CHAPTER 7

TRIANGLES

7.1 CONCEPT OF SIMILARITY

In earlier classes, we have learnt about congruent figures. Two geometric figures having the same shape and size are known as congruent figures. Note that congruent figures are alike in every respect. In this chapter, we shall study about similarity of geometric figures. Geometric figures having the same shape but different sizes are known as similar figures. Two congruent figures are always similar but similar figures need not be congruent as discussed in the following illustrations.

ILLUSTRATION 1 *Any two line segments are always similar but they need not be congruent. They are congruent, if their lengths are equal.*

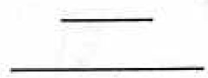


Fig. 7.1

ILLUSTRATION 2 *Any two circles are similar but not necessarily congruent. They are congruent if their radii are equal.*

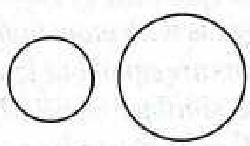


Fig. 7.2

ILLUSTRATION 3 (i) *Any two squares are similar. (See Fig. 7.3(i)).*

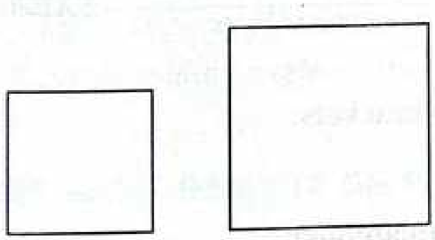


Fig. 7.3(i)

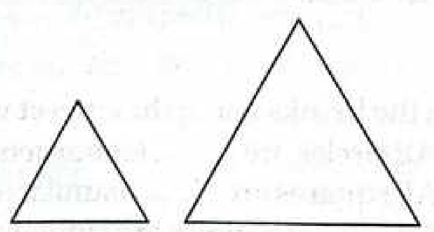


Fig. 7.3(ii)

(ii) *Any two equilateral triangles are similar. (see Fig. 7.3(ii)).*

If two figures are similar one can be obtained from the other either by shrinking or by stretching, without changing its shape. There is one-to-one correspondence between the parts of two similar figures [Fig. 7.3 (ii)].

7.2 SIMILAR POLYGONS

DEFINITION *Two polygons are said to be similar to each other, if*

(i) *their corresponding angles are equal, and*

(ii) *the lengths of their corresponding sides are proportional.*

If two polygons $ABCDE$ and $PQRST$ are similar, then from the above definition it follows that:

Angle at A = Angle at P, Angle at B = Angle at Q, Angle at C = Angle at R,

Angle at D = Angle at S, Angle at E = Angle at T

and, $$\frac{AB}{PQ}=\frac{BC}{QR}=\frac{CD}{RS}=\frac{DE}{ST}=\frac{EA}{TP}$$

If two polygons $ABCDE$ and $PQRST$, are similar, we write $ABCDE \sim PQRST$. Here, the symbol '$\sim$' stands for 'is similar to'.

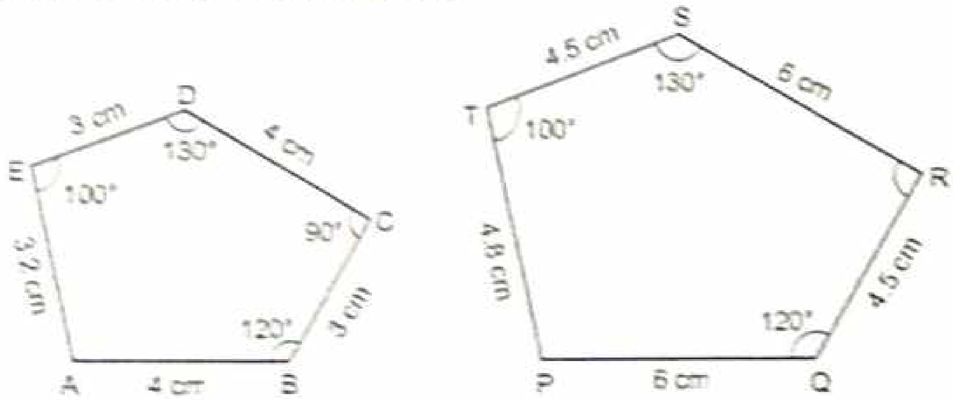


Fig. 7.4

NOTE 1 *It should be noted that for the similarity of polygons with more than three sides, the two conditions given in the definition are independent of each other i. e., either of the two conditions without the other is not sufficient for polygons with more than three sides to be similar. In other words, if the corresponding angles of two polygons are equal but lengths of their corresponding sides are not proportional, the polygons need not be similar. Similarly, if the corresponding angles of two polygons are not equal but length of the the corresponding sides are proportional, then the polygons need not be similar.*

NOTE 2 *Triangles are special type of polygons. In case of triangles, if either of the two conditions given in the above definition holds, then the other holds automatically.*

EXERCISE 7.1

LEVEL-1

1. Fill in the blanks using the correct word given in brackets:

   (i) All circles are ______ (congruent, similar).

   (ii) All squares are ______ (similar, congruent).

   (iii) All ______ triangles are similar (isosceles, equilaterals):

   (iv) Two triangles are similar, if their corresponding angles are ______ (proportional, equal)

   (v) Two triangles are similar, if their corresponding sides are ______ (proportional, equal)

   (vi) Two polygons of the same number of sides are similar, if (a) their corresponding angles are and (b) their corresponding sides are ______ (equal, proportional).

[NCERT]

2. Write the truth value (T/F) of each of the following statements:
   (i) Any two similar figures are congruent.
   (ii) Any two congruent figures are similar.
   (iii) Two polygons are similar, if their corresponding sides are proportional.
   (iv) Two polygons are similar if their corresponding angles are proportional.
   (v) Two triangles are similar if their corresponding sides are proportional.
   (vi) Two triangles are similar if their corresponding angles are equal.

ANSWERS

1. (i) similar	(ii) similar	(iii) equilateral
(iv) equal	(v) proportional	(vi) equal, proportional
2. (i) False	(ii) True	(iii) False
(iv) False	(v) True	(vi) True

7.3 SIMILAR TRIANGLES AND THEIR PROPERTIES

DEFINITION *Two triangles are said to be similar, if their*

(i) *corresponding angles are equal and,*
(ii) *corresponding sides are proportional.*

It follows from this definition that two trianges ABC and DEF are similar, if

(i) $\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$ and, (ii) $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

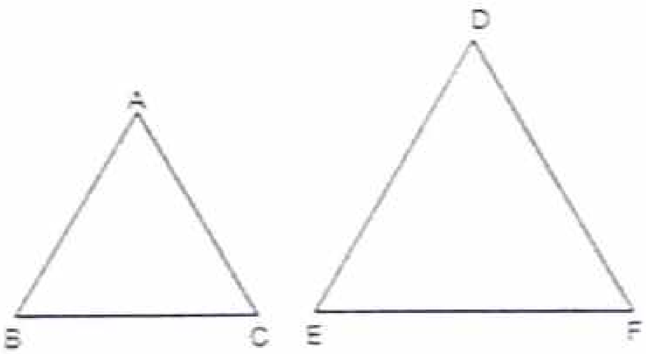


Fig. 7.5

NOTE 1 *In the later part of this chapter we shall show that the two conditions given in the above definition are not independent. In fact, if either of the two conditions holds, then the other holds automatically. So any one of the two conditions can be used to define similar triangles.*

NOTE 2 *If corresponding angles of two triangles are equal, then they are known as equiangular triangles.*

7.4 SOME BASIC RESULTS ON PROPORTIONALITY

In this section, we shall discuss some basic results on proportionality.

Let us first do the following activity.

ACTIVITY Draw any angle $\angle XAY$ and mark points P_1, P_2, D, P_3 and B on its arm AX such that $AP_1 = P_1P_2 = P_2D = DP_3 = P_3B = 1$ unit.

Through point B, draw any line intersecting arm AY at point C. Also, through point D, draw a line parallel to BC to intersect AC at E.

We have,

$$AD = AP_1 + P_1P_2 + P_2D = 3 \text{ units}$$

and, $$DB = DP_3 + P_3B = 2 \text{ units.}$$

$$\therefore \quad \frac{AD}{DB} = \frac{3}{2}$$

Now, measure AE and EC and find $\frac{AE}{EC}$.

You will find that

$$\frac{AE}{EC} = \frac{3}{2}$$

$$\therefore \quad \frac{AD}{DB} = \frac{AE}{EC}$$

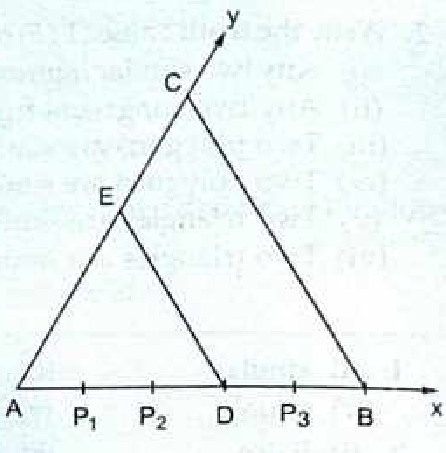


Fig. 7.6

Thus, we observe that in ΔABC if $DE \parallel BC$, then

$$\frac{AD}{DB} = \frac{AE}{EC}$$

We prove this result as a theorem known as basic proportionality theorem or Thale's Theorem as given below.

THEOREM 1 *(Basic proportionality Theorem or Thales Theorem) If a line is drawn parallel to one side of a triangle intersecting the other two sides, then it divides the two sides in the same ratio.*

[NCERT, CBSE 2002 C, 2005, 2006 C, 2007, 2008, 2009, 2010]

GIVEN A triangle ABC in which $DE \parallel BC$, and intersects AB in D and AC in E.

TO PROVE $\frac{AD}{DB} = \frac{AE}{EC}$

CONSTRUCTION Join BE, CD and draw $EF \perp BA$ and $DG \perp CA$.

PROOF Since EF is perpendicular to AB. Therefore, EF is the height of triangles ADE and DBE.

Now, $$\text{Area}(\Delta ADE) = \frac{1}{2}(\text{base} \times \text{height}) = \frac{1}{2}(AD \,.\, EF)$$

and, $$\text{Area}(\Delta DBE) = \frac{1}{2}(\text{base} \times \text{height}) = \frac{1}{2}(DB \,.\, EF)$$

$$\therefore \quad \frac{\text{Area}(\Delta ADE)}{\text{Area}(\Delta DBE)} = \frac{\frac{1}{2}(AD \,.\, EF)}{\frac{1}{2}(DB \,.\, EF)} = \frac{AD}{DB} \quad \text{...(i)}$$

Similarly, we have

$$\frac{\text{Area}(\Delta ADE)}{\text{Area}(\Delta DEC)} = \frac{\frac{1}{2}(AE \,.\, DG)}{\frac{1}{2}(EC \,.\, DG)} = \frac{AE}{EC} \quad \text{...(ii)}$$

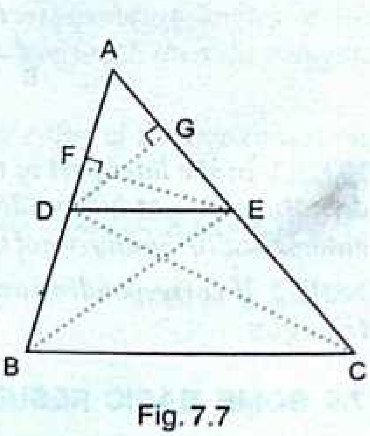


Fig. 7.7

But, ΔDBE and ΔDEC are on the same base DE and between the same parallels DE and BC.

$$\therefore \quad \text{Area}(\Delta DBE) = \text{Area}(\Delta DEC)$$

$$\Rightarrow \quad \frac{1}{\text{Area}(\Delta DBE)} = \frac{1}{\text{Area}(\Delta DEC)} \qquad \text{[Taking reciprocals of both sides]}$$

$\Rightarrow \quad \frac{\text{Area}(\Delta ADE)}{\text{Area}(\Delta DBE)} = \frac{\text{Area}(\Delta ADE)}{\text{Area}(\Delta DEC)}$ [Multiplying both sides by Area (Δ *ADE*)]

$\Rightarrow \quad \frac{AD}{DB} = \frac{AE}{EC}$ [Using (i) and (ii)]

Q.E.D.

COROLLARY *If in a Δ ABC, a line DE || BC, intersects AB in D and AC in E, then:*

(i) $\frac{AB}{AD} = \frac{AC}{AE}$ **[NCERT]** (ii) $\frac{AB}{DB} = \frac{AC}{EC}$

PROOF (i) From the basic proportionality theorem, we have

$$\frac{AD}{DB} = \frac{AE}{EC}$$

$\Rightarrow \quad \frac{DB}{AD} = \frac{EC}{AE}$ [Taking reciprocals of both sides]

$\Rightarrow \quad 1 + \frac{DB}{AD} = 1 + \frac{EC}{AE}$ [Adding 1 on both sides]

$\Rightarrow \quad \frac{AD + DB}{AD} = \frac{AE + EC}{AE}$

$\Rightarrow \quad \frac{AB}{AD} = \frac{AC}{AE}$

(ii) From the basic proportionality theorem, we have

$$\frac{AD}{DB} = \frac{AE}{EC}$$

$\Rightarrow \quad \frac{AD}{DB} + 1 = \frac{AE}{EC} + 1$ [Adding 1 on both sides]

$\Rightarrow \quad \frac{AD + DB}{DB} = \frac{AE + EC}{EC}$

$\Rightarrow \quad \frac{AB}{DB} = \frac{AC}{EC}$

Q.E.D.

The above results can be summarised as follows:

SUMMARY If in a Δ *ABC*, *DE* || *BC*, and intersects *AB* in *D* and *AC* in *E*, then we have

(i) $\frac{AD}{DB} = \frac{AE}{EC}$ (ii) $\frac{DB}{AD} = \frac{EC}{AE}$ (iii) $\frac{AB}{AD} = \frac{AC}{AE}$

(iv) $\frac{AD}{AB} = \frac{AE}{AC}$ (v) $\frac{AB}{DB} = \frac{AC}{EC}$ (vi) $\frac{DB}{AB} = \frac{EC}{AC}$

Let us now perform the following activity.

ACTIVITY Draw any angle *XAY* and mark points B_1, B_2, B_3, B_4 and *B* on its arm *AX* such that $AB_1 = B_1B_2 = B_2B_3 = B_3B_4 = B_4B = 1$ unit. Also, mark points C_1, C_2, C_3, C_4 and C on arm *AY* such that $AC_1 = C_1C_2 = C_2C_3 = C_3C_4 = C_4C = 1$ unit. Join $B_1C_1, B_2C_2, B_3C_3, B_4C_4$ and *BC* as shown in Fig. 7.8.

We observe that

$AB_1 = 1$ unit, $AC_1 = 1$ unit,

$B_1B = 4$ units, $C_1C = 4$ units,

$$\therefore \quad \frac{AB_1}{B_1B} = \frac{AC_1}{C_1C}\left(=\frac{1}{4}\right)$$

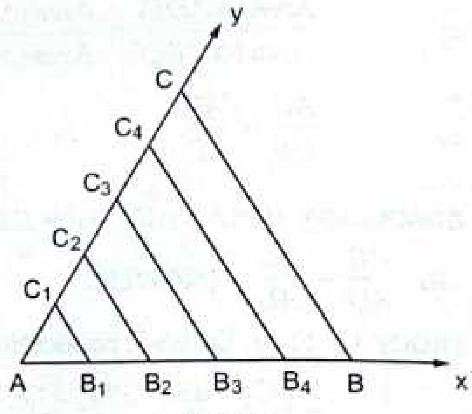


Fig. 7.8

You can also see that B_1C_1 and BC are parallel to each other.

Similarly, we observe that

$$\frac{AB_2}{B_2B} = \frac{AC_2}{C_2C}\left(=\frac{2}{3}\right) \text{ and } B_2C_2 \parallel BC$$

$$\frac{AB_3}{B_3B} = \frac{AC_3}{C_3C}\left(=\frac{3}{2}\right) \text{ and } B_3C_3 \parallel BC$$

$$\frac{AB_4}{B_4B} = \frac{AC_4}{C_4C}\left(=\frac{4}{1}\right) \text{ and } B_4C_4 \parallel BC.$$

It follows from the above activity that if a line divides two sides of a triangle in the same ratio, then it is parallel to the third side of the triangle.

This fact is stated and proved as a theorem given below and it is the converse of the basic proportionality theorem.

