -a-b)$

(iii) $A(at_1^2, 2at_1)$ *and* $B(at_2^2, 2at_2)$

SOLUTION (i) Here, $x_1 = -6$, $y_1 = 7$ and $x_2 = -1$, $y_2 = -5$

$$\therefore \quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\Rightarrow \quad PQ = \sqrt{(-1+6)^2 + (-5-7)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

(ii) Using distance formula, we obtain

$$RS = \sqrt{(a-b-a-b)^2 + (-a-b-a+b)^2} = \sqrt{4b^2 + 4a^2} = 2\sqrt{a^2 + b^2}$$

(iii) Using distance formula, we obtain

$$AB = \sqrt{\left(at_2^2 - at_1^2\right)^2 + \left(2at_2 - 2at_1\right)^2}$$

$$\Rightarrow \quad AB = \sqrt{a^2(t_2 - t_1)^2(t_2 + t_1)^2 + 4a^2(t_2 - t_1)^2}$$

$$\Rightarrow \quad AB = a(t_2 - t_1)\sqrt{(t_2 + t_1)^2 + 4}$$

EXAMPLE 2 *If the point (x, y) is equidistant from the points $(a+b, b-a)$ and $(a-b, a+b)$, prove that $bx = ay$.*

SOLUTION Let $P(x, y)$, $Q(a+b, b-a)$ and $R(a-b, a+b)$ be the given points. Then,

$$PQ = PR \qquad \text{[Given]}$$

$$\Rightarrow \quad \sqrt{\{x-(a+b)\}^2 + \{y-(b-a)\}^2} = \sqrt{\{x-(a-b)\}^2 + \{y-(a+b)\}^2}$$

$$\Rightarrow \quad \{x-(a+b)\}^2 + \{y-(b-a)\}^2 = \{x-(a-b)\}^2 + \{y-(a+b)\}^2$$

$$\Rightarrow \quad x^2 - 2x(a+b) + (a+b)^2 + y^2 - 2y(b-a) + (b-a)^2$$
$$= x^2 + (a-b)^2 - 2x(a-b) + y^2 - 2y(a+b) + (a+b)^2$$
$$\Rightarrow \quad -2x(a+b) - 2y(b-a) = -2x(a-b) - 2y(a+b)$$
$$\Rightarrow \quad ax + bx + by - ay = ax - bx + ay + by$$
$$\Rightarrow \quad 2bx = 2ay \Rightarrow bx = ay$$

REMARK *We know that a point which is equidistant from points P and Q lies on the perpendicular bisector of PQ. Therefore, $bx = ay$ is the equation of the perpendicular bisector of PQ.*

EXAMPLE 3 *Find the equation of the perpendicular bisector of AB, where A and B are the points (3, 6) and (–3, 4) respectively. Also, find its point of intersecction with (i) x-axis (ii) y-axis.*

SOLUTION Let $P(x, y)$ be any point on the perpendicular bisector of AB. Then,

$$PA = PB$$
$$\Rightarrow \quad \sqrt{(x-3)^2 + (y-6)^2} = \sqrt{(x+3)^2 + (y-4)^2}$$
$$\Rightarrow \quad (x-3)^2 + (y-6)^2 = (x+3)^2 + (y-4)^2$$
$$\Rightarrow \quad x^2 - 6x + 9 + y^2 - 12y + 36 = x^2 + 6x + 9 + y^2 - 8y + 16$$
$$\Rightarrow \quad 12x + 4y - 20 = 0$$
$$\Rightarrow \quad 3x + y - 5 = 0 \quad \ldots(i)$$

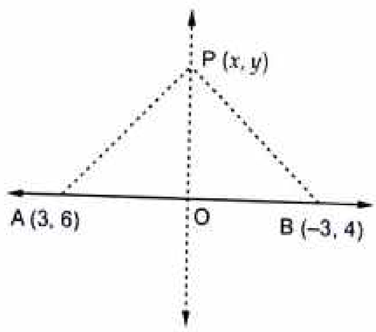


Fig. 6.7

Hence, the equation of the perpendicular bisector of AB is $3x + y - 5 = 0$.

(i) We know that the coordinates of any point on x-axis are of the form $(x, 0)$. In other words, y-coordinate of every point on x-axis is zero. So, putting $y = 0$ in (i), we get

$$3x - 5 = 0 \Rightarrow x = \frac{5}{3}$$

Thus, the perpendicular bisector of AB cuts x-axis at $(5/3, 0)$.

(ii) The coordinates of any point on y-axis are of the form $(0, y)$. Putting $x = 0$ in (i), we get

$$y - 5 = 0 \Rightarrow y = 5$$

Thus, the perpendicular bisector of AB intersects y-axis at $(0, 5)$.

EXAMPLE 4 *Find the value of x, if the distance between the points (x, –1) and (3, 2) is 5.*

SOLUTION Let $P(x, -1)$ and $Q(3, 2)$ be the given points. Then,

$$PQ = 5$$
$$\Rightarrow \quad \sqrt{(x-3)^2 + (-1-2)^2} = 5 \quad \text{[Given]}$$
$$\Rightarrow \quad (x-3)^2 + 9 = 5^2$$

$\Rightarrow \quad x^2 - 6x + 18 = 25$

$\Rightarrow \quad x^2 - 6x - 7 = 0 \Rightarrow (x-7)(x+1) = 0 \Rightarrow x = 7$ or, $x = -1$

EXAMPLE 5 *If the points A (4, 3) and B (x, 5) are on the circle with centre O (2, 3), find the value of x.* **[CBSE 2009]**

SOLUTION Since A and B lie on the circle having centre O.

$\therefore \quad OA = OB$ [Each equal to radius]

$\Rightarrow \quad \sqrt{(4-2)^2 + (3-3)^2} = \sqrt{(x-2)^2 + (5-3)^2}$

$\Rightarrow \quad 2 = \sqrt{(x-2)^2 + 4}$

$\Rightarrow \quad 4 = (x-2)^2 + 4 \Rightarrow (x-2)^2 = 0 \Rightarrow x - 2 = 0 \Rightarrow x = 2.$

EXAMPLE 6 *Find a point on x-axis which is equidistant from A (2, –5) and B (–2, 9).* **[NCERT, CBSE 2009, 2017]**

SOLUTION We know that a point on x-axis is of the form $(x, 0)$. So, let $P(x, 0)$ be the point equidistant from $A(2, -5)$ and $B(-2, 9)$. Then,

$PA = PB$

$\Rightarrow \quad \sqrt{(x-2)^2 + (0+5)^2} = \sqrt{(x+2)^2 + (0-9)^2}$

$\Rightarrow \quad (x-2)^2 + 25 = (x+2)^2 + 81$

$\Rightarrow \quad x^2 - 4x + 4 + 25 = x^2 + 4x + 4 + 81 \Rightarrow -8x = 56 \Rightarrow x = -7$

Hence, the required point is $(-7, 0)$.

EXAMPLE 7 *Find a point on the y-axis which is equidistant from the point A (6, 5) and B (–4, 3).* **[NCERT, CBSE 2017]**

SOLUTION We know that a point on y-axis is of the form $(0, y)$. So, let the required point be $P(0, y)$. Then,

$PA = PB$

$\Rightarrow \quad \sqrt{(0-6)^2 + (y-5)^2} = \sqrt{(0+4)^2 + (y-3)^2}$

$\Rightarrow \quad 36 + (y-5)^2 = 16 + (y-3)^2$

$\Rightarrow \quad 36 + y^2 - 10y + 25 = 16 + y^2 - 6y + 9 \Rightarrow 4y = 36 \Rightarrow y = 9$

So, the required point is $(0, 9)$.

EXAMPLE 8 *The x-coordinate of a point P is twice its y-coordinate. If P is equidistant from Q (2, –5) and R (–3, 6), then find the coordinates of P.* **[CBSE 2010, 2016]**

SOLUTION Let the coordinates of P be (x, y). It is given that $x = 2y$. It is also given that

$PQ = PR$

$\Rightarrow \quad \sqrt{(x-2)^2 + (y+5)^2} = \sqrt{(x+3)^2 + (y-6)^2}$

$\Rightarrow \quad \sqrt{(2y-2)^2 + (y+5)^2} = \sqrt{(2y+3)^2 + (y-6)^2}$

$\Rightarrow \quad \sqrt{5y^2 + 2y + 29} = \sqrt{5y^2 + 45}$

$\Rightarrow \quad 5y^2 + 2y + 29 = 5y^2 + 45 \Rightarrow 2y = 16 \Rightarrow y = 8$

Hence, the coordinates of P are $(16, 8)$.

EXAMPLE 9 *Do the points A (3, 2), B (–2, –3) and C (2, 3) form a triangle? If so, name the type of triangle formed.* **[NCERT]**

SOLUTION Using distance formula, we obtain

$$AB = \sqrt{(-2-3)^2 + (-3-2)^2} = \sqrt{25+25} = \sqrt{50}$$

$$BC = \sqrt{(2+2)^2 + (3+3)^2} = \sqrt{16+36} = \sqrt{52}$$

and, $$AC = \sqrt{(2-3)^2 + (3-2)^2} = \sqrt{1+1} = \sqrt{2}$$

Clearly, $AB + BC > AC$, $AC + BC > AB$ and $AB + AC > BC$. Therefore, points A, B and C form a triangle. We also observe that $BC^2 = AB^2 + AC^2$. Therefore, ΔABC is a right triangle, right angled at A.

EXAMPLE 10 *Show that the points $(a, a), (-a, -a)$ and $(-\sqrt{3}a, \sqrt{3}a)$ are the vertices of an equilateral triangle. Also, find its area.* **[CBSE 2015]**

SOLUTION Let $A(a, a)$, $B(-a, -a)$ and $C(-\sqrt{3}a, \sqrt{3}a)$ be the given points. Then,

$$AB = \sqrt{(-a-a)^2 + (-a-a)^2} = \sqrt{4a^2 + 4a^2} = 2\sqrt{2}\,a$$

$$BC = \sqrt{(-\sqrt{3}\,a + a)^2 + (\sqrt{3}\,a + a)^2}$$

$$\Rightarrow \quad BC = \sqrt{a^2(1-\sqrt{3})^2 + a^2(\sqrt{3}+1)^2}$$

$$\Rightarrow \quad BC = a\sqrt{(1-\sqrt{3})^2 + (1+\sqrt{3})^2}$$

$$\Rightarrow \quad BC = a\sqrt{1+3-2\sqrt{3}+1+3+2\sqrt{3}} = a\sqrt{8} = 2\sqrt{2}\,a$$

and, $$AC = \sqrt{(-\sqrt{3}\,a - a)^2 + (\sqrt{3}\,a - a)^2}$$

$$\Rightarrow \quad AC = \sqrt{a^2(\sqrt{3}+1)^2 + a^2(\sqrt{3}-1)^2}$$

$$\Rightarrow \quad AC = a\sqrt{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2}$$

$$\Rightarrow \quad AC = a\sqrt{3+1+2\sqrt{3}+3+1-2\sqrt{3}} = a\sqrt{8} = 2\sqrt{2}\,a$$

Clearly, we have

$$AB = BC = AC$$

Hence, the triangle ABC formed by the given points is an equilateral triangle.

Now,

$$\text{Area of } \Delta ABC = \frac{\sqrt{3}}{4}(\text{Side})^2$$

$$= \frac{\sqrt{3}}{4} \times AB^2 = \frac{\sqrt{3}}{4} \times (2\sqrt{2}a)^2 \text{ sq. units} = 2\sqrt{3}a^2 \text{ sq. units}$$

EXAMPLE 11 *Show that the points $(1, -1), (5, 2)$ and $(9, 5)$ are collinear.* **[CBSE 2006 C]**

SOLUTION Let $A(1, -1)$, $B(5, 2)$ and $(9, 5)$ be the given points. Then, we have

$$AB = \sqrt{(5-1)^2 + (2+1)^2} = \sqrt{16+9} = 5$$

$$BC = \sqrt{(5-9)^2 + (2-5)^2} = \sqrt{16+9} = 5$$

and, $$AC = \sqrt{(1-9)^2 + (-1-5)^2} = \sqrt{64+36} = 10$$

Clearly, $AC = AB + BC$. Hence, A, B, C are collinear points.

EXAMPLE 12 *Show that four points (0, −1), (6, 7), (−2, 3) and (8, 3) are the vertices of a rectangle. Also, find its area.* **[CBSE 2013]**

SOLUTION Let $A(0-1)$, $B(6,7)$, $C(-2,3)$ and $D(8,3)$ be the given points. Then,

$$AD = \sqrt{(8-0)^2 + (3+1)^2} = \sqrt{64+16} = 4\sqrt{5}$$

$$BC = \sqrt{(6+2)^2 + (7-3)^2} = \sqrt{64+16} = 4\sqrt{5}$$

$$AC = \sqrt{(-2-0)^2 + (3+1)^2} = \sqrt{4+16} = 2\sqrt{5}$$

and, $BD = \sqrt{(8-6)^2 + (3-7)^2} = \sqrt{4+16} = 2\sqrt{5}$

$\therefore$ $AD = BC$ and $AC = BD$

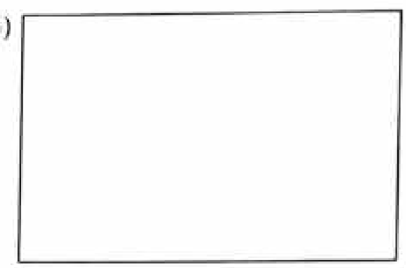


Fig. 6.8

So, $ADBC$ is a parallelogram.

Now, $AB = \sqrt{(6-0)^2 + (7+1)^2} = \sqrt{36+64} = 10$ and, $CD = \sqrt{(8+2)^2 + (3-3)^2} = 10$

Clearly, $AB^2 = AD^2 + DB^2$ and $CD^2 = CB^2 + BD^2$.

Hence, $ADBC$ is a rectangle.

Area of rectangle $ADBC = AD \times DB = (4\sqrt{5} \times 2\sqrt{5})$ sq. units $= 40$ sq. units

EXAMPLE 13 *Show that* $A(6,4)$, $B(5,-2)$ *and* $C(7,-2)$ *are the vertices of an isosceles triangle. Also, find the length of the median through A.* **[CBSE 2010]**

SOLUTION We have

$$AB = \sqrt{(6-5)^2 + (4+2)^2} = \sqrt{37}, \quad AC = \sqrt{(6-7)^2 + (4+2)^2} = \sqrt{37}$$

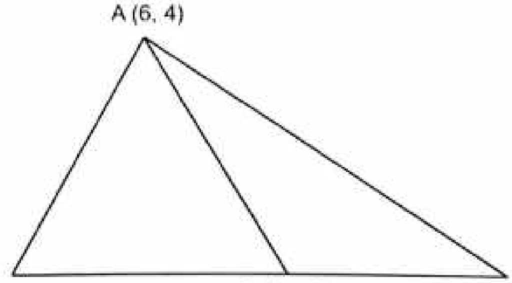


Fig. 6.9

$\therefore$ $AB = AC$

So, ΔABC is isosceles.

Let D be the mid-point of BC. Then, coordinates of D are $\left(\frac{5+7}{2}, \frac{-2-2}{2}\right)$ i.e. $(6, -2)$.

$$\therefore \quad AD = \sqrt{(6-6)^2 + (4+2)^2} = \sqrt{36} = 6$$

EXAMPLE 14 *If* $P(2,-1)$, $Q(3,4)$, $R(-2,3)$ *and* $S(-3,-2)$ *be four points in a plane, show that PQRS is a rhombus but not a square. Find the area of the rhombus.* **[CBSE 2013]**

SOLUTION The given points are $P(2,-1)$, $Q(3,4)$, $R(-2,3)$ and $S(-3,-2)$.
We have,

$$PQ = \sqrt{(3-2)^2 + (4+1)^2} = \sqrt{1^2 + 5^2} = \sqrt{26} \text{ units}$$

$$QR = \sqrt{(-2-3)^2 + (3-4)^2} = \sqrt{25+1} = \sqrt{26} \text{ units}$$

$$RS = \sqrt{(-3+2)^2 + (-2-3)^2} = \sqrt{1+25} = \sqrt{26} \text{ units}$$

$$SP = \sqrt{(-3-2)^2 + (-2+1)^2} = \sqrt{26} \text{ units}$$

$$PR = \sqrt{(-2-2)^2 + (3+1)^2} = \sqrt{16+16} = 4\sqrt{2} \text{ units}$$

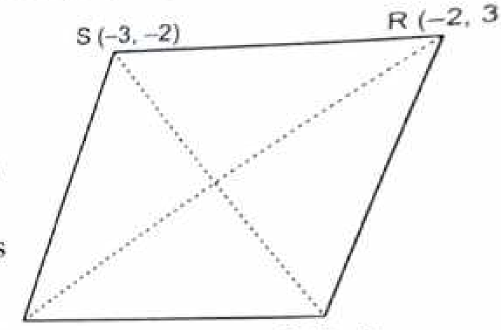


Fig. 6.10

and, $QS = \sqrt{(-3-3)^2 + (-2-4)^2} = \sqrt{36+36} = 6\sqrt{2}$ units

$\therefore$ $PQ = QR = RS = SP = \sqrt{26}$ units

and, $PR \neq QS$

This means that $PQRS$ is a quadrilateral whose sides are equal but diagonals are not equal. Thus, $PQRS$ is a rhombus but not a square.

Now, Area of rhombus $PQRS = \frac{1}{2} \times$ (Product of lengths of diagonals)

$$= \frac{1}{2} \times (PR \times QS) = \left(\frac{1}{2} \times 4\sqrt{2} \times 6\sqrt{2}\right) \text{sq. units} = 24 \text{ sq. units}$$

EXAMPLE 15 *Find the coordinates of the centre of the circle passing through the points (0, 0), (–2, 1) and (–3, 2). Also, find its radius.*

SOLUTION Let $P(x, y)$ be the centre of the circle passing through the points $O(0, 0)$, $A(-2, 1)$ and $B(-3, 2)$. Then,

$OP = AP = BP$

Now, $OP = AP$

$\Rightarrow$ $OP^2 = AP^2$

$\Rightarrow$ $x^2 + y^2 = (x+2)^2 + (y-1)^2$

$\Rightarrow$ $x^2 + y^2 = x^2 + y^2 + 4x - 2y + 5$

$\Rightarrow$ $4x - 2y + 5 = 0$ ...(i)

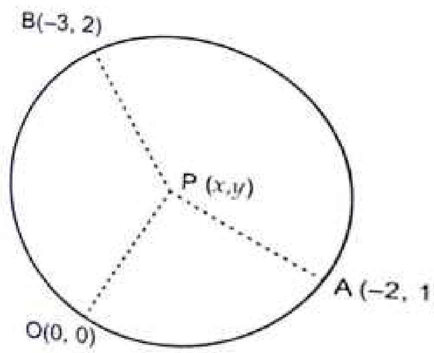


Fig. 6.11

and, $OP = BP$

$\Rightarrow$ $OP^2 = BP^2$

$\Rightarrow$ $x^2 + y^2 = (x+3)^2 + (y-2)^2$

$\Rightarrow$ $x^2 + y^2 = x^2 + y^2 + 6x - 4y + 13$

$\Rightarrow$ $6x - 4y + 13 = 0$ ...(ii)

On solving equations (i) and (ii), we get: $x = \frac{3}{2}$ and $y = \frac{11}{2}$.

Thus, the coordinates of the centre are $\left(\frac{3}{2}, \frac{11}{2}\right)$

$\therefore$ Radius $= OP = \sqrt{x^2 + y^2} = \sqrt{\frac{9}{4} + \frac{121}{4}} = \frac{1}{2}\sqrt{130}$ units.

EXAMPLE 16 *If $(-4, 0)$ and $(4, 0)$ are two vertices of an equilateral triangle, find the coordinates of its third vertex.* **[CBSE 2014]**

SOLUTION Let $C(x, y)$ be the third vertex of triangle ABC having two vertices at $A(-4, 0)$ and $B(4, 0)$. Since ΔABC is equilateral. Therefore,

$AC = BC = AB$

Now, $AC = BC$

$\Rightarrow \sqrt{(x+4)^2 + (y-0)^2} = \sqrt{(x-4)^2 + (y-0)^2}$

$\Rightarrow (x+4)^2 + y^2 = (x-4)^2 + y^2$

$\Rightarrow 16x = 0$

$\Rightarrow x = 0$

Again,

$AC = BC = AB$

$\Rightarrow AC = AB$

$\Rightarrow \sqrt{(x+4)^2 + (y-0)^2} = \sqrt{(4+4)^2 + 0^2}$

$\Rightarrow (x+4)^2 + y^2 = 64$

$\Rightarrow (0+4)^2 + y^2 = 64$ $[\because x = 0]$

$\Rightarrow y^2 = 48$

$\Rightarrow y = \pm 4\sqrt{3}$

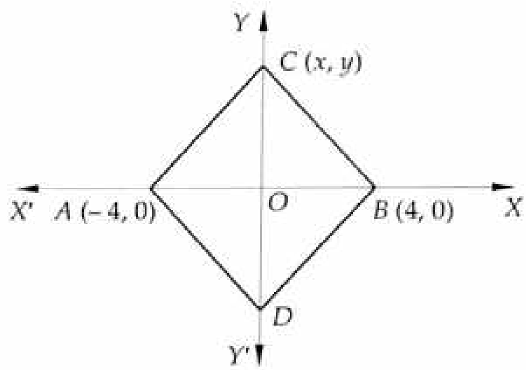


Fig. 6.12

Hence, the coordinates of the third vertex are $C(0, 4\sqrt{3})$ and $D(0, -4\sqrt{3})$.

EXAMPLE 17 *Points $A(-1, y)$ and $B(5, 7)$ lie on a circle with centre $O(2, -3y)$. Find the values of y. Hence, find the radius of the circle.* **[CBSE 2014]**

SOLUTION Since O is the centre of the circle and A, B are points on its circumference.

$\therefore OA = OB =$ Radius

$\Rightarrow OA = OB$

$\Rightarrow \sqrt{(2+1)^2 + (-3y-y)^2} = \sqrt{(2-5)^2 + (-3y-7)^2}$

$\Rightarrow 9 + 16y^2 = 9 + (3y+7)^2$

$\Rightarrow 16y^2 = 9y^2 + 42y + 49$

$\Rightarrow 7y^2 - 42y - 49 = 0$

$\Rightarrow y^2 - 6y - 7 = 0$

$\Rightarrow (y-7)(y+1) = 0 \Rightarrow y = -1, 7$

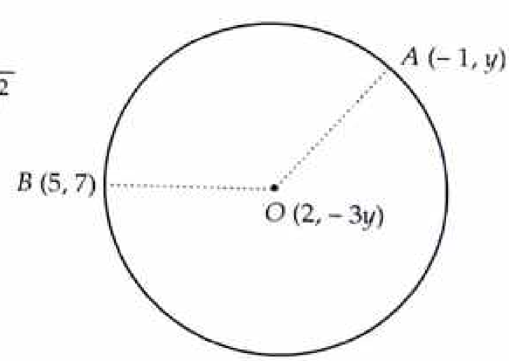


Fig. 6.13

CASE I *When* $y = -1$: In this case

The coordinates of O, A and B are $O(2, 3)$, $A(-1, -1)$ and $B(5, 7)$ respectively.

$\therefore$ Radius $= OA = \sqrt{(2+1)^2 + (3+1)^2} = 5$

CASE II *When* $y = 7$ *:* In this case

The coordinates O, A and B are $O(2,-21)$, $A(-1,7)$ and $B(5,7)$ respectively.

$\therefore$ $$\text{Radius} = OA = \sqrt{(2+1)^2 + (-21-7)^2} = \sqrt{9+784} = \sqrt{793}$$

EXAMPLE 18 *If* $A(5,2), B(2,-2)$ *and* $C(-2,t)$ *are the vertices of right angled triangle w* $\angle B = 90°$*, then find the value of* t. [CBSE 201

SOLUTION Using Pythagoras theorem in right triangle ABC, we obtain

$$AC^2 = AB^2 + BC^2$$

$\Rightarrow$ $$(5+2)^2 + (2-t)^2 = \{(5-2)^2 + (2+2)^2\} + \{(2+2)^2 + (-2-t)^2\}$$

$\Rightarrow$ $$49 + (4 - 4t + t^2) = (9+16) + (16 + 4 + 4t + t^2)$$

$\Rightarrow$ $$t^2 - 4t + 53 = t^2 + 4t \times 45$$

$\Rightarrow$ $$-8t = -8$$

$\Rightarrow$ $$t = 1$$

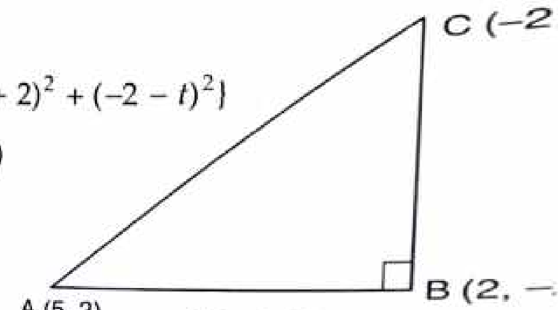


Fig. 6.14

LEVEL-2

EXAMPLE 19 *If P and Q are two points whose coordinates are* $(at^2, 2at)$ *and* $\left(\frac{a}{t^2}, \frac{2a}{t}\right)$ *respectivel and S is the point* $(a, 0)$*. Show that* $\frac{1}{SP} + \frac{1}{SQ}$ *is independent of* t.