THEOREM 2 *(Converse of Basic Proportionality Theorem) If a line divides any two sides of a triangle in the same ratio, then the line must be paralled to the third side.* **[NCERT]**

GIVEN A $\Delta\, ABC$ and a line l intersecting AB in D and AC in E, such that $\frac{AD}{DB} = \frac{AE}{EC}$

TO PROVE $l : BC$ i.e. $DE \parallel BC$

PROOF If possible, let DE be not parallel to BC. Then, there must be another line parallel to BC. Let $DF \parallel BC$.

Since $DF \parallel BC$. Therefore, from Basic Proportionality Theorem, we get

$$\frac{AD}{DB} = \frac{AF}{FC} \quad \text{...(i)}$$

But, $$\frac{AD}{DB} = \frac{AE}{EC} \quad \text{(Given)} \quad \text{...(ii)}$$

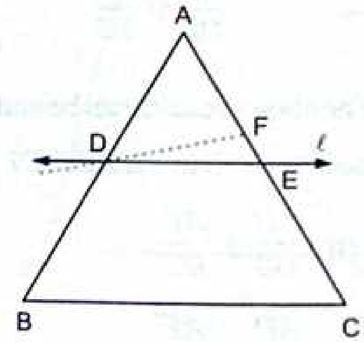


Fig. 7.9

From (i) and (ii), we get

$$\frac{AF}{FC} = \frac{AE}{EC}$$

$$\Rightarrow \quad \frac{AF}{FC} + 1 = \frac{AE}{EC} + 1 \quad \text{[Adding 1 on both sides]}$$

$$\Rightarrow \quad \frac{AF + FC}{FC} = \frac{AE + EC}{EC}$$

$$\Rightarrow \quad \frac{AC}{FC} = \frac{AC}{EC}$$

$$\Rightarrow \quad FC = EC$$

This is possible only when F and E coincide i.e. DF is the line l itself. But, $DF \parallel BC$.

Hence, $l \parallel BC$.

Q.E.D.

We shall now discuss some examples which will illustrate the applications of the results discussed so far.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **BASED ON THE RESULT THAT THE LINE DRAWN PARALLEL TO ONE SIDE OF A TRIANGLE INTERSECTING THE OTHER TWO SIDES DIVIDES THEM IN THE SAME RATIO.**

EXAMPLE 1 *In Fig 7.10, PQ is parallel to MN. If* $\frac{KP}{PM} = \frac{4}{13}$ *and* $KN = 20.4$ *cm. Find KQ.*

SOLUTION In $\Delta\ KMN$, we have

$PQ \parallel MN$

$\therefore \quad \frac{KP}{PM} = \frac{KQ}{QN}$ [By Thale's Theorem]

$\Rightarrow \quad \frac{KP}{PM} = \frac{KQ}{KN - KQ}$

$\Rightarrow \quad \frac{4}{13} = \frac{KQ}{20.4 - KQ}$

$\Rightarrow \quad 4(20.4 - KQ) = 13\ KQ$

$\Rightarrow \quad 81.6 - 4\ KQ = 13\ KQ$

$\Rightarrow \quad 17\ KQ = 81.6$

$\Rightarrow \quad KQ = \frac{81.6}{17} = 4.8$ cm

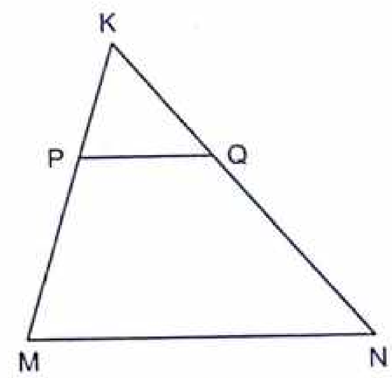


Fig. 7.10

EXAMPLE 2 *In a given* $\Delta\ ABC$, $DE \parallel BC$ *and* $\frac{AD}{DB} = \frac{3}{5}$. *If* $AC = 5.6$, *find AE.*

SOLUTION In $\Delta\ ABC$, we have

$DE \parallel BC$

$\therefore \quad \frac{AD}{DB} = \frac{AE}{EC}$ [By Thale's Theorem]

$\Rightarrow \quad \frac{AD}{DB} = \frac{AE}{AC - AE}$

$\Rightarrow \quad \frac{3}{5} = \frac{AE}{5.6 - AE}$ $[\because AC = 5.6]$

$\Rightarrow \quad 3(5.6 - AE) = 5AE$

$\Rightarrow \quad 16.8 - 3AE = 5AE$

$\Rightarrow \quad 8AE = 16.8$

$\Rightarrow \quad AE = \frac{16.8}{8}$ cm $= 2.1$ cm.

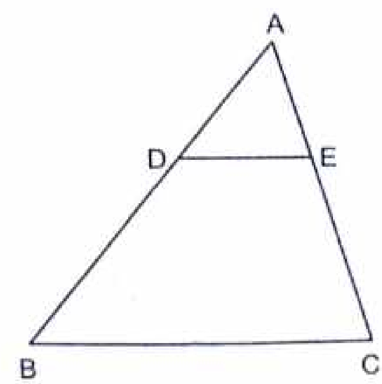


Fig. 7.11

EXAMPLE 3 *In Fig. 7.12, DE || BC. If AD = x, DB = x - 2, AE = x + 2 and EC = x - 1, find the value of x.*

SOLUTION In ΔABC, we have

$DE \parallel BC$

$\therefore \quad \frac{AD}{DB} = \frac{AE}{EC}$ [By Thale's Theorem]

$\Rightarrow \quad \frac{x}{x-2} = \frac{x+2}{x-1}$

$\Rightarrow \quad x(x-1) = (x-2)(x+2)$

$\Rightarrow \quad x^2 - x = x^2 - 4 \Rightarrow x = 4$

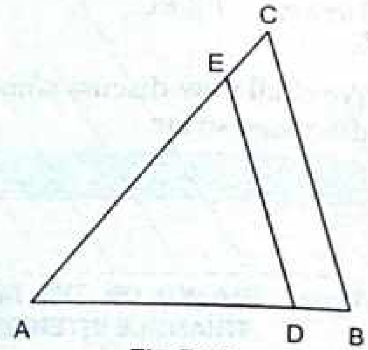


Fig. 7.12

EXAMPLE 4 *In Fig. 7.13, LM || AB. If AL = x – 3, AC = 2x, BM = x – 2 and BC = 2x + 3, find the value of x.*

SOLUTION In ΔABC, we have

$LM \parallel AB$

$\therefore \quad \frac{AL}{LC} = \frac{BM}{MC}$ [By Thaley's Theorem]

$\Rightarrow \quad \frac{AL}{AC - AL} = \frac{BM}{BC - BM}$

$\Rightarrow \quad \frac{x-3}{2x-(x-3)} = \frac{x-2}{(2x+3)-(x-2)}$

$\Rightarrow \quad \frac{x-3}{x+3} = \frac{x-2}{x+5}$

$\Rightarrow \quad (x-3)(x+5) = (x-2)(x+3)$

$\Rightarrow \quad x^2 + 2x - 15 = x^2 + x - 6$

$\Rightarrow \quad x = 9$

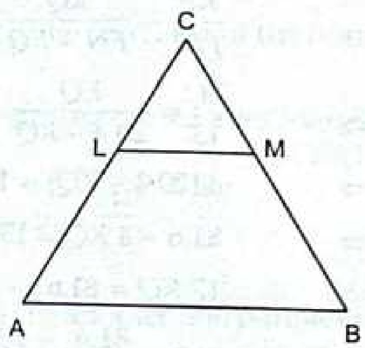


Fig. 7.13

EXAMPLE 5 *In Fig. 7.14, if ST || QR. Find PS.*

SOLUTION In ΔPRQ, we have

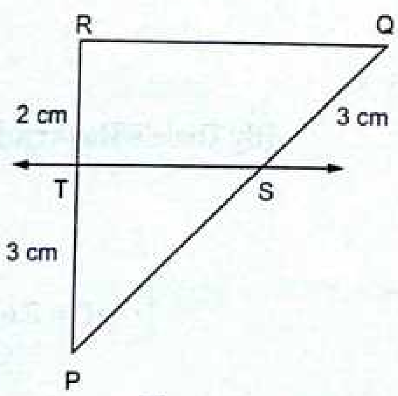


Fig. 7.14

$ST \parallel QR$

$\Rightarrow \quad \frac{PS}{QS} = \frac{PT}{RT}$ [By Thale's Theorem]

$$\Rightarrow \quad \frac{PS}{3} = \frac{3}{2}$$

$$\Rightarrow \quad PS = \frac{9}{2} \text{ cm}$$

EXAMPLE 6 *In Fig. 7.15 (i) and (ii), PQ || BC. Find QC in (i) and AQ in (ii).*

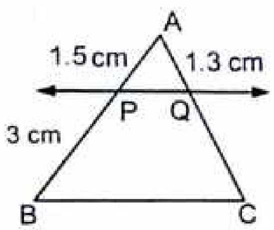


Fig. 7.15 (i)

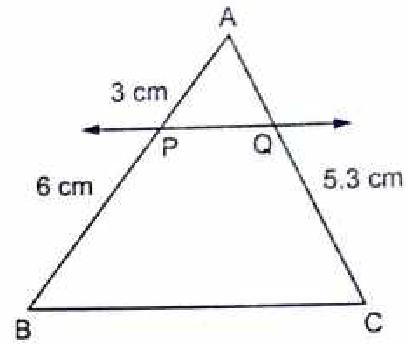


Fig. 7.15 (ii)

SOLUTION In Fig. 7.15 (i), we have

$PQ \parallel BC$

Therefore, by basic proportionality theorem, we have

$$\frac{AP}{PB} = \frac{AQ}{QC}$$

$$\Rightarrow \quad \frac{1.5}{3} = \frac{1.3}{QC}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{1.3}{QC}$$

$$\Rightarrow \quad QC = 2.6 \text{ cm}$$

In Fig. 7.15 (ii), it is given that $PQ \parallel BC$.

Therefore, by basic proportionality theorem, we have

$$\frac{AP}{PB} = \frac{AQ}{QC}$$

$$\Rightarrow \quad \frac{3}{6} = \frac{AQ}{5.3}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{AQ}{5.3}$$

$$\Rightarrow \quad AQ = \frac{5.3}{2} = 2.65 \text{ cm}$$

EXAMPLE 7 *In Fig. 7.16, if PQ || BC and PR || CD. Prove that (i)* $\frac{AR}{AD} = \frac{AQ}{AB}$ *(ii)* $\frac{QB}{AQ} = \frac{DR}{AR}$ [CBSE 2010]

SOLUTION In $\Delta\, ABC$, we have

$PQ \parallel BC$ [Given]

Therefore, by basic proportionality theorem, we have

$$\frac{AQ}{AB} = \frac{AP}{AC} \quad \ldots \text{(i)}$$

In ΔACD, we have

$PR \parallel CD$

Therefore, by basic proportionality theorem, we have

$$\frac{AP}{AC} = \frac{AR}{AD} \quad ...(ii)$$

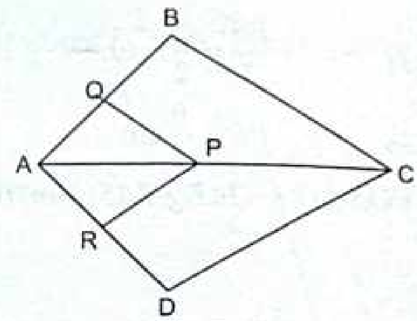


Fig. 7.16

From (i) and (ii), we obtain that

$$\frac{AQ}{AB} = \frac{AR}{AD} \text{ or, } \frac{AR}{AD} = \frac{AQ}{AB}$$

(ii) From (i) we have

$$\frac{AQ}{AB} = \frac{AR}{AD}$$

$$\Rightarrow \quad \frac{AB}{AQ} = \frac{AD}{AR}$$

$$\Rightarrow \quad \frac{AQ + QB}{AQ} = \frac{AR + RD}{AR} = 1 + \frac{QB}{AQ}$$

$$= 1 + \frac{RD}{AR} \Rightarrow \frac{QB}{AQ} = \frac{DR}{AR}$$

EXAMPLE 8 *In Fig. 7.17, DE $\parallel$ AC and DC $\parallel$ AP. Prove that* $\frac{BE}{EC} = \frac{BC}{CP}$ **[CBSE 2005]**

SOLUTION In ΔBPA, we have

$DC \parallel AP$ [Given]

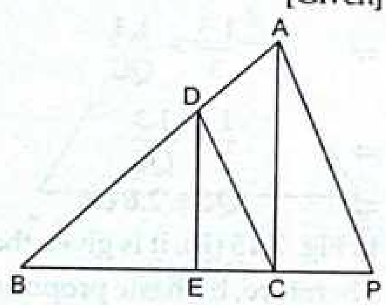


Fig. 7.17

Therefore, by basic proportionality theorem, we have

$$\frac{BC}{CP} = \frac{BD}{DA} \quad ...(i)$$

In ΔBCA, we have

$DE \parallel AC$ [Given]

Therefore, by basic proportionality theorem, we have

$$\frac{BE}{EC} = \frac{BD}{DA} \quad ...(ii)$$

From (i) and (ii), we get

$$\frac{BC}{CP} = \frac{BE}{EC} \text{ or, } \frac{BE}{EC} = \frac{BC}{CP}$$

Type II **PROBLEMS BASED UPON THE CONVERSE OF PROPORTIONALITY THEOREM**

EXAMPLE 9 *D and E are respectively the points on the sides AB and AC of a ΔABC such that $AB = 5.6\,cm$, $AD = 1.4\,cm$, $AC = 7.2\,cm$ and $AE = 1.8\,cm$, show that DE $\parallel$ BC.*

SOLUTION We have,

$AB = 5.6$ cm, $AD = 1.4$ cm, $AC = 7.2$ cm and $AE = 1.8$ cm.

$\therefore \quad BD = AB - AD = (5.6 - 1.4)$ cm $= 4.2$ cm

and,

$EC = AC - AE = (7.2 - 1.8)$ cm $= 5.4$ cm

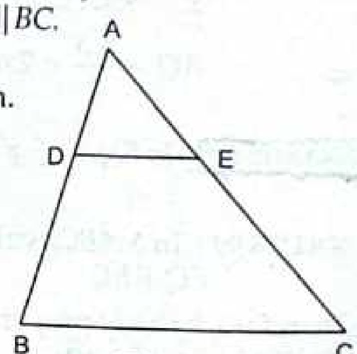


Fig. 7.18

Now, $\frac{AD}{DB} = \frac{1.4}{4.2} = \frac{1}{3}$ and $\frac{AE}{EC} = \frac{1.8}{5.4} = \frac{1}{3}$

$$\Rightarrow \quad \frac{AD}{DB} = \frac{AE}{EC}$$

Thus, DE divides sides AB and AC of ΔABC in the same ratio. Therefore, by the converse of Basic Pro-portionality Theorem, we have

$$DE \parallel BC$$

EXAMPLE 10 *Any point X inside ΔDEF is joined to its vertices. From a point P in DX, PQ is drawn parallel to DE meeting XE at Q and QR is drawn parallel to EF meeting XF in R. Prove that $PR \parallel DF$.*
[NCERT, CBSE 2002]

GIVEN A ΔDEF and a point X inside it. Point X is joined to the vertices D, E and F. P is any point on DX. $PQ \parallel DE$ and $QR \parallel EF$.

TO PROVE $PR \parallel DF$

CONSTRUCTION Join PR.

PROOF In ΔXED, we have

$$PQ \parallel DE$$

$$\therefore \quad \frac{XP}{PD} = \frac{XQ}{QE} \quad \text{...(i)} \quad \text{[By Thale's Theorem]}$$

In ΔXEF, we have

$$QR \parallel EF$$

$$\therefore \quad \frac{XQ}{QE} = \frac{XR}{RF} \quad \text{...(ii)} \quad \text{[By Thale's Theorem]}$$

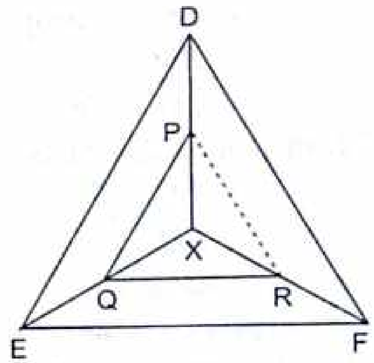


Fig. 7.19

From (i) and (ii), we have

$$\frac{XP}{PD} = \frac{XR}{RF}$$

Thus, in ΔXFD, points R and P are dividing sides XF and XD in the same ratio. Therefore, by the converse of Basic Proportionality Theorem, we have

$$PR \parallel DF$$

EXAMPLE 11 *In a ΔABC, D and E are points on sides AB and AC respectively such that $BD = CE$. If $\angle B = \angle C$, show that $DE \parallel BC$.*

SOLUTION In ΔABC, we have

$$\angle B = \angle C$$

$$\Rightarrow \quad AC = AB \quad \text{[Sides opposite to equal angles are equal]}$$

$$\Rightarrow \quad AE + EC = AD + DB$$

$$\Rightarrow \quad AE + CE = AD + BD$$

$$\Rightarrow \quad AE + CE = AD + CE \quad [\because BD = CE]$$

$$\Rightarrow \quad AE = AD$$

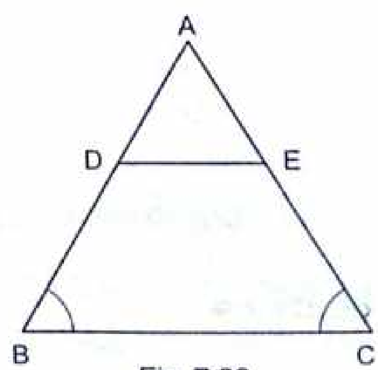


Fig. 7.20

Thus, we have

$$AD = AE \text{ and } BD = CE$$

$$\therefore \quad \frac{AD}{BD} = \frac{AE}{CE}$$

$$\Rightarrow \quad \frac{AD}{DB} = \frac{AE}{EC}$$

$$\Rightarrow \quad DE \parallel BC \quad \text{[By the converse of Thale's Theorem]}$$

EXAMPLE 12 *In Fig. 7.21, if DE || AQ and DF || AR. Prove that EF || QR.* **[NCERT, CBSE 2008]**

SOLUTION In ΔPQA, we have

$DE \parallel AQ$ [Given]

Therefore, by basic proportionality theorem, we have

$$\frac{PE}{EQ} = \frac{PD}{DA} \quad \ldots(i)$$

In ΔPAR, we have

$DF \parallel AD$ [Given]

Therefore, by basic proportionality theorem, we have

$$\frac{PD}{DA} = \frac{PF}{FR} \quad \ldots(ii)$$

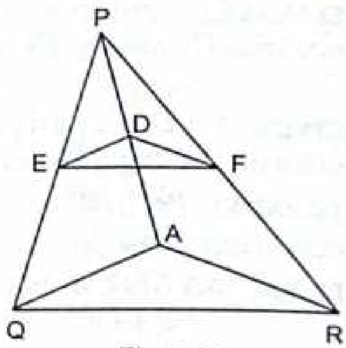


Fig. 7.21

From (i) and (ii), we have

$$\frac{PE}{EQ} = \frac{PF}{FR}$$

$\Rightarrow$ $EF \parallel QR$ [By the converse of Basic Proportionality Theorem]

EXAMPLE 13 *In Fig. 7.22, A, B and C are points on OP, OQ and OR respectively such that AB || PQ and BC || QR. Show that AC || PR.* **[NCERT]**

SOLUTION In ΔOPQ, we have

$AB \parallel PQ$

$$\Rightarrow \quad \frac{OA}{AP} = \frac{OB}{BQ} \quad \ldots(i)$$

In ΔOQR, we have

$BC \parallel QR$

$$\Rightarrow \quad \frac{OB}{BQ} = \frac{OC}{CR} \quad \ldots(ii)$$

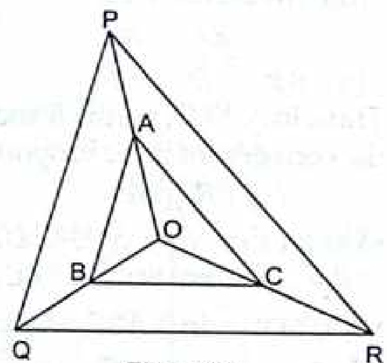


Fig. 7.22

From (i) and (ii), we get

$$\frac{OA}{AP} = \frac{OC}{CR}$$

Thus, A and C are points on sides OP and OR respectively of ΔOPR, such that

$$\frac{OA}{AP} = \frac{OC}{CR}$$

$\Rightarrow$ $AC \parallel PR$ [Using the converse of BPT]

LEVEL-2

Type I BASED UPON BASIC PROPORTIONALITY THEOREM

EXAMPLE 14 *In Fig. 7.23, if EF || DC || AB, prove that* $\frac{AE}{ED} = \frac{BF}{FC}$.

GIVEN $EF \parallel DC \parallel AB$ in the given figure.

TO PROVE $\frac{AE}{ED} = \frac{BF}{FC}$

CONSTRUCTION Produce DA and CB to meet at P (say).

PROOF In ΔPEF, we have

$AB \parallel EF$

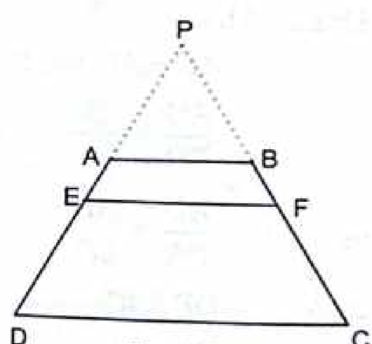


Fig. 7.23

$\therefore \quad \frac{PA}{AE} = \frac{PB}{BF}$ [By Thale's Theorem]

$\Rightarrow \quad \frac{PA}{AE} + 1 = \frac{PB}{BF} + 1$ [Adding 1 on both sides]

$\Rightarrow \quad \frac{PA + AE}{AE} = \frac{PB + BF}{BF}$

$\Rightarrow \quad \frac{PE}{AE} = \frac{PF}{BF}$ ...(i)

In ΔPDC, we have

$EF \parallel DC$

$\therefore \quad \frac{PE}{ED} = \frac{PF}{FC}$ [By Basic Proportionality Theorem] ...(ii)

On dividing equation (i) by equation (ii), we get

$$\frac{\frac{PE}{AE}}{\frac{PE}{ED}} = \frac{\frac{PF}{BF}}{\frac{PF}{FC}}$$

$\Rightarrow \quad \frac{ED}{AE} = \frac{FC}{BF}$

$\Rightarrow \quad \frac{AE}{ED} = \frac{BF}{FC}$

EXAMPLE 15 *Let X be any point on the side BC of a triangle ABC. If XM, XN are drawn parallel to BA and CA meeting CA, BA in M, N respectively; MN meets BC produced in T, prove that* $TX^2 = TB \times TC$.

SOLUTION In ΔTXM, we have

$XM \parallel BN$

$\therefore \quad \frac{TB}{TX} = \frac{TN}{TM}$ ...(i)

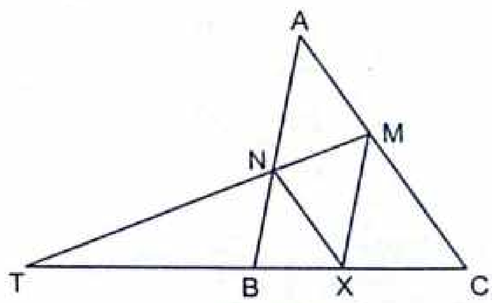


Fig. 7.24

In ΔTMC, we have

$XN \parallel CM$

$\therefore \quad \frac{TX}{TC} = \frac{TN}{TM}$ ...(ii)

From equations (i) and (ii), we get

$$\frac{TB}{TX} = \frac{TX}{TC}$$

$\Rightarrow \quad TX^2 = TB \times TC$

EXAMPLE 16 *ABCD is a parallelogram, P is a point on side BC and DP when produced meets AB produced at L. Prove that*

(i) $\frac{DP}{PL} = \frac{DC}{BL}$ (ii) $\frac{DL}{DP} = \frac{AL}{DC}$

GIVEN A parallelogram $ABCD$ in which P is a point on side BC such that DP produced meets AB produced at L.

TO PROVE

(i) $\frac{DP}{PL} = \frac{DC}{BL}$ (ii) $\frac{DL}{DP} = \frac{AL}{DC}$

PROOF (i) In ΔALD, we have

$BP \parallel AD$

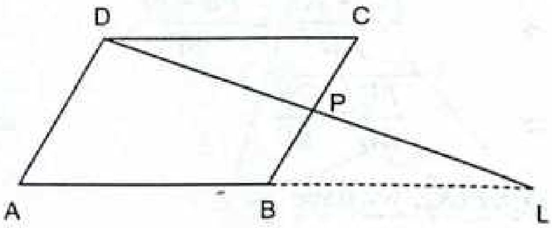


Fig. 7.25

$\therefore \quad \frac{LB}{BA} = \frac{LP}{PD}$

$\Rightarrow \quad \frac{BL}{AB} = \frac{PL}{DP}$

$\Rightarrow \quad \frac{BL}{DC} = \frac{PL}{DP}$ $[\because AB = DC]$

$\Rightarrow \quad \frac{DP}{PL} = \frac{DC}{BL}$ [Taking reciprocals of both sides]

(ii) From (i), we have

$\frac{DP}{PL} = \frac{DC}{BL}$

$\Rightarrow \quad \frac{PL}{DP} = \frac{BL}{DC}$ [Taking reciprocals of both sides]

$\Rightarrow \quad \frac{PL}{DP} = \frac{BL}{AB}$ $[\because DC = AB]$

$\Rightarrow \quad \frac{PL}{DP} + 1 = \frac{BL}{AB} + 1$ [Adding 1 on both sides]

$\Rightarrow \quad \frac{DP + PL}{DP} = \frac{BL + AB}{AB}$

$\Rightarrow \quad \frac{DL}{DP} = \frac{AL}{AB}$

$\Rightarrow \quad \frac{DL}{DP} = \frac{AL}{DC}$ $[\because AB = DC]$

EXAMPLE 17 *In Fig. 7.26, $EF \parallel AB \parallel DC$. Prove that* $\frac{AE}{ED} = \frac{BF}{FC}$ **[NCERT]**

SOLUTION We have,

$EF \parallel AB \parallel DC$

$\Rightarrow \quad EP \parallel DC$

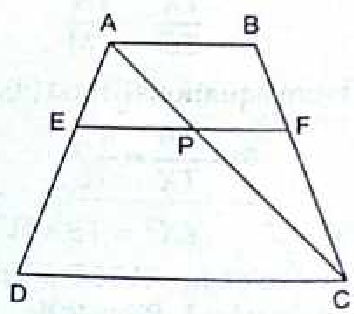


Fig. 7.26

Thus, in ΔADC, we have

$EP \parallel DC$

Therefore, by basic proportionality theorem, we have

$\frac{AE}{ED} = \frac{AP}{PC}$ ...(i)

Again, $EF \parallel AB \parallel DC$

$\Rightarrow \quad FP \parallel AB$

Thus, in $\Delta\, CAB$, we have

$$FP \parallel BA$$

Therefore, by basic proportionality theorem, we have

$$\frac{BF}{FC} = \frac{AP}{PC} \quad \dots \text{(ii)}$$

From (i) and (ii), we obtain $\dfrac{AE}{ED} = \dfrac{BF}{FC}$

EXAMPLE 18 *In Fig. 7.27, $DE \parallel BC$ and $CD \parallel EF$. Prove that* $AD^2 = AB \times AF$. **[CBSE 2007]**

SOLUTION In $\Delta\, ABC$, we have

$$DE \parallel BC$$

$$\Rightarrow \quad \frac{AB}{AD} = \frac{AC}{AE} \quad \dots \text{(i)}$$

In $\Delta\, ADC$, we have

$$FE \parallel DC$$

$$\Rightarrow \quad \frac{AD}{AF} = \frac{AC}{AE} \quad \dots \text{(ii)}$$

From (i) and (ii), we get

$$\frac{AB}{AD} = \frac{AD}{AF}$$

$$\Rightarrow \quad AD^2 = AB \times AF$$

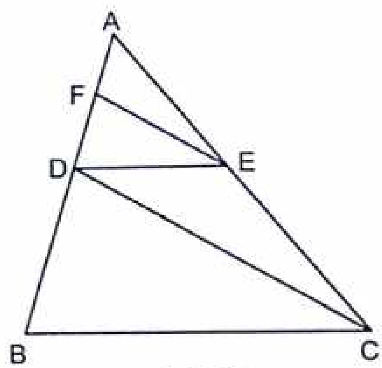


Fig. 7.27

Type II ON THALE'S THEOREM AND ITS CONVERSE

EXAMPLE 19 *Two triangles ABC and DBC lie on the same side of the base BC. From a point P on BC, $PQ \parallel AB$ and $PR \parallel BD$ are drawn. They meet AC in Q and DC in R respectively. Prove that $QR \parallel AD$*

GIVEN Two triangles ABC and DBC lie on the same side of the base BC. Points P, Q and R are points on BC, AC and CD respectively such that $PR \parallel BD$ and $PQ \parallel AB$.

TO PROVE $QR \parallel AD$

PROOF In $\Delta\, ABC$, we have

$$PQ \parallel AB$$

$$\therefore \quad \frac{CP}{PB} = \frac{CQ}{QA} \quad \dots \text{(i) [By Basic Proportionality Theorem]}$$

In $\Delta\, BCD$, we have

$$PR \parallel BD$$

$$\therefore \quad \frac{CP}{PB} = \frac{CR}{RD} \quad \dots \text{(ii) [By Thale's Theorem]}$$

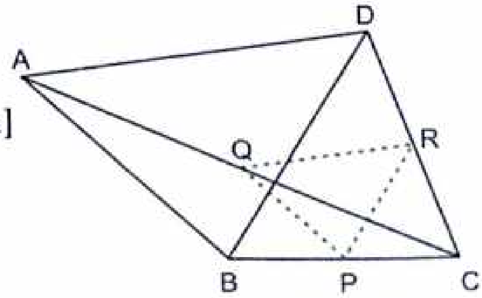


Fig. 7.28

From (i) and (ii), we have

$$\frac{CQ}{QA} = \frac{CR}{RD}$$

Thus, in $\Delta\, ACD$, Q and R are points on AC and CD respectively such that

$$\frac{CQ}{QA} = \frac{CR}{RD}$$

$$\Rightarrow \quad QR \parallel AD \quad \text{[By the converse of Basic Proportionality Theorem]}$$

EXAMPLE 20 *ABCD is a quadrilateral; P, Q, R and S are the points of trisection of sides AB, BC, CD and DA respectively and are adjacent to A and C; prove that PQRS is a parallelogram.*

GIVEN A quadrilateral $ABCD$ in which P, Q, R and S are the points of trisection of sides AB, BC, CD and DA respectively and are adjacent to A and C.

TO PROVE $PQRS$ is a parallelogram i. e., $PQ \parallel SR$ and $QR \parallel PS$.

CONSTRUCTION Join AC.

PROOF Since P, Q, R and S are the points of trisection of AB, BC, CD and DA respectively.

$\therefore \quad BP = 2\,PA, BQ = 2\,QC, DR = 2\,RC$ and, $DS = 2\,SA$

In $\Delta\, ADC$, we have

$$\frac{DS}{SA} = \frac{2SA}{SA} = 2 \text{ and, } \frac{DR}{RC} = \frac{2RC}{RC} = 2$$

$$\Rightarrow \quad \frac{DS}{SA} = \frac{DR}{RC}$$

$\Rightarrow$ S and R divide the sides DA and DC respectively in the same ratio.

$\Rightarrow$ $SR \parallel AC$ [By the converse of Thale's Theorem] ...(i)

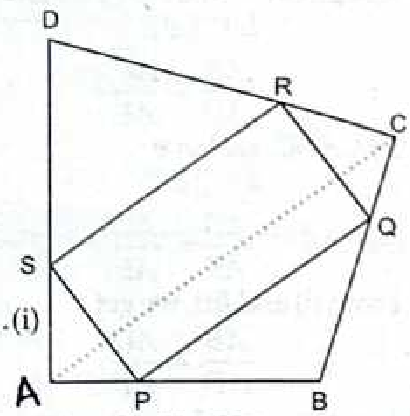


Fig. 7.29

In $\Delta\, ABC$, we have

$$\frac{BP}{PA} = \frac{2\,PA}{PA} = 2 \text{ and } \frac{BQ}{QC} = \frac{2\,QC}{QC} = 2$$

$$\Rightarrow \quad \frac{BP}{PA} = \frac{BQ}{QC}$$

$\Rightarrow$ P and Q divide the sides BA and BC respectively in the same ratio.

$\Rightarrow$ $PQ \parallel AC$ [By the converse of Thale's Theorem] ...(ii)

From (i) and (ii), we obtain

$$SR \parallel AC \text{ and } PQ \parallel AC \Rightarrow SR \parallel PQ$$

Similarly, by joining BD, we can prove that $QR \parallel PS$. Hence, $PQRS$ is a parallelogram.