SOLUTION Using distance formula, we obtain

$$SP = \sqrt{(at^2 - a)^2 + (2at - 0)^2} = a\sqrt{(t^2-1)^2 + 4t^2} = a(t^2+1)$$

and, $$SQ = \sqrt{\left(\frac{a}{t^2} - a\right)^2 + \left(\frac{2a}{t} - 0\right)^2}$$

$\Rightarrow$ $$SQ = \sqrt{\frac{a^2(1-t^2)^2}{t^4} + \frac{4a^2}{t^2}} = \frac{a}{t^2}\sqrt{(1-t^2)^2 + 4t^2} = \frac{a}{t^2}\sqrt{(1+t^2)^2} = \frac{a}{t^2}(1+t^2)$$

$\therefore$ $$\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a(t^2+1)} + \frac{t^2}{a(t^2+1)}$$

$\Rightarrow$ $$\frac{1}{SP} + \frac{1}{SQ} = \frac{1+t^2}{a(t^2+1)} = \frac{1}{a},$$ which is independent of t.

EXAMPLE 20 *If two vertices of an equilateral triangle be* $(0,0)$, $(3, \sqrt{3})$*, find the third vertex.*

SOLUTION $O(0,0)$ and $A(3, \sqrt{3})$ be the given points and let $B(x, y)$ be the third vertex of equilateral ΔOAB. Then,

$$OA = OB = AB$$

$\Rightarrow$ $$OA^2 = OB^2 = AB^2$$

We have, $$OA^2 = (3-0)^2 + (\sqrt{3} - 0)^2 = 12,$$

$$OB^2 = x^2 + y^2$$

and, $AB^2 = (x-3)^2 + (y-\sqrt{3})^2$

$\Rightarrow$ $AB^2 = x^2 + y^2 - 6x - 2\sqrt{3}\,y + 12$

$\therefore$ $OA^2 = OB^2 = AB^2$

$\Rightarrow$ $OA^2 = OB^2$ and $OB^2 = AB^2$

$\Rightarrow$ $x^2 + y^2 = 12$

and, $x^2 + y^2 = x^2 + y^2 - 6x - 2\sqrt{3}\,y + 12$

$\Rightarrow$ $x^2 + y^2 = 12$ and $6x + 2\sqrt{3}\,y = 12$

$\Rightarrow$ $x^2 + y^2 = 12$ and $3x + \sqrt{3}\,y = 6$

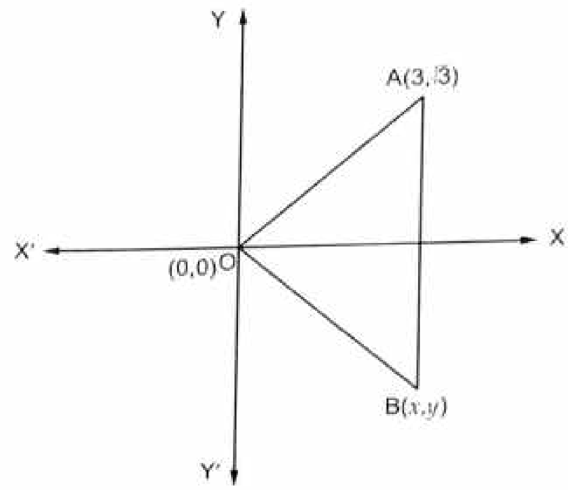


Fig. 6.15

$\Rightarrow$ $x^2 + \left(\frac{6-3x}{\sqrt{3}}\right)^2 = 12$ $\left[\because\ 3x + \sqrt{3}\,y = 6 \therefore\ y = \frac{6-3x}{\sqrt{3}}\right]$

$\Rightarrow$ $3x^2 + (6-3x)^2 = 36$

$\Rightarrow$ $12x^2 - 36x = 0$

$\Rightarrow$ $x = 0, 3$

$\therefore$ $x = 0 \Rightarrow \sqrt{3}\,y = 6 \Rightarrow y = \frac{6}{\sqrt{3}} = 2\sqrt{3}$ $\left[\text{Putting } x = 0 \text{ in } 3x + \sqrt{3}y = 6\right]$

and, $x = 3 \Rightarrow 9 + \sqrt{3}\,y = 6 \Rightarrow y = \frac{6-9}{\sqrt{3}} = -\sqrt{3}$ $\left[\text{Putting } x = 3 \text{ in } 3x + \sqrt{3}y = 6\right]$

Hence, the coordinates of the third vertex B are $(0, 2\sqrt{3})$ or, $(3, -\sqrt{3})$.

EXAMPLE 21 *Find the coordinates of the circumcentre of the triangle whose vertices are (8, 6), (8, –2) and (2, –2). Also, find its circum-radius.*

SOLUTION Recall that the circumcentre of a triangle is equidistant from the vertices of a triangle. Let $A\,(8, 6)$, $B\,(8, -2)$ and $C\,(2, -2)$ be the vertices of the given triangle and let $P\,(x, y)$ be the circumcentre of this triangle. Then,

$PA = PB = PC$

$\Rightarrow$ $PA^2 = PB^2 = PC^2$

Now, $PA^2 = PB^2$

$\Rightarrow$ $(x-8)^2 + (y-6)^2 = (x-8)^2 + (y+2)^2$

$\Rightarrow$ $x^2 + y^2 - 16x - 12y + 100 = x^2 + y^2 - 16x + 4y + 68$

$\Rightarrow$ $16y = 32$

$\Rightarrow$ $y = 2$

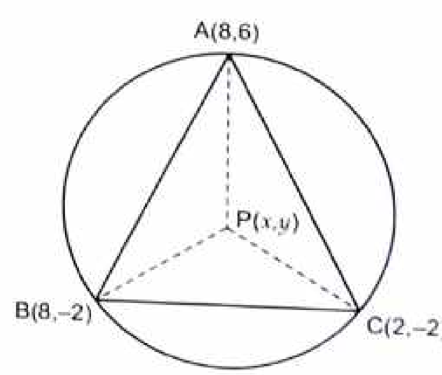


Fig. 6.16

and, $PB^2 = PC^2$

$\Rightarrow$ $(x-8)^2 + (y+2)^2 = (x-2)^2 + (y+2)^2$

$\Rightarrow$ $x^2 + y^2 - 16x + 4y + 68 = x^2 + y^2 - 4x + 4y + 8$

$\Rightarrow$ $12x = 60$

$\Rightarrow$ $x = 5$

So, the coordinates of the circumcentre P are (5, 2).

Also, Circum-radius $= PA = PB = PC = \sqrt{(5-8)^2 + (2-6)^2} = 5$

EXAMPLE 22 *Let the opposite angular points of a square be (3, 4) and (1, −1). Find the coordinates of the remaining angular points.*

SOLUTION Let $ABCD$ be a square and let A (3, 4) and C (1, −1) be the given angular points. Let $B(x, y)$ be the unknown vertex.

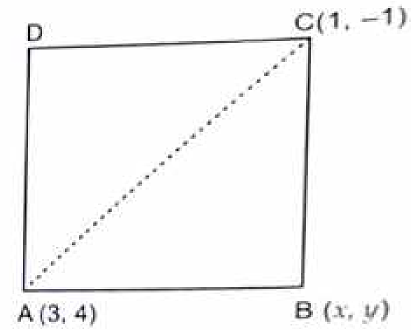


Fig. 6.17

Then, $AB = BC$

$\Rightarrow \quad AB^2 = BC^2$

$\Rightarrow \quad (x-3)^2 + (y-4)^2 = (x-1)^2 + (y+1)^2$

$\Rightarrow \quad 4x + 10y - 23 = 0$

$\Rightarrow \quad x = \dfrac{23 - 10y}{4}$ ...(i)

In right-angled triangle ABC, we have

$$AB^2 + BC^2 = AC^2$$

$\Rightarrow \quad (x-3)^2 + (y-4)^2 + (x-1)^2 + (y+1)^2 = (3-1)^2 + (4+1)^2$ ...(ii)

$\Rightarrow \quad x^2 + y^2 - 4x - 3y - 1 = 0$

Substituting the value of x from (i) into (ii), we get

$$\left(\frac{23-10y}{4}\right)^2 + y^2 - (23 - 10y) - 3y - 1 = 0$$

$\Rightarrow \quad 4y^2 - 12y + 5 = 0 \Rightarrow (2y-1)(2y-5) = 0 \Rightarrow y = \dfrac{1}{2}$ or, $\dfrac{5}{2}$

Putting $y = \dfrac{1}{2}$ and $y = \dfrac{5}{2}$ respectively in (i), we get $x = \dfrac{9}{2}$ and $x = \dfrac{-1}{2}$ respectively.

Hence, the required vertices of the square are (9/2, 1/2) and (− 1/2, 5/2).

EXAMPLE 23 *Prove that the points (−3, 0), (1, −3) and (4, 1) are the vertices of an isosceles right-angled triangle. Find the area of this triangle.*

SOLUTION Let $A(-3, 0)$, $B(1, -3)$ and $C(4, 1)$ be the given points. Then,

$$AB = \sqrt{\{1-(-3)\}^2 + (-3-0)^2} = \sqrt{4^2 + (-3)^2} = \sqrt{16+9} = 5 \text{ units}$$

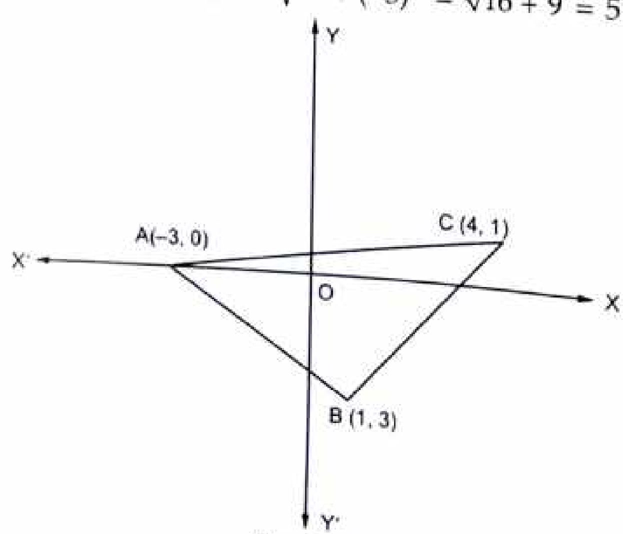


Fig. 6.18

$$BC = \sqrt{(4-1)^2 + (1+3)^2} = \sqrt{9+16} = 5 \text{ units}$$

and, $CA = \sqrt{(4+3)^2 + (1-0)^2} = \sqrt{49+1} = 5\sqrt{2}$ units.

Clearly, $AB = BC$. Therefore, ΔABC is isosceles.

Also, $AB^2 + BC^2 = 25 + 25 = (5\sqrt{2})^2 = CA^2$

$\therefore$ ΔABC is right-angled at B.

Thus, ΔABC is a right-angled isosceles triangle.

Now, Area of $\Delta ABC = \frac{1}{2}(\text{Base} \times \text{Height}) = \frac{1}{2}(AB \times BC) = \left(\frac{1}{2} \times 5 \times 5\right)$ sq. units $= \frac{25}{2}$ sq. units

EXERCISE 6.2

LEVEL-1

1. Find the distance between the following pair of points :
   (i) $(-6, 7)$ and $(-1, -5)$
   (ii) $(a+b, b+c)$ and $(a-b, c-b)$
   (iii) $(a \sin\alpha, -b\cos\alpha)$ and $(-a\cos\alpha, b\sin\alpha)$
   (iv) $(a, 0)$ and $(0, b)$
2. Find the value of a when the distance between the points $(3, a)$ and $(4, 1)$ is $\sqrt{10}$.
3. If the points $(2, 1)$ and $(1, -2)$ are equidistant from the point (x, y), show that $x + 3y = 0$.
4. Find the values of x, y if the distances of the point (x, y) from $(-3, 0)$ as well as from $(3, 0)$ are 4.
5. The length of a line segment is of 10 units and the coordinates of one end-point are $(2, -3)$. If the abscissa of the other end is 10, find the ordinate of the other end.
6. Show that the points $(-4, -1)$, $(-2, -4)$, $(4, 0)$ and $(2, 3)$ are the vertices points of a rectangle. [CBSE 2006 C]
7. Show that the points $A(1, -2)$, $B(3, 6)$, $C(5, 10)$ and $D(3, 2)$ are the vertices of a parallelogram.
8. Prove that the points $A(1, 7)$, $B(4, 2)$, $C(-1, -1)$ and $D(-4, 4)$ are the vertices of a square. [NCERT]
9. Prove that the points $(3, 0)$, $(6, 4)$ and $(-1, 3)$ are vertices of a right-angled isosceles triangle. [CBSE 2006 C]
10. Prove that $(2, -2)$, $(-2, 1)$ and $(5, 2)$ are the vertices of a right angled triangle. Find the area of the triangle and the length of the hypotenuse. [CBSE 2016]
11. Prove that the points $(2a, 4a)$, $(2a, 6a)$ and $(2a + \sqrt{3}a, 5a)$ are the vertices of an equilateral triangle.
12. Prove that the points $(2, 3)$, $(-4, -6)$ and $(1, 3/2)$ do not form a triangle.
13. The points $A(2, 9)$, $B(a, 5)$ and $C(5, 5)$ are the vertices of a triangle ABC right angled at B. Find the values of a and hence the area of ΔABC. [NCERT EXEMPLAR]
14. Show that the quadrilateral whose vertices are $(2, -1)$, $(3, 4)$, $(-2, 3)$ and $(-3, -2)$ is a rhombus.
15. Two vertices of an isosceles triangle are $(2, 0)$ and $(2, 5)$. Find the third vertex if the length of the equal sides is 3.

16. Which point on x-axis is equidistant from $(5, 9)$ and $(-4, 6)$?
17. Prove that the points $(-2, 5)$, $(0, 1)$ and $(2, -3)$ are collinear.
18. The coordinates of the point P are $(-3, 2)$. Find the coordinates of the point Q which lies on the line joining P and origin such that $OP = OQ$.
19. Which point on y-axis is equidistant from $(2, 3)$ and $(-4, 1)$?
20. The three vertices of a parallelogram are $(3, 4)$, $(3, 8)$ and $(9, 8)$. Find the fourth vertex.
21. Find a point which is equidistant from the points $A(-5, 4)$ and $B(-1, 6)$. How many such points are there? **[NCERT EXEMPLAR]**
22. The centre of a circle is $(2a, a-7)$. Find the values of a if the circle passes through the point $(11, -9)$ and has diameter $10\sqrt{2}$ units. **[NCERT EXEMPLAR]**
23. Ayush starts walking from his house to office. Instead of going to the office directly, he goes to a bank first, from there to his daughter's school and then reaches the office. What is the extra distance travelled by Ayush in reaching the office? (Assume that all distances covered are in straight lines). If the house is situated at $(2, 4)$, bank at $(5, 8)$, school at $(13, 14)$ and office at $(13, 26)$ and coordinates are in kilometers. **[NCERT EXEMPLAR]**
24. Find the value of k, if the point $P(0, 2)$ is equidistant from $(3, k)$ and $(k, 5)$.
25. If $(-4, 3)$ and $(4, 3)$ are two vertices of an equilateral triangle, find the coordinates of the third vertex, given that the origin lies in the (i) interior, (ii) exterior of the triangle. **[NCERT EXEMPLAR]**
26. Show that the points $(-3, 2)$, $(-5, -5)$, $(2, -3)$ and $(4, 4)$ are the vertices of a rhombus. Find the area of this rhombus.
27. Find the coordinates of the circumcentre of the triangle whose vertices are $(3, 0)$, $(-1, -6)$ and $(4, -1)$. Also, find its circumradius.
28. Find a point on the x-axis which is equidistant from the points $(7, 6)$ and $(-3, 4)$. **[CBSE 2005]**
29. (i) Show that the points $A(5, 6)$, $B(1, 5)$, $C(2, 1)$ and $D(6, 2)$ are the vertices of a square. **[CBSE 2004]**
    (ii) Prove that the points $A(2, 3)$, $B(-2, 2)$, $C(-1, -2)$, and $D(3, -1)$ are the vertices of a square $ABCD$. **[CBSE 2013]**
    (iii) Name the type of triangle PQR formed by the points $P(\sqrt{2}, \sqrt{2})$, $Q(-\sqrt{2}, -\sqrt{2})$ and $R(-\sqrt{6}, \sqrt{6})$ **[NCERT EXEMPLAR]**
30. Find the point on x-axis which is equidistant from the points $(-2, 5)$ and $(2, -3)$. **[CBSE 2004]**
31. Find the value of x such that $PQ = QR$ where the coordinates of P, Q and R are $(6, -1)$, $(1, 3)$ and $(x, 8)$ respectively. **[CBSE 2005]**
32. Prove that the points $(0, 0)$, $(5, 5)$ and $(-5, 5)$ are the vertices of a right isosceles triangle. **[CBSE 2005]**
33. If the point $P(x, y)$ is equidistant from the points $A(5, 1)$ and $B(1, 5)$, prove that $x = y$. **[CBSE 2005]**
34. If $Q(0, 1)$ is equaidistant from $P(5, -3)$ and $R(x, 6)$, find the values of x. Also, find the distances QR and PR. **[NCERT]**
35. Find the values of y for which the distance between the points $P(2, -3)$ and $Q(10, y)$ is 10 units. **[NCERT]**
36. If the point $P(k-1, 2)$ is equidistant from the points $A(3, k)$ and $B(k, 5)$, find the values of k. **[CBSE 2014]**

37. If the point $A(0, 2)$ is equidistant from the points $B(3, p)$ and $C(p, 5)$, find p. Also, find the length of AB. [CBSE 2014]
38. Name the quadrilateral formed, if any, by the following points, and give reasons for your answers:
    (i) $A(-1, -2), B(1, 0), C(-1, 2), D(-3, 0)$
    (ii) $A(-3, 5), B(3, 1), C(0, 3), D(-1, -4)$
    (iii) $A(4, 5), B(7, 6), C(4, 3), D(1, 2)$ [NCERT]
39. Find the equation of the perpendicular bisector of the line segment joining points (7, 1) and (3, 5). [NCERT]
40. Prove that the points (3, 0), (4, 5), (–1, 4) and (–2, –1), taken in order, form a rhombus. Also, find its area. [NCERT]
41. In the seating arrangement of desks in a classroom three students Rohini, Sandhya and Bina are seated at $A(3, 1)$, $B(6, 4)$ and $C(8, 6)$. Do you think they are seated in a line?
42. Find a point on y-axis which is equidistant from the points (5, – 2) and (– 3, 2). [CBSE 2009]
43. Find a relation between x and y such that the point (x, y) is equidistant from the points (3, 6) and (–3, 4). [NCERT]
44. If a point $A(0, 2)$ is equidistant from the points $B(3, p)$ and $C(p, 5)$, then find the value of p. [CBSE 2012, 2013]
45. Prove that the points (7, 10), (–2, 5) and (3, – 4) are the vertices of an isosceles right triangle. [CBSE 2013]
46. If the point $P(x, 3)$ is equidistant from the points $A(7, -1)$ and $B(6, 8)$, find the value of x and find the distance AP. [CBSE 2014]
47. If $A(3, y)$ is equidistant from points $P(8, -3)$ and $Q(7, 6)$, find the vaue of y and find the distance AQ. [CBSE 2014]
48. If (0, – 3) and (0, 3) are the two vertices of an equilateral triangle, find the coordinates of its third vertex. [CBSE 2014]
49. If the point $P(2, 2)$ is equidistant from the points $A(-2, k)$ and $B(-2k, -3)$, find k. Also, find the length of AP. [CBSE 2014]
50. Show that ΔABC, where $A(-2, 0)$, $B(2, 0)$, $C(0, 2)$ and ΔPQR, where $P(-4, 0)$, $Q(4, 0)$, $R(0, 4)$ are similar. [CBSE 2017]

LEVEL-2

51. An equilateral triangle has two vertices at the points (3, 4) and (– 2, 3), find the coordinates of the third vertex.
52. Find the circumcentre of the triangle whose vertices are (–2, –3), (– 1, 0), (7, – 6).
53. Find the angle subtended at the origin by the line segment whose end points are (0, 100) and (10, 0).
54. Find the centre of the circle passing through (5, – 8), (2, – 9) and (2, 1).
55. If two opposite vertices of a square are (5, 4) and (1, –6), find the coordinates of its remaining two vertices.
56. Find the centre of the circle passing through (6, – 6), (3, –7) and (3, 3).
57. Two opposite vertices of a square are (–1, 2) and (3, 2). Find the coordinates of other two vertices.

ANSWERS

1. (i) 13 (ii) $2\sqrt{2}b$ (iii) $\sqrt{a^2 + b^2}(\sin\alpha + \cos\alpha)$ (iv) $\sqrt{a^2 + b^2}$
2. 4, –2

4. $x = 0, y = \pm\sqrt{7}$ 5. 3, –9 10. $\frac{25}{2}$ sq. units, $5\sqrt{2}$

13. $a = 2$, Area = 6 sq. units. 15. $\left(2-\frac{\sqrt{11}}{2}, \frac{5}{2}\right), \left(2+\frac{\sqrt{11}}{2}, \frac{5}{2}\right)$ 16. (3, 0) 18. (3, –2)

19. (0, – 1) 20. (9, 4)

21. (–3, 5). Infinite number of points. Infact all the points which are solutions of the equation $2x + y + 1 = 0$.

22. $a = 5, 3$ 23. 2.4 km 24. 1

25. (i) $(0, 3 - 4\sqrt{3})$ (ii) $(0, 3 + 4\sqrt{3})$ 26. 45 sq. units

27. (1, –3), $\sqrt{13}$ units 28. (3, 0) 29. (iii) Equilateral

30. (–2, 0) 31. 5, – 3

34. $x = -4, 4$; $QR = \sqrt{41}$ units ; $PR = \sqrt{82}, 9\sqrt{2}$ units 35. $y = 3, -9$

37. $p = 1, \sqrt{10}$ 36. $k = 1, 5$

38. (i) Square (ii) Not a quadrilateral (iii) parallelogram

39. $x - y = 2$ 40. 24 sq. units 41. Yes. 42. (0, –2)

43. $3x + y = 5$ 44. $p = 1$ 46. $x = 2, \sqrt{41}$ units

47. $y = 1, AQ = \sqrt{41}$ units 48. $(3\sqrt{3}, 0), (-3\sqrt{3}, 0)$ 49. $k = -1, -3$; $AP = 5$

51. $\left(\frac{1+\sqrt{3}}{2}, \frac{7-5\sqrt{3}}{2}\right), \left(\frac{1-\sqrt{3}}{2}, \frac{7+5\sqrt{3}}{2}\right)$ 52. (3, –3) 53. 90°

54. (2, –4) 55. (8, – 3), and (–2, 1) 56. (3, –2) 57. (1, 0) and (1, 4)

6.4 SECTION FORMULAE

Let A and B be two points in the plane of the paper as shown in Fig. 6.19 and P be a point on the segment joining A and B such that $AP : BP = m : n$. Then, we say that the point P divides segment AB internally in the ratio $m : n$.

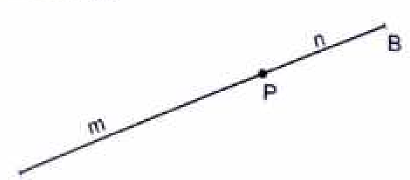


Fig. 6.19

If P is a point on AB produced such that $AP : BP = m : n$, then point P is said to divide AB externally in the ratio $m : n$

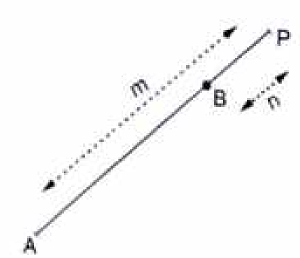


Fig. 6.20

In this section, we shall develop a formula, generally known as section formula, for finding the coordinates of P when we are given the coordinates of A and B and the ratio in which P divides AB internally.

THEOREM *Prove that the coordinates of the point which divides the line segment joining the points (x_1, y_1) and (x_2, y_2) internally in the ratio $m : n$ are given by*

$$\left(x = \frac{mx_2 + nx_1}{m+n}, y = \frac{my_2 + ny_1}{m+n}\right)$$

PROOF Let O be the origin and let OX and OY be the x-axis and y-axis respectively. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the given points. Let (x, y) be the coordinates of the point P which divides AB internally in the ratio $m : n$. Draw $AL \perp OX$, $BM \perp OX$, $PN \perp OX$. Also, draw AH and PK perpendiculars from A and P on PN and BM respectively. Then,

$$OL = x_1, ON = x, OM = x_2, AL = y_1, PN = y \text{ and } BM = y_2$$

$$\therefore \quad AH = LN = ON - OL = x - x_1, \; PH = PN - HN = PN - AL = y - y_1,$$

$$PK = NM = OM - ON = x_2 - x$$

and, $\quad BK = BM - MK = BM - PN = y_2 - y$

Clearly, triangle AHP and PKB are similar.

$$\therefore \quad \frac{AP}{BP} = \frac{AH}{PK} = \frac{PH}{BK}$$

$$\Rightarrow \quad \frac{m}{n} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$$

Now, $\quad \dfrac{m}{n} = \dfrac{x - x_1}{x_2 - x}$

$$\Rightarrow \quad mx_2 - mx = nx - nx_1$$

$$\Rightarrow \quad mx + nx = mx_2 + nx_1$$

$$\Rightarrow \quad x = \frac{mx_2 + nx_1}{m+n}$$

and, $\quad \dfrac{m}{n} = \dfrac{y - y_1}{y_2 - y}$

$$\Rightarrow \quad my_2 - my = ny - ny_1$$

$$\Rightarrow \quad my + ny = my_2 + ny_1$$

$$\Rightarrow \quad y = \frac{my_2 + ny_1}{m+n}$$

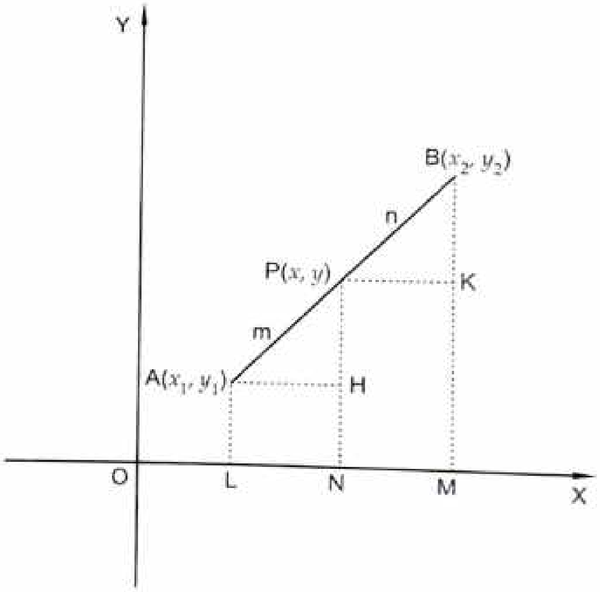


Fig. 6.21

Thus, the coordinates of P are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

NOTE 1 *If P is the mid-point of AB, then it divides AB in the ratio $1 : 1$, so its coordinates are*

$$\left(\frac{1 \cdot x_1 + 1 \cdot x_2}{1+1}, \frac{1 \cdot y_1 + 1 \cdot y_2}{1+1}\right) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

NOTE 2 *Fig. 6.22 will help to remember the section formula.*

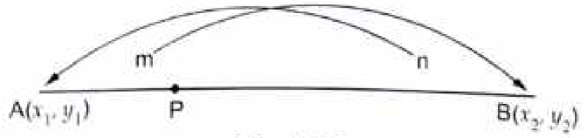


Fig. 6.22

NOTE 3 *The ratio m : n can also be written as* $\frac{m}{n}:1$, or $\lambda:1$, *where* $\lambda=\frac{m}{n}$.

So, the coordinates of point P dividing the line segment joining the points $A(x_1, y_1)$ *and* $B(x_2, y_2)$ *are*

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)=\left(\frac{\frac{m}{n}x_2+x_1}{\frac{m}{n}+1}, \frac{\frac{m}{n}y_2+y_1}{\frac{m}{n}+1}\right)=\left(\frac{\lambda x_2+x_1}{\lambda+1}, \frac{\lambda y_2+y_1}{\lambda+1}\right)$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE SECTION POINT WHEN THE SECTION RATIO IS GIVEN**

EXAMPLE 1 *Find the coordinates of the point which divides the line segment joining the points (6, 3) and (–4, 5) in the ratio 3 : 2 internally.*

SOLUTION Let $P(x, y)$ be the required point. Then,

$$x=\frac{3\times -4+2\times 6}{3+2} \text{ and } y=\frac{3\times 5+2\times 3}{3+2}$$

$$\Rightarrow \quad x=0 \text{ and } y=\frac{21}{5}$$

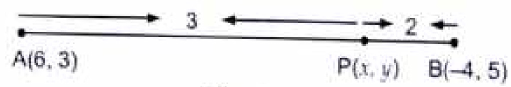


Fig. 6.23

So, the coordinates of P are $(0, 21/5)$.

EXAMPLE 2 *Find the coordinates of points which trisect the line segment joining (1, –2) and (–3, 4).* **[CBSE 2017]**

SOLUTION Let $A(1, -2)$ and $B(-3, 4)$ be the given points. Let the points of trisection be P and Q. Then, $AP = PQ = QB = \lambda$ (say).

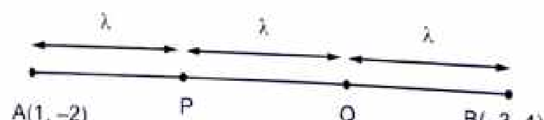


Fig. 6.24

$$\therefore \quad PB = PQ + QB = 2\lambda \text{ and } AQ = AP + PQ = 2\lambda$$

$$\Rightarrow \quad AP : PB = \lambda : 2\lambda = 1 : 2 \text{ and } AQ : QB = 2\lambda : \lambda = 2 : 1$$

So, P divides AB internally in the ratio 1 : 2 while Q divides internally in the ratio 2 : 1. Thus, the coordinates of P and Q are

$$P\left(\frac{1\times -3+2\times 1}{1+2}, \frac{1\times 4+2\times -2}{1+2}\right)=P\left(\frac{-1}{3}, 0\right)$$

$$Q\left(\frac{2\times -3+1\times 1}{2+1}, \frac{2\times 4+1\times(-2)}{2+1}\right)=Q\left(\frac{-5}{3}, 2\right) \text{ respectively}$$

Hence, the two points of trisection are $(-1/3, 0)$ and $(-5/3, 2)$.

REMARK *As Q is the mid-point of BP. So, the coordinates of Q can also be obtained by using mid-point formula.*

Type II ON FINDING THE SECTION RATIO OR AN END POINT OF THE SEGMENT WHEN THE SECTION POINT IS GIVEN

EXAMPLE 3 *In what ratio does the x-axis divide the line segment joining the points (2, –3) and (5, 6)? Also, find the coordinates of the point of intersection.*

SOLUTION Let the required ratio be $\lambda : 1$. Then, the coordinates of the point of division are,

$$R\left(\frac{5\lambda + 2}{\lambda + 1}, \frac{6\lambda - 3}{\lambda + 1}\right)$$

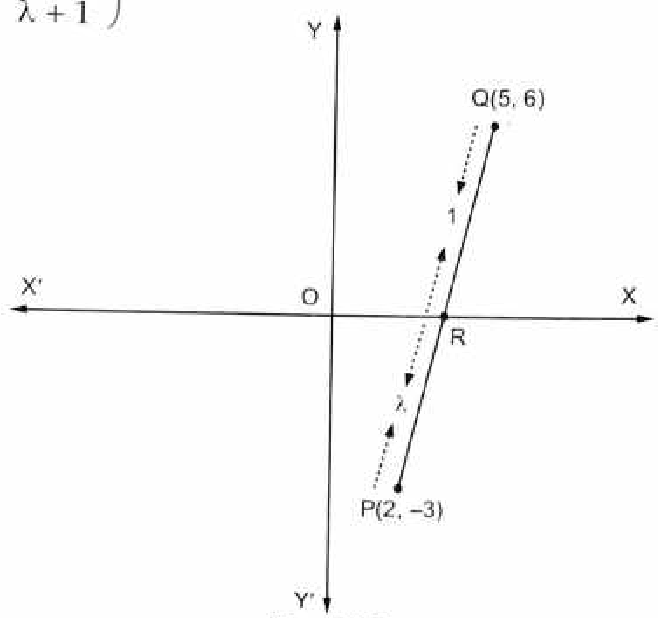


Fig. 6.25

But, it is a point on x-axis on which y-coordinates of every point is zero.

$$\therefore \qquad \frac{6\lambda - 3}{\lambda + 1} = 0 \Rightarrow \lambda = \frac{1}{2}$$

Thus, the required ratio is $\frac{1}{2} : 1$ or, $1 : 2$.

Putting $\lambda = 1/2$ in the coordinates of R, we find that its coordinates are (3, 0).

EXAMPLE 4 *In what ratio does the y-axis divide the line segment joining the point P (– 4, 5) and Q (3, –7)? Also, find the coordinates of the point of intersection.*

SOLUTION Suppose x-axis divides PQ in the ratio $\lambda : 1$. Then, the coordinates of the point of division are

$$R\left(\frac{3\lambda - 4}{\lambda + 1}, \frac{-7\lambda + 5}{\lambda + 1}\right)$$

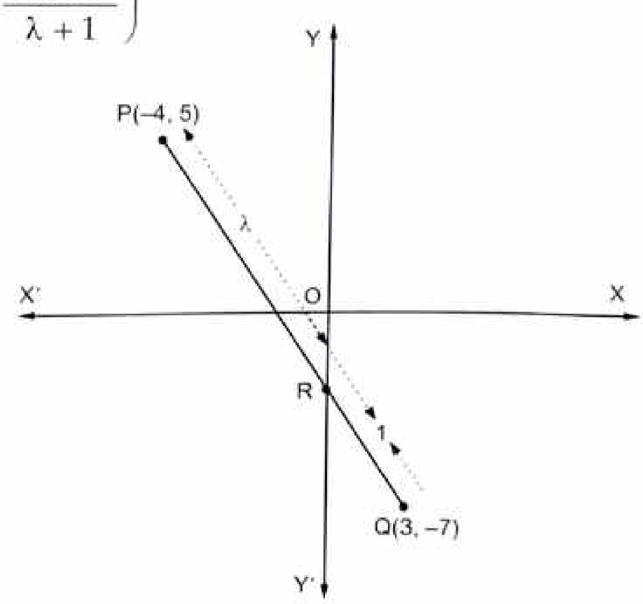


Fig. 6.26

Since R lies on y-axis and x-coordinate of every point on y-axis is zero.

$$\therefore \quad \frac{3\lambda - 4}{\lambda + 1} = 0 \Rightarrow 3\lambda - 4 = 0 \Rightarrow \lambda = \frac{4}{3}$$

Hence, the required ratio is $\frac{4}{3} : 1$ i.e., $4:3$

Putting $\lambda = 4/3$ in the coordinates of R, we find that its coordinates are $\left(0, \frac{-13}{7}\right)$.

EXAMPLE 5 *In what ratio does the point C (3/5, 11/5) divide the line segment joining the points A (3, 5) and B (−3, −2)?*

SOLUTION Let the point C divide AB in the ratio $\lambda : 1$. Then, the coordinates of C are

$$\left(\frac{-3\lambda + 3}{\lambda + 1}, \frac{-2\lambda + 5}{\lambda + 1}\right)$$

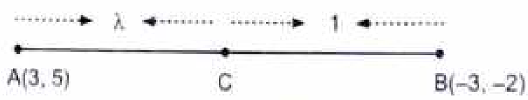


Fig. 6.27

But, the coordinates of C are given as (3/5, 11/5).

$$\therefore \quad \frac{-3\lambda + 3}{\lambda + 1} = \frac{3}{5} \text{ and } \frac{-2\lambda + 5}{\lambda + 1} = \frac{11}{5}$$

$$\Rightarrow \quad -15\lambda + 15 = 3\lambda + 3 \text{ and } -10\lambda + 25 = 11\lambda + 11$$

$$\Rightarrow \quad 18\lambda = 12 \text{ and } 21\lambda = 14$$

$$\Rightarrow \quad \lambda = \frac{2}{3}$$

Hence, the point C divides AB in the ratio $2:3$.

EXAMPLE 6 *If the point C (−1, 2) divides internally the line segment joining A (2, 5) and B in ratio 3 : 4, find the coordinates of B.*

SOLUTION Let the coordinates of B be (α, β). It is given that $AC : BC = 3 : 4$. So, the co-ordinates of C are

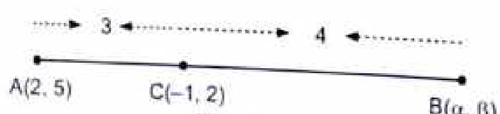


Fig. 6.28

$$\left(\frac{3\alpha + 4 \times 2}{3 + 4}, \frac{3\beta + 4 \times 5}{3 + 4}\right) = \left(\frac{3\alpha + 8}{7}, \frac{3\beta + 20}{7}\right)$$

But, the coordinates of C are (−1, 2).

$$\therefore \quad \frac{3\alpha + 8}{7} = -1 \text{ and } \frac{3\beta + 20}{7} = 2$$

$$\Rightarrow \quad \alpha = -5 \text{ and } \beta = -2$$

Thus, the coordinates of B are (−5, −2).

EXAMPLE 7 *Determine the ratio in which the line $3x + y - 9 = 0$ divides the segment joining the points (1, 3) and (2, 7).* **[CBSE 2008]**

SOLUTION Suppose the line $3x + y - 9 = 0$ divides the line segment joining A (1, 3) and B (2, 7) in the ratio $k : 1$ at point C. Then, the coordinates of C are

$$\left(\frac{2k+1}{k+1}, \frac{7k+3}{k+1}\right)$$

But, C lies on $3x + y - 9 = 0$. Therefore,

$$3\left(\frac{2k+1}{k+1}\right) + \frac{7k+3}{k+1} - 9 = 0 \Rightarrow 6k+3+7k+3-9k-9=0 \Rightarrow k = \frac{3}{4}$$

So, the required ratio is 3 : 4 internally.

EXAMPLE 8 *Find the ratio in which the point (–3, p) divides the line segment joining the points (–5, –4) and (–2, 3). Hence, find the value of p.* **[CBSE 2016]**

SOLUTION Suppose the point $P(-3, p)$ divides the line segment joining points $A(-5, -4)$ and $B(-2, 3)$ in the ratio $k : 1$.

Then, the coordinates of P are $\left(\frac{-2k-5}{k+1}, \frac{3k-4}{k+1}\right)$

But, the coordinates of P are given as $(-3, p)$.

$$\therefore \quad \frac{-2k-5}{k+1} = -3 \text{ and } \frac{3k-4}{k+1} = p$$

$$\Rightarrow \quad -2k - 5 = -3k - 3 \text{ and } \frac{3k-4}{k+1} = p$$

$$\Rightarrow \quad k = 2 \text{ and } p = \frac{3k-4}{k+1}$$

$$\Rightarrow \quad k = 2 \text{ and } p = 2/3$$

Hence, the ratio is 2 : 1 and $p = 2/3$.

Type III ON DETERMINATION OF THE TYPE OF A GIVEN QUADRILATERAL

EXAMPLE 9 *Prove that the points (–2, –1), (1, 0), (4, 3) and (1, 2) are the vertices of a parallelogram. Is it a rectangle?*

SOLUTION Let the given point be A, B, C and D respectively. Then,

$$\text{Coordinates of the mid-point of } AC \text{ are } \left(\frac{-2+4}{2}, \frac{-1+3}{2}\right) = (1, 1)$$

$$\text{Coordinates of the mid-point of } BD \text{ are } \left(\frac{1+1}{2}, \frac{0+2}{2}\right) = (1, 1)$$

Thus, AC and BD have the same mid-point. Hence, $ABCD$ is a parallelogram.
Now, we shall see whether $ABCD$ is a rectangle or not.
We have,

$$AC = \sqrt{(4-(-2))^2 + (3-(-1))^2} = 2\sqrt{13}$$

and, $$BD = \sqrt{(1-1)^2 + (0-2)^2} = 2$$

Clearly, $AC \neq BD$. So, $ABCD$ is not a rectangle.

EXAMPLE 10 *Prove that (4, –1), (6, 0), (7, 2) and (5, 1) are the vertices of a rhombus. Is it a square?*

SOLUTION Let the given points be A, B, C and D respectively. Then,

$$\text{Coordinates of the mid-point of } AC \text{ are } \left(\frac{4+7}{2}, \frac{-1+2}{2}\right) = \left(\frac{11}{2}, \frac{1}{2}\right)$$

Coordinates of the mid-point of BD are $\left(\frac{6+5}{2}, \frac{0+1}{2}\right) = \left(\frac{11}{2}, \frac{1}{2}\right)$

Thus, AC and BD have the same mid-point.
Hence, $ABCD$ is a parallelogram.
Now,

$$AB = \sqrt{(6-4)^2 + (0+1)^2} = \sqrt{5},\ BC = \sqrt{(7-6)^2 + (2-0)^2} = \sqrt{5}$$

$\therefore \quad AB = BC$

So, $ABCD$ is a parallelogram whose adjacent sides are equal.
Hence, $ABCD$ is a rhombus.
We have,

$$AC = \sqrt{(7-4)^2 + (2+1)^2} = 3\sqrt{2}, \text{ and, } BD = \sqrt{(6-5)^2 + (0-1)^2} = \sqrt{2}$$

Clearly, $AC \neq BD$. So, $ABCD$ is not a square.

Type IV ON FINDING THE UNKNOWN VERTEX FROM GIVEN POINTS

EXAMPLE 11 *The three vertices of a parallelogram taken in order are (–1, 0), (3, 1) an (2, 2) respectively. Find the coordinates of the fourth vertex.*

SOLUTION Let $A(-1, 0)$, $B(3, 1)$, $C(2, 2)$ and $D(x, y)$ be the vertices of a parallelogram ABC taken in order. Since, the diagonals of a parallelogram bisect each other.

$\therefore$ Coordinates of the mid-point of AC = Coordinates of the mid-point of BD

$$\Rightarrow \quad \left(\frac{-1+2}{2}, \frac{0+2}{2}\right) = \left(\frac{3+x}{2}, \frac{1+y}{2}\right)$$

$$\Rightarrow \quad \left(\frac{1}{2}, 1\right) = \left(\frac{3+x}{2}, \frac{y+1}{2}\right)$$

$$\Rightarrow \quad \frac{3+x}{2} = \frac{1}{2} \text{ and } \frac{y+1}{2} = 1$$

$$\Rightarrow \quad x = -2 \text{ and } y = 1$$

Hence, the fourth vertex of the parallelogram is $(-2, 1)$.

EXAMPLE 12 *If the points $A(6, 1)$, $B(8, 2)$, $C(9, 4)$ and $D(p, 3)$ are the vertices of a parallelogram taken in order, find the value of p.*

SOLUTION We know that the diagonals of a parallelogram bisect each other. So, coordinates of the mid-point of diagonal AC are same as the coordinates of the mid-point of diagonal BD.

$$\therefore \quad \left(\frac{6+9}{2}, \frac{1+4}{2}\right) = \left(\frac{8+p}{2}, \frac{2+3}{2}\right)$$

$$\Rightarrow \quad \left(\frac{15}{2}, \frac{5}{2}\right) = \left(\frac{8+p}{2}, \frac{5}{2}\right)$$

$$\Rightarrow \quad \frac{15}{2} = \frac{8+p}{2} \Rightarrow 15 = 8 + p \Rightarrow p = 7$$

EXAMPLE 13 *If $A(-2, -1)$, $B(a, 0)$, $C(4, b)$ and $D(1, 2)$ are the vertices of a parallelogram, find the values of a and b.* [CBSE 2006 C, 2018]

SOLUTION We know that the diagonals of a parallelogram bisect each other. Therefore, the coordinates of the mid-point of AC are same as the coordinates of the mid-point of BD i.e.,

$$\left(\frac{-2+4}{2}, \frac{-1+b}{2}\right) = \left(\frac{a+1}{2}, \frac{0+2}{2}\right)$$

$$\Rightarrow \quad \left(1, \frac{b-1}{2}\right) = \left(\frac{a+1}{2}, 1\right)$$

$$\Rightarrow \quad \frac{a+1}{2} = 1 \text{ and } \frac{b-1}{2} = 1$$

$$\Rightarrow \quad a+1 = 2 \text{ and } b-1 = 2$$

$$\Rightarrow \quad a = 1 \text{ and } b = 3$$

Hence, $a = 1$ and $b = 3$

EXAMPLE 14 *If the coordinates of the mid-points of the sides of a triangle are (1, 2) (0, –1) and (2, –1). Find the coordinates of its vertices.*

SOLUTION Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC. Let $D(1, 2)$, $E(0, -1)$, and $F(2, -1)$ be the mid-points of sides BC, CA and AB respectively.

Since D is the mid-point of BC.

$$\therefore \quad \frac{x_2+x_3}{2} = 1 \text{ and } \frac{y_2+y_3}{2} = 2$$

$$\Rightarrow \quad x_2+x_3 = 2 \text{ and } y_2+y_3 = 4 \qquad \ldots \text{(i)}$$

Similarly, E and F are the mid-points of CA and AB respectively.

$$\therefore \quad \frac{x_1+x_3}{2} = 0 \text{ and } \frac{y_1+y_3}{2} = -1$$

$$\Rightarrow \quad x_1+x_3 = 0 \text{ and } y_1+y_3 = -2 \qquad \ldots \text{(ii)}$$

$$\text{and,} \quad \frac{x_1+x_2}{2} = 2 \text{ and } \frac{y_1+y_2}{2} = -1$$

$$\Rightarrow \quad x_1+x_2 = 4 \text{ and } y_1+y_2 = -2 \qquad \ldots \text{(iii)}$$

From (i), (ii) and (iii), we get

$$(x_2+x_3)+(x_1+x_3)+(x_1+x_2) = 2+0+4 \text{ and, } (y_2+y_3)+(y_1+y_3)+(y_1+y_2) = 4-2-2$$

$$\Rightarrow \quad 2(x_1+x_2+x_3) = 6 \text{ and } 2(y_1+y_2+y_3) = 0 \qquad \ldots \text{(iv)}$$

$$\Rightarrow \quad x_1+x_2+x_3 = 3 \text{ and } y_1+y_2+y_3 = 0$$

From (i) and (iv), we get

$$x_1+2 = 3 \text{ and } y_1+4 = 0$$

$$\Rightarrow \quad x_1 = 1 \text{ and } y_1 = -4$$

So, the coordinates of A are (1, –4)

From (ii) and (iv), we get

$$x_2+0 = 3 \text{ and } y_2-2 = 0$$

$$\Rightarrow \quad x_2 = 3 \text{ and } y_2 = 2$$

So, coordinates of B are (3, 2)

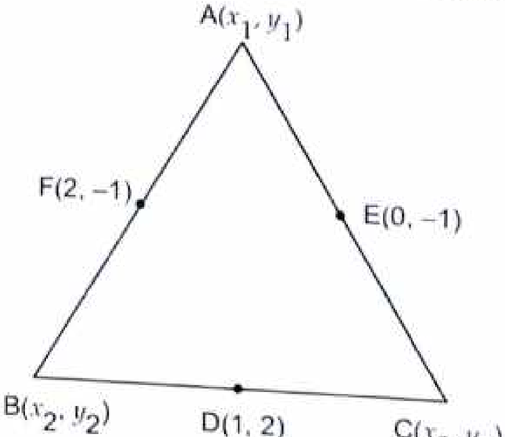


Fig. 6.29

From (iii) and (iv), we get

$$x_3 + 4 = 3 \text{ and } y_3 - 2 = 0$$

$$\Rightarrow \quad x_3 = -1 \text{ and } y_3 = 2$$

So, coordinates of C are (–1, 2)

Hence, the vertices of the triangle ABC are $A(1,-4)$, $B(3,2)$ and $C(-1,2)$.

EXAMPLE 15 *The coordinates of one end point of a diameter of a circle are (4, –1) coordinates of the centre of the circle are (1, –3). Find the coordinates of the other end of the di*

SOLUTION Let AB be a diameter of the circle having its centre at C (1, –3) such tl coordinates of one end A are (4, –1).

Let the coordinates of B be (x, y).

Since C is the mid-point of AB. Therefore, the coordinates of C are $\left(\frac{x+4}{2}, \frac{y-1}{2}\right)$.

But, the coordiantes of C are given to be (1, –3).

$$\therefore \quad \frac{x+4}{2} = 1 \text{ and } \frac{y-1}{2} = -3$$

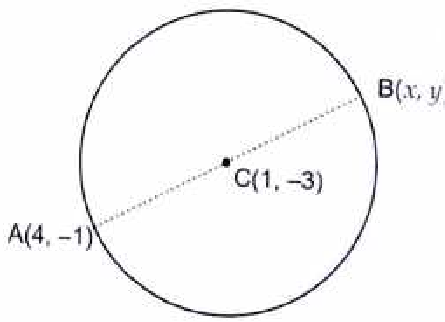


Fig. 6.30

$$\Rightarrow \quad x + 4 = 2 \text{ and } y - 1 = -6$$

$$\Rightarrow \quad x = -2 \text{ and } y = -5$$

Hence, the coordinates of B are (–2, –5).

EXAMPLE 16 *Find the lengths of the medians of a ΔABC whose vertices are A (7, –3), B (5, C (3, –1).*

SOLUTION Let D, E, F be the mid-points of the sides BC, CA and AB respectively. Ther coordinates of D, E and F are

$$D\left(\frac{5+3}{2}, \frac{3-1}{2}\right) = D(4,1),\ E\left(\frac{3+7}{2}, \frac{-1-3}{2}\right) = E(5,-2)$$

and, $$F\left(\frac{7+5}{2}, \frac{-3+3}{2}\right) = F(6,0)$$

$$\therefore \quad AD = \sqrt{(7-4)^2 + (-3-1)^2} = \sqrt{9+16} = 5 \text{ units}$$

$$BE = \sqrt{(5-5)^2 + (-2-3)^2} = \sqrt{0+25} = 5 \text{ units}$$

and, $$CF = \sqrt{(6-3)^2 + (0+1)^2} = \sqrt{9+1} = \sqrt{10} \text{ units.}$$

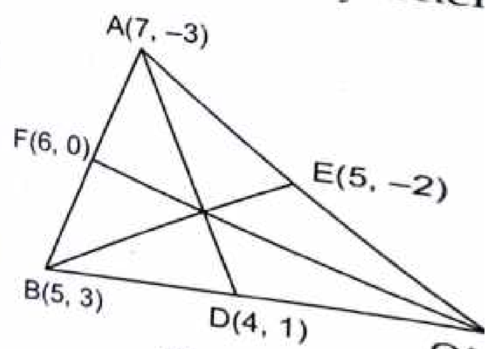


Fig. 6.31

EXAMPLE 17 *If A (5,–1), B (–3,–2) and C (–1,8) are the vertices of triangle ABC, fina length of median through A and the coordinates of the centroid.* **[CBSE 200**

SOLUTION Let AD be the median through the vertex A of ΔABC. Then, D is the mid-poi BC. So, the coordinates of D are $\left(\frac{-3-1}{2}, \frac{-2+8}{2}\right)$ i.e., (– 2, 3).