EXAMPLE 21 *Let ABC be a triangle and D and E be two points on side AB such that AD = BE. If DP || BC and EQ || AC, then prove that PQ || AB.*

SOLUTION In $\Delta\, ABC$, we have

$DP \parallel BC$ and $EQ \parallel AC$

$$\therefore \quad \frac{AD}{DB} = \frac{AP}{PC} \text{ and } \frac{BE}{EA} = \frac{BQ}{QC}$$

$$\Rightarrow \frac{AD}{DB} = \frac{AP}{PC} \text{ and } \frac{AD}{DB} = \frac{BQ}{QC} \quad \begin{bmatrix} EA = ED + DA = ED + BE = BD \\ (\because AD = BE) \end{bmatrix}$$

$$\Rightarrow \frac{AP}{PC} = \frac{BQ}{QC}$$

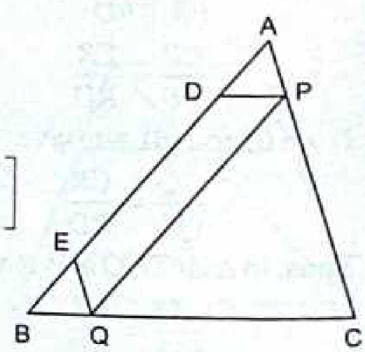


Fig. 7.30

$\Rightarrow$ In a $\Delta\, ABC$, P and Q divide sides CA and CB respectively in the same ratio.

$\Rightarrow$ $PQ \parallel AB$.

EXAMPLE 22 *In Fig. 7.31, ABC is a triangle in which AB = AC. Points D and E are points on the sides AB and AC respectively such that AD = AE. Show that the points B, C, E and D are concyclic.*

SOLUTION In order to prove that the points B, C, E and D are concyclic, it is sufficient to show that $\angle ABC + \angle CED = 180°$ and $\angle ACB + \angle BDE = 180°$.

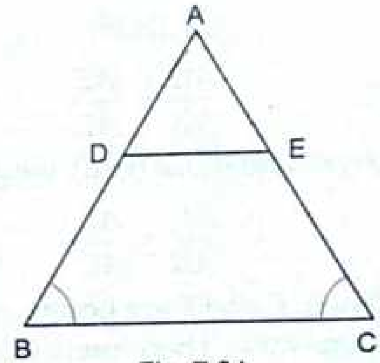


Fig. 7.31

In $\Delta\, ABC$, we have

$AB = AC$ and $AD = AE$

$\Rightarrow$ $AB - AD = AC - AE$

$\Rightarrow$ $DB = EC$

Thus, we have

$AD = AE$ and $DB = EC$

$\Rightarrow$ $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

$\Rightarrow$ $DE \parallel BC$ [By the converse of Thale's Theorem]

$\Rightarrow$ $\angle ABC = \angle ADE$ [Corresponding angles]

$\Rightarrow$ $\angle ABC + \angle BDE = \angle ADE + \angle BDE$ [Adding $\angle BDE$ on both sides]

$\Rightarrow$ $\angle ABC + \angle BDE = 180°$

$\Rightarrow$ $\angle ACB + \angle BDE = 180°$ [$\because AB = AC \therefore \angle ABC = \angle ACB$]

Again, $DE \parallel BC$

$\Rightarrow$ $\angle ACB = \angle AED$

$\Rightarrow$ $\angle ACB + \angle CED = \angle AED + \angle CED$ [Adding $\angle CED$ on both sides]

$\Rightarrow$ $\angle ACB + \angle CED = 180°$

$\Rightarrow$ $\angle ABC + \angle CED = 180°$ [$\because \angle ABC = \angle ACB$]

Thus, $BDEC$ is quadrilateral such that $\angle ACB + \angle BDE = 180°$

and $\angle ABC + \angle CED = 180°$

Therefore, $BDEC$ is a cyclic quadrilateral. Hence, B, C, E and D are concyclic points.

EXAMPLE 23 *The side BC of a triangle ABC is bisected at D; O is any point in AD. BO and CO produced meet AC and AB in E and F respectively and AD is produced to X so that D is the mid-point of OX. Prove that AO : AX = AF : AB and show that FE || BC.*

SOLUTION Join BX and CX.

We have,

$BD = CD$ and $OD = DX$.

Thus, BC and OX bisect each other.

$\Rightarrow$ $OBXC$ is a parallelogram.

$\Rightarrow$ $BX \parallel CO$ and $CX \parallel BO$

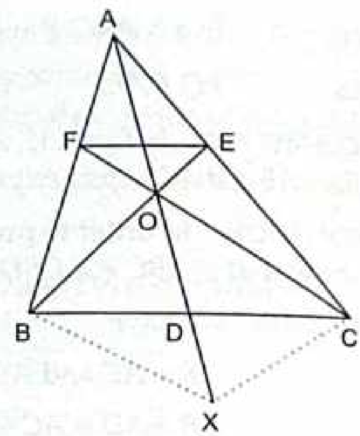


Fig. 7.32

$\Rightarrow \quad BX \parallel CF$ and $CX \parallel BE$

$\Rightarrow \quad BX \parallel OF$ and $CX \parallel OE$

In ΔABX, we have

$BX \parallel OF$

$\Rightarrow \quad \frac{AO}{AX} = \frac{AF}{AB}$ ...(i)

In ΔACX, we have

$CX \parallel OE$

$\Rightarrow \quad \frac{AO}{AX} = \frac{AE}{AC}$ ...(ii)

From equations (i) (ii), we get

$$\frac{AF}{AB} = \frac{AE}{AC}$$

Thus, E and F are points on AB and AC such that they divide AB and AC respectively in the same ratio. Therefore, by the converse of Thale's Theorem $FE \parallel BC$.

EXAMPLE 24 *In Fig. 7.33, if $\frac{AD}{DC} = \frac{BE}{EC}$ and $\angle CDE = \angle CED$, prove that ΔCAB is isosceles.*

SOLUTION In ΔABC, we have

$\frac{AD}{DC} = \frac{BE}{EC}$ [Given]

Therefore, by the converse of basic proportionality theorem, we have,

$DE \parallel AB$

$\Rightarrow \quad \angle CDE = \angle CAB$ and $\angle CED = \angle CBA$ [Corresponding angles]

But, $\angle CDE = \angle CED$ [Given]

$\therefore \quad \angle CAB = \angle CBA$

$\Rightarrow \quad \angle A = \angle B$

$\Rightarrow \quad BC = AC$ [$\because$ Sides opposite to equal angles are equal]

$\Rightarrow \quad \Delta CAB$ is isosceles.

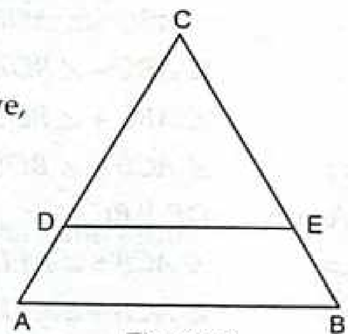


Fig. 7.33

EXAMPLE 25 *In Fig. 7.34, $\frac{PS}{SQ} = \frac{PT}{TR}$ and $\angle PST = \angle PRQ$. Prove that ΔPQR is an isosceles triangle.* **[NCERT]**

SOLUTION We have,

$\frac{PS}{SQ} = \frac{PT}{TR}$

$\Rightarrow \quad ST \parallel QR$ [By using the converse of Basic Proportionality Theorem]

$\Rightarrow \quad \angle PST = \angle PQR$ [Corresponding angles]

$\Rightarrow \quad \angle PRQ = \angle PQR$ [$\because \angle PST = \angle PRQ$ (Given)]

$\Rightarrow \quad PQ = PR$ [$\because$ Sides opposite to equal angles are equal]

$\Rightarrow \quad \Delta PQR$ is isosceles.

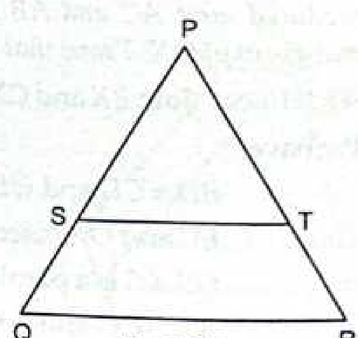


Fig. 7.34

EXERCISE 7.2

LEVEL-1

1. In a ΔABC, D and E are points on the sides AB and AC respectively such that $DE \parallel BC$.
   (i) If $AD = 6$ cm, $DB = 9$ cm and $AE = 8$ cm, find AC.
   (ii) If $\frac{AD}{DB} = \frac{3}{4}$ and $AC = 15$ cm, find AE.
   (iii) If $\frac{AD}{DB} = \frac{2}{3}$ and $AC = 18$ cm, find AE.
   (iv) If $AD = 4$, $AE = 8$, $DB = x - 4$, and $EC = 3x - 19$, find x
   (v) If $AD = 8$ cm, $AB = 12$ cm and $AE = 12$ cm, find CE.
   (vi) If $AD = 4$ cm, $DB = 4.5$ cm and $AE = 8$ cm, find AC.
   (vii) If $AD = 2$ cm, $AB = 6$ cm and $AC = 9$ cm, find AE.
   (viii) If $\frac{AD}{BD} = \frac{4}{5}$ and $EC = 2.5$ cm, find AE.
   (ix) If $AD = x$, $DB = x - 2$, $AE = x + 2$ and $EC = x - 1$, find the value of x.
   (x) If $AD = 8x - 7$, $DB = 5x - 3$, $AE = 4x - 3$ and $EC = (3x - 1)$, find the value of x.
   (xi) If $AD = 4x - 3$, $AE = 8x - 7$, $BD = 3x - 1$ and $CE = 5x - 3$, find the volume x. **[CBSE 2002]**
   (xii) If $AD = 2.5$ cm, $BD = 3.0$ cm and $AE = 3.75$ cm, find the length of AC. **[CBSE 2006C]**

2. In a ΔABC, D and E are points on the sides AB and AC respectively. For each of the following cases show that $DE \parallel BC$:
   (i) $AB = 12$ cm, $AD = 8$ cm, $AE = 12$ cm and $AC = 18$ cm.
   (ii) $AB = 5.6$ cm, $AD = 1.4$ cm, $AC = 7.2$ cm and $AE = 1.8$ cm.
   (iii) $AB = 10.8$ cm, $BD = 4.5$ cm, $AC = 4.8$ cm and $AE = 2.8$ cm.
   (iv) $AD = 5.7$ cm, $BD = 9.5$ cm, $AE = 3.3$ cm and $EC = 5.5$ cm.

3. In a ΔABC, P and Q are points on sides AB and AC respectively, such that $PQ \parallel BC$. If $AP = 2.4$ cm, $AQ = 2$ cm, $QC = 3$ cm and $BC = 6$ cm, find AB and PQ.

4. In a ΔABC, D and E are points on AB and AC respectively such that $DE \parallel BC$. If $AD = 2.4$ cm, $AE = 3.2$ cm, $DE = 2$ cm and $BC = 5$ cm, find BD and CE. **[CBSE 2001C]**

5. In Fig. 7.35, state if $PQ \parallel EF$.

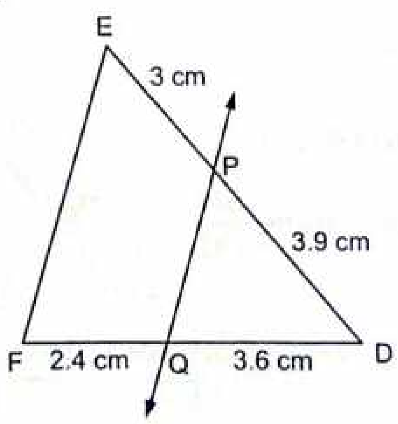


Fig. 7.35

6. M and N are points on the sides PQ and PR respectively of a ΔPQR. For each of the following cases, state whether $MN \parallel QR$:
   (i) $PM = 4$ cm, $QM = 4.5$ cm, $PN = 4$ cm, $NR = 4.5$ cm
   (ii) $PQ = 1.28$ cm, $PR = 2.56$ cm, $PM = 0.16$ cm, $PN = 0.32$ cm

LEVEL-2

7. In three line segments OA, OB, and OC, points L, M, N respectively are so chosen that $LM \parallel AB$ and $MN \parallel BC$ but neither of L, M, N nor of A, B, C are collinear. Show that $LN \parallel AC$.

8. If D and E are points on sides AB and AC respectively of a ΔABC such that $DE \parallel BC$ and $BD = CE$. Prove that ΔABC is isosceles. **[CBSE 2007, 2009]**

ANSWERS

1. (i) 20 cm (ii) 6.43 cm (iii) 7.2 cm (iv) 11 cm (v) 6 cm (vi) 17 cm (vii) 3 cm (viii) 2 cm (xi) $x = 4$ (x) $x = 1$ (xi) 1 (xii) 8.25 cm
3. $AB = 6$ cm, $PQ = 2.4$ cm
4. $DB = 3.6$ cm, $CE = 4.8$ cm
5. No
6. (i) Yes (ii) Yes

HINT TO SELECTED PROBLEM

8. By Thale's Theorem, we obtain $\frac{AD}{BD} = \frac{AE}{EC} \Rightarrow AD = AE$

But, $BD = CE$ and $AD = AE$

$\therefore \quad AD + BD = AE + CE \Rightarrow AB = AC$

7.5 INTERNAL AND EXTERNAL BISECTORS OF AN ANGLE OF A TRIANGLE

In this section, we will derive some properties of internal and external bisectors of an angle of a triangle. These, properties will be stated and proved as theorems.

Let us first perform the following activity.

ACTIVITY Draw any angle $\angle XAY$ and mark points P_1, P_2, P_3, P_4, P_5 and B on its arm AX such that $AP_1 = P_1P_2 = P_2P_3 = P_3P_4 = P_4P_5 = P_5B = 1$ unit. Also, mark points Q_1, Q_2, Q_3 and C on arm AY such that $AQ_1 = Q_1Q_2 = Q_2Q_3 = Q_3C = 1$ unit. Join BC.

We have,

$$\frac{AB}{AC} = \frac{6}{4} = \frac{3}{2}$$

Drow bisector of $\angle XAY$ to intersect BC at D.

Measure lengths BD and DC and compute $\frac{BD}{DC}$.

You will find that

$$\frac{BD}{DC} = \frac{3}{2}$$

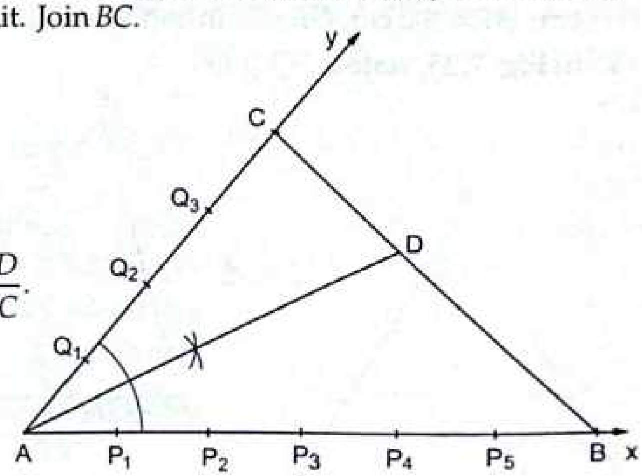


Fig. 7.36

$$\therefore \quad \frac{AB}{AC} = \frac{BD}{DC}$$

This means that the bisector of $\angle A$ of ΔABC divides opposite side BC in the ratio $AB : AC$.

This fact is stated and proved as a theorem given below.

THEOREM 1 *The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.*

GIVEN A ΔABC in which AD is the internal bisector of $\angle A$ and meets BC in D.

TO PROVE $\dfrac{BD}{DC} = \dfrac{AB}{AC}$

CONSTRUCTION Draw $CE \parallel DA$ to meet BA produced in E.

PROOF Since $CE \parallel DA$ and AC cuts them.