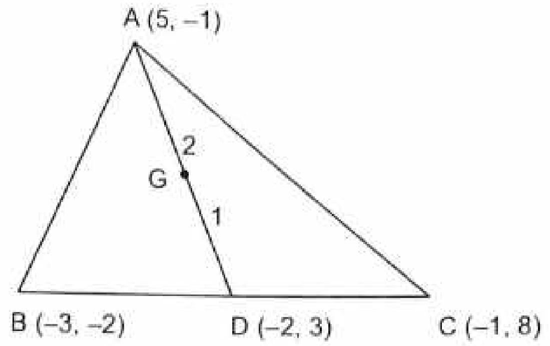


Fig. 6.32

$\therefore \quad AD = \sqrt{(5+2)^2 + (-1-3)^2} = \sqrt{49+16} = \sqrt{65}$ units

Let G be the centroid of $\Delta\ ABC$. Then, G lies on median AD and divides it in the ratio 2:1. So, coordinates of G are

$$\left(\frac{2\times -2 + 1\times 5}{2+1}, \frac{2\times 3 + 1\times -1}{2+1}\right) = \left(\frac{-4+5}{3}, \frac{6-1}{3}\right) = \left(\frac{1}{3}, \frac{5}{3}\right)$$

LEVEL-2

EXAMPLE 18 *Point P divides the line segment joining the points A (–1, 3) and B (9, 8) such that $\frac{AP}{BP} = \frac{k}{1}$. If P lies on the line $x - y + 2 = 0$, find the value of k.* **[CBSE 2010]**

SOLUTION It is given that P divides the line segment joining A (–1, 3) and B (9, 8) in the ratio $k:1$. So, coordinates of P are $\left(\frac{9k-1}{k+1}, \frac{8k+3}{k+1}\right)$.

$P\left(\frac{9k-1}{k+1}, \frac{8k+3}{k+1}\right)$ lines on the line $x - y + 2 = 0$

Fig. 6.33

$\therefore \quad \frac{9k-1}{k+1} - \frac{8k+3}{k+1} + 2 = 0$

$\Rightarrow \quad 9k - 1 - 8k - 3 + 2k + 2 = 0$

$\Rightarrow \quad 3k - 2 = 0$

$\Rightarrow \quad k = 2/3$

EXAMPLE 19 *Point P divides the line segment joining the points A (2, 1) and B (5, –8) such that $\frac{AP}{AB} = \frac{1}{3}$. If P lies on the line $2x - y + k = 0$, find the value of k.* **[CBSE 2010]**

SOLUTION We have,

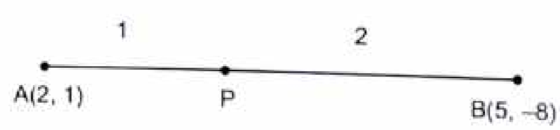


Fig. 6.34

$\frac{AP}{AB} = \frac{1}{3}$

$\Rightarrow \quad \frac{AP}{AP + PB} = \frac{1}{3}$

$\Rightarrow \quad 3AP = AP + BP$

$\Rightarrow \quad 2AP = BP$

$\Rightarrow \quad \frac{AP}{BP} = \frac{1}{2}$

So, P divides AB in the ratio 1 : 2.

$\therefore$ Coordinates of P are $\left(\frac{1\times 5+2\times 2}{1+2}, \frac{1\times -8+2\times 1}{1+2}\right)=(3, 2)$

Since, $P(3, 2)$ lies on the line $2x-y+k=0$.

$\therefore$ $2\times 3-2+k=0 \Rightarrow k=-4$

EXAMPLE 20 *The vertices of a ΔABC are $A(5,5)$, $B(1,5)$ and $C(9,1)$. A line is drawn to intersect sides AB and AC at P and Q respectively, such that $\frac{AP}{AB}=\frac{AQ}{AC}=\frac{3}{4}$. Find the length of the line segment PQ.* **[CBSE 2014]**

SOLUTION We have,

$$\frac{AP}{AB}=\frac{AQ}{AC}=\frac{3}{4}$$

$$\Rightarrow \frac{AP}{AP+PB}=\frac{AQ}{AQ+QC}=\frac{3}{4}$$

$$\Rightarrow \frac{AP}{AP+PB}=\frac{3}{4}, \frac{AQ}{AQ+QC}=\frac{3}{4}$$

$$\Rightarrow 4\,AP=3\,AP+3\,PB \text{ and } 4\,AQ=3\,AQ+3\,QC$$

$$\Rightarrow AP=3\,PB \text{ and } AQ=3\,QC$$

$$\Rightarrow \frac{AP}{PB}=\frac{3}{1} \text{ and } \frac{AQ}{QC}=\frac{3}{1}$$

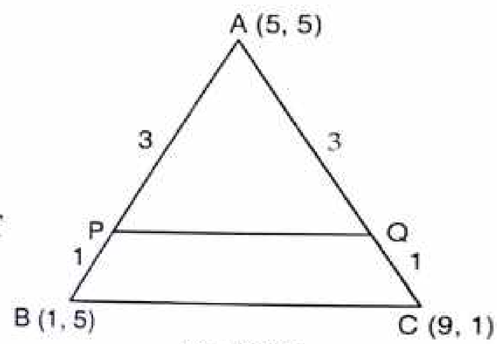


Fig. 6.35

$\Rightarrow$ P and Q divide AB and AC respectively in the same ratio 3 : 1

Thus, the coordinates of P and Q are

$$\left(\frac{3\times 1+1\times 5}{3+1}, \frac{3\times 5+1\times 5}{3+1}\right)=(2, 5) \text{ and } \left(\frac{3\times 9+1\times 5}{3+1}, \frac{3\times 1+1\times 5}{3+1}\right)=(8, 2)$$

$\therefore$ $PQ=\sqrt{(2-8)^2+(5-2)^2}=\sqrt{45}=3\sqrt{5}$ units

EXERCISE 6.3

LEVEL-1

1. Find the coordinates of the point which divides the line segment joining $(-1, 3)$ and $(4, -7)$ internally in the ratio 3 : 4.
2. Find the points of trisection of the line segment joining the points:
   (i) $(5, -6)$ and $(-7, 5)$, (ii) $(3, -2)$ and $(-3, -4)$, (iii) $(2, -2)$ and $(-7, 4)$. **[NCERT]**
3. Find the coordinates of the point where the diagonals of the parallelogram formed by joining the points $(-2, -1)$, $(1, 0)$, $(4, 3)$ and $(1, 2)$ meet.
4. Prove that the points $(3, -2)$, $(4, 0)$, $(6, -3)$ and $(5, -5)$ are the vertices of a parallelogram.
5. If $P(9a-2, -b)$ divides the line segment joining $A(3a+1, -3)$ and $B(8a, 5)$ in the ratio 3 : 1, find the values of a and b. **[NCERT EXEMPLAR]**
6. If (a, b) is the mid-point of the line segment joining the points $A(10, -6)$, $B(k, 4)$ and $a-2b=18$, find the value of k and the distance AB. **[NCERT EXEMPLAR]**

7. Find the ratio in which the point $(2, y)$ divides the line segment joining the points $A(-2, 2)$ and $B(3, 7)$. Also, find the value of y. [CBSE 2009]

8. If $A(-1, 3)$, $B(1, -1)$ and $C(5, 1)$ are the vertices of a triangle ABC, find the length of the median through A.

9. If the points P, $Q(x, 7)$, R, $S(6, y)$ in this order divide the line segment joining $A(2, p)$ and $B(7, 10)$ in 5 equal parts, find x, y and p. [CBSE 2015]

10. If a vertex of a triangle be $(1, 1)$ and the middle points of the sides through it be $(-2, 3)$ and $(5, 2)$, find the other vertices.

11. (i) In what ratio is the line segment joining the points $(-2, -3)$ and $(3, 7)$ divided by the y-axis? Also, find the coordinates of the point of division. [CBSE 2006C]

(ii) In what ratio is the line segment joining $(-3, -1)$ and $(-8, -9)$ divided at the point $(-5, -21/5)$?

12. If the mid-point of the line joining $(3, 4)$ and $(k, 7)$ is (x, y) and $2x + 2y + 1 = 0$ find the value of k. [NCERT EXEMPLAR]

13. Find the ratio in which the points $P(3/4, 5/12)$ divides the line segments joining the points $A(1/2, 3/2)$ and $B(2, -5)$. [CBSE 2015]

14. Find the ratio in which the line segment joining $(-2, -3)$ and $(5, 6)$ is divided by (i) x-axis (ii) y-axis. Also, find the coordinates of the point of division in each case. [CBSE 2013]

15. Prove that the points $(4, 5)$, $(7, 6)$, $(6, 3)$, $(3, 2)$ are the vertices of a parallelogram. Is it a rectangle.

16. Prove that $(4, 3)$, $(6, 4)$, $(5, 6)$ and $(3, 5)$ are the angular points of a square.

17. Prove that the points $(-4, -1)$, $(-2, -4)$, $(4, 0)$ and $(2, 3)$ are the vertices of a rectangle.

18. Find the lengths of the medians of a triangle whose vertices are $A(-1,3)$, $B(1,-1)$ and $C(5, 1)$.

19. Find the ratio in which the line segment joining the points $A(3,-3)$ and $B(-2,7)$ is divided by x-axis. Also, find the coordinates of the point of division. [CBSE 2014]

20. Find the ratio in which the point $P(x, 2)$ divides the line segment joining the points $A(12,5)$ and $B(4,-3)$. Also, find the value of x. [CBSE 2014]

21. Find the ratio in which the point $P(-1, y)$ lying on the line segment joining $A(-3, 10)$ and $B(6, -8)$ divides it. Also find the value of y. [CBSE 2013]

22. Find the coordinates of a point A, where AB is a diameter of the circle whose centre is $(2, -3)$ and B is $(1, 4)$. [NCERT]

23. If the points $(-2, -1)$, $(1, 0)$, $(x, 3)$ and $(1, y)$ form a parallelogram, find the values of x and y.

24. The points $A(2,0)$, $B(9,1)$, $C(11,6)$ and $D(4,4)$ are the vertices of a quadrilateral $ABCD$. Determine whether $ABCD$ is a rhombus or not.

25. In what ratio does the point $(-4, 6)$ divide the line segment joining the points $A(-6, 10)$ and $B(3, -8)$? [NCERT, CBSE 2017]

26. Find the ratio in which the y-axis divides the line segment joining the points $(5, -6)$ and $(-1, -4)$. Also, find the coordinates of the point of division. [CBSE 2010, 2016]

27. Show that $A(-3, 2)$, $B(-5, -5)$, $C(2, -3)$ and $D(4, 4)$ are the vertices of a rhombus.

28. Find the lengths of the medians of a ΔABC having vertices at $A(0,-1)$, $B(2, 1)$ and $C(0, 3)$.

29. Find the ratio in which $P(4, m)$ divides the line segment joining the points $A(2, 3)$ and $B(6, -3)$. Hence, find m. [CBSE 2018]

30. Find the coordinates of the points which divide the line segment joining the points $(-4, 0)$ and $(0, 6)$ in four equal parts.

31. Show that the mid-point of the line segment joining the points $(5, 7)$ and $(3, 9)$ is also the mid-point of the line segment joining the points $(8, 6)$ and $(0, 10)$.

32. Find the distance of the point $(1, 2)$ from the mid-point of the line segment joining the points $(6, 8)$ and $(2, 4)$.

33. If A and B are $(1, 4)$ and $(5, 2)$ respectively, find the coordinates of P when $AP/BP = 3/4$.

34. Show that the points A $(1, 0)$, B $(5, 3)$, C $(2, 7)$ and D $(-2, 4)$ are the vertices of a parallelogram.

35. Determine the ratio in which the point P $(m, 6)$ divides the join of $A(-4, 3)$ and $B(2, 8)$. Also, find the value of m. [CBSE 2004]

36. Determine the ratio in which the point $(-6, a)$ divides the join of $A(-3, 1)$ and $B(-8, 9)$. Also find the value of a. [CBSE 2004]

37. $ABCD$ is a rectangle formed by joining the points A $(-1, -1)$, B $(-1, 4)$, C $(5, 4)$ and D $(5, -1)$. P, Q, R and S are the mid-points of sides AB, BC, CD and DA respectively. Is the quadrilateral $PQRS$ a square? a rectangle? or a rhombus? Justify your answer. [NCERT]

38. Points P, Q, R and S divide the line segment joining the points A $(1, 2)$ and B $(6, 7)$ in 5 equal parts. Find the coordinates of the points P, Q and R. [CBSE 2014]

39. If A and B are two points having coordinates $(-2, -2)$ and $(2, -4)$ respectively, find the coordinates of P such that $AP = \frac{3}{7} AB$. [NCERT, CBSE 2008, 2009]

40. Find the coordinates of the points which divide the line segment joining A $(-2, 2)$ and B $(2, 8)$ into four equal parts. [NCERT]

LEVEL-2

41. Three consecutive vertices of a parallelogram are $(-2, -1)$, $(1, 0)$ and $(4, 3)$. Find the fourth vertex.

42. The points $(3, -4)$ and $(-6, 2)$ are the extremities of a diagonal of a parallelogram. If the third vertex is $(-1, -3)$. Find the coordinates of the fourth vertex.

43. If the coordinates of the mid-points of the sides of a triangle are $(1, 1)$, $(2, -3)$ and $(3, 4)$, find the vertices of the triangle.

44. Determine the ratio in which the straight line $x - y - 2 = 0$ divides the line segment joining $(3, -1)$ and $(8, 9)$.

45. Three vertices of a parallelogram are $(a + b, a - b)$, $(2a + b, 2a - b)$, $(a - b, a + b)$. Find the fourth vertex.

46. If two vertices of a parallelogram are $(3, 2)$, $(-1, 0)$ and the diagonals cut at $(2, -5)$, find the other vertices of the parallelogram.

47. If the coordinates of the mid-points of the sides of a triangle are $(3, 4)$, $(4, 6)$ and $(5, 7)$, find its vertices. [CBSE 2008]

48. The line segment joining the points P $(3, 3)$ and Q $(6, -6)$ is trisected at the points A and B such that A is nearer to P. If A also lies on the line given by $2x + y + k = 0$, find the value of k. [CBSE 2009]

49. If three consecutive vertices of a parallelogram are $(1, -2)$, $(3, 6)$ and $(5, 10)$, find its fourth vertex.

50. If the points A $(a, -11)$, B $(5, b)$, C $(2, 15)$ and D $(1, 1)$ are the vertices of a parallelogram $ABCD$, find the values of a and b.

51. If the coordinates of the mid-points of the sides of a triangle be (3, –2), (–3, 1) and (4, –3), then find the coordinates of its vertices.

52. The line segment joining the points (3, –4) and (1, 2) is trisected at the points P and Q. If the coordinates of P and Q are $(p, -2)$ and $(5/3, q)$ respectively. Find the values of p and q. **[CBSE 2005]**

53. The line joining the points (2, 1) and (5,–8) is trisected at the points P and Q. If point P lies on the line $2x - y + k = 0$. Find the value of k. **[CBSE 2005]**

54. $A(4, 2)$, $B(6, 5)$ and $C(1, 4)$ are the vertices of ΔABC.
   (i) The median from A meets BC in D. Find the coordinates of the point D.
   (ii) Find the coordinates of point P on AD such that $AP : PD = 2 : 1$.
   (iii) Find the coordinates of the points Q and R on medians BE and CF respectively such that $BQ : QE = 2 : 1$ and $CR : RF = 2 : 1$.
   (iv) What do you observe? **[NCERT, CBSE, 2009, 10]**

55. If the points $A(6, 1)$, $B(8, 2)$, $C(9, 4)$ and $D(k, p)$ are the vertices of a parallelogram taken in order, then find the values of k and p.

56. A point P divides the line segment joining the points $A(3, -5)$ and $B(-4, 8)$ such that $\frac{AP}{PB} = \frac{k}{1}$. If P lies on the line $x + y = 0$, then find the value of k. **[CBSE 2012]**

57. The mid-point P of the line segment joining the points $A(-10, 4)$ and $B(-2, 0)$ lies on the line segment joining the points $C(-9, -4)$ and $D(-4, y)$. Find the ratio in which P divides CD. Also, find the value of y. **[CBSE 2014]**

58. If the point $C(-1, 2)$ divides internally the line segment joining the points $A(2, 5)$ and $B(x, y)$ in the ratio $3 : 4$, find the value of $x^2 + y^2$. **[CBSE 2016]**

59. $ABCD$ is a parallelogram with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Find the coordinates of the fourth vertex D in terms of x_1, x_2, x_3, y_1, y_2 and y_3.
[NCERT EXEMPLAR]

60. The points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of ΔABC.
   (i) The median from A meets BC at D. Find the coordinates of the point D.
   (ii) Find the coordinates of the point P on AD such that $AP : PD = 2 : 1$.
   (iii) Find the points of coordinates Q and R on medians BE and CF respectively such that $BQ : QE = 2 : 1$ and $CR : RF = 2 : 1$.
   (iv) What are the coordinates of the centroid of the triangle ABC?
[NCERT EXEMPLAR]

ANSWERS

1. $(8/7, -9/7)$
2. (i) $(1, -7/3)$, $(-3, 4/3)$ (ii) $(1, -8/3)$, $(-1, -10/3)$ (iii) $(-1, 0)$, $(-4, 2)$
3. $(1, 1)$
5. $a = 1, b = -3$
6. $k = 22$, $AB = 2\sqrt{61}$
7. $4 : 1$, $y = 6$
8. 5
9. $x = 4, y = 9, p = 5$
10. $(-5, 5)$, $(9, 3)$
11. (i) $2 : 3$ internally; $(0, 1)$ (ii) $2 : 3$ internally
12. $k = -15$
13. $1 : 5$
14. (i) $2 : 5; \left(0, \frac{-3}{7}\right)$ (ii) $2 : 5; \left(0, \frac{-3}{7}\right)$
15. No
18. $AD = 5$, $BE = \sqrt{10}$, $CF = 5$
19. $3 : 7$, $(3/2, 0)$

20. $3:5; x=9$ 21. $2:7, 6$ 22. $(3,-10)$ 23. $x=4, y=2$

24. No 25. $2:7$

28. $AD=\sqrt{10}$ units, $BE=2$ units, $CF=\sqrt{10}$ units

29. $1:1, m=0$

30. $(-3,1.5), (-2,3), (-1,4.5)$ 32. 5 units 33. $\left(\frac{19}{7}, \frac{22}{7}\right)$

35. $3:2, m=\frac{-2}{5}$ 36. $3:2, a=5$

38. $P(2,3), Q(3,4), R(4,5)$ 39. $\left(\frac{-2}{7}, \frac{-20}{7}\right)$ 40. $\left(-1, \frac{7}{2}\right), (0,5), \left(1, \frac{13}{2}\right)$

41. $(1,2)$ 42. $(-2,1)$ 43. $(4,0), (2,8), (0,-6)$

44. $2:3$ internally 45. $(-b, b)$ 46. $(1,-12), (5,-10)$

47. $(6,9), (4,5), (2,3)$ 48. -8 49. $(3,2)$

50. $a=4, b=3$ 51. $A(-2,0), B(10,-6), C(-4,2)$ 52. $p=\frac{7}{3}, q=0$

53. $k=-8$ 55. $k=7, p=3$ 56. $1/2$ 57. $3:2, y=6$ 58. 29

59. $(x_1+x_3-x_2, y_1+y_3-y_2)$

60. (i) $\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$ (ii) $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{4}\right)$

(iii) $Q\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$, $R\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

HINT TO THE SELECTED PROBLEM

39. We have, $AP=\frac{3}{7}AB$. Also, $AP+BP=AB$

$\therefore \quad \frac{3}{7}AB+BP=AB \Rightarrow BP=\frac{4}{7}AB$

Hence, $AP:BP=3:4$

6.5 SOME APPLICATIONS OF SECTION FORMULA

In this section, we shall discuss an application of the section formula learnt in the previous section to find the coordinates of the centroid of a triangle in terms of the coordinates of its vertices.

THEOREM *Prove that the coordinates of the centroid of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) are*

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

Also, deduce that the medians of a triangle are concurrent.

[NCERT, CBSE 2004]

PROOF Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC whose medians are AD, BE and CF respectively. So D, E and F are respectively the mid-points of BC, CA and AB.

Coordinates of D are $\left(\dfrac{x_2 + x_3}{2}, \dfrac{y_2 + y_3}{2}\right)$

Coordinates of a point dividing AD in the ratio 2 : 1 are

$$\left(\frac{1.x_1 + 2\left(\frac{x_2 + x_3}{2}\right)}{1+2}, \frac{1.y_1 + 2\left(\frac{y_2 + y_3}{2}\right)}{1+2}\right) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

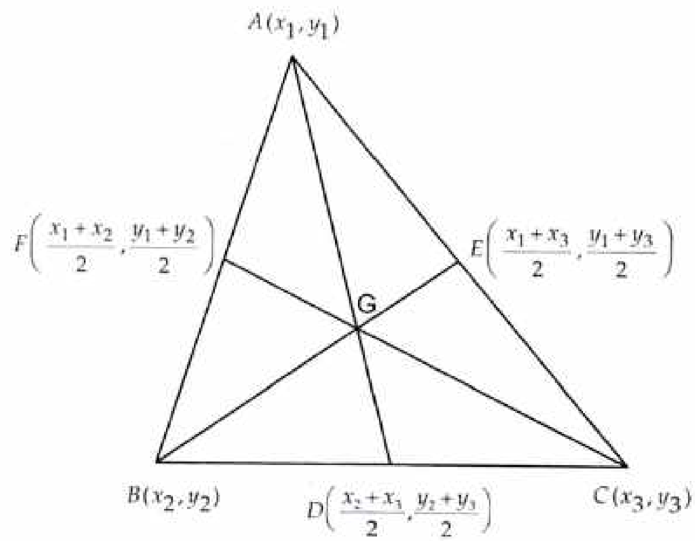


Fig. 6.36

The coordinates of E are $\left(\dfrac{x_1 + x_3}{2}, \dfrac{y_1 + y_3}{2}\right)$

The coordinates of a point dividing BE in the ratio 2 : 1 are

$$\left(\frac{1.x_2 + \frac{2(x_1 + x_3)}{2}}{1+2}, \frac{1.y_2 + \frac{2(y_1 + y_3)}{2}}{1+2}\right) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Similarly the coordinates of a point dividing CF in the ratio 2 : 1 are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$

Thus, the point having coordinates $\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$ is common to AD, BE and CF and divides them in the ratio 1 : 2.

Hence, medians of a triangle are concurrent and the coordinates of the centroid are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the coordinates of the centroid of a triangle whose vertices are (0, 6), (8, 12) and (8, 0).*

SOLUTION We know that the coordinates of the centroid of a triangle whose angular points are $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

So, the coordinates of the centroid of a triangle whose vertices are (0, 6), (8, 12) and (8, 0) are

$$\left(\frac{0+8+8}{3}, \frac{6+12+0}{3}\right) \text{ or, } \left(\frac{16}{3}, 6\right)$$

EXAMPLE 2 *If $x - 2y + k = 0$ is a median of the triangle whose vertices are at points A (– 1, 3), B (0, 4) and C (– 5, 2) find the value of k.*

SOLUTION The coordinates of the centroid G of ΔABC are

$$\left(\frac{-1+0-5}{3}, \frac{3+4+2}{3}\right) \text{i.e.} (-2, 3)$$

Since G lies on the median $x - 2y + k = 0$. So, coordinates of G satisfy its equation.

$$\therefore \quad -2 - 6 + k = 0 \Rightarrow k = 8.$$

EXAMPLE 3 *If the coordinates of the mid-points of the sides of a triangle are (1, 1), (2, –3) and (3, 4). Find its centroid.*

SOLUTION Let P (1, 1), Q (2, –3), R (3, 4) be the mid-points of sides AB, BC and CA respectively of triangle ABC. Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of triangle. ABC. Then,

P is the mid-point of BC

$$\Rightarrow \quad \frac{x_1 + x_2}{2} = 1, \frac{y_1 + y_2}{2} = 1$$

$$\Rightarrow \quad x_1 + x_2 = 2 \text{ and } y_1 + y_2 = 2 \quad \text{...(i)}$$

Q is the mid-point of BC

$$\Rightarrow \quad \frac{x_2 + x_3}{2} = 2, \frac{y_2 + y_3}{2} = -3$$

$$\Rightarrow \quad x_2 + x_3 = 4 \text{ and } y_2 + y_3 = -6 \quad \text{...(ii)}$$

R is the mid-point of AC

$$\Rightarrow \quad \frac{x_1 + x_3}{2} = 3 \text{ and } \frac{y_1 + y_3}{2} = 4$$

$$\Rightarrow \quad x_1 + x_3 = 6 \text{ and } y_1 + y_3 = 8 \quad \text{...(iii)}$$

From (i), (ii) and (iii), we get

$$x_1 + x_2 + x_2 + x_3 + x_1 + x_3 = 2 + 4 + 6$$

and, $$y_1 + y_2 + y_2 + y_3 + y_1 + y_3 = 2 - 6 + 8$$

$$\Rightarrow \quad x_1 + x_2 + x_3 = 6 \text{ and } y_1 + y_2 + y_3 = 2 \quad \text{...(iv)}$$

The coordinates of the centroid of ΔABC are

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)=\left(\frac{6}{3}, \frac{2}{3}\right)=\left(2, \frac{2}{3}\right) \qquad \text{[Using (iv)]}$$

EXAMPLE 4 *Two vertices of a triangle are (3, –5) and (–7, 4). If its centroid is (2, –1), find the third vertex.*

SOLUTION Let the coordinates of the third vertex be (x, y). Then,

$$\frac{x+3-7}{3}=2 \text{ and } \frac{y-5+4}{3}=-1$$

$\Rightarrow$ $x-4=6$ and $y-1=-3$

$\Rightarrow$ $x=10$ and $y=-2$

Thus, the coordinates of the third vertex are (10, –2).