$\therefore \quad \angle 2 = \angle 3$ ...(i) [Alternate angles]

and, $\angle 1 = \angle 4$ ...(ii) [Corresponding angles]

But, $\angle 1 = \angle 2$ [$\because$ AD is the bisector of $\angle A$]

From (i) and (ii), we get

$\angle 3 = \angle 4$

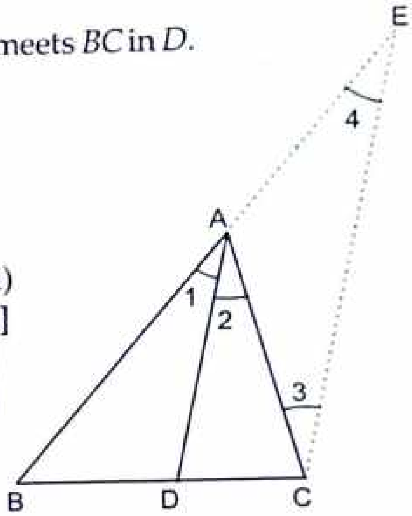


Fig. 7.37

Thus, in ΔACE, we have

$\angle 3 = \angle 4$

$\Rightarrow \quad AE = AC$ ...(iii) [Sides opposite to equal angles are equal]

Now, in ΔBCE, we have

$DA \parallel CE$

$\Rightarrow \quad \dfrac{BD}{DC} = \dfrac{BA}{AE}$ [Using Basic Proportionality Theorem]

$\Rightarrow \quad \dfrac{BD}{DC} = \dfrac{AB}{AC}$ [$\because BA = AB$ and $AE = AC$ (From (iii)]

Hence, $\dfrac{BD}{DC} = \dfrac{AB}{AC}$

In order to see whether the converse of the above theorem is true on not. Let us perform the following activity.

ACTIVITY Draw any angle $\angle XAY$ and mark points P_1, P_2, P_3, P_4, P_5 and B on its arm AX such that $AP_1 = P_1P_2 = P_2P_3 = P_3P_4 = P_4P_5 = P_5B = 1$ unit. Also, mark points Q_1, Q_2 and C on arm AY such that $AQ_1 = Q_1Q_2 = Q_2C = 1$ unit. Join BC.

Compute $AB : AC$.

We have,

$AB = 6$ units and $AC = 3$ units.

$$\therefore \quad \frac{AB}{AC} = \frac{6}{3} = \frac{2}{1}$$

Divide BC into 3 (= 2 + 1) equal parts and mark the points of division as R and D.

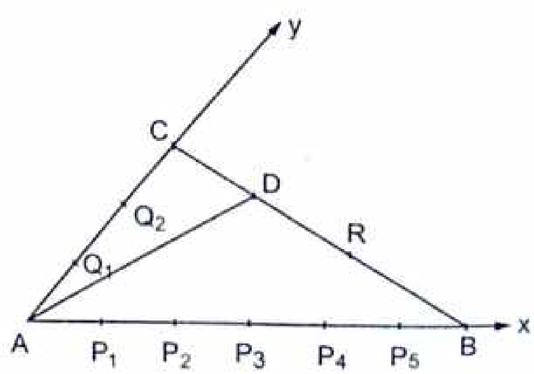


Fig. 7.38

We have,

$BD = BR + RD = 2$ units and $CD = 1$ unit.

$\therefore \quad \frac{BD}{CD} = \frac{2}{1}$

That is D divides BC in the ratio 2 : 1.

Join AD and measure $\angle XAD$ and $\angle YAD$.

You will find that $\angle XAD$ and $\angle YAD$. That is, AD is the bisector of $\angle BAC$ of ΔABC.

This means that if D is a point on side BC of ΔABC such that it divides BC in the ratio $AB : AC$. Then, AD is the bisector of $\angle A$ of ΔABC.

We state and prove this fact as a theorem given below.

THEOREM 2 *In a triangle ABC, if D is a point on BC such that* $\frac{BD}{DC} = \frac{AB}{AC}$, *prove that AD is the bisector of* $\angle A$.

OR

In a triangle ABC, if D is a point on BC such that D divides BC in the ratio AB : AC, then AD is the bisector of $\angle A$.

OR

If a line through one vertex of a triangle divides the opposite sides in the ratio of other two sides, then the line bisects the angle at the vertex.

GIVEN A ΔABC, in which D is a point on BC such that $\frac{BD}{DC} = \frac{AB}{AC}$

TO PROVE AD is the bisector of $\angle A$.

CONSTRUCTION Produce BA to E such that $AE = AC$. Join EC.

PROOF In ΔACE, we have

$AE = AC$ [By construction]

$\Rightarrow \quad \angle 3 = \angle 4$ ...(i)

Now, $\quad \frac{BD}{DC} = \frac{AB}{AC}$

$\Rightarrow \quad \frac{BD}{DC} = \frac{AB}{AE}$ [$\because AC = AE$]

Thus, in ΔBCE, we have

$\frac{BD}{DC} = \frac{BA}{AE}$

Therefore, by the converse of Basic Proportionality Theorem, we have

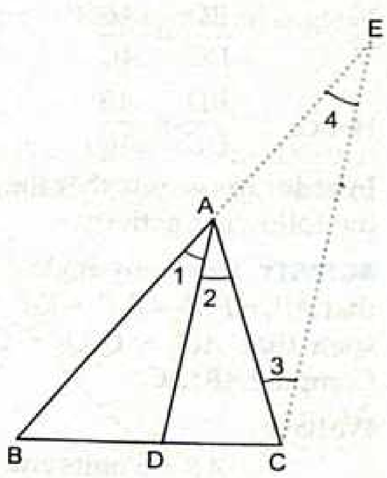


Fig. 7.39

$DA \parallel CE$

$\Rightarrow \quad \angle 1 = \angle 4$ ...(ii) [Corresponding angles]

and, $\quad \angle 2 = \angle 3$ ...(iii) [Alternate angles]

But, $\quad \angle 3 = \angle 4$ [From (i)]

$\therefore \quad \angle 1 = \angle 2$ [From (ii) and (iii)]

Hence, AD is the bisector of $\angle A$.

Q.E.D.

REMARK *In the previous two theorems we have seen that the internal bisector of an angle of a triangle divides the opposite side in the ratio of the sides containing the angle and vice-versa. In the following theorem, we shall prove that the bisector of the exterior of an angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle.*

THEOREM 3 *The external bisector of an angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle.*

GIVEN A ΔABC, in which AD is the bisector of the exterior of angle $\angle A$ and intersects BC produced in D.

TO PROVE $\frac{BD}{CD} = \frac{AB}{AC}$

CONSTRUCTION Draw $CE \parallel DA$ meeting AB in E.

PROOF Since $CE \parallel DA$ and AC intersects them.

$\therefore \quad \angle 1 = \angle 3 \quad \ldots(i)$

Also, $CE \parallel DA$ and BK intersects them.

$\therefore \quad \angle 2 = \angle 4 \quad \ldots(ii)$

But, $\quad \angle 1 = \angle 2 \quad$ [$\because$ AD is the bisector of $\angle CAK \therefore \angle 1 = \angle 2$]

$\therefore \quad \angle 3 = \angle 4 \quad$ [From (i) and (ii)]

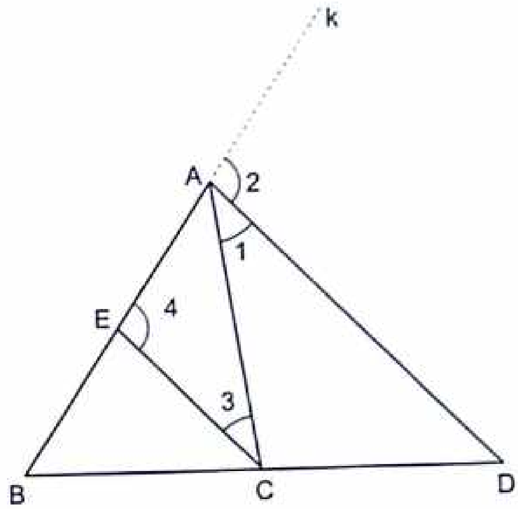


Fig. 7.40

Thus, in ΔACE, we have

$\angle 3 = \angle 4$

$\Rightarrow \quad AE = AC \quad$ [$\because$ Sides opposite to equal angles in a Δ are equal] $\quad \ldots(iii)$

Now, in ΔBAD, we have

$EC \parallel AD$

$\therefore \quad \frac{BD}{CD} = \frac{BA}{EA} \quad$ [Using corollary of Basic Proportionality Theorem]

$\Rightarrow \quad \frac{BD}{CD} = \frac{AB}{AE} \quad$ [$\because BA = AB$ and $EA = AE$]

$\Rightarrow \quad \frac{BD}{CD} = \frac{AB}{AC} \quad$ [$\because AE = AC$, From (iii)]

Q.E.D.

The following examples will illustrate the applications of the above results.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *In Fig 7.41, AD is the bisector of $\angle A$. If $BD = 4$ cm, $DC = 3$ cm and $AB = 6$ cm, determine AC.*

SOLUTION In ΔABC, AD is the bisector of $\angle A$.

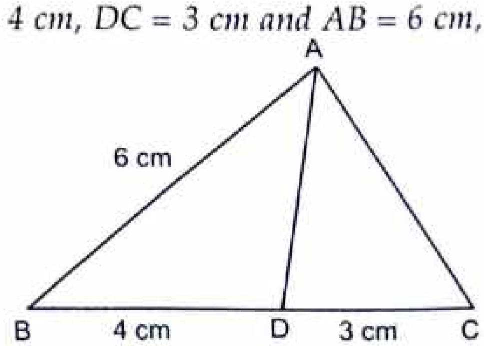


Fig. 7.41

$\therefore \quad \frac{BD}{DC} = \frac{AB}{AC}$

$\Rightarrow \quad \frac{4}{3} = \frac{6}{AC}$

$\Rightarrow \quad 4\,AC = 18$

$\Rightarrow \quad AC = \frac{9}{2} \text{ cm} = 4.5 \text{ cm}.$

EXAMPLE 2 *In Fig. 7.42, AD is the bisector of ∠ BAC. If AB = 10 cm. AC = 14 cm and BC = 6 cm, find BD and DC.*

SOLUTION Let $BD = x$ cm. Then, $DC = (6-x)$ cm.

Since AD is the bisector of $\angle A$.

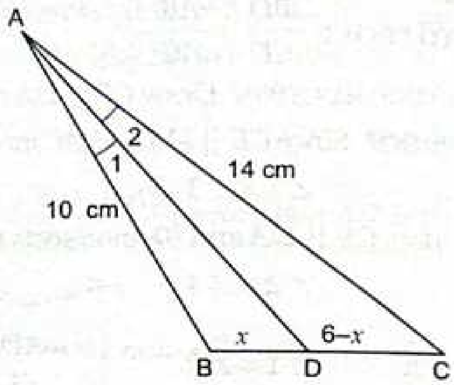


Fig. 7.42

$$\therefore \quad \frac{AB}{AC} = \frac{BD}{DC}$$

$$\Rightarrow \quad \frac{10}{14} = \frac{x}{6-x}$$

$$\Rightarrow \quad \frac{5}{7} = \frac{x}{6-x}$$

$$\Rightarrow \quad 30 - 5x = 7x \qquad [\because AC = 5.6]$$

$$\Rightarrow \quad 12x = 30$$

$$\Rightarrow \quad x = \frac{5}{2} = 2.5 \text{ cm}$$

$$\Rightarrow \quad BD = 2.5 \text{ cm and, } DC = (6-x) \text{ cm} = (6-2.5) \text{ cm} = 3.5 \text{ cm}$$

EXAMPLE 3 *If the diagonal BD of a quadrilateral ABCD bisects both ∠ B and ∠ D, show that* $\frac{AB}{BC} = \frac{AD}{CD}$.

GIVEN A quadrilateral $ABCD$ in which the diagonal BD bisects $\angle B$ and $\angle D$.

TO PROVE $\frac{AB}{BC} = \frac{AD}{CD}$

CONSTRUCTION Join AC intersecting BD in O.

PROOF In $\Delta\, ABC$, BO is the bisector of $\angle B$.

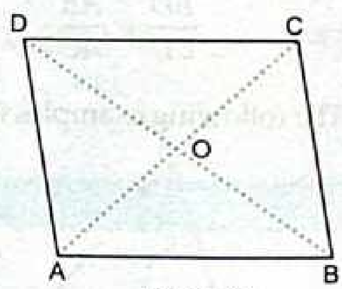


Fig. 7.43

$$\therefore \quad \frac{AO}{OC} = \frac{BA}{BC}$$

$$\Rightarrow \quad \frac{OA}{OC} = \frac{AB}{BC} \qquad \ldots(i)$$

In $\Delta\, ADC$, DO is the bisector of $\angle D$.

$$\therefore \quad \frac{AO}{OC} = \frac{DA}{DC}$$

$$\Rightarrow \quad \frac{OA}{OC} = \frac{AD}{CD} \qquad \ldots(ii)$$

From (i) and (ii), we get $\frac{AB}{BC} = \frac{AD}{CD}$

EXAMPLE 4 *If the bisector of an angle of a triangle bisects the opposite side, prove that the triangle is isosceles.* **[CBSE 2002]**

GIVEN In $\Delta\, ABC$, the bisector AD of $\angle A$ bisects the side BC.

TO PROVE $AB = AC$

PROOF In ΔABC, AD is the bisector of $\angle A$.

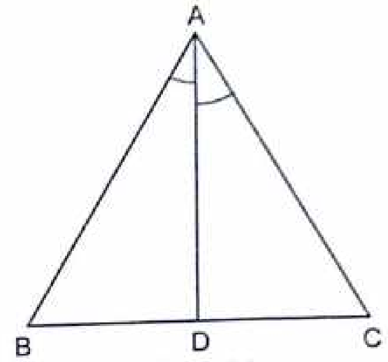


Fig. 7.44

$$\therefore \quad \frac{AB}{AC} = \frac{BD}{DC}$$

$$\Rightarrow \quad \frac{AB}{AC} = 1. \quad [D \text{ is the mid-point of } BC \therefore BD = DC]$$

$$\Rightarrow \quad AB = AC$$

Hence, the triangle ABC is isosceles.

EXAMPLE 5 *In ΔABC, the bisector of $\angle B$ meets AC at D. A line $PQ \parallel AC$ meets AB, BC and BD at P, Q and R respectively. Show that*

(i) $PR \cdot BQ = QR \cdot BP$ (ii) $AB \times CQ = BC \times AP$

GIVEN ΔABC in which BD is the bisector of $\angle B$ and a line $PQ \parallel AC$ meets AB, BC and BD at P, Q and R respectively.

TO PROVE (i) $PR \cdot BQ = QR \cdot BP$ (ii) $AB \times CQ = BC \times AP$

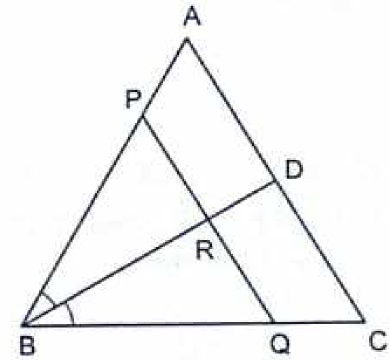


Fig. 7.45

PROOF (i) In ΔBQP, BR is the bisector of $\angle B$.

$$\therefore \quad \frac{BQ}{BP} = \frac{QR}{PR}$$

$$\Rightarrow \quad BQ \cdot PR = BP \cdot QR$$

$$\Rightarrow \quad PR \cdot BQ = QR \cdot BP$$

(i) In ΔABC, we have

$$PQ \parallel AC \quad \text{[Given]}$$

$$\Rightarrow \quad \frac{AB}{AP} = \frac{CB}{CQ} \quad \text{[By Thale's Theorem]}$$

$$\Rightarrow \quad AB \times CQ = BC \cdot AP$$

EXAMPLE 6 *In ΔABC, if AD is the bisector of $\angle A$, prove that:*

$$\frac{\text{Area}(\Delta ABD)}{\text{Area}(\Delta ACD)} = \frac{AB}{AC}$$

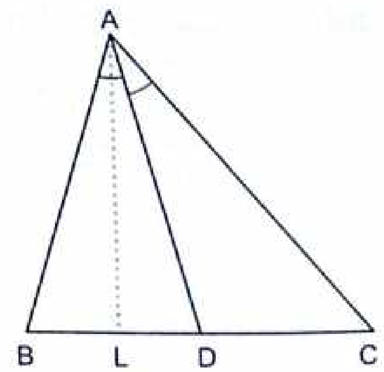


Fig. 7.46

SOLUTION In ΔABC, AD is the bisector of $\angle A$.

$$\therefore \quad \frac{AB}{AC} = \frac{BD}{DC} \quad \text{...(i)}$$

From A, draw $AL \perp BC$.

$$\therefore \quad \frac{\text{Area}(\Delta ABD)}{\text{Area}(\Delta ACD)} = \frac{(1/2)BD \cdot AL}{(1/2)DC \cdot AL}$$

$$\Rightarrow \quad \frac{\text{Area}(\Delta ABD)}{\text{Area}(\Delta ACD)} = \frac{BD}{DC}$$

$$\Rightarrow \quad \frac{\text{Area}(\Delta ABD)}{\text{Area}(\Delta ACD)} = \frac{AB}{AC} \quad \text{[From (i)]}$$

EXAMPLE 7 *The bisectors of the angles B and C of a triangle ABC, meet the opposite sides in D and E respectively. If $DE \parallel BC$, prove that the triangle is isosceles.*

GIVEN A ΔABC in which the bisectors of $\angle B$ and $\angle C$ meet the sides AC and AB at D and E respectively.