LEVEL-2

EXAMPLE 5 *Use analytical geometry to prove that the mid-point of the hypotenuse of a right-angled triangle is equidistant from its vertices.*

SOLUTION Let AOB be a right-angled triangle with base OA taken along x-axis and the perpendicular OB taken along y-axis. Let $OA = a$ and $OB = b$.

Let D be the mid-point of the hypotenuse AB. Then, the coordinates of A, B and D are respectively $(a, 0)$, $(0, b)$ and $(a/2, b/2)$.

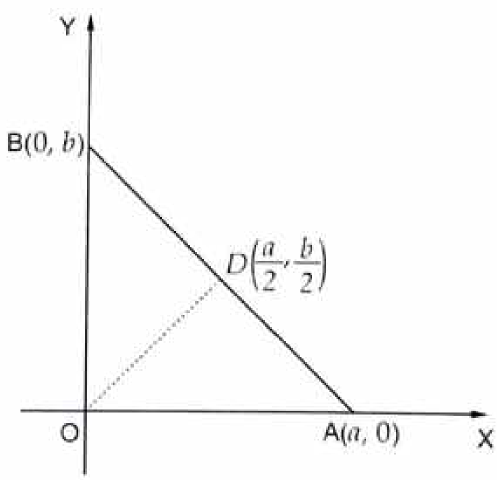


Fig. 6.37

Now, $DO=\sqrt{\left(\frac{a}{2}-0\right)^2+\left(\frac{b}{2}-0\right)^2}=\frac{1}{2}\sqrt{a^2+b^2}$,

$DA=\sqrt{\left(\frac{a}{2}-a\right)^2+\left(\frac{b}{2}-0\right)^2}=\frac{1}{2}\sqrt{a^2+b^2}$

and, $DB=\sqrt{\left(\frac{a}{2}-0\right)^2+\left(\frac{b}{2}-b\right)^2}=\frac{1}{2}\sqrt{a^2+b^2}$

Hence, $DA = DB = DC$ i.e., D is equidistant from the vertices of triangle ABC.

EXAMPLE 6 *Using analytical geometry, prove that the diagonals of a rhombus are perpendicular to each other.*

SOLUTION Let $OABC$ be a rhombus such that OA is along x-axis. Let BL and CM be perpendiculars from B and C respectively on x-axis.

Clearly, triangles ABL and OCM are congruent.

$\therefore \quad OM = AL$ and $CM = BL$

Let the coordinates of A and C be $(x_1, 0)$ and (x_2, y_2) respectively. Then, $OM = x_2$ and $OA = x_1$.

$\therefore \quad OL = OA + AL = OA + OM = x_1 + x_2$ and $BL = CM = y_2$

So, the coordinates of B are $(x_1 + x_2, y_2)$.

Now, $\quad OA = OC \Rightarrow OA^2 = OC^2 \Rightarrow x_1^2 = x_2^2 + y_2^2 \quad \ldots(i)$

In order to prove that the diagonals OB and AC are mutually perpendicular, it is sufficient to show that $\angle ODA = \pi/2$.

Since the diagonals of a rhombus bisect each other. Therefore, coordinates of D are $\left(\frac{x_1 + x_2}{2}, \frac{y_2}{2}\right)$.

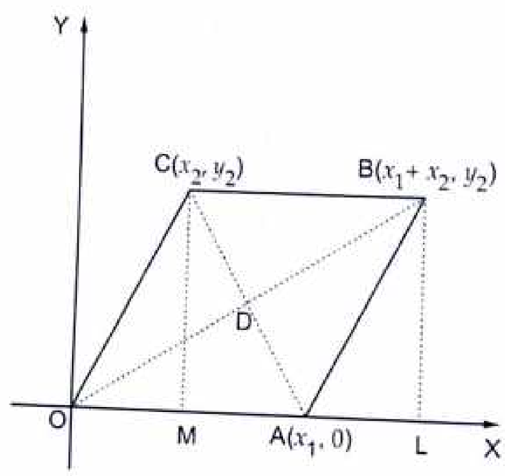


Fig. 6.38

Now,

$$OD^2 = \left(\frac{x_1 + x_2}{2} - 0\right)^2 + \left(\frac{y_2}{2} - 0\right)^2 = \left(\frac{x_1 + x_2}{2}\right)^2 + \left(\frac{y_2}{2}\right)^2$$

$$AD^2 = \left(\frac{x_1 + x_2}{2} - x_1\right)^2 + \left(\frac{y_2}{2} - 0\right)^2 \Rightarrow AD^2 = \left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2}{2}\right)^2$$

and, $\quad OA^2 = (x_1 - 0)^2 + (0 - 0)^2 = x_1^2$

$$\therefore \quad OD^2 + AD^2 = \left(\frac{x_1 + x_2}{2}\right)^2 + \left(\frac{y_2}{2}\right)^2 + \left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2}{y}\right)^2$$

$$\Rightarrow \quad OD^2 + AD^2 = \frac{1}{4}\left\{2x_1^2 + 2x_2^2 + 2y_2^2\right\}$$

$$\Rightarrow \quad OD^2 + AD^2 = \frac{1}{2}(x_1^2 + x_2^2 + y_2^2) = \frac{1}{2}(x_1^2 + x_1^2)$$

$$\Rightarrow \quad OD^2 + AD^2 = x_1^2 = OA^2 \quad \text{[Using (i)]}$$

$\therefore \quad \Delta\, ODA$, is a right-angled triangle such that $\angle ODA = \pi/2$.

Hence, the diagonals of a rhombus are at right angles.

EXAMPLE 7 *Prove that the diagonals of a rectangle bisect each other and are equal.*

SOLUTION Let $OACB$ be a rectangle such that OA is a along x-axis and OB is along y-axis. Let $OA = a$ and $OB = b$.

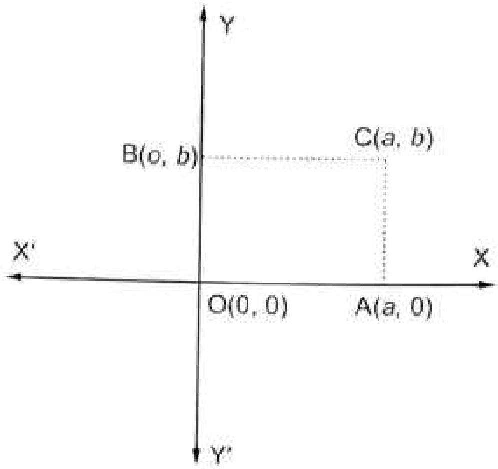


Fig. 6.39

Then, the coordinates of A and B are $(a, 0)$ and $(0, b)$ respectively.
Since, $OACB$ is a rectangle. Therefore,

$$AC = Ob \Rightarrow AC = b$$

Thus, we have

$$OA = a \text{ and } AC = b$$

So, the coordinates of C are (a, b).

The coordinates of the mid-point of OC are $\left(\frac{a+0}{2}, \frac{b+0}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right)$

Also, the coordinates of the mid-points of AB are $\left(\frac{a+0}{2}, \frac{0+b}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right)$

Clearly, coordinates of the mid-point of OC and AB are same.
Hence, OC and AB bisect each other.

Also, $\quad OC = \sqrt{a^2 + b^2}$ and $AB = \sqrt{(a-0)^2 + (0-b)^2} = \sqrt{a^2 + b^2}$

$\therefore \quad OC = AB$

EXERCISE 6.4

LEVEL-1

1. Find the centroid of the triangle whose vertices are:
   (i) (1, 4), (–1, –1), (3, –2) (ii) (– 2, 3), (2, –1), (4, 0)
2. Two vertices of a triangle are (1, 2), (3, 5) and its centroid is at the origin. Find the coordinates of the third vertex.
3. Find the third vertex of a triangle, if two of its vertices are at (–3, 1) and (0, –2) and the centroid is at the origin.
4. A (3, 2) and B (–2, 1) are two vertices of a triangle ABC whose centroid G has the coordinates $(5/3, -1/3)$. Find the coordinates of the third vertex C of the triangle. **[CBSE 2004]**
5. If (–2, 3), (4, –3) and (4, 5) are the mid-points of the sides of a triangle, find the coordinates of its centroid.

LEVEL-2

6. Prove analytically that the line segment joining the middle points of two sides of a triangle is equal to half of the third side.

7. Prove that the lines joining the middle points of the opposite sides of a quadrilateral and the join of the middle points of its diagonals meet in a point and bisect one another.
8. If G be the centroid of a triangle ABC and P be any other point in the plane, prove that $PA^2 + PB^2 + PC^2 = GA^2 + GB^2 + GC^2 + 3GP^2$.
9. If G be the centroid of a triangle ABC, prove that:
$AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2)$
10. In Fig. 6.40, a right triangle BOA is given. C is the mid-point of the hypotenuse AB. Show that it is equidistant from the vertices O, A and B.

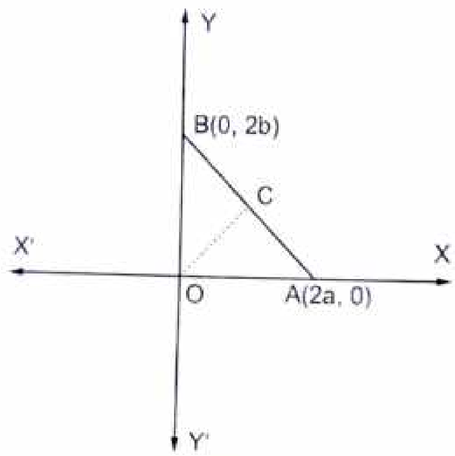


Fig. 6.40

ANSWERS

1.(i) $\left(1, \frac{1}{3}\right)$ (ii) $\left(\frac{4}{3}, \frac{2}{3}\right)$ 2. $(-4, -7)$ 3. $(3, 1)$

4. $(4, -4)$ 5. $\left(2, \frac{5}{3}\right)$

6.6 AREA OF A TRIANGLE

In earlier classes, we have computed the area of a triangle by using the formula

$$\text{Area of a triangle} = \frac{1}{2} \times \text{Base} \times \text{Altitude}$$

In class IX, we have used Heron's formula to find the area of a triangle when the lengths of its sides are given. In this section, we will find the area of a triangle in terms of the coordinates of its vertices. We can find the lengths of three sides of triangle by using distance formula and then, we can use the Heron's formula. But, this becomes tedious, particularly when the lengths of the sides are irrational numbers. That is why, we prefer to compute the area in terms of the coordinates of the vertices of the triangle. In the following theorem, we state and prove the same.

THEOREM *The area of a triangle, the coordinates of whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is*

$$\frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

PROOF Let ABC be a triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Draw AL, BM and CN perpendiculars from A, B, C on the x-axis.

Clearly, $ABML$, $ALNC$ and $BMNC$ are all trapeziums.

We know that

$$\text{Area of trapezium} = \frac{1}{2}\text{ (Sum of parallel sides) (Distance between them)}$$

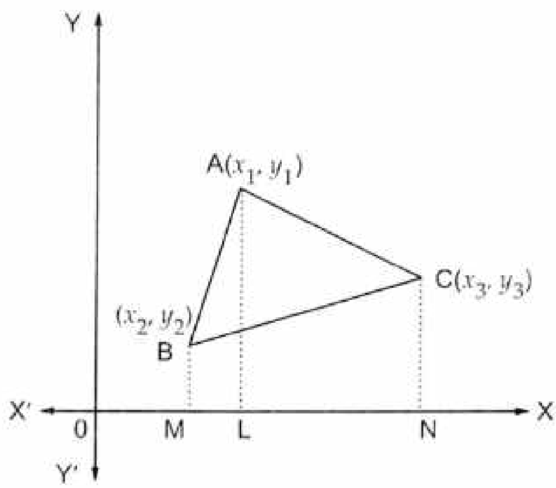


Fig. 6.41

We have,

$$\text{Area of } \Delta ABC = \text{Area of trapezium } ABML + \text{Area of trapezium } ALNC - \text{Area of trapezium } BMNC$$

Let Δ denote the area of ΔABC. Then,

$$\Delta = \frac{1}{2}(BM + AL)(ML) + \frac{1}{2}(AL + CN)(LN) - \frac{1}{2}(BM + CN)(MN)$$

$$\Rightarrow \quad \Delta = \left|\frac{1}{2}(y_2 - y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)\right|$$

$$\Rightarrow \quad \Delta = \left|\frac{1}{2}\left\{x_1(y_2 + y_1 - y_1 - y_3) + x_2(-y_2 - y_1 + y_2 + y_3) + x_3(y_1 + y_3 - y_2 - y_3\right\}\right|$$

$$\Rightarrow \quad \Delta = \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

REMARK 1 *To find the area of a polygon we divide it in triangles and take numerical value of the area of each of the triangles.*

REMARK 2 *The area of ΔABC can also be computed by using the following steps:*

STEP I *Write the coordinates of the vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ in three columns as shown below and augment the coordinates of $A(x_1, y_1)$ as fourth column.*

STEP II *Draw broken parallel lines pointing down wards from left to right and right to left.*

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{array}$$

STEP III *Compute the sum of the products of numbers at the ends of the lines pointing downwards from left to right and subtract from this sum the sum of the products of numbers at the ends of the lines pointing downward from right to left i.e., compute*

$$(x_1 y_2 + x_2 y_3 + x_3 y_1) - (x_2 y_1 + x_3 y_2 + x_1 y_3)$$

STEP IV *Find the absolute of the number obtained in step III and take its half to obtain the area.*

REMARK 3 *Three points* $A(x_1, y_1)$, $B(x_2, y_2)$ *and* $C(x_3, y_3)$ *are collinear iff*

Area of $\Delta ABC = 0$ i.e., $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE AREA OF A TRIANGLE WHEN COORDINATES OF ITS VERTICES AR GIVEN**

EXAMPLE 1 *Find the area of a triangle whose vertices are A (3, 2), B (11, 8) and C (8, 12).*

SOLUTION Let $A = (x_1, y_1) = (3, 2)$, $B = (x_2, y_2) = (11, 8)$ and $C = (x_3, y_3) = (8, 12)$ be th given points. Then,

$$\text{Area of } \Delta ABC = \frac{1}{2}\left|\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}\right|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}\left|\{3(8-12) + 11(12-2) + 8(2-8)\}\right|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}|(-12 + 110 - 48)| = 25 \text{ sq. units}$$

ALITER We have,

$$\begin{array}{cccc} 3 & 11 & 8 & 3 \\ 2 & 8 & 12 & 2 \end{array}$$

$$\therefore \quad \text{Area of } \Delta ABC = \frac{1}{2}|(3\times 8 + 11\times 12 + 8\times 2) - (11\times 2 + 8\times 8 + 3\times 12)|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}|(24 + 132 + 16) - (22 + 64 + 36|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}|172 - 122| = 25 \text{ sq. units}$$

EXAMPLE 2 *Find the area of the triangle formed by the points A (5, 2), B (4, 7) and C (7, −4).*

SOLUTION Here, $x_1 = 5, y_1 = 2, x_2 = 4, y_2 = 7, x_3 = 7$ and $y_3 = -4$

$$\therefore \quad \text{Area of } \Delta ABC = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}|5(7+4) + 4(-4-2) + 7(2-7)|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}|(5\times 11 + 4\times -6 + 7\times -5)|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}|(55 - 24 - 35)| = \frac{1}{2}|-4| = 2 \text{ sq. units}$$

ALITER We have,

$$\begin{array}{cccc} 5 & 4 & 7 & 5 \\ 2 & 7 & -4 & 2 \end{array}$$

$$\therefore \quad \text{Area of } \Delta ABC = \frac{1}{2}|(5\times 7 + 4\times -4 + 7\times 2) - (4\times 2 + 7\times 7 + 5\times -4)|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}|(35 - 16 + 14) - (8 + 49 - 20)|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}|33-(37)| = \frac{1}{2}|-4| = 2 \text{ sq. units}$$

EXAMPLE 3 *Prove that the area of triangle whose vertices are $(t, t-2)$, $(t+2, t+2)$ and $(t+3, t)$ is independent of t.* **[CBSE 2016]**

SOLUTION Let $A = (x_1, y_1) = (t, t-2)$, $B = (x_2, y_2) = (t+2, t+2)$ and $C = (x_3, y_3) = (t+3, t)$ be the vertices of the given triangle. Then,

$$\therefore \quad \text{Area of } \Delta ABC = \frac{1}{2}\left|x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)\right|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}\left|\{t(t+2-t) + (t+2)(t-t+2) + (t+3)(t-2-t-2)\}\right|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}\left|\{(2t+2t+4-4t-12)\}\right| = |-4| = 4 \text{ sq. units}$$

Clearly, area of ΔABC is independent of t.

ALITER We have,

$$\begin{matrix} t & & t+2 & & t+3 & & t \\ t-2 & & t+2 & & t & & t-2 \end{matrix}$$

$$\therefore \quad \text{Area of } \Delta ABC = \frac{1}{2}\left|\{t(t+2) + (t+2)t + (t+3)(t-2)\} - \{(t+2)(t-2) + (t+3)(t+2) + t \times t\}\right|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}\left|(t^2+2t+t^2+2t+t^2+t-6) - (t^2-4+t^2+5t+6+t^2)\right|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}\left|(3t^2+5t-6) - (3t^2+5t+2)\right|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}|-6-2|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = 4 \text{ sq. units}$$

Hence, Area of ΔABC is independent of t.

EXAMPLE 4 *Find the area of the triangle formed by joining the mid-points of the sides of the triangle whose vertices are $(0, -1)$, $(2, 1)$ and $(0, 3)$. Find the ratio of area of the triangle formed to the area of the given triangle.* **[NCERT]**

SOLUTION Let $A(0, -1)$, $B(2, 1)$ and $C(0, 3)$ be the vertices of ΔABC. Let D, E, F be the mid-points of sides BC, CA and AB respectively. Then, the coordinates of D, E and F are $(1, 2)$, $(0, 1)$ and $(1, 0)$ respectively.

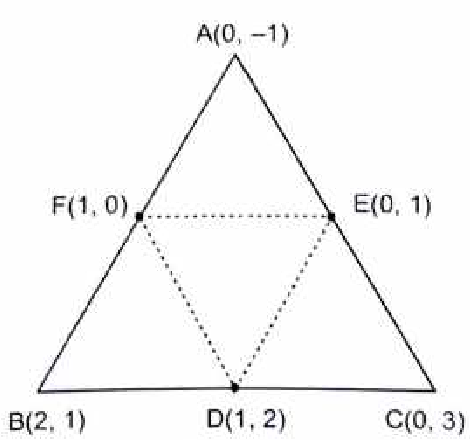


Fig. 6.42

Now,

$$\text{Area of } \Delta ABC = \frac{1}{2}\left| x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2) \right|$$

$\Rightarrow$ $$\text{Area of } \Delta ABC = \frac{1}{2}\left| 0 (1 - 3) + 2 (3 - (-1) + 0 (0 - 1) \right|$$

$\Rightarrow$ $$\text{Area of } \Delta ABC = \frac{1}{2}\left| 0 + 8 + 0 \right| = 4 \text{ sq. units}$$

$$\text{Area of } \Delta DEF = \frac{1}{2}\left| x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2) \right|$$

$\Rightarrow$ $$\text{Area of } \Delta DEF = \frac{1}{2}\left| 1 (1 - 0) + 0 (0 - 2) + 1 (2 - 1) \right|$$

$\Rightarrow$ $$\text{Area of } \Delta DEF = \frac{1}{2}\left| 1 + 1 \right| = 1 \text{ sq. units}$$

$\therefore$ $$\text{Area of } \Delta DEF : \text{Area of } \Delta ABC = 1 : 4$$

EXAMPLE 5 *If $P(1, 2)$, $Q(1, 0)$ and $R(0, 1)$ are the mid-points of the sides AB, BC ar respectively of ΔABC, find the coordinates of the vertices A, B and C, and hence find th of ΔABC.*

[CBSE

SOLUTION Let the coordinates of the vertices A, B and C of ΔABC be (x_1, y_1), $(x_2, y_2$ (x_3, y_3) respectively. It is given that $P(1, 2)$ is the mid-point of AB.

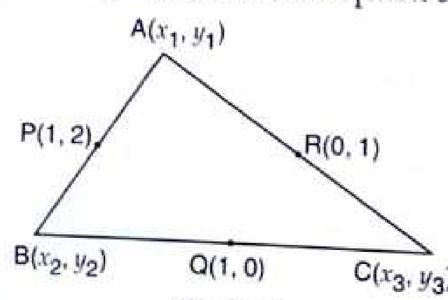


Fig. 6.43

$\therefore$ $$\frac{x_1 + x_2}{2} = 1 \text{ and } \frac{y_1 + y_2}{2} = 2$$

$\Rightarrow$ $x_1 + x_2 = 2$ ...(i) and $y_1 + y_2 = 4$

$Q(1, 0)$ is the mid-point of BC.

$\therefore$ $$\frac{x_2 + x_3}{2} = 1 \text{ and } \frac{y_2 + y_3}{2} = 0$$

$\Rightarrow$ $x_2 + x_3 = 2$ ...(iii) and $y_2 + y_3 = 0$ ...

Point $R(0, 1)$ is the mid-point of AC.

$\therefore$ $$\frac{x_1 + x_3}{2} = 0 \text{ and } \frac{y_1 + y_3}{2} = 1$$

$\Rightarrow$ $x_1 + x_3 = 0$ ...(v) and $y_1 + y_3 = 2$ ...(

Adding (i), (iii) and (v), we obtain

$$2(x_1 + x_2 + x_3) = 2 + 2 + 0$$

$\Rightarrow$ $$x_1 + x_2 + x_3 = 2$$

Adding (ii), (iv) and (vi), we obtain

$$2(y_1 + y_2 + y_3) = 4 + 0 + 2 \Rightarrow y_1 + y_2 + y_3 = 3 \quad ...(viii)$$

Subtracting (i), (iii) and (v) respectively from (vii), we obtain

$$x_3 = 0, x_1 = 0, x_2 = 2$$

Subtracting (ii), (iv) and (vi) respectively from (viii), we obtain

$$y_3 = -1, y_1 = 3, y_2 = 1$$

Hence, the coordinates of the vertices of ΔABC are $A(0, 3)$, $B(2, 1)$ and $C(0, -1)$.

$$\therefore \quad \text{Area of } \Delta ABC = \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

$$= \frac{1}{2}\left|0(1-(-1)) + 2(-1-3) + 0(3-1)\right| = 4 \text{ sq. units}$$

EXAMPLE 6 *The vertices of ΔABC are $A(4, 6)$, $B(1, 5)$ and $C(7, 2)$. A line is drawn to intersect sides AB and AC at D and E respectively such that $\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4}$. Calculate the area of ΔADE and compare it with the area of ΔABC.* **[NCERT]**

SOLUTION We have,

$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4}$$

$$\Rightarrow \quad \frac{AB}{AD} = \frac{AC}{AE} = 4$$

$$\Rightarrow \quad \frac{AD + DB}{AD} = \frac{AE + EC}{AE} = 4$$

$$\Rightarrow \quad 1 + \frac{DB}{AD} = 1 + \frac{EC}{AE} = 4$$

$$\Rightarrow \quad \frac{DB}{AD} = \frac{EC}{AE} = 3$$

$$\Rightarrow \quad \frac{AD}{DB} = \frac{AE}{EC} = \frac{1}{3}$$

$$\Rightarrow \quad AD : DB = AE : EC = 1 : 3$$

$\Rightarrow$ D and E divide AB and AC respectively in the ratio 1 : 3.

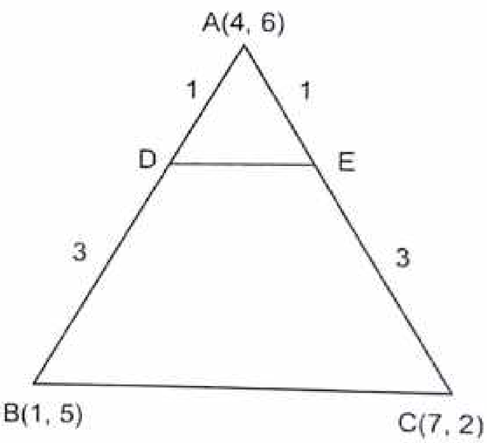


Fig. 6.44

So, the coordinates of D and E are

$$\left(\frac{1+12}{1+3}, \frac{5+18}{1+3}\right) = \left(\frac{13}{4}, \frac{23}{4}\right) \text{ and } \left(\frac{7+12}{1+3}, \frac{2+18}{1+3}\right) = \left(\frac{19}{4}, 5\right) \text{ respectively.}$$

We have,

$$\begin{matrix} 4 & & \frac{13}{4} & & \frac{19}{4} & & 4 \\ & \times & & \times & & \times & \\ 6 & & \frac{23}{4} & & 5 & & 6 \end{matrix}$$

$$\therefore \quad \text{Area of } \Delta ADE = \frac{1}{2}\left|\left(4 \times \frac{23}{4} + \frac{13}{4} \times 5 + \frac{19}{4} \times 6\right) - \left(\frac{13}{4} \times 6 + \frac{19}{4} \times \frac{23}{4} + 4 \times 5\right)\right|$$

$\Rightarrow$ Area of $\Delta\, ABC = \frac{1}{2}\left|\left(\frac{92}{4}+\frac{65}{4}+\frac{114}{4}\right)-\left(\frac{78}{4}+\frac{437}{16}+20\right)\right|$

$\Rightarrow$ Area of $\Delta\, ABC = \frac{1}{2}\left|\frac{271}{4}-\frac{1069}{16}\right| = \frac{1}{2}\times\frac{15}{16}=\frac{15}{32}$ sq. units

Also, we have

$$\begin{array}{cccc} 4 & 1 & 7 & 4 \\ 6 & 5 & 2 & 6 \end{array}$$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}|(4\times 5+1\times 2+7\times 6)-(1\times 6+7\times 5+4\times 2)|$

$\Rightarrow$ Area of $\Delta\, ABC = \frac{1}{2}|(20+2+42)-(6+35+8)|$

$\Rightarrow$ Area of $\Delta\, ABC = \frac{1}{2}|64-49| = \frac{15}{2}$ sq. units

$\therefore$ $\frac{\text{Area of } \Delta\, ADE}{\text{Area of } \Delta\, ABC} = \frac{15/32}{15/2} = \frac{1}{16}$

Hence, Area of $\Delta\, ADE$: Area of $\Delta\, ABC = 1:16$

EXAMPLE 7 *If A (4, −6), B (3, −2) and C (5, 2) are the vertices of $\Delta\, ABC$, then verify the fact a median of a triangle ABC divides it into two triangles of equal areas.* **[CBSE 2013, 2(**

SOLUTION Let D be the mid-point of BC. Then, the coordinates of D are (4, 0).
We have,

$$\begin{array}{cccc} 4 & 3 & 5 & 4 \\ -6 & -2 & 2 & -6 \end{array}$$

$\therefore$ Area of $\Delta\, ABC = \frac{1}{2}|(4\times -2+3\times 2+5\times -6)-(3\times -6+5\times -2+4\times 2)|$

$\Rightarrow$ Area of $\Delta\, ABC = \frac{1}{2}|(-8+6-30)-(-18-10+8)|$

$\Rightarrow$ Area of $\Delta\, ABC = \frac{1}{2}|-32+20| = 6$ sq. units

Also, we have

$$\begin{array}{cccc} 4 & 3 & 4 & 4 \\ -6 & -2 & 0 & -6 \end{array}$$

$\therefore$ Area of $\Delta\, ABD = \frac{1}{2}\left|\begin{array}{l}\{(4\times(-2)+3\times 0+4\times(-6)\} \\ -\{3\times(-6)+4\times(-2)+4\times 0\}\end{array}\right|$

$\Rightarrow$ Area of $\Delta\, ABD = \frac{1}{2}|(-8+0-24)-(-18-8+0)|$

$\Rightarrow$ Area of $\Delta\, ABD = \frac{1}{2}|(-32+26| = 3$ sq. units

$\therefore$ $\frac{\text{Area of } \Delta\, ABC}{\text{Area of } \Delta\, ABD} = \frac{6}{3} = \frac{2}{1}$

$\Rightarrow$ Area of $\Delta\, ABC = 2$ (Area of $\Delta\, ABD$)

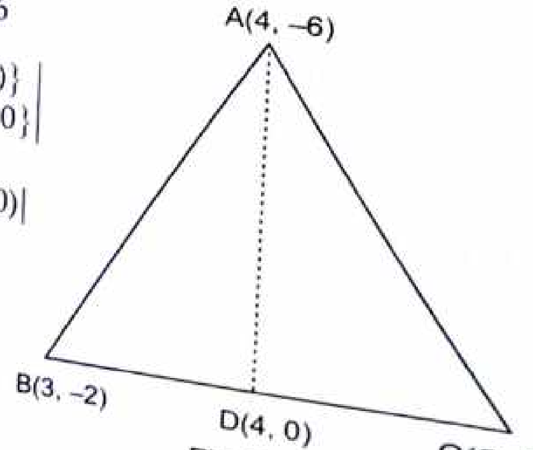


Fig. 6.45

EXAMPLE 8 *Find the area of the triangle ABC with A (1, – 4) and mid-points of sides through A being (2, – 1) and (0, –1).* **[NCERT EXEMPLAR, CBSE 2015]**

SOLUTION Let the coordinates of B and C be (x_1, y_1) and (x_2, y_2) respectively. It is given that the points E and F are the mid-points of AB and AC respectively.

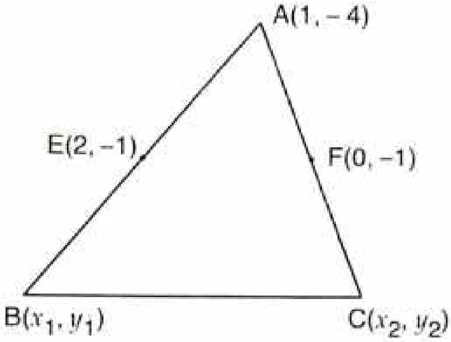


Fig. 6.46

$$\frac{x_1+1}{2}=2, \frac{y_1-4}{2}=-1 \text{ and } \frac{x_2+1}{2}=0, \frac{y_2-4}{2}=-1$$

$\Rightarrow \quad x_1 = 3, y_1 = 2$ and $x_2 = -1, y_2 = 2$

Thus, the coordinates of B and C are (3, 2) and (–1, 2) respectively.

$$\begin{matrix} 1 & 3 & -1 & 1 \\ -4 & 2 & 2 & -4 \end{matrix}$$

$\therefore \quad$ Area of $\Delta ABC = \frac{1}{2}|(2+6+4)-(-12-2+2)|$

$$= \frac{1}{2}|12-(-12)| = 12 \text{ sq. units}$$

Type II **ON FINDING THE AREA OF A QUADRILATERAL WHEN COORDINATES OF ITS VERTICES ARE GIVEN**

EXAMPLE 9 *Find the area of the quadrilateral ABCD whose vertices are respectively A (1, 1), B (7, –3) C(12, 2) and D (7, 21).* **[CBSE 2017]**

SOLUTION We have,

Area of quadrilateral $ABCD$ = | Area of ΔABC | + | Area of ΔACD |

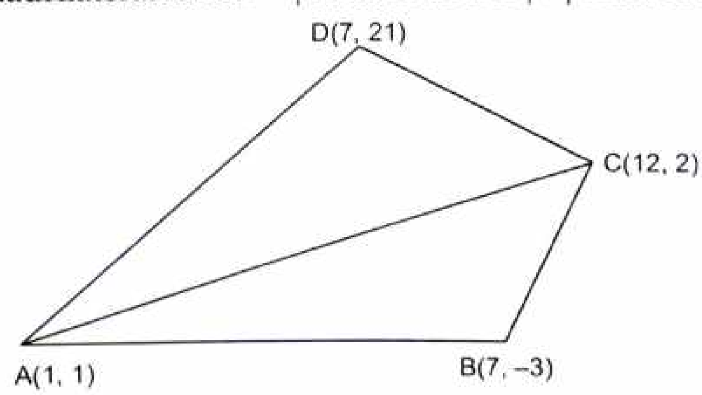


Fig. 6.47

We have,

$$\begin{matrix} 1 & 7 & 12 & 1 \\ 1 & -3 & 2 & 1 \end{matrix}$$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}|(1 \times -3 + 7 \times 2 + 12 \times 1) - (7 \times 1 + 12 \times (-3) + 1 \times 2)|$

$\Rightarrow$ Area of $\Delta ABC = \frac{1}{2}|(-3 + 14 + 12) - (7 - 36 + 2)|$

$\Rightarrow$ Area of $\Delta ABC = \frac{1}{2}|23 + 27| = 25$ sq. units

Also, we have

$$\begin{array}{cccc} 1 & 12 & 7 & 1 \\ 1 & 2 & 21 & 1 \end{array}$$

$\therefore$ Area of $\Delta ACD = \frac{1}{2}|(1 \times 2 + 12 \times 21 + 7 \times 1) - (12 \times 1 + 7 \times 2 + 1 \times 21)|$

$\Rightarrow$ Area of $\Delta ACD = \frac{1}{2}|(2 + 252 + 7) - (12 + 14 + 21)|$

$\Rightarrow$ Area of $\Delta ACD = \frac{1}{2}|261 - 47| = 107$ sq. units

$\therefore$ Area of quadrilateral $ABCD = 25 + 107 = 132$ sq. units

Type III ON COLLINEARITY OF THREE POINTS

FORMULA *Three points* $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) *are collinear iff*

$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$$

or,

$$(x_1y_2 + x_2y_3 + x_3y_1) - (x_1y_3 + x_3y_2 + x_2y_1) = 0$$

EXAMPLE 10 *Prove that the points (2, –2), (–3, 8) and (–1, 4) are collinear.*

SOLUTION Let Δ be the area of the triangle formed by the given points.

We have,

$$\begin{array}{cccc} 2 & -3 & -1 & 2 \\ -2 & 8 & 4 & -2 \end{array}$$

$\therefore$ $\Delta = \frac{1}{2}|\{2 \times 8 + (-3) \times 4 + (-1) \times (-2)\} - \{(-3) \times (-2) + (-1) \times 8 + 2 \times 4\}|$

$\Rightarrow$ $\Delta = \frac{1}{2}|(16 - 12 + 2) - (6 - 8 + 8)|$

$\Rightarrow$ $\Delta = \frac{1}{2}|6 - 6| = 0$

Hence, given points are collinear.

EXAMPLE 11 *Prove that the points (a, b + c), (b, c + a) and (c, a + b) are collinear.* [CBSE

SOLUTION Let Δ be the area of the triangle formed by the points $(a, b + c), (b, c + a)$ and $(c, a + b)$.

We have,

$$\begin{array}{cccc} a & b & c & a \\ b+c & c+a & a+b & b+c \end{array}$$

$\therefore$ $\Delta = \frac{1}{2}|\{a(c + a) + b(a + b) + c(b + c)\} - \{b(b + c) + c(c + a) + a(a + b)\}|$

$$\Rightarrow \quad \Delta = \frac{1}{2}\left|(ac + a^2 + ab + b^2 + bc + c^2) - (b^2 + bc + c^2 + ca + a^2 + ab)\right|$$

$$\Rightarrow \quad \Delta = 0$$

Hence, the given points are collinear.

Type IV ON FINDING THE DESIRED RESULT OR UNKNOWN WHEN THREE POINTS ARE COLLINEAR

EXAMPLE 12 *For what value of k are the points (k, 2 – 2k) (–k + 1, 2k) and (–4 – k, 6 – 2k) are collinear?*

SOLUTION Given points will be collinear, if area of the triangle formed by them is zero. We have,

$$\begin{matrix} k & -k+1 & -4-k & k \\ 2-2k & 2k & 6-2k & 2-2k \end{matrix}$$

i.e.,

$$\left|\left\{2k^2 + (-k+1)(6-2k) + (-4-k)(2-2k)\right\} - \left\{(-k+1)(2-2k) + (-4-k)(2k) + k(6-2k)\right\}\right| = 0$$

$$\Rightarrow \quad \left|(2k^2 + 6 - 8k + 2k^2 + 2k^2 + 6k - 8) - (2 - 4k + 2k^2 - 8k - 2k^2 + 6k - 2k^2)\right| = 0$$

$$\Rightarrow \quad (6k^2 - 2k - 2) - (-2k^2 - 6k + 2) = 0$$

$$\Rightarrow \quad 8k^2 + 4k - 4 = 0$$

$$\Rightarrow \quad 2k^2 + k - 1 = 0 \Rightarrow (2k-1)(k+1) = 0 \Rightarrow k = 1/2 \text{ or, } k = -1$$

Hence, the given points are collinear for $k = 1/2$ or, $k = -1$.

EXAMPLE 13 *For what value of x will the points (x, –1), (2, 1) and (4, 5) lie on a line?* **[CBSE 2013]**

SOLUTION Given points will be collinear if the area of the triangle formed by them is zero. We have,

$$\begin{matrix} x & 2 & 4 & x \\ -1 & 1 & 5 & -1 \end{matrix}$$

$\therefore$ Area of the triangle = 0

$$\Rightarrow \quad \left|\left\{x \times 1 + 2 \times 5 + 4 \times (-1)\right\} - \left\{(2 \times -1 + 4 \times 1 + x \times 5\right\}\right| = 0$$

$$\Rightarrow \quad (x + 10 - 4) - (-2 + 4 + 5x) = 0$$

$$\Rightarrow \quad (x + 6) - (5x + 2) = 0$$

$$\Rightarrow \quad -4x + 4 = 0$$

$$\Rightarrow \quad x = 1$$

Hence, the given points lie on a line, if $x = 1$.

EXAMPLE 14 *Find the condition that the point (x, y) may lie on the line joining (3, 4) and (–5, –6).*

SOLUTION Since the point $P(x, y)$ lies on the line joining A (3, 4) and B (–5, –6). Therefore, P, A and B are collinear points.

$$\begin{matrix} x & 3 & -5 & x \\ y & 4 & -6 & y \end{matrix}$$

$$\therefore \quad \left\{4x + 3 \times -6 + (-5) \times y\right\} - \left\{3y + (-5) \times 4 + x \times (-6)\right\} = 0$$

$$\Rightarrow \quad (4x - 18 - 5y) - (3y - 6x - 20) = 0$$

$$\Rightarrow \quad 10x - 8y + 2 = 0 \Rightarrow 5x - 4y + 1 = 0$$

Hence, the point (x, y) lies on the line joining $(3, 4)$ and $(-5, -6)$, if $5x - 4y + 1 = 0$.

EXAMPLE 15 *If $P(x, y)$ is any point on the line joining the points $A(a, 0)$ and $B(0, b)$, then show that $\frac{x}{a} + \frac{y}{b} = 1$.* **[CBSE 2009]**

SOLUTION It is given that the point $P(x, y)$ lies on the line segment joining points $A(a, 0)$ and $B(0, b)$. Therefore, points $P(x, y)$, $A(a, 0)$ and $B(0, b)$ are collinear points.

$$\begin{matrix} x & a & 0 & x \\ y & 0 & b & y \end{matrix}$$

$$\therefore \quad (x \times 0 + a \times b + 0 \times y) - (a \times y + 0 \times 0 + x \times b) = 0$$

$$\Rightarrow \quad ab - (ay + bx) = 0$$

$$\Rightarrow \quad ab = ay + bx$$

$$\Rightarrow \quad \frac{ab}{ab} = \frac{ay}{ab} + \frac{bx}{ab} \qquad \text{[Dividing throughout by } ab\text{]}$$

$$\Rightarrow \quad 1 = \frac{y}{b} + \frac{x}{a} \text{ or } \frac{x}{a} + \frac{y}{b} = 1.$$

EXAMPLE 16 *If the points (p, q), (m, n) and $(p - m, q - n)$ are collinear, show that $pn = qm$.* **[CBSE 2010]**

SOLUTION Given points are collinear. Therefore

$$\begin{matrix} p & m & p-m & p \\ q & n & q-n & q \end{matrix}$$

$$\{p \times n + m(q - n) + (p - m)q\} - \{m \times q + (p - m)n + p(q - n)\} = 0$$

$$\Rightarrow \quad (pn + qm - mn + pq - mq) - (mq + pn - mn + pq - pn) = 0$$

$$\Rightarrow \quad (pn + pq - mn) - (mq - mn + pq) = 0$$

$$\Rightarrow \quad pn - mq = 0$$

$$\Rightarrow \quad pn = qm$$

EXAMPLE 17 *Find k so that the point $P(-4, 6)$ lies on the line segment joining $A(k, 10)$ and $B(3, -8)$. Also, find the ratio in which P divides AB.* **[CBSE 2010]**

SOLUTION If $P(-4, 6)$ lies on the line segment joining $A(k, 10)$ and $B(3, -8)$, then P, A and B are collinear.

$$\begin{matrix} -4 & k & 3 & -4 \\ 6 & 10 & -8 & 6 \end{matrix}$$

$$\therefore \quad (-4 \times 10 + k \times -8 + 3 \times 6) - (6k + 30 + -4 \times -8) = 0$$

$$\Rightarrow \quad (-40 - 8k + 18) - (6k + 30 + 32) = 0$$

$$\Rightarrow \quad (-22 - 8k) - (6k + 62) = 0$$

$$\Rightarrow \quad -14k - 84 = 0$$

$$\Rightarrow \quad k = -6$$

Suppose P divides AB in the ratio $\lambda : 1$. Then, the coordinates of P are $\left(\frac{3\lambda - 6}{\lambda + 1}, \frac{-8\lambda + 10}{\lambda + 1}\right)$. But, the coordinates of P are $(-4, 6)$.

$$\therefore \quad \frac{3\lambda - 6}{\lambda + 1} = -4 \text{ and } \frac{-8\lambda + 10}{\lambda + 1} = 6$$

$$\Rightarrow \quad \lambda = \frac{2}{7}$$

Fig. 6.48

Hence, P divides AB in the ratio $\frac{2}{7} : 1$ or $2 : 7$.

EXAMPLE 18 *If the points $A(1, -2)$, $B(2, 3)$, $C(-3, 2)$ and $D(-4, -3)$ are the vertices of parallelogram ABCD, then taking AB as the base, find the height of the parallelogram.*

[CBSE 2013]

SOLUTION Let $DM = h$ be the height of parallelogram $ABCD$ when AB is taken as the base. From Fig. 6.49,

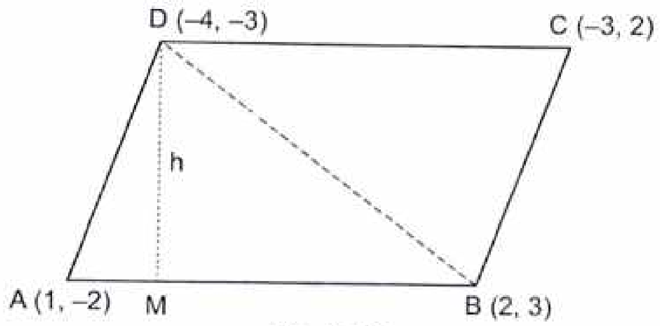


Fig. 6.49

$$\text{Area of } \Delta ABD = \frac{1}{2}(AB \times DM)$$

$$\Rightarrow \quad \text{Area of } \Delta ABD = \frac{1}{2} AB \times h$$

$$\Rightarrow \quad h = \frac{2\,(\text{Area of } \Delta ABD)}{AB} \qquad \ldots(i)$$

Using distance formula, we obtain

$$AB = \sqrt{(2-1)^2 + (3+2)^2} = \sqrt{26}$$

The coordinates of vertices of ΔABD are $A(1, -2)$, $B(2, 3)$ and $D(-4, -3)$.

$$\begin{matrix} 1 & 2 & -4 & 1 \\ -2 & 3 & -3 & -2 \end{matrix}$$

$$\therefore \quad \text{Area of } \Delta ABD = \frac{1}{2}\left|\{1 \times 3 + 2(-3) + (-4)(-2)\} - \{1 \times (-3) + (-4) \times 3 + 2 \times (-2)\}\right|$$

$$= \frac{1}{2}\left|(3 - 6 + 8) - (-3 - 12 - 4)\right| = \frac{1}{2}(5 + 19) = 12 \text{ sq. units}$$

Substituting the values of AB and area of ΔABD in (i), we obtain

$$h = \frac{2 \times 12}{\sqrt{26}} = \frac{24}{\sqrt{26}} \text{ Units}$$

EXAMPLE 19 *Three vertices of a parallelogram ABCD are A(3, –4), B(–1, –3) and C(–6, 2). Find the coordinates of vertex D and find the area of parallellogram ABCD.* **[CBSE 2013]**

SOLUTION Let (x, y) be the coordinates of vertex D. We know that the diagonals of a parallelogram bisect each other. Therefore, mid-points of diagonals AC and BD are same. Consequently, the coordinates of their mid-points are same and hence,

$$\left(\frac{x-1}{2}, \frac{y-3}{2}\right) = \left(\frac{3-6}{2}, \frac{-4+2}{2}\right)$$

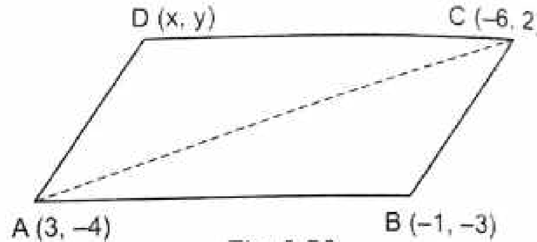


Fig. 6.50

$$\Rightarrow \quad \left(\frac{x-1}{2}, \frac{y-3}{2}\right) = \left(-\frac{3}{2}, -1\right)$$

$$\Rightarrow \quad \frac{x-1}{2} = -\frac{3}{2} \text{ and } \frac{y-3}{2} = -1$$

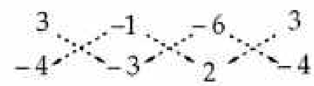
$$\Rightarrow \quad x - 1 = -3 \text{ and } y - 3 = -2$$

$$\Rightarrow \quad x = -2 \text{ and } y = 1$$

Hence, the coordinates of D are $(-2, 1)$.

We know that each diagonal of a parallelogram divides it in two triangles of equal area.

$\therefore$ Area of parallelogram $ABCD = 2$ (Area of ΔABC) ...(i)



Now, Area of $\Delta ABC = \frac{1}{2}\left|\{3 \times (-3) + (-1) \times 2 + (-6)(-4)\} - \{3 \times 2 + (-6)(-3) + (-1)(-4)\}\right|$

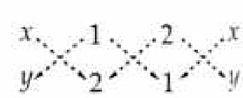
$$= \frac{1}{2}\left|(-9 - 2 + 24) - (6 + 18 + 4)\right| = \frac{1}{2}(13 - 28) = \frac{15}{2} \text{ square units.}$$

Substituting this in (i), we obtain

$$\text{Area of parallelogram } ABCD = 2 \times \frac{15}{2} = 15 \text{ square units.}$$

EXAMPLE 20 *If the area of ΔABC formed by A (x, y), B(1, 2) and C(2, 1) is 6 square units, then prove that $x + y = 15$ or, $x + y + 9 = 0$.* **[CBSE 2013]**

SOLUTION We have,



$$\text{Area of } \Delta ABC = 6$$

$$\Rightarrow \quad \frac{1}{2}\left|(2x + 1 + 2y) - (x + 4 + y)\right| = 6$$

$$\Rightarrow \quad |x + y - 3| = 12$$

$$\Rightarrow \quad x + y - 3 = \pm 12$$

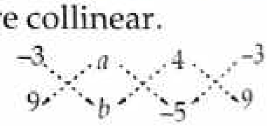
$$\Rightarrow \quad x + y - 15 = 0 \text{ or, } x + y + 9 = 0$$

$$\Rightarrow \quad x + y = 15 \text{ or, } x + y + 9 = 0$$

EXAMPLE 21 *If the points $P(-3, 9)$, $Q(a, b)$ and $R(4, -5)$ are collinear and $a + b = 1$, find the values of a and b.* **[CBSE 2014]**

SOLUTION It is given that the points $P(-3, 9)$, $Q(a, b)$ and $R(4, -5)$ are collinear.

$$\therefore \quad \text{Area of } \Delta PQR = 0$$



$$\Rightarrow \quad \left|\{-3b - 5a + 36\} - \{15 + 4b + 9a\}\right| = 0$$

$\Rightarrow \quad |(-14a - 7b + 21)| = 0$

$\Rightarrow \quad 14a + 7b - 21 = 0$

$\Rightarrow \quad 2a + b - 3 = 0$ ...(i)

It is given that $a + b = 1$ ...(ii)

Solving (i) and (ii), we obtain $a = 2$ and $b = -1$.

LEVEL-2

Type I ON FINDING THE AREA OF A TRIANGLE

EXAMPLE 22 *If D, E and F are the mid-points of sides BC, CA and AB respectively of a ΔABC, then using coordinate geometry prove that*

$$\text{Area of } \Delta DEF = \frac{1}{4} (\text{Area of } \Delta ABC)$$

SOLUTION Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of ΔABC. Then, the coordinates of D, E and F are $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)\left(\frac{x_1 + x_3}{2}, \frac{y_1 + y_3}{2}\right)$ and $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ respectively.