TO PROVE $AB = AC$

CONSTRUCTION Join DE

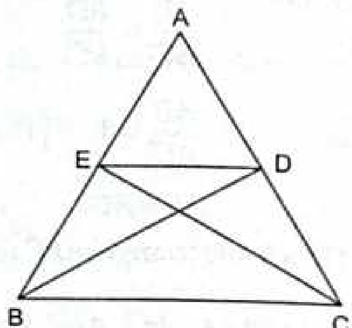


Fig. 7.47

PROOF In ΔABC, BD is the bisector of $\angle B$.

$$\therefore \quad \frac{AB}{BC} = \frac{AD}{DC} \quad \ldots(i)$$

In ΔABC, CE is the bisector of $\angle C$.

$$\therefore \quad \frac{AC}{BC} = \frac{AE}{BE} \quad \ldots(ii)$$

Now, $DE \parallel BC$

$$\Rightarrow \quad \frac{AE}{BE} = \frac{AD}{DC} \quad \text{[By Thale's Theorem]} \quad \ldots(iii)$$

From (iii), we find the RHS of (i) and (ii) are equal. Therefore, their LHS are also equal i. e.,

$$\frac{AB}{BC} = \frac{AC}{BC}$$

$$\Rightarrow \quad AB = AC$$

Hence, ΔABC is isosceles.

EXAMPLE 8 *BO and CO are respectively the bisectors of $\angle B$ and $\angle C$ of ΔABC. AO produced meets BC at P. Show that*

(i) $\frac{AB}{BP} = \frac{AO}{OP}$ (ii) $\frac{AC}{CP} = \frac{AO}{OP}$ (iii) $\frac{AB}{AC} = \frac{BP}{PC}$

(iv) *AP is the bisector of $\angle BAC$.*

SOLUTION (i) In ΔABP, BO is the bisector of $\angle B$

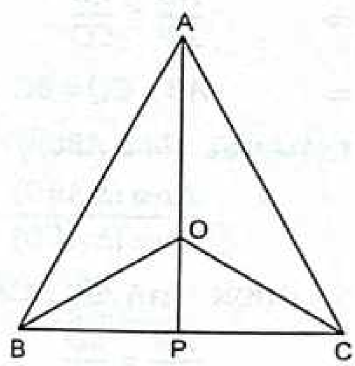


Fig. 7.48

$$\therefore \quad \frac{AB}{BP} = \frac{AO}{OP}$$

(ii) In ΔACP, OC is the bisector of $\angle C$

$$\therefore \quad \frac{AC}{CP} = \frac{AO}{OP}$$

(iii) We have, proved that

$$\frac{AB}{BP} = \frac{AO}{OP} \text{ and } \frac{AC}{CP} = \frac{AO}{OP}$$

$$\Rightarrow \quad \frac{AB}{BP} = \frac{AC}{CP}$$

$$\Rightarrow \quad \frac{AB}{AC} = \frac{BP}{PC}$$

(iv) As proved above that in ΔABC, we have

$$\frac{AB}{AC} = \frac{BP}{CP} \Rightarrow AP \text{ is the bisector of } \angle BAC.$$

LEVEL-2

EXAMPLE 9 *The bisector of interior $\angle A$ of ΔABC meets BC in D, and the bisector of exterior $\angle A$ meets BC produced in E. Prove that* $\frac{BD}{BE} = \frac{CD}{CE}$.

GIVEN In ΔABC, AD and AE are respectively the bisectors of the interior and exterior angles at A

TO PROVE $\frac{BD}{BE} = \frac{CD}{CE}$

PROOF Since AD is the internal bisector of $\angle A$ meeting BC at D.

$$\therefore \quad \frac{AB}{AC} = \frac{BD}{DC} \quad \ldots(i)$$

Since AE is the external bisector of $\angle A$ meeting BC produced in E.

$$\therefore \quad \frac{AB}{AC} = \frac{BE}{CE} \quad \ldots(ii)$$

From (i) and (ii), we get

$$\frac{BD}{DC} = \frac{BE}{CE}$$

$$\Rightarrow \quad \frac{BD}{BE} = \frac{CD}{CE}$$

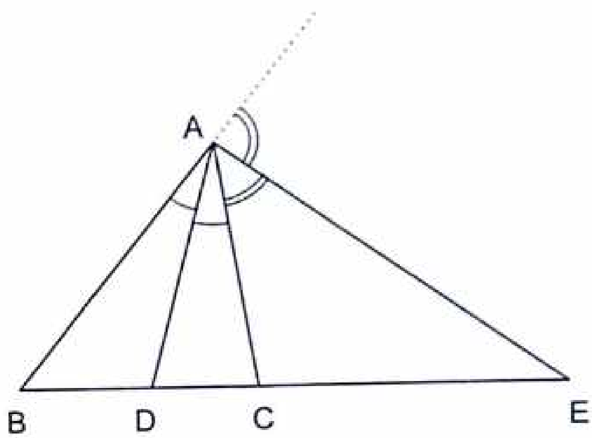


Fig. 7.49

EXAMPLE 10 *ABCD is a quadrilateral in which $AB = AD$. The bisector of $\angle BAC$ and $\angle CAD$ intersect the sides BC and CD at the points E and F respectively. Prove that $EF \parallel BD$.*

GIVEN A quadrilateral $ABCD$ in which $AB = AD$ and the bisectors of $\angle BAC$ and $\angle CAD$ meet the sides BC and CD at E and F respectively.

TO PROVE $EF \parallel BD$

CONSTRUCTION Join AC, BD and EF.

PROOF In ΔCAB, AE is the bisector of $\angle BAC$.

$$\therefore \quad \frac{AC}{AB} = \frac{CE}{BE} \quad \ldots(i)$$

In ΔACD, AF is the bisector of $\angle CAD$.

$$\therefore \quad \frac{AC}{AD} = \frac{CF}{DF}$$

$$\Rightarrow \quad \frac{AC}{AB} = \frac{CF}{DF} \quad [\because AD = AB] \quad \ldots(ii)$$

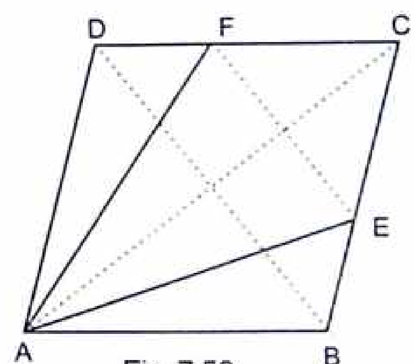


Fig. 7.50

From (i) and (ii), we get

$$\frac{CE}{BE} = \frac{CF}{DF}$$

$$\Rightarrow \quad \frac{CE}{EB} = \frac{CF}{FD}$$

Thus, in ΔCBD, E and F divide the sides CB and CD respectively in the same ratio. Therefore, by the converse of Thale's Theorem, we have $EF \parallel BD$.

EXAMPLE 11 *O is any point inside a triangle ABC. The bisector of $\angle AOB$, $\angle BOC$ and $\angle COA$ meet the sides AB, BC and CA in point D, E and F respectively. Show that*

$$AD \times BE \times CF = DB \times EC \times FA$$

SOLUTION In ΔAOB, OD is the bisector of $\angle AOB$.

$$\therefore \quad \frac{OA}{OB} = \frac{AD}{DB} \quad \ldots(i)$$

In ΔBOC, OE is the bisector of $\angle BOC$.

$$\therefore \quad \frac{OB}{OC} = \frac{BE}{EC} \quad \ldots(ii)$$

In ΔCOA, OF is the bisector of $\angle COA$.

$$\therefore \quad \frac{OC}{OA} = \frac{CF}{FA} \quad \ldots(iii)$$

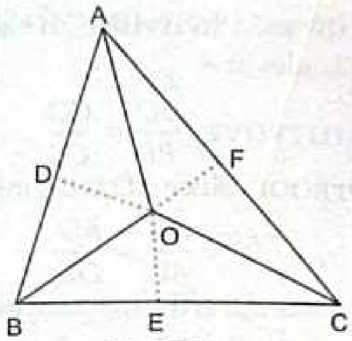


Fig. 7.51

Multiplying the corresponding sides of (i), (ii) and (iii), we get

$$\frac{OA}{OB} \times \frac{OB}{OC} \times \frac{OC}{OA} = \frac{AD}{DB} \times \frac{BE}{EC} \times \frac{CF}{FA}$$

$$\Rightarrow \quad 1 = \frac{AD}{DB} \times \frac{BE}{EC} \times \frac{CF}{FA}$$

$$\Rightarrow \quad DB \times EC \times FA = AD \times BE \times CF$$

$$\Rightarrow \quad AD \times BE \times CF = DB \times EC \times FA$$

EXAMPLE 12 *AD is a median of ΔABC. The bisector of $\angle ADB$ and $\angle ADC$ meet AB and AC in E and F respectively. Prove that $EF \parallel BC$.*

GIVEN In ΔABC, AD is the median and DE and DF are the bisectors of $\angle ADB$ and $\angle ADC$ respectively, meeting AB and AC in E and F respectively.

TO PROVE $EF \parallel BC$

PROOF In ΔADB, DE is the bisector of $\angle ADB$.

$$\therefore \quad \frac{AD}{DB} = \frac{AE}{EB} \quad \ldots(i)$$

In ΔADC, DF is the bisector of $\angle ADC$.

$$\therefore \quad \frac{AD}{DC} = \frac{AF}{FC}$$

$$\Rightarrow \quad \frac{AD}{DB} = \frac{AF}{FC} \quad \ldots(ii) \quad \left[\begin{array}{l}\because AD \text{ is the median} \\ \therefore BD = DC\end{array}\right]$$

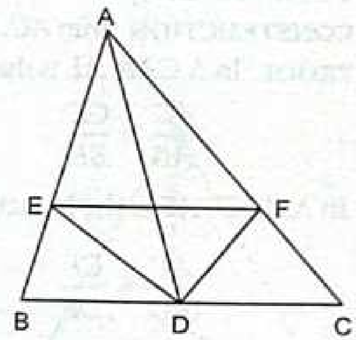


Fig. 7.52

From (i) and (ii), we get

$$\frac{AE}{EB} = \frac{AF}{FC}$$

Thus, in ΔABC, line segment EF divides the sides AB and AC in the same ratio.

Hence, EF is parallel to BC.

EXAMPLE 13 *In ΔABC, D is the mid-point of BC and ED is the bisector of the $\angle ADB$ and EF is drawn parallel to BC cutting AC in F. Prove that $\angle EDF$ is a right angle.*

GIVEN A ΔABC in which D is the mid-point of side BC and ED is the bisector of $\angle ADB$, meeting AB in E. EF is drawn parallel to BC meeting AC in F.

TO PROVE $\angle EDF$ is a right angle.

PROOF In $\Delta\, ADB$, DE is the bisector of $\angle ADB$.

$$\therefore \quad \frac{AD}{DB} = \frac{AE}{EB}$$

$$\Rightarrow \quad \frac{AD}{DC} = \frac{AE}{EB} \quad [\because D \text{ is the mid-point of } BC \therefore DB = DC] \quad \ldots(i)$$

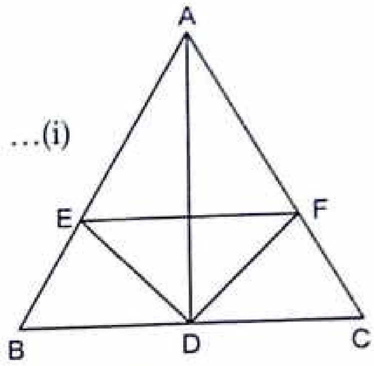


Fig. 7.53

In $\Delta\, ABC$, we have

$$EF \parallel BC$$

$$\Rightarrow \quad \frac{AE}{EB} = \frac{AF}{FC} \quad \ldots(ii)$$

From (i) and (ii), we get

$$\frac{AD}{DC} = \frac{AF}{FC}$$

$\Rightarrow$ In $\Delta\, ADC$, DF divides AC in the ratio $AD : DC$

$\Rightarrow$ DF is the bisector of $\angle ADC$

Thus, DE and DF are the bisectors of adjacent supplementary angles $\angle ADB$ and $\angle ADC$ respectively.

Hence, $\angle EDF$ is a right angle.

EXAMPLE 14 *In $\Delta\, ABC$, $\angle B = 2\angle C$ and the bisector of $\angle B$ intersects AC at D. Prove that* $\frac{BD}{DA} = \frac{BC}{BA}$.

SOLUTION In $\Delta\, ABC$, bisector of $\angle B$ meets AC at D.

$$\therefore \quad \frac{CD}{AD} = \frac{BC}{BA}$$

$$\Rightarrow \quad \frac{BD}{AD} = \frac{BC}{BA} \quad \left[\begin{array}{l}\text{In } \Delta BCD, \text{ we have} \\ \angle DBC = \angle BCD = C \therefore BD = CD\end{array}\right]$$

$$\Rightarrow \quad \frac{BD}{DA} = \frac{BC}{BA}$$

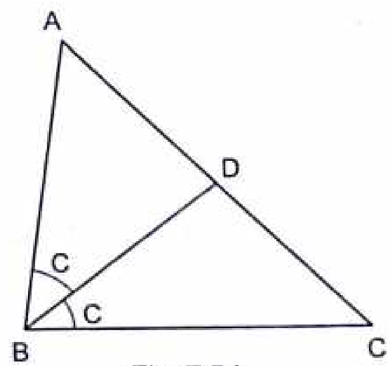


Fig. 7.54

EXAMPLE 15 *In Fig. 7.55, $\angle BAC = 90°$, AD is its bisector. If $DE \perp AC$, prove that* $DE \times (AB + AC) = AB \times AC$.

SOLUTION It is given that AD is the bisector of $\angle A$ of $\Delta\, ABC$.

$$\therefore \quad \frac{AB}{AC} = \frac{BD}{DC}$$

$$\Rightarrow \quad \frac{AB}{AC} + 1 = \frac{BD}{DC} + 1 \quad [\text{Adding 1 on both sides}]$$

$$\Rightarrow \quad \frac{AB + AC}{AC} = \frac{BD + DC}{DC}$$

$$\Rightarrow \quad \frac{AB + AC}{AC} = \frac{BC}{DC} \quad \ldots(i)$$

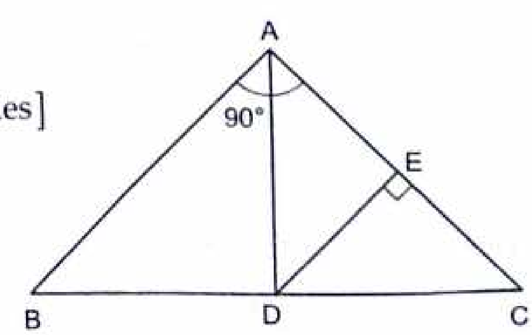


Fig. 7.55

In Δ's CDE and CBA, we have

$\angle DCE = \angle BCA = \angle C$ [Common]

$\angle BAC = \angle DEC$ [Each equal to 90°]

So, by AA-criterion of similarity, we have

$\Delta CDE \sim \Delta CBA$

$\Rightarrow \quad \frac{CD}{CB} = \frac{DE}{BA}$

$\Rightarrow \quad \frac{AB}{DE} = \frac{BC}{DC}$ ...(ii)

From (i) and (ii), we obtain

$$\frac{AB + AC}{AC} = \frac{AB}{DE} \Rightarrow DE \times (AB + AC) = AB \times AC$$

LEVEL-2

EXAMPLE 16 *In a quadrilateral ABCD, if bisectors of the $\angle ABC$ and $\angle ADC$ meet on the diagonal AC, prove that the bisectors of $\angle BAD$ and $\angle BCD$ will meet on the diagonal BD.*

GIVEN $ABCD$ is a quadrilateral in which the bisectors of $\angle ABC$ and $\angle ADC$ meet on the diagonal AC at P.

TO PROVE Bisectors of $\angle BAD$ and $\angle BCD$ meet on the diagonal BD

CONSTRUCTION Join BP and DP. Let the bisector of $\angle BAD$ meet BD at Q. Join AQ and CQ.

PROOF In order to prove that the bisectors of $\angle BAD$ and $\angle BCD$ meet on the diagonal BD. It is sufficient to prove that CQ is the bisector of $\angle BCD$. For which we will prove that Q divides BD in the ratio $BC : DC$.

In ΔABC, BP is the bisector of $\angle ABC$.

$\therefore \quad \frac{AB}{BC} = \frac{AP}{PC}$ ...(i)

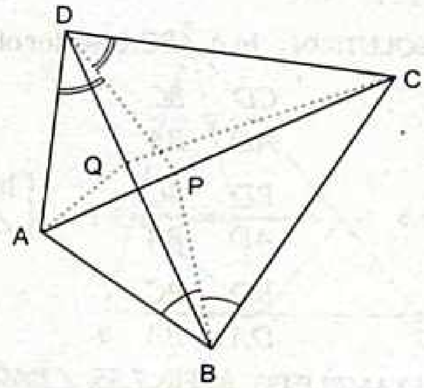


Fig. 7.56

In ΔACD, DP is the bisector of $\angle ADC$.

$\therefore \quad \frac{AD}{DC} = \frac{AP}{PC}$ ...(ii)

From (i) and (ii), we get

$\frac{AB}{BC} = \frac{AD}{DC}$

$\Rightarrow \quad \frac{AB}{AD} = \frac{BC}{DC}$ ...(iii)

In ΔABD, AQ is the bisector of $\angle BAD$. [By construction]

$\therefore \quad \frac{AB}{AD} = \frac{BQ}{DQ}$ ...(iv)

From (iii) and (iv), we get : $\frac{BC}{DC} = \frac{BQ}{DQ}$.

Thus, in ΔCBD, Q divides BD in the ratio $CB : CD$. Therefore, CQ is the bisectors of $\angle BCD$. Hence, bisectors of $\angle BAD$ and $\angle BCD$ meet on the diagonal BD.

EXERCISE 7.3

LEVEL-1

1. In a ΔABC, AD is the bisector of $\angle A$, meeting side BC at D.
   - (i) If $BD = 2.5$ cm, $AB = 5$ cm and $AC = 4.2$ cm, find DC.
   - (ii) If $BD = 2$ cm, $AB = 5$ cm and $DC = 3$ cm, find AC.
   - (iii) If $AB = 3.5$ cm, $AC = 4.2$ cm and $DC = 2.8$ cm, find BD.
   - (iv) If $AB = 10$ cm, $AC = 14$ cm and $BC = 6$ cm, find BD and DC.
   - (v) If $AC = 4.2$ cm, $DC = 6$ cm and $BC = 10$ cm, find AB.
   - (vi) If $AB = 5.6$ cm, $AC = 6$ cm and $DC = 3$ cm, find BC. [CBSE 2001C]
   - (vii) If $AD = 5.6$ cm, $BC = 6$ cm and $BD = 3.2$ cm, find AC. [CBSE 2001C]
   - (viii) If $AB = 10$ cm, $AC = 6$ cm and $BC = 12$ cm, find BD and DC. [CBSE 2001]
2. In Fig. 7.57, AE is the bisector of the exterior $\angle CAD$ meeting BC produced in E. If $AB = 10$ cm, $AC = 6$ cm and $BC = 12$ cm, find CE.

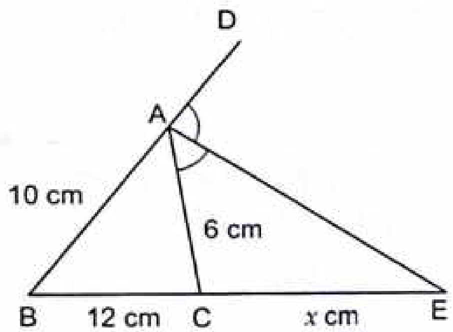


Fig. 7.57

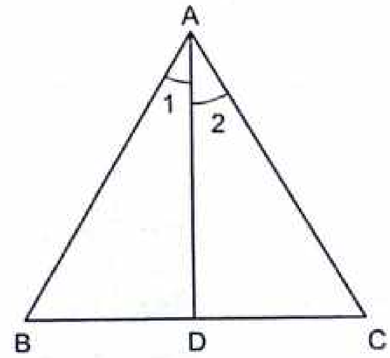


Fig. 7.58

3. In Fig. 7.58, ΔABC is a triangle such that $\frac{AB}{AC} = \frac{BD}{DC}$, $\angle B = 70°$, $\angle C = 50°$. Find $\angle BAD$.
4. In Fig. 7.59, check whether AD is the bisector of $\angle A$ of ΔABC in each of the following:
   - (i) $AB = 5$ cm, $AC = 10$ cm, $BD = 1.5$ cm and $CD = 3.5$ cm
   - (ii) $AB = 4$ cm, $AC = 6$ cm, $BD = 1.6$ cm and $CD = 2.4$ cm
   - (iii) $AB = 8$ cm, $AC = 24$ cm, $BD = 6$ cm and $BC = 24$ cm
   - (iv) $AB = 6$ cm, $AC = 8$ cm, $BD = 1.5$ cm and $CD = 2$ cm
   - (v) $AB = 5$ cm, $AC = 12$ cm, $BD = 2.5$ cm and $BC = 9$ cm

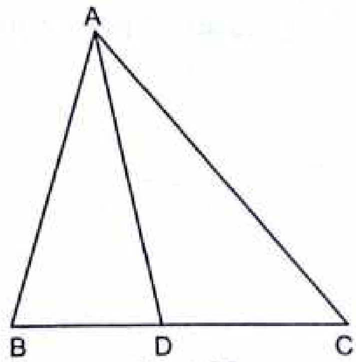


Fig. 7.59

5. In Fig. 7.60, AD bisects $\angle A$, $AB = 12$ cm, $AC = 420$ cm and $BD = 5$ cm, determine CD.

LEVEL-2

6. In ΔABC (Fig. 7.60), if $\angle 1 = \angle 2$, prove that $\frac{AB}{AC} = \frac{BD}{DC}$.

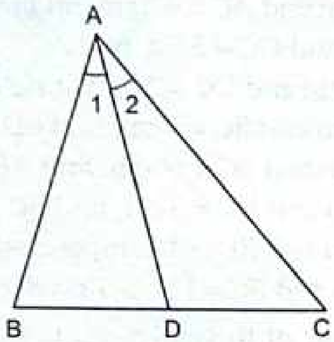


Fig. 7.60

7. D, E and F are the points on sides BC, CA and AB respectively of ΔABC such that AD bisects $\angle A$, BE bisects $\angle B$ and CF bisects $\angle C$. If $AB = 5$ cm, $BC = 8$ cm and $CA = 4$ cm, determine AF, CE and BD.

ANSWERS

1. (i) 2.1 cm (ii) 7.5 cm (iii) 2.3 cm (iv) $BD = 2.5$ cm, $DC = 3.5$ cm (v) 2.8 cm (vi) 5.8 cm (vii) 4.9 cm (viii) 7.5 cm, 4.5 cm
2. 18
3. 30°
4. (i) No (ii) Yes (iii) Yes (iv) Yes (v) No
5. 8.33 cm
7. $AF = \frac{5}{3}$ cm, $CE = \frac{32}{13}$ cm, $BD = \frac{40}{9}$ cm

HINTS TO SELECTED PROBLEMS

2. Since AE is the bisector of the exterior $\angle CAD$.

$$\therefore \quad \frac{BE}{CE} = \frac{AB}{AC} \Rightarrow \frac{12 + x}{x} = \frac{10}{6} \Rightarrow x = 18.$$

7. Since AD is the bisector of $\angle A$.

$$\therefore \quad \frac{AB}{AC} = \frac{BD}{CD}$$

$$\Rightarrow \quad \frac{5}{4} = \frac{BD}{BC - BD} \Rightarrow \frac{5}{4} = \frac{BD}{8 - BD} \Rightarrow 40 - 5\,BD = 4\,BD \Rightarrow 9\,BD = 40 \Rightarrow BD = \frac{40}{9} \text{ cm.}$$

Since BE is the bisector of $\angle B$.

$$\therefore \quad \frac{AB}{BC} = \frac{AE}{CE} \Rightarrow \frac{AB}{BC} = \frac{AC - CE}{CE} \Rightarrow \frac{5}{8} = \frac{4 - CE}{CE} \Rightarrow 13\,CE = 32 \Rightarrow CE = \frac{32}{13} \text{ cm}.$$

Since CF is the bisector of the $\angle C$.

$$\therefore \quad \frac{BC}{CA} = \frac{BF}{AF}$$

$$\Rightarrow \quad \frac{8}{4} = \frac{AB - AF}{AF} \Rightarrow 2 = \frac{5 - AF}{AF} \Rightarrow 3\,AF = 5 \Rightarrow AF = \frac{5}{3} \text{ cm.}$$

7.6 MORE ON BASIC PROPORTIONALITY THEOREM

In this section, we shall discuss some more applications of basic proportionality theorem.

THEOREM 1 *The line drawn from the mid-point of one side of a triangle parallel to another side bisects the third side.* [NCERT]

GIVEN A $\Delta\, ABC$ in which D is the mid-point of side AB and the line DE is drawn parallel to BC, meeting AC in E.

TO PROVE E is the mid-point of AC i. e., $AE = EC$.

PROOF In $\Delta\, ABC$, we have

$$DE \parallel BC$$

$$\Rightarrow \quad \frac{AD}{DB} = \frac{AE}{EC} \quad \text{[By Thale's Theorem]} \quad \ldots\text{(i)}$$

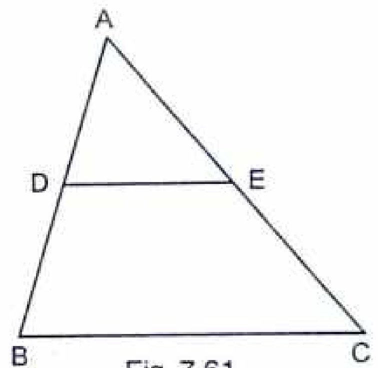


Fig. 7.61

But, D is the mid-point of AB.

$$\Rightarrow \quad AD = DB$$

$$\Rightarrow \quad \frac{AD}{DB} = 1 \quad \ldots\text{(ii)}$$

From (i) and (ii), we get

$$\frac{AE}{EC} = 1 \Rightarrow AE = EC.$$

Hence, E bisects AC.

Q.E.D.

THEOREM 2 *The line joining the mid-points of two sides of a triangle is parallel to the third side.* [NCERT]

GIVEN A $\Delta\, ABC$ in which D and E are mid-points of sides AB and AC respectively.

TO PROVE $DE \parallel BC$.

PROOF Since D and E are mid-points of AB and AC respectively.

$$\therefore \quad AD = DB \text{ and } AE = EC$$

$$\Rightarrow \quad \frac{AD}{DB} = 1 \text{ and } \frac{AE}{EC} = 1$$

$$\Rightarrow \quad \frac{AD}{DB} = \frac{AE}{EC}$$

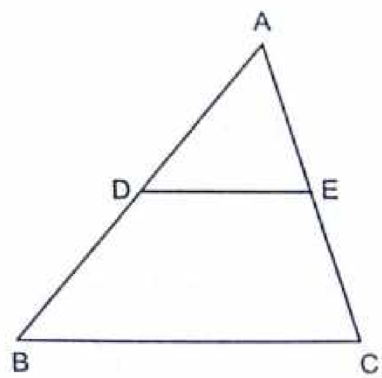


Fig. 7.62

Thus, the line DE divides the sides AB and AC of $\Delta\, ABC$ in the same ratio. Therefore, by the converse of Basic Proportionality Theorem, we obtain $DE \parallel BC$.

Q.E.D.

THEOREM 3 *Prove that the diagonals of a trapezium divide each other proportionally.* [NCERT]

GIVEN A trapezium $ABCD$ in which the diagonals AC and BD intersect at E.

TO PROVE $\frac{DE}{EB} = \frac{CE}{EA}$

CONSTRUCTION Draw $EF \parallel BA \parallel CD$, meeting AD in F.

PROOF In ΔABD, we have

$FE \parallel AB$

$\Rightarrow \quad \frac{DE}{EB} = \frac{DF}{FA}$ [By Thale's Theorem] ...(i)

In ΔCDA, we have

$FE \parallel DC$

$\Rightarrow \quad \frac{CE}{EA} = \frac{DF}{FA}$ [By Thale's Theorem] ...(ii)

From (i) and (ii), we get

$$\frac{DE}{EB} = \frac{CE}{EA}$$

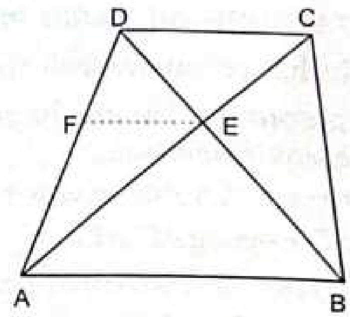


Fig. 7.63

Q.E.D.

THEOREM 4 *If the diagonals of a quadrilateral divide each other proportionally, then it is a trapezium.* **[NCERT, CBSE 2005]**

GIVEN A quadrilateral $ABCD$ whose diagonals AC and BD intersect at E such that $\frac{DE}{EB} = \frac{CE}{EA}$.

TO PROVE Quadrilateral $ABCD$ is a trapezium. For this it is sufficient to prove that $AB \parallel DC$.

CONSTRUCTION Draw $EF \parallel BA$, meeting AD in F.

PROOF In ΔABD, we have

$EF \parallel BA$

$\Rightarrow \quad \frac{DF}{FA} = \frac{DE}{EB}$ [By Thale's Theorem] ...(i)

But, $\quad \frac{DE}{EB} = \frac{CE}{EA}$ [Given] ...(ii)

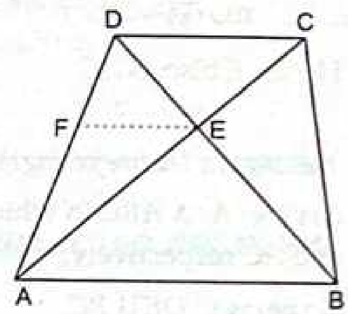


Fig. 7.64

From (i) and (ii), we get

$$\frac{DF}{FA} = \frac{CE}{EA}$$

Thus, in ΔDCA, E and F are points on CA and DA respectively such that

$$\frac{DF}{FA} = \frac{CE}{EA}$$

Therefore, by the converse of Basic Proportionality Theorem, we have

$FE \parallel DC$

But, $\quad FE \parallel BA$ [By construction]

$\therefore \quad DC \parallel BA \Rightarrow AB \parallel DC$

Hence, $ABCD$, is a trapezium.

Q.E.D.

THEOREM 5 *Any line parallel to the parallel sides of a trapezium divides the non-parallel sides proportionally.*

GIVEN A trapezium $ABCD$ in which $DC \parallel AB$ and EF is a line parallel to DC and AB.

TO PROVE $\frac{AE}{ED} = \frac{BF}{FC}$

CONSTRUCTION Join AC, meeting EF in G.

PROOF In ΔADC, we have

$EG \parallel DC$

$\Rightarrow \frac{AE}{ED} = \frac{AG}{GC}$ [By Thale's Theorem] ...(i)

In ΔABC, we have

$GF \parallel AB$

$\Rightarrow \frac{AG}{GC} = \frac{BF}{FC}$ [By Thale's Theorem] ...(ii)

From (i) and (ii), we get : $\frac{AE}{ED} = \frac{BF}{FC}$

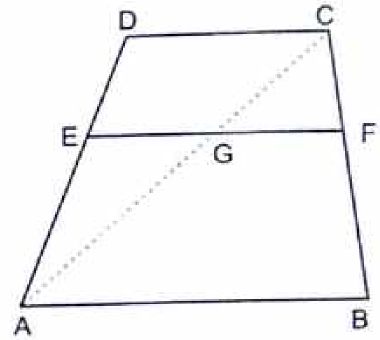


Fig. 7.65

Q.E.D.

THEOREM 6 *If three or more parallel lines are intersected by two transversals, prove that the intercepts made by them on the transversals are proportional.*

GIVEN Three parallel lines l, m, n which are cut by the transversals AB and CD in P, Q, R and E, F, G respectively.

TO PROVE $\frac{PQ}{QR} = \frac{EF}{FG}$

CONSTRUCTION Draw $PL \parallel CD$ meeting the lines m and n in M and L respectively.

PROOF Since $PE \parallel MF$ and $PM \parallel EF$.

$\therefore$ $PMFE$ is a parallelogram

$\Rightarrow PM = EF$ ...(i)

Also, $MF \parallel LG$ and $ML \parallel FG$.

$\therefore$ $MLGF$ is a parallelogram

$\Rightarrow ML = FG$ ...(ii)

Now, in ΔPRL, we have

$QM \parallel RL$

$\Rightarrow \frac{PQ}{QR} = \frac{PM}{ML}$ [By Thale's Theorem]

$\Rightarrow \frac{PQ}{QR} = \frac{EF}{FG}$ [Using (i) and (ii)]

Hence, $\frac{PQ}{QR} = \frac{EF}{FG}$. Q.E.D.