We have,

$$\Delta_1 = \text{Area of } \Delta ABC = \frac{1}{2} |x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)|$$

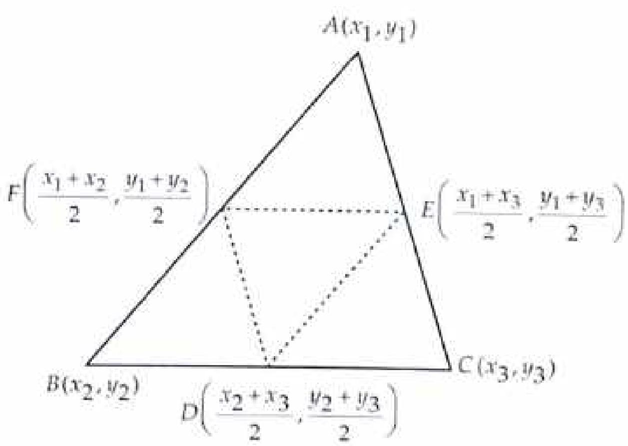


Fig. 6.51

$$\Delta_2 = \text{Area of } \Delta DEF = \frac{1}{2} \left| \left(\frac{x_2 + x_3}{2}\right)\left(\frac{y_1 + y_3}{2} - \frac{y_1 + y_2}{2}\right) + \left(\frac{x_1 + x_3}{2}\right)\left(\frac{y_1 + y_2}{2} - \frac{y_2 + y_3}{2}\right) + \left(\frac{x_1 + x_2}{2}\right)\left(\frac{y_2 + y_3}{2} - \frac{y_1 + y_3}{2}\right) \right|$$

$$\Rightarrow \quad \Delta_2 = \frac{1}{8} |(x_2 + x_3)(y_3 - y_2) + (x_1 + x_3)(y_1 - y_3) + (x_1 + x_2)(y_2 - y_1)|$$

$$\Rightarrow \quad \Delta_2 = \frac{1}{8} |x_1 (y_1 - y_3 + y_2 - y_1) + x_2 (y_3 - y_2 + y_2 - y_1) + x_3 (y_3 - y_2 + y_1 - y_3)|$$

$$\Rightarrow \quad \Delta_2 = \frac{1}{8} |x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)|$$

$$\Rightarrow \qquad \Delta_2 = \frac{1}{4}(\text{Area of } \Delta ABC) = \frac{1}{4}\Delta_1$$

Hence, Area of $\Delta DEF = \frac{1}{4}$ (Area of ΔABC)

Type II **MIXED PROBLEMS BASED UPON THE CONCEPT OF AREA OF A TRIANGLE**

EXAMPLE 23 *If the vertices of a triangle have integral coordinates, prove that the triangle cannot be equilateral.*

SOLUTION Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle ABC, where $x_i, y_i, i = 1, 2, 3$ are integers. Then, the area of ΔABC is given by

$$\Delta = \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

$$\Rightarrow \qquad \Delta = \text{A rational number} \qquad [\because x_i, y_i \text{ are integers}]$$

If possible, let the triangle ABC be an equilateral triangle, then its area is given by

$$\Delta = \frac{\sqrt{3}}{4}(\text{Side})^2 = \frac{\sqrt{3}}{4}(AB)^2 \qquad [\because AB = BC = CA]$$

$$\Rightarrow \qquad \Delta = \frac{\sqrt{3}}{4}(\text{A positive integer}) \qquad \begin{bmatrix} \because \text{ vertices are integers} \\ \therefore AB^2 \text{ is a positive integer} \end{bmatrix}$$

$$\Rightarrow \qquad \Delta = \text{An irrational number}$$

This is a contradiction to the fact that the area is a rational number.

Hence, the triangle cannot be equilateral.

EXAMPLE 24 *If the coordinates of two points A and B are (3, 4) and (5, −2) respectively. Find the coordinates of any point P, if PA = PB and Area of $\Delta PAB = 10$.*

SOLUTION Let the coordinates of P be (x, y). Then,

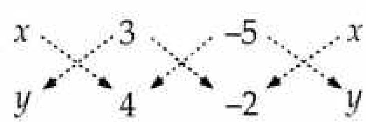
$$PA = PB$$

$$\Rightarrow \qquad PA^2 = PB^2$$

$$\Rightarrow \qquad (x-3)^2 + (y-4)^2 = (x-5)^2 + (y+2)^2$$

$$\Rightarrow \qquad x - 3y - 1 = 0 \qquad \ldots(\text{i})$$

Now, Area of $\Delta PAB = 10$



$$\Rightarrow \qquad \frac{1}{2}\left|(4x + 3 \times -2 + 5y) - (3y + 20 - 2x)\right| = 10$$

$$\Rightarrow \qquad \left|(4x + 5y - 6) - (-2x + 3y + 20)\right| = 20$$

$$\Rightarrow \qquad \left|6x + 2y - 26\right| = 20$$

$$\Rightarrow \qquad 6x + 2y - 26 = \pm 20$$

$$\Rightarrow \qquad 6x + 2y - 46 = 0 \text{ or, } 6x + 2y - 6 = 0$$

$$\Rightarrow \qquad 3x + y - 23 = 0 \text{ or, } 3x + y - 3 = 0 \qquad \ldots(\text{ii})$$

Solving $x - 3y - 1 = 0$ and $3x + y - 23 = 0$ we get $x = 7, y = 2$.

Solving $x - 3y - 1 = 0$ and $3x + y - 3 = 0$, we get $x = 1, y = -0$.
Thus, the coordinates of P are $(7, 2)$ or $(1, 0)$.

EXAMPLE 25 *The coordinates of A, B, C are (6, 3), (–3, 5) and (4, –2) respectively and P is any point (x, y). Show that the ratio of the areas of triangles PBC and ABC is* $\left|\frac{x+y-2}{7}\right|$.

SOLUTION We have,

$$\begin{array}{cccc} x & -3 & 4 & x \\ y & 5 & -2 & y \end{array}$$

$\therefore$ Area of $\Delta PBC = \frac{1}{2}|(5x + 6 + 4y) - (-3y + 20 - 2x)|$

$\Rightarrow$ Area of $\Delta PBC = \frac{1}{2}|5x + 6 + 4y + 3y - 20 + 2x|$

$\Rightarrow$ Area of $\Delta PBC = \frac{1}{2}|7x + 7y - 14|$

$\Rightarrow$ Area of $\Delta PBC = \frac{7}{2}|x + y - 2|$

$\Rightarrow$ Area of $\Delta ABC = \frac{7}{2}|6 + 3 - 2|$ [Replacing x by 6 and $y = 3$ in Area of ΔPBC]

$\Rightarrow$ Area of $\Delta ABC = \frac{49}{2}$

$$\therefore \quad \frac{\text{Area of } \Delta PBC}{\text{Area of } \Delta ABC} = \frac{\frac{7}{2}|x+y-2|}{\frac{49}{2}} = \frac{|x+y-2|}{7} = \left|\frac{x+y-2}{7}\right|$$

EXERCISE 6.5

LEVEL-1

1. Find the area of a triangle whose vertices are
   (i) $(6, 3), (-3, 5)$ and $(4, -2)$ (ii) $(at_1^2, 2at_1), (at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$
   (iii) $(a, c + a), (a, c)$ and $(-a, c - a)$
2. Find the area of the quadrilaterals, the coordinates of whose vertices are
   (i) $(-3, 2), (5, 4), (7, -6)$ and $(-5, -4)$ (ii) $(1, 2), (6, 2), (5, 3)$ and $(3, 4)$
   (iii) $(-4, -2), (-3, -5), (3, -2), (2, 3)$ [NCERT] [CBSE 2009]
3. The four vertices of a quadrilateral are $(1, 2), (-5, 6), (7, -4)$ and $(k, -2)$ taken in order. If the area of the quadrilateral is zero, find the value of k.
4. The vertices of ΔABC are $(-2, 1), (5, 4)$ and $(2, -3)$ respectively. Find the area of the triangle and the length of the altitude through A.
5. Show that the following sets of points are collinear.
   (a) $(2, 5), (4, 6)$ and $(8, 8)$ (b) $(1, -1), (2, 1)$ and $(4, 5)$.
6. Find the area of a quadrilateral $ABCD$, the coordinates of whose varities are $A(-3, 2)$, $B(5, 4)$, $C(7, -6)$ and $(-5, -4)$. [CBSE 2016]

7. In ΔABC, the coordinates of vertex A are $(0, -1)$ and $D(1, 0)$ and $E(0, 1)$ respectively the mid-points of the sides AB and AC. If F is the mid-point of side BC, find the area of ΔDEF. **[CBSE 2016]**

8. Find the area of the triangle PQR with $Q(3, 2)$ and the mid-points of the sides through Q being $(2, -1)$ and $(1, 2)$. **[CBSE 2015]**

9. If $P(-5, -3), Q(-4, -6), R(2, -3)$ and $S(1, 2)$ are the vertices of a quadrilateral $PQRS$, find its area. **[CBSE 2015]**

10. If $A(-3, 5), B(-2, -7), C(1, -8)$ and $D(6, 3)$ are the vertices of a quadrilateral $ABCD$, find its area. **[CBSE 2014, 2018]**

11. For what value of a the point $(a, 1), (1, -1)$ and $(11, 4)$ are collinear? **[CBSE 2017]**

12. Prove that the points $(a, b), (a_1, b_1)$ and $(a - a_1, b - b_1)$ are collinear if $ab_1 = a_1 b$.

13. If the vertices of a triangle are $(1, -3), (4, p)$ and $(-9, 7)$ and its area is 15 sq. units, findthe value(s) of p. **[CBSE 2012]**

14. If (x, y) be on the line joining the two points $(1, -3)$ and $(-4, 2)$, prove that $x + y + 2 = 0$.

15. Find the value of k if points $(k, 3), (6, -2)$ and $(-3, 4)$ are collinear. **[CBSE 2008]**

16. Find the value of k, if the points $A(7, -2), B(5, 1)$ and $C(3, 2k)$ are collinear. **[CBSE 2010]**

17. If the point $P(m, 3)$ lies on the line segment joining the points $A\left(-\frac{2}{5}, 6\right)$ and $B(2, 8)$, find the value of m. **[CBSE 2010]**

18. If $R(x, y)$ is a point on the line segment joining the points $P(a, b)$ and $Q(b, a)$, then prove that $x + y = a + b$. **[CBSE 2010]**

19. Find the value of k, if the points $A(8, 1), B(3, -4)$ and $C(2, k)$ are collinear. **[CBSE 2010]**

20. Find the value of a for which the area of the triangle formed by the points $A(a, 2a), B(-2, 6)$ and $C(3, 1)$ is 10 square units.

21. If $a \neq b \neq 0$, prove that the points $(a, a^2), (b, b^2), (0, 0)$ are never collinear. **[CBSE 2017]**

22. The area of a triangle is 5 sq. units. Two of its vertices are at $(2, 1)$ and $(3, -2)$. If the third vertex is $(7/2, y)$, find y. **[CBSE 2017]**

LEVEL-2

23. Prove that the points $(a, 0), (0, b)$ and $(1, 1)$ are collinear if, $\frac{1}{a} + \frac{1}{b} = 1$.

24. The point A divides the join of $P(-5, 1)$ and $Q(3, 5)$ in the ratio $k : 1$. Find the two values of k for which the area of ΔABC where B is $(1, 5)$ and $C(7, -2)$ is equal to 2 units.

25. The area of a triangle is 5. Two of its vertices are $(2, 1)$ and $(3, -2)$. The third vertex lies on $y = x + 3$. Find the third vertex.

26. If $a \neq b \neq c$, prove that the points $(a, a^2), (b, b^2), (c, c^2)$ can never be collinear.

27. Four points $A(6, 3), B(-3, 5), C(4, -2)$ and $D(x, 3x)$ are given in such a way that $\frac{\Delta DBC}{\Delta ABC} = \frac{1}{2}$, find x.

28. If three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ lie on the same line, prove that

$$\frac{y_2 - y_3}{x_2 x_3} + \frac{y_3 - y_1}{x_3 x_1} + \frac{y_1 - y_2}{x_1 x_2} = 0.$$

29. Find the area of a parallelogram $ABCD$ if three of its vertices are $A(2, 4)$, $B(2 + \sqrt{3}, 5)$ and $C(2, 6)$. **[CBSE 2013]**

30. Find the value (s) of k for which the points $(3k - 1, k - 2), (k, k - 7)$ and $(k - 1, -k - 2)$ are collinear. **[CBSE 2014]**

31. If the points $A(-1, -4)$, $B(b, c)$ and $C(5, -1)$ are collinear and $2b + c = 4$, find the values of b and c. **[CBSE 2014]**

32. If the points $A(-2, 1)$, $B(a, b)$ and $C(4, -1)$ are collinear and $a - b = 1$, find the values of a and b. **[CBSE 2014]**

33. If the points $A(1, -2)$, $B(2, 3)$, $C(a, 2)$ and $D(-4, -3)$ form a parallelogram, find the value of a and height of the parallelogram taking AB as base. **[NCERT EXEMPLAR]**

34. $A(6, 1)$, $B(8, 2)$ and $C(9, 4)$ are three vertices of a parallelogram $ABCD$. If E is the mid-point of DC, find the area of ΔADE. **[NCERT EXEMPLAR]**

35. If $D(-1/5, 5/2)$, $E(7, 3)$ and $F(7/2, 7/2)$ are the mid-points of sides of ΔABC, find the area of ΔABC. **[NCERT EXEMPLAR]**

ANSWERS

1. (i) $\frac{49}{2}$ sq. units (ii) $a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$ (iii) a^2
2. (i) 80 sq. units (ii) $\frac{11}{2}$ sq. units (iii) 28 sq. units
3. $k = 3$
4. 20 sq. units. $\frac{40}{\sqrt{58}}$
6. 85 sq. units
7. 4 sq. units, 1 sq. unit
8. 12 sq. units
9. 13 sq. units
10. 72 sq. units
11. $a = 5$
13. $p = -3, -9$
15. $k = \frac{-3}{2}$
16. $k = 2$
17. $m = -4$
19. $k = -5$
20. $a = 0, \frac{8}{3}$
22. $y = \frac{13}{2}, -\frac{27}{2}$
24. $k = 7$ or $\frac{31}{9}$
25. $(7/2, 13/2)$ or $(-3/2, 3/2)$
27. $\frac{11}{8}, \frac{-3}{8}$
29. $2\sqrt{3}$ sq. units.
30. $k = 0, 3$
31. $b = 3, c = -2$
32. $a = 1, b = 0$
33. $a = -3, h = \frac{12\sqrt{2}}{\sqrt{13}}$
34. $\frac{3}{4}$ sq. units
35. 11 sq. units

HINTS TO SELECTED PROBLEMS

1. (ii) Let Δ be the area of the triangle formed by the points $(at_1^2, 2at_1), (at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$. Then,

We have,

$$\begin{matrix} at_1^2 & & at_2^2 & & at_3^2 & & at_1^2 \\ & \times & & \times & & \times & \\ 2at_1 & & 2at_2 & & 2at_3 & & 2at_1 \end{matrix}$$

$\therefore \quad \Delta=\frac{1}{2}\left|(at_1^2 \times 2at_2+at_2^2 \times 2at_3+at_3^2 \times 2at_1)-(at_2^2 \times 2at_1+at_3^2 \times 2at_2+at_1^2 \times 2at_3\right.$

$\Rightarrow \quad \Delta=\frac{1}{2}\left|(2a^2 t_1^2 t_2+2a^2 t_2^2 t_3+2a^2 t_3^2 t_1)-(2a^2 t_1 t_2^2+2a^2 t_2 t_3^2+2a^2 t_1\right.$

$\Rightarrow \quad \Delta=a^2\left|(t_1^2 t_2+t_2^2 t_3+t_3^2 t_1)-(t_1 t_2^2+t_2 t_3^2+t_3 t_1^2)\right|$

$\Rightarrow \quad \Delta=a^2\left|(t_1^2 t_2-t_1^2 t_3)+(t_2^2 t_3-t_2 t_3^2)+(t_3^2 t_1-t_1 t_2^2)\right|$

$\Rightarrow \quad \Delta=a^2\left|t_1^2(t_2-t_3)+t_2 t_3(t_2-t_3)-t_1(t_2^2-t_3^3)\right|$

$\Rightarrow \quad \Delta=a^2\left|(t_2-t_3)\{t_1^2+t_2 t_3-t_1(t_2+t_3)\}\right|$

$\Rightarrow \quad \Delta=a^2\left|(t_2-t_3)\{t_1^2+t_2 t_3-t_1 t_2-t_1 t_3\}\right|$

$\Rightarrow \quad \Delta=a^2\left|(t_2-t_3)\{(t_1^2-t_1 t_2)-(t_1 t_3-t_2 t_3)\}\right|$

$\Rightarrow \quad \Delta=a^2\left|(t_2-t_3)\{t_1(t_1-t_2)-t_3(t_1-t_2)\}\right|$

$\Rightarrow \quad \Delta=a^2\left|(t_2-t_3)(t_1-t_2)(t_1-t_3)\right|$

$\Rightarrow \quad \Delta=a^2\left|(t_1-t_2)(t_2-t_3)(t_3-t_1)\right|$

(iii) Let Δ be the area of the triangle formed by the points $(a, c+a)$, (a, c) and $(-a$
We have,

$$\begin{matrix} a & & a & & -a & & a \\ c+a & & c & & c-a & & c+a \end{matrix}$$

$\therefore \quad \Delta=\frac{1}{2}\left|\{ac+a(c-a)-a(c+a)\}-\{a(c+a)-ac+a(c-a)\}\right|$

$\Rightarrow \quad \Delta=\frac{1}{2}\left|\{ac+ac-a^2-ac-a^2\}-\{ac+a^2-ac+ac-a^2\}\right|$

$\Rightarrow \quad \Delta=\frac{1}{2}\left|(ac-2a^2)-(ac)\right|$

$\Rightarrow \quad \Delta=\frac{1}{2}\left|-2a^2\right|=a^2$ sq. units

2. (i) Let $A(-3,2)$, $B(5,4)$, $C(7,-6)$ and $D(-5,-4)$ be the vertices of the quadrila
We have,

$$\begin{matrix} -3 & & 5 & & 7 & & -3 \\ 2 & & 4 & & -6 & & 2 \end{matrix}$$

$\therefore \quad$ Area of $\Delta ABC=\frac{1}{2}|(-12-30+14)-(10+28+18)|$

$\Rightarrow \quad$ Area of $\Delta ABC=\frac{1}{2}|-28-56|=42$ sq. units

We have,

$$\begin{matrix} -3 & & 7 & & -5 & & -3 \\ 2 & & -6 & & -4 & & 2 \end{matrix}$$

$\therefore$ Area of $\Delta\, ACD = \frac{1}{2}|(18 - 28 - 10) - (14 + 30 + 12)|$

$\Rightarrow$ Area of $\Delta\, ACD = \frac{1}{2}|-20 - 56| = 38$ sq. units

$\therefore$ Area of quadrilateral $ABCD = (42 + 38) = 80$ sq. units

(ii) Let A (1, 2), B (6, 2), C (5, 3) and D (3, 4) be the vertices of the given quadrilateral. We have,

$$\begin{matrix} 1 & 6 & 5 & 1 \\ 2 & 2 & 3 & 2 \end{matrix}$$

$\therefore$ Area of $\Delta\, ABC = \frac{1}{2}|(2 + 18 + 10) - (12 + 10 + 3)|$

$\Rightarrow$ Area of $\Delta\, ABC = \frac{1}{2}|30 - 25| = \frac{5}{2}$ sq. units

Also, we have

$$\begin{matrix} 1 & 5 & 3 & 1 \\ 2 & 3 & 4 & 2 \end{matrix}$$

$\therefore$ Area of $\Delta\, ACD = \frac{1}{2}|(3 + 20 + 6) - (10 + 9 + 4)| = 3$ sq. units

$\therefore$ Area of quadrilateral $ABCD = \left(\frac{5}{2} + 3\right)$ sq. units $= \frac{11}{2}$ sq. units

3. Let $A\,(1, 2)$, $B\,(-5, 6)$, $C\,(7, -4)$ and $D\,(k, -2)$ be the vertices of the quadrilateral. We have,

$$\begin{matrix} 1 & -5 & 7 & 1 \\ 2 & 6 & -4 & 2 \end{matrix}$$

$\therefore$ Area of $\Delta\, ABC = \frac{1}{2}|\{(6 + 20 + 14) - (-10 + 42 - 4)\}|$

$\Rightarrow$ Area of $\Delta\, ABC = \frac{1}{2}(40 - 28) = 6$ sq. units

Also, we have

$$\begin{matrix} 1 & 7 & k & 1 \\ 2 & -4 & -2 & 2 \end{matrix}$$

Area of $\Delta\, ACD = \frac{1}{2}\{(-4 - 14 + 2k) - (14 - 4k - 2)\}$

$\Rightarrow$ Area of $\Delta\, ACD = \frac{1}{2}\{(2k - 18) - (12 - 4k)\}$

$\Rightarrow$ Area of $\Delta\, ACD = \frac{1}{2}(6k - 30) = (3k - 15)$

$\therefore$ Area of quadrilateral $ABCD = 6 + 3k - 15 = 3k - 9$

It is given that the area of quadrilateral is zero.

$\therefore$ $3k - 9 = 0 \Rightarrow k = 3$

4. We have,

$$\begin{matrix} -2 & 5 & 2 & -2 \\ 1 & 4 & -3 & 1 \end{matrix}$$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}\left|(-8-15+2)-(5+8+6)\right| = 20$ sq. units

We have, $BC = \sqrt{(5-2)^2+(4+3)^2} = \sqrt{58}$

Now, Area of $\Delta ABC = \frac{1}{2} BC \times$ (Length of the altitude through A)

$$\Rightarrow \quad 20 = \frac{1}{2} \times \sqrt{58} \times \text{Length of the altitude through } A$$

$$\therefore \quad \text{Length of the altitude through} = \frac{40}{\sqrt{58}}$$

11. Points $(a, 1), (1, -1)$ and $(11, 4)$ will be collinear, if

$$\begin{matrix} a & & 1 & & 11 & & a \\ 1 & & -1 & & 4 & & 1 \end{matrix}$$

$$(a \times -1 + 1 \times 4 + 11 \times 1) - (1 \times 1 + 11 \times -1 + a \times 4) = 0$$

$\Rightarrow \quad (-a+15) - (1-11+4a) = 0$

$\Rightarrow \quad -a+15+10-4a = 0$

$\Rightarrow \quad -5a+25 = 0$

$\Rightarrow \quad a = 5$

12. Points $(a,b), (a_1, b_1)$ and $(a-a_1, b-b_1)$ will be collinear, if

$$\begin{matrix} a & & a_1 & & a-a_1 & & a \\ b & & b_1 & & b-b_1 & & b \end{matrix}$$

$$\{ab_1 + a_1(b-b_1) + (a-a_1)b\} - \{a_1b + (a-a_1)b_1 + a(b-b_1)\} = 0$$

$\Rightarrow \quad (ab_1 + a_1b - a_1b_1 + ab - a_1b) - (a_1b + ab_1 - a_1b_1 + ab - ab_1) = 0$

$\Rightarrow \quad (ab_1 - a_1b_1 + ab) - (a_1b - a_1b_1 + ab) = 0$

$\Rightarrow \quad ab_1 - a_1b = 0$

$\Rightarrow \quad ab_1 = a_1b$

23. Points $(a, 0), (0, b)$ and $(1, 1)$ are collinear.

$$\begin{matrix} a & & 0 & & 1 & & a \\ 0 & & b & & 1 & & 0 \end{matrix}$$

$\therefore \quad (ab + 0 \times 1 + 1 \times 0) - (0 \times 0 + 1 \times b + a \times 1) = 0$

$\Rightarrow \quad ab - a - b = 0 \Rightarrow ab = a + b \Rightarrow 1 = \frac{a}{ab} + \frac{b}{ab} \Rightarrow \frac{1}{a} + \frac{1}{b} = 1$

24. It is given that the point A divides the join of $P(-5, 1)$ and $Q(3, 5)$ $k : 1$. So, the coordinates of A are

$$\left(\frac{3k-5}{k+1}, \frac{5k+1}{k+1}\right)$$

We have,

$$\begin{matrix} \frac{3k-5}{k+1} & & 1 & & 7 & & \frac{3k-5}{k+1} \\ & \times & & \times & & \times & \\ \frac{5k+1}{k+1} & & 5 & & -2 & & \frac{5k+1}{k+1} \end{matrix}$$

$\therefore$ Area of $\Delta ABC = 2$ sq. units

$$\Rightarrow \quad \frac{1}{2}\left|\left\{\frac{3k-5}{k+1}\times 5 - 2 + 7\times\frac{5k+1}{k+1}\right\} - \left\{\frac{5k+1}{k+1}\times 1 + 35 - 2\times\frac{3k-5}{k+1}\right\}\right| = 2$$

$$\Rightarrow \quad \frac{1}{2}\left|\left(\frac{15k-25}{k+1} - 2 + \frac{35k+7}{k+1}\right) - \left(\frac{5k+1}{k+1} + 35 - \frac{6k-10}{k+1}\right)\right| = 2$$

$$\Rightarrow \quad \frac{1}{2}\left|\frac{(15k-25-2k-2+35k+7)-(5k+1+35k+35-6k+10)}{k+1}\right| = 2$$

$$\Rightarrow \quad \frac{1}{2}\left|\frac{(48k-20)-(34k+46)}{k+1}\right| = 2$$

$$\Rightarrow \quad \frac{1}{2}\left|\frac{14k-66}{k+1}\right| = 2$$

$$\Rightarrow \quad \left|\frac{7k-33}{k+1}\right| = 2$$

$$\Rightarrow \quad \frac{7k-33}{k+1} = \pm 2$$

$$\Rightarrow \quad 7k - 33 = \pm 2(k+1)$$

$$\Rightarrow \quad 7k - 33 = 2k + 2, 7k - 33 = -2k - 2$$

$$\Rightarrow \quad 5k = 35, \; 9k = 31$$

$$\Rightarrow \quad k = 7, k = \frac{31}{9}$$

25. Let the third vertex be $A(x, y)$. Other two vertices of the triangle are $B(2, 1)$ and $C(3, -2)$.

We have,

$$\begin{matrix} x & & 2 & & 3 & & x \\ & \times & & \times & & \times & \\ y & & 1 & & -2 & & y \end{matrix}$$

$\therefore$ Area of $\Delta ABC = 5$ sq. units

$$\Rightarrow \quad \frac{1}{2}|(x-4+3y)-(2y+3-2x)| = 5$$

$$\Rightarrow \quad \frac{1}{2}|x-4+3y-2y-3+2x)| = 5$$

$$\Rightarrow \quad \frac{1}{2}|3x+y-7| = 5$$

$$\Rightarrow \quad 3x + y - 7 = \pm 10$$

$$\Rightarrow \quad 3x + y - 17 = 0 \text{ or, } 3x + y + 3 = 0$$

It is given that the vertex $A(x, y)$ lies on $y = x + 3$.