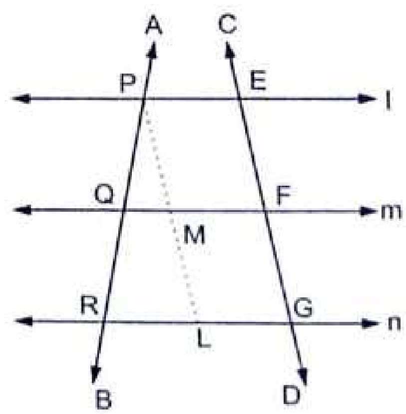


Fig. 7.66

COROLLARY *If three or more parallel straight lines make equal intercepts on a given transversal, prove that they will make equal intercepts on any other transversal.*

PROOF Let, l, m, n be three parallel lines which make equal intercepts PQ and QR on a transversal AB (see Fig. 7.66). Let CD be any other transversal cutting l, m and n at E, F and G respectively. Then,

$$\frac{PQ}{QR} = \frac{EF}{FG}$$ [By Theorem 6] ...(i)

But, $PQ = QR. \Rightarrow \frac{PQ}{QR} = 1$ ...(ii)

From (i) and (ii), we get

$$\frac{EF}{FG} = 1 \Rightarrow EF = FG$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Prove that the line segments joining the mid-points of the adjacent sides of a quadrilateral form a parallelogram.*

GIVEN A quadrilateral $ABCD$ in which P, Q, R and S are the mid-points of sides AB, BC, CD and DA respectively.

TO PROVE $PQRS$ is a parallelogram.

CONSTRUCTION Join AC.

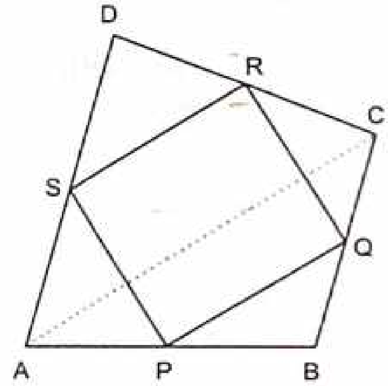


Fig. 7.67

PROOF In ΔABC, P and Q are the mid-points of AB and BC respectively.

$\therefore$ $PQ \parallel AC$ ...(i)

In ΔACD, R and S are the mid-points of CD and DA respectively.

$\therefore$ $SR \parallel AC$ ...(ii)

From (i) and (ii), we have

$PQ \parallel AC$ and $SR \parallel AC$

$\Rightarrow$ $PQ \parallel SR$

Similarly, by considering triangles ABD and BCD, we can prove that

$PS \parallel QR$

Hence, $PQRS$ is a parallelogram.

LEVEL-2

EXAMPLE 2 *In Fig. 7.68, P is the mid-point of BC and Q is the mid-point of AP. If BQ when produced meets AC at R, prove that* $RA = \frac{1}{3}CA$. **[CBSE 2006C]**

GIVEN A ΔABC in which P is the mid-point of BC, Q is the mid-point of BR and, Q is also the mid-point of AP such that BQ produced meets AC at R.

TO PROVE $RA = \frac{1}{3}CA$.

CONSTRUCTION Draw $PS \parallel BR$, meeting AC at S.

PROOF In ΔBCR, P is the mid-point of BC and $PS \parallel BR$.

$\therefore$ S is the mid-point of CR.

$\Rightarrow$ $CS = SR$ ...(i)

In ΔAPS, Q is the mid-point of AP and $QR \parallel PS$.

$\therefore$ R is the mid-point of AS.

$\Rightarrow$ $AR = RS$ ...(ii)

From (i) and (ii), we get

$AR = RS = SC$

$\Rightarrow$ $AC = AR + RS + SC = 3\,AR$

$\Rightarrow$ $AR = \frac{1}{3}AC = \frac{1}{3}CA$

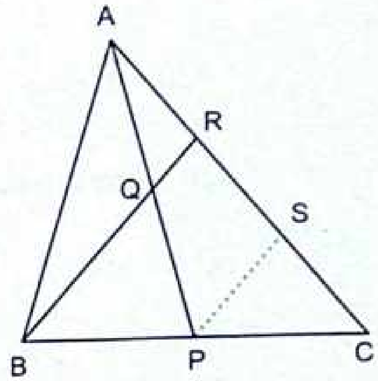


Fig. 7.68

EXAMPLE 3 *In Fig. 7. 69, AB || DC. Find the value of x.*

SOLUTION Since the diagonals of a trapezium divide each other proportionally.

$\therefore$ $\frac{AO}{OC} = \frac{BO}{OD}$ ...(iii)

$\Rightarrow$ $\frac{3x-19}{x-5} = \frac{x-3}{3}$

$\Rightarrow$ $3(3x-19) = (x-5)(x-3)$

$\Rightarrow$ $9x - 57 = x^2 - 8x + 15$

$\Rightarrow$ $x^2 - 17x + 72 = 0$

$\Rightarrow$ $(x-8)(x-9) = 0$

$\Rightarrow$ $x - 8 = 0$ or, $x - 9 = 0 \Rightarrow x = 8$ or, $x = 9$

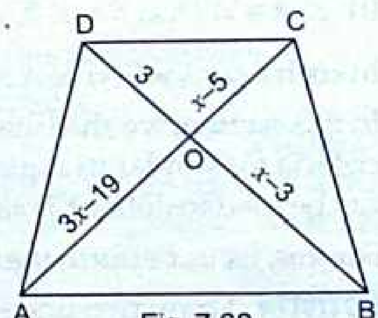


Fig. 7.69

EXERCISE 7.4

LEVEL-1

1. (i) In Fig. 7. 70, if $AB \parallel CD$, find the value of x.

   (ii) In Fig. 7. 71, if $AB \parallel CD$, find the value of x.

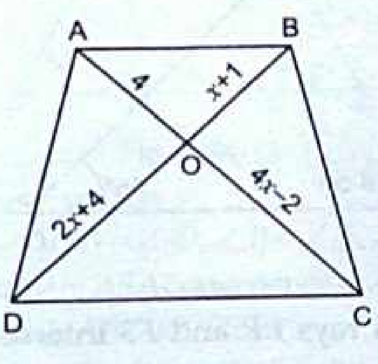


Fig. 7.70

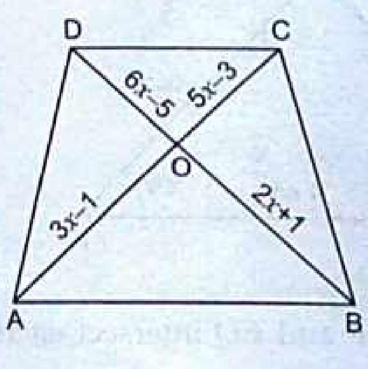


Fig. 7.71

   (iii) In Fig. 7.72, $AB \parallel CD$. If $OA = 3x - 19$, $OB = x - 4$, $OC = x - 3$ and $OD = 4$, find x. **[CBSE 2000C]**

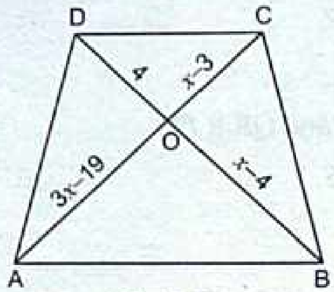


Fig. 7.72

ANSWERS

1. (i) 3 (ii) 2 (iii) 11 or, 8

7.7 CRITERIA FOR SIMILARITY OF TRIANGLES

In section 7. 3, we have defined similarity of two triangles. Let us recall that two triangles are similar iff (i) their corresponding angles are equal and (ii) their corresponding sides are proportional. In other words, two triangles *ABC* and *DEF* are similar, if

(i) $\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$ and, (ii) $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$

In such a case, we write $\Delta\, ABC \sim \Delta\, DEF$

In this section, we shall make use of the theorems discussed in earlier sections to derive some criteria for similar triangles which in turn will imply that either of the above two conditions can be used to define the similarity of two triangles.

For this, let us perform the following activity:

ACTIVITY Draw two line segments *BC* and *EF* of two different lengths, say 6 cm and 4 cm respectively. At *B* and *C* construct angles of some measures say 65° and 45° respectively. Also, construct angles of 65° and 45° at *E* and *F* respectively.

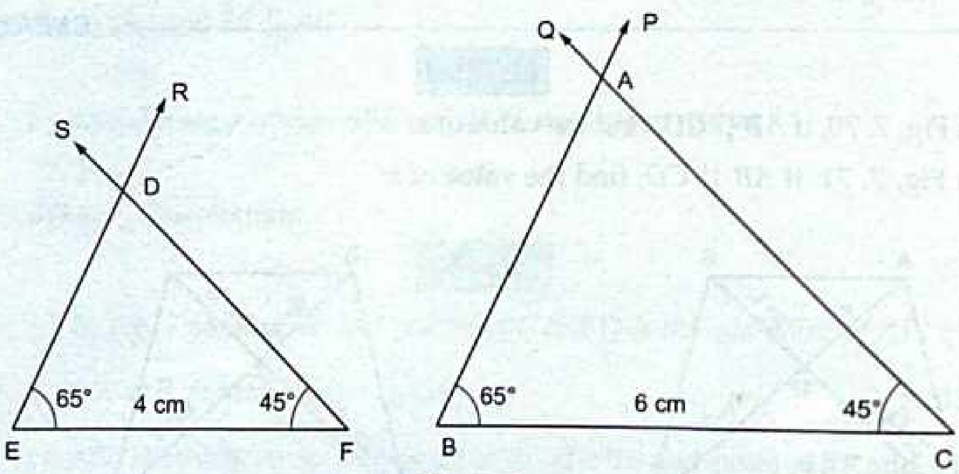


Fig. 7.73

Suppose rays *BP* and *CQ* intersect each other at *A* and rays *ER* and *FS* intersect each other at *D*.

We have,

$$\angle A = 180° - (\angle B + \angle C) = 180° - 110° = 70°$$

and, $\angle D = 180° - (\angle E + \angle F) = 180° - 110° = 70°$

In triangles ABC and DEF, we observe that

$$\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$$

That is, corresponding angles of these two triangles are equal. We also observe that

$$\frac{BC}{EF} = \frac{6}{4} = \frac{3}{2} = 1.5$$

Now, measure AB, DE, CA and FD and compute $\frac{AB}{DE}$ and $\frac{CA}{FD}$

You will find that $\frac{AB}{DE} = \frac{CA}{FD} = 1.5$

Thus, $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$

It follows from this activity that if corresponding angles of two triangles are equal, then their corresponding sides are in the same ratio.

Thus, we have following criterion for similarity of two triangles.

EQUIANGULAR TRIANGLES *Two triangles are said to be equiangular, if their corresponding angles are equal.*

THEOREM 1 *(AAA Similarity Criterion) If two triangles are equiangular, then they are similar.*

GIVEN Two triangles ABC and DEF such that $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$.

TO PROVE $\Delta ABC \sim \Delta DEF$ **[NCERT]**

PROOF Recall that two triangles are similar iff their corresponding angles are equal and the corresponding sides are proportional. Since corresponding angles are given equal, we must prove that the corresponding sides are proportional i.e.,

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

For this purpose we divide the proof into three parts.

CASE I *When $AB = DE$.*

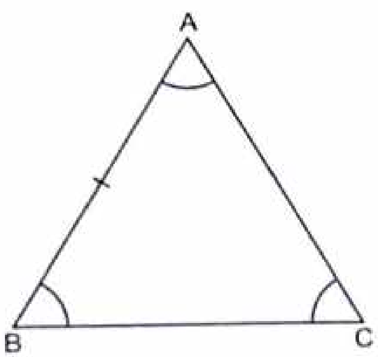


Fig. 7.74

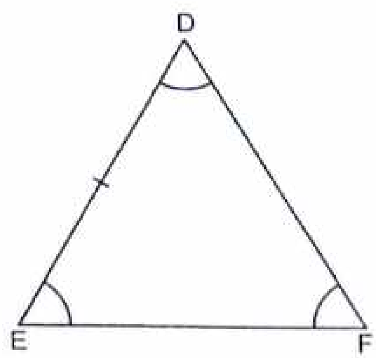


Fig. 7.75

In this case, we have

$$\angle A = \angle D, \angle B = \angle E, \angle C = \angle F \text{ and } AB = DE$$

Therefore, by ASA congruence criterion, we have

$$\Delta ABC \cong \Delta DEF$$

$\Rightarrow$ $AB = DE$, $BC = EF$ and $AC = DF$

$\Rightarrow$ $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

Hence, $\Delta ABC \sim \Delta DEF$.

CASE II *When* $AB < DE$.

Mark a point P on the line DE and Q on the line DF such that $AB = DP$ and $AC = DQ$. Join PQ.

In triangles ABC and DPQ, we have

$AB = DP$, $\angle A = \angle D$ and $AC = DQ$

$\therefore$ $\Delta ABC \cong \Delta DPQ$ [By SAS criterion of congruence]

$\Rightarrow$ $\angle B = \angle DPQ$

But, $\angle B = \angle E = \angle DEF$

$\therefore$ $\angle DPQ = \angle DEF$

$\Rightarrow$ $PQ \parallel EF$ [$\because$ Corresponding angles are equal]

$\Rightarrow$ $\dfrac{DP}{DE} = \dfrac{DQ}{DF}$ [By corollary of Thale's Theorem]

$\Rightarrow$ $\dfrac{AB}{DE} = \dfrac{AC}{DF}$

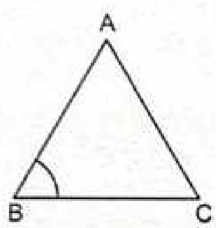


Fig. 7.76

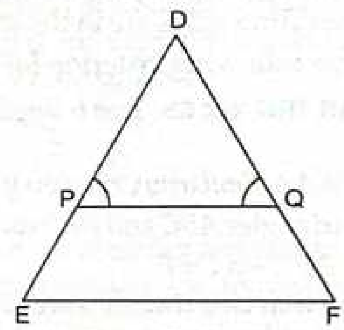


Fig. 7.77

Similarly, we can prove that

$$\frac{AB}{DE} = \frac{BC}{EF}$$

$\therefore$ $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$

Hence, $\Delta ABC \sim \Delta DEF$.

CASE III *When* $AB > DE$.

Mark a point P on the line DE produced and Q on the line DF produced such that $DP = AB$ and $DQ = AC$. Join PQ.

In triangles ABC and DPQ, we have

$AB = DP$, $AC = DQ$ and $\angle A = \angle D$.

$\therefore$ $\Delta ABC \cong \Delta DPQ$ [By SAS criterion of congruence]

$\Rightarrow$ $\angle B = \angle DPQ$

But, $\angle B = \angle E = \angle DEF$

$\therefore$ $\angle DPQ = \angle DEF$

$\Rightarrow$ $PQ \parallel EF$ [$\because$ Corresponding angles are equal]

$\Rightarrow$ $\dfrac{DE}{DP} = \dfrac{DF}{DQ}$ [By corollary of Thale's Theorem]

$\Rightarrow$ $\dfrac{DE}{AB} = \dfrac{DF}{AC}$ [$\because AB = DP$ and $AC = DQ$]

$$\Rightarrow \quad \frac{AB}{DE} = \frac{AC}{DF}$$

Similarly, we can prove that

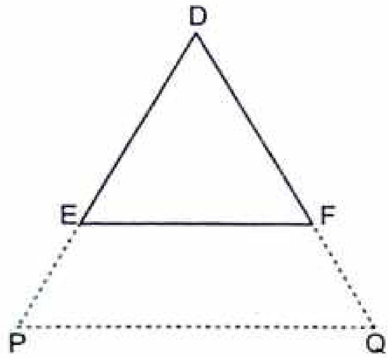


Fig. 7.78

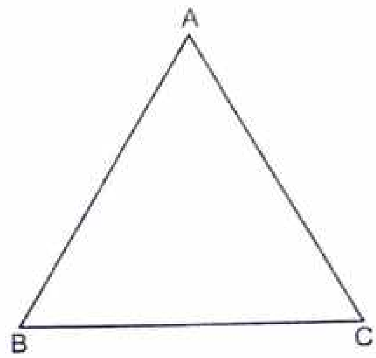


Fig. 7.79

$$\frac{AB}{DE} = \frac{BC}{EF}$$

$$\therefore \quad \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

Hence, $\Delta ABC \sim \Delta DEF$.

Q.E.D.

REMARK *It follows