Solving $3x + y - 17 = 0$ and $y = x + 3$, we get $x = \frac{7}{2}$ and $y = \frac{13}{2}$

Solving $3x + y + 3 = 0$ and $y = x + 3$, we get $x = \frac{-3}{2}$ and $y = \frac{3}{2}$

Hence, the coordinates of the third vertex are $\left(\frac{7}{2}, \frac{13}{2}\right)$ or, $\left(\frac{-3}{2}, \frac{3}{2}\right)$.

26. Let Δ be the area of the triangle formed by the points (a, a^2), (b, b^2) and (c, c^2)

$$\begin{matrix} a & & b & & c & & a \\ & \times & & \times & & \times & \\ a^2 & & b^2 & & c^2 & & a^2 \end{matrix}$$

$\therefore \quad \Delta = \frac{1}{2}\left|(ab^2 + bc^2 + ca^2) - (a^2b + b^2c + c^2a)\right|$

$\Rightarrow \quad \Delta = \frac{1}{2}\left|(a^2c - a^2b) + (ab^2 - ac^2) + (bc^2 - b^2c)\right|$

$\Rightarrow \quad \Delta = \frac{1}{2}\left|-a^2(b - c) + a(b^2 - c^2) - bc(b - c)\right|$

$\Rightarrow \quad \Delta = \frac{1}{2}\left|(b - c)\left\{-a^2 + a(b + c) - bc\right\}\right|$

$\Rightarrow \quad \Delta = \frac{1}{2}\left|(b - c)\left(-a^2 + ab + ac - bc\right)\right|$

$\Rightarrow \quad \Delta = \frac{1}{2}\left|(b - c)\left\{-a(a - b) + c(a - b)\right\}\right|$

$\Rightarrow \quad \Delta = \frac{1}{2}\left|(b - c)(a - b)(c - a)\right|$

It is given that $a \neq b \neq c$.

$\therefore \quad \Delta \neq 0$

Hence, given points are never collinear.

27. We have,

$$\begin{matrix} x & & -3 & & 4 & & x \\ & \times & & \times & & \times & \\ 3x & & 5 & & -2 & & 3x \end{matrix}$$

$\therefore \quad$ Area of $\Delta DBC = \frac{1}{2}\left|(5x + 6 + 12x) - (-9x + 20 - 2x)\right|$

$\Rightarrow \quad$ Area of $\Delta DBC = \frac{1}{2}\left|(28x - 14)\right| = |14x - 7| = 7|2x - 1|$

Also, we have

$$\begin{matrix} 6 & & -3 & & 4 & & 6 \\ & \times & & \times & & \times & \\ 3 & & 5 & & -2 & & 3 \end{matrix}$$

$\therefore \quad$ Area of $\Delta ABC = \frac{1}{2}\left|(30 + 6 + 12) - (-9 + 20 - 12)\right|$

$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}|48+1| = \frac{49}{2}$

Now,

$$\frac{\text{Area of } \Delta DBC}{\text{Area of } \Delta ABC} = \frac{1}{2}$$

$\Rightarrow \quad \dfrac{7|2x-1|}{\dfrac{49}{2}} = \dfrac{1}{2}$

$\Rightarrow \quad |2x-1| = \dfrac{7}{4}$

$\Rightarrow \quad 2x - 1 = \pm\dfrac{7}{4}$

$\Rightarrow \quad 2x = \dfrac{11}{4} \text{ or, } 2x = -\dfrac{3}{4}$

$\Rightarrow \quad x = \dfrac{11}{8} \text{ or } x = -\dfrac{3}{8}$

28. Points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear.

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_1 \\ y & y_2 & y_3 & y_1 \end{array}$$

$\therefore \quad (x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3) = 0$

$\Rightarrow \quad (x_1y_2 - x_1y_3) + (x_2y_3 - x_2y_1) + (x_3y_1 - x_3y_2) = 0$

$\Rightarrow \quad x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$

$\Rightarrow \quad \dfrac{x_1(y_2 - y_3)}{x_1x_2x_3} + \dfrac{x_2(y_3 - y_1)}{x_1x_2x_3} + \dfrac{x_3(y_1 - y_2)}{x_1x_2x_3} = 0$

$\Rightarrow \quad \dfrac{y_2 - y_3}{x_2x_3} + \dfrac{y_3 - y_1}{x_1x_3} + \dfrac{y_1 - y_2}{x_1x_2} = 0$

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

Answer each of the following questions either in one word or one sentence or as per requirement of the questions:

1. Write the distance between the points $A(10\cos\theta, 0)$ and $B(0, 10\sin\theta)$.
2. Write the perimeter of the triangle formed by the points $O(0, 0)$, $A(a, 0)$ and $B(0, b)$.
3. Write the ratio in which the line segment joining points (2, 3) and (3, –2) is divided by X axis.
4. What is the distance between the points $(5\sin 60°, 0)$ and $(0, 5\sin 30°)$?
5. If $A(-1, 3)$, $B(1, -1)$ and $C(5, 1)$ are the vertices of a triangle ABC, what is the length of the median through vertex A?
6. If the distance between points $(x, 0)$ and $(0, 3)$ is 5, what are the values of x?
7. What is the area of the triangle formed by the points $O(0, 0)$, $A(6, 0)$ and $B(0, 4)$?
8. Write the coordinates of the point dividing line segment joining points (2, 3) and (3, 4) internally in the ratio 1 : 5.

9. If the centroid of the triangle formed by points $P(a, b)$, $Q(b, c)$ and $R(c, a)$ is at the origin, what is the value of $a + b + c$?

10. In Q. No. 9, what is the value of $\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab}$?

11. Write the coordinates of a point on X-axis which is equidistant from the points $(-3, 4)$ and $(2, 5)$.

12. If the mid-point of the segment joining $A(x, y+1)$ and $B(x+1, y+2)$ is $C(3/2, 5/2)$, find x, y.

13. Two vertices of a triangle have coordinates $(-8, 7)$ and $(9, 4)$. If the centroid of the triangle is at the origin, what are the coordinates of the third vertex?

14. Write the coordinates the reflections of point $(3, 5)$ in X and Y-axes.

15. If points Q and R reflections of point $P(-3, 4)$ in X and Y axes respectively, what is QR?

16. Write the formula for the area of the triangle having its vertices at (x_1, y_1), (x_2, y_2) and (x_3, y_3).

17. Write the condition of collinearity of points (x_1, y_1), (x_2, y_2) and (x_3, y_3).

18. Find the values of x for which the distance between the point $P(2, -3)$ and $Q(x, 5)$ is 10.

19. Write the ratio in which the line segment doining the points $A(3, -6)$ and $B(5, 3)$ is divided by X-axis.

20. Find the distance between the points $(-8/2, 2)$ and $(2/5, 2)$. **[CBSE 2009]**

21. Find the value of a so that the point $(3, a)$ lies on the line represented by $2x - 3y + 5 = 0$. **[CBSE 2009]**

22. What is the distance between the points $A(c, 0)$ and $B(0, -c)$? **[CBSE 2010]**

23. If $P(2, 6)$ is the mid-point of the line segment joining $A(6, 5)$ and $B(4, y)$, find y. **[CBSE 2010]**

24. If the distance between the points $(3, 0)$ and $(0, y)$ is 5 units and y is positive, then what is the value of y? **[CBSE 2010]**

25. If $P(x, 6)$ is the mid-point of the line segment joining $A(6, 5)$ and $B(4, y)$, find y. **[CBSE 2010]**

26. If $P(2, p)$ is the mid-point of the line segment joining the points $A(6, -5)$ and $B(-2, 11)$, find the value of p. **[CBSE 2010]**

27. If $A(1, 2)$, $B(4, 3)$ and $C(6, 6)$ are the three vertices of a parallelogram $ABCD$, find the coordinates of fourth vertex D. **[CBSE 2010]**

28. What is the distance between the points $A(\sin\theta - \cos\theta, 0)$ and $B(0, \sin\theta + \cos\theta)$? **[CBSE 2015]**

29. What are the coordinates of the point where the perpendicular bisector of the line segment joining the points $A(1, 5)$, and $B(4, 6)$ cuts the y-axis?

30. Find the area of the triangle with vertices $(a, b+c)$, $(b, c+a)$ and $(c, a+b)$.

31. If the points $A(1, 2)$, $O(0, 0)$, and $C(a, b)$ are collinear, then find $a : b$.

32. Find the coordinates of the point which is equidistant from the three vertices $A(2x, 0)$, $O(0, 0)$ and $B(0, 2y)$ of ΔAOB.

33. If the distance between the points $(4, k)$, and $(1, 0)$ is 5, then what can be the possible value of k? **[CBSE 2017]**

34. Find the distance of a point $P(x, y)$ from the origin. **[CBSE 2018]**

ANSWERS

1. 10 **2.** $\frac{1}{2}ab$ **3.** 3 : 2 **4.** 5 **5.** 5 **6.** ± 4

7. 12 sq. units **8.** (13/6, 19/6) **9.** 0 **10.** 3

11. (2/5, 0) **12.** (1, 1) **13.** (− 1, − 11) **14.** (3, −5), (−3, 5) **15.** 10

16. $\frac{1}{2}\left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right|$

17. $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$ **18.** 8, − 4 **19.** 2 : 1

20. 2 **21.** $\frac{1}{3}$ **22.** $\sqrt{2}c$ **23.** 7 **24.** 4 **25.** 7

26. 3 **27.** (3, 6) **28.** $\sqrt{2}$ **29.** (0, 13) **30.** 0 **31.** 1 : 2

32. (x, y) **33.** ± 4 **34.** $\sqrt{x^2 + y^2}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The distance between the points $(\cos\theta, \sin\theta)$ and $(\sin\theta - \cos\theta)$ is

(a) $\sqrt{3}$ (b) $\sqrt{2}$ (c) 2 (d) 1

2. The distance between the points $(a\cos 25°, 0)$ and $(0, a\cos 65°)$ is

(a) a (b) $2a$ (c) $3a$ (d) None of these

3. If x is a positive integer such that the distance between points $P(x, 2)$ and $Q(3, -6)$ is 10 units, then $x =$

(a) 3 (b) − 3 (c) 9 (d) − 9

4. The distance between the points $(a\cos\theta + b\sin\theta, 0)$ and $(0, a\sin\theta - b\cos\theta)$ is

(a) $a^2 + b^2$ (b) $a + b$ (c) $a^2 - b^2$ (d) $\sqrt{a^2 + b^2}$

5. If the distance between the points $(4, p)$ and $(1, 0)$ is 5, then $p =$

(a) ± 4 (b) 4 (c) − 4 (d) 0

6. A line segement is of length 10 units. If the coordinates of its one end are (2, − 3) and the abscissa of the other end is 10, then its ordinate is

(a) 9, 6 (b) 3, − 9 (c) − 3, 9 (d) 9, − 6

7. The perimeter of the triangle formed by the points (0, 0), (1, 0) and (0, 1) is

(a) $1 \pm \sqrt{2}$ (b) $\sqrt{2} + 1$ (c) 3 (d) $2 + \sqrt{2}$

8. If $A(2, 2)$, $B(-4, -4)$ and $C(5, -8)$ are the vertices of a triangle, then the length of the median through vertex C is

(a) $\sqrt{65}$ (b) $\sqrt{117}$ (c) $\sqrt{85}$ (d) $\sqrt{113}$

9. If three points $(0, 0)$, $(3, \sqrt{3})$ and $(3, \lambda)$ form an equilateral triangle, then $\lambda =$

(a) 2 (b) − 3 (c) − 4 (d) None of these

10. If the points $(k, 2k)$, $(3k, 3k)$ and $(3, 1)$ are collinear, then k

(a) $\frac{1}{3}$ (b) $-\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

11. The coordinates of the point on X-axis which are equidistant from the poin $(-3, 4)$ and $(2, 5)$ are
(a) $(20, 0)$ (b) $(-23, 0)$ (c) $(4/5, 0)$ (d) None of these

12. If $(-1, 2)$, $(2, -1)$ and $(3, 1)$ are any three vertices of a parallelogram, then
(a) $a = 2, b = 0$ (b) $a = -2, b = 0$
(c) $a = -2, b = 6$ (d) $a = 6, b = 2$

13. If $A(5, 3)$, $B(11, -5)$ and $P(12, y)$ are the vertices of a right triangle right angled at P, the $y =$
(a) $-2, 4$ (b) $-2, 4$ (c) $2, -4$ (d) $2, 4$

14. The area of the triangle formed by $(a, b+c)$, $(b, c+a)$ and $(c, a+b)$ is
(a) $a+b+c$ (b) abc (c) $(a+b+c)^2$ (d) 0

15. If $(x, 2)$, $(-3, -4)$ and $(7, -5)$ are collinear, then $x =$
(a) 60 (b) 63 (c) -63 (d) -60

16. If points $(t, 2t)$, $(-2, 6)$ and $(3, 1)$ are collinear, then $t =$
(a) $\frac{3}{4}$ (b) $\frac{4}{3}$ (c) $\frac{5}{3}$ (d) $\frac{3}{5}$

17. If the area of the triangle formed by the points $(x, 2x)$, $(-2, 6)$ and $(3, 1)$ is 5 square units, then $x =$
(a) $2/3$ (b) $3/5$ (c) 3 (d) 5

18. If points $(a, 0)$, $(0, b)$ and $(1, 1)$ are collinear, then $\frac{1}{a} + \frac{1}{b} =$
(a) 1 (b) 2 (c) 0 (d) -1

19. If the centroid of a triangle is $(1, 4)$ and two of its vertices are $(4, -3)$ and $(-9, 7)$, then the area of the triangle is
(a) 183 sq. units (b) $\frac{183}{2}$ sq. units (c) 366 sq. units (d) $\frac{183}{4}$ sq. units

20. The line segment joining points $(-3, -4)$, and $(1, -2)$ is divided by y-axis in the ratio
(a) $1:3$ (b) $2:3$ (c) $3:1$ (d) $2:3$

21. The ratio in which $(4, 5)$ divides the join of $(2, 3)$ and $(7, 8)$ is
(a) $-2:3$ (b) $-3:2$ (c) $3:2$ (d) $2:3$

22. The ratio in which the x-axis divides the segment joining $(3, 6)$ and $(12, -3)$ is
(a) $2:1$ (b) $1:2$ (c) $-2:1$ (d) $1:-2$

23. If the centroid of the triangle formed by the points (a, b), (b, c) and (c, a) is at the origin, then $a^3 + b^3 + c^3 =$
(a) abc (b) 0 (c) $a+b+c$ (d) $3\,abc$

24. If points $(1, 2)$, $(-5, 6)$ and $(a, -2)$ are collinear, then $a =$
(a) -3 (b) 7 (c) 2 (d) -2

25. If the centroid of the triangle formed by $(7, x)$, $(y, -6)$ and $(9, 10)$ is at $(6, 3)$, then $(x, y) =$
(a) $(4, 5)$ (b) $(5, 4)$ (c) $(-5, -2)$ (d) $(5, 2)$

26. The distance of the point (4, 7) from the x-axis is

(a) 4 (b) 7 (c) 11 (d) $\sqrt{65}$

27. The distance of the point (4, 7) from the y-axis is

(a) 4 (b) 7 (c) 11 (d) $\sqrt{65}$

28. If P is a point on x-axis such that its distance from the origin is 3 units, then the coordinates of a point Q on OY such that $OP = OQ$, are

(a) (0, 3) (b) (3, 0) (c) (0, 0) (d) (0, – 3)

29. If the point $(x, 4)$ lies on a circle whose centre is at the origin and radius is 5, then $x =$

(a) ± 5 (b) ± 3 (c) 0 (d) ± 4

30. If the point $P(x, y)$ is equidistant from $A(5, 1)$ and $B(-1, 5)$, then

(a) $5x = y$ (b) $x = 5y$ (c) $3x = 2y$ (d) $2x = 3y$

31. If points $A(5, p)$, $B(1, 5)$, $C(2, 1)$ and $D(6, 2)$ form a square $ABCD$, then $p =$

(a) 7 (b) 3 (c) 6 (d) 8

32. The coordinates of the circumcentre of the triangle formed by the points $O(0, 0)$, $A(a, 0)$ and $B(0, b)$ are

(a) (a, b) (b) $(a/2, b/2)$ (c) $(b/2, a/2)$ (d) (b, a)

33. The coordinates of a point on x-axis which lies on the perpendicular bisector of the line segment joining the points (7, 6) and (–3, 4) are

(a) (0, 2) (b) (3, 0) (c) (0, 3) (d) (2, 0)

34. If the centroid of the triangle formed by the points $(3, -5)$, $(-7, 4)$, $(10, -k)$ is at the point $(k, -1)$, then $k =$

(a) 3 (b) 1 (c) 2 (d) 4

35. If (–2, 1) is the centroid of the triangle having its vertices at $(x, 2)$, $(10, -2)$, $(-8, y)$, then x, y satisfy the relation

(a) $3x + 8y = 0$ (b) $3x - 8y = 0$ (c) $8x + 3y = 0$ (d) $8x = 3y$

36. The coordinates of the fourth vertex of the rectangle formed by the points (0, 0), (2, 0), (0, 3) are

(a) (3, 0) (b) (0, 2) (c) (2, 3) (d) (3, 2)

37. The length of a line segment joining $A(2, -3)$ and B is 10 units. If the abscissa of B is 10 units, then its ordinates can be

(a) 3 or –9 (b) –3 or 9 (c) 6 or 27 (d) –6 or –27

38. The ratio in which the line segment joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ is divided by x-axis is

(a) $y_1 : y_2$ (b) $-y_1 : y_2$ (c) $x_1 : x_2$ (d) $-x_1 : x_2$

39. The ratio in which the line segment joining points $A(a_1, b_1)$ and $B(a_2, b_2)$ is divided by y-axis is

(a) $-a_1 : a_2$ (b) $a_1 : a_2$ (c) $b_1 : b_2$ (d) $-b_1 : b_2$

40. If the line segment joining the points (3, –4), and (1, 2) is trisected at points $P(a, -2)$ ar $Q\left(\frac{5}{3}, b\right)$. Then,

(a) $a = \frac{8}{3}, b = \frac{2}{3}$ (b) $a = \frac{7}{3}, b = 0$ (c) $a = \frac{1}{3}, b = 1$ (d) $a = \frac{2}{3}, b = \frac{1}{3}$

41. If the coordinates of one end of a diameter of a circle are (2, 3) and the coordinates of i centre are (– 2, 5), then the coordinates of the other end of the diameter are

(a) (–6, 7) (b) (6, –7) (c) (6, 7) (d) (– 6, – 7) **[CBSE 2012**

42. The coordinates of the point P dividing the line segment joining the points $A(1, 3)$ an $B(4, 6)$ in the ratio 2 : 1 are

(a) (2, 4) (b) (3, 5) (c) (4, 2) (d) (5, 3) **[CBSE 2012**

43. In Fig. 6.52, the area of ΔABC (in square units) is

(a) 15 (b) 10 (c) 7.5 (d) 2.5 **[CBSE 2013**

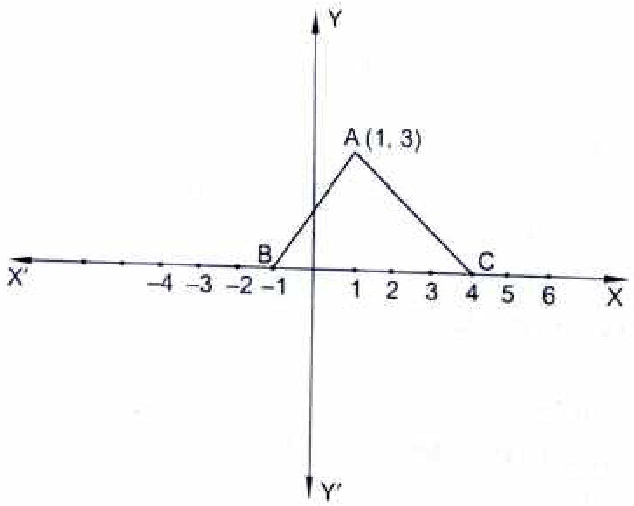


Fig. 6.52

44. The point on the x-axis which is equidistant from points (–1, 0) and (5, 0) is

(a) (0, 2) (b) (2, 0) (c) (3, 0) (d) (0, 3) **[CBSE 2013]**

45. If $A(4, 9)$, $B(2, 3)$ and $C(6, 5)$ are the vertices of ΔABC, then the length of median through C is

(a) 5 units (b) $\sqrt{10}$ units (c) 25 units (d) 10 units **[CBSE 2014]**

46. If $P(2, 4)$, $Q(0, 3)$, $R(3, 6)$ and $S(5, y)$ are the vertices of a parallelogram $PQRS$, then the value of y is

(a) 7 (b) 5 (c) – 7 (d) – 8 **[CBSE 2014]**

47. If $A(x, 2)$, $B(-3, -4)$ and $C(7, -5)$ are collinear, then the value of x is

(a) – 63 (b) 63 (c) 60 (d) – 60 **[CBSE 2014]**

48. The perimeter of a triangle with vertices (0, 4) and (0, 0) and (3, 0) is

(a) $7 + \sqrt{5}$ (b) 5 (c) 10 (d) 12 **[CBSE 2014]**

49. If the point P (2, 1) lies on the line segment joining points A (4, 2) and B (8, 4), then

(a) $AP = \frac{1}{3}AB$ (b) $AP = BP$ (c) $PB = \frac{1}{3}AB$ (d) $AP = \frac{1}{2}AB$

50. A line intersects the y-axis and x-axis at P and Q, respectively. If (2, –5) is the mid-point of PQ, then the coordinates of P and Q are, respectively

(a) (0, –5) and (2, 0) (b) (0, 10) and (–4, 0)
(c) (0, 4) and (–10, 0) (d) (0, –10) and (4, 0)

ANSWERS

1. (b)	2. (a)	3. (c)	4. (d)	5. (a)	6. (b)
7. (d)	8. (c)	9. (d)	10. (b)	11. (d)	12. (c)
13. (c)	14. (d)	15. (c)	16. (b)	17. (a)	18. (a)
19. (b)	20. (c)	21. (d)	22. (a)	23. (d)	24. (b)
25. (d)	26. (b)	27. (a)	28. (a)	29. (b)	30. (d)
31. (c)	32. (b)	33. (b)	34. (c)	35. (a)	36. (c)
37. (a)	38. (b)	39. (a)	40. (b)	41. (a)	42. (b)
43. (c)	44. (b)	45. (b)	46. (a)	47. (a)	48. (d)
49. (d)	50. (d)				

SUMMARY

1. The abscissa and ordinate of a given point are the distances of the point from y-axis and x-axis respectively.
2. The coordinates of any point on x-axis are of the form $(x, 0)$.
3. The coordinates of any point on y-axis are of the form $(0, y)$.
4. The distance between points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

5. Distance of a point $P(x, y)$ from the origin $O(0, 0)$ is given by

$$OP = \sqrt{x^2 + y^2}$$

6. The coordinates of the point which divides the join of points $P(x_1, y_1)$ and $Q(x_2, y_2)$ internally in the ratio $m : n$ are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right).$$

7. The coordinates of the mid-point of the line segment joining the points $P(x_1, y_1)$ and (x_2, y_2) are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
8. The coordinates of the centroid of triangle formed by the points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$

9. The area of the triangle formed by the points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3,$

$$\frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

or, $\frac{1}{2}\left|(x_1 y_2 + x_2 y_3 \; x_3 y_1) - (x_1 y_3 + x_2 y_1 + x_3 y_2)\right|$

10. If points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear, then

$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$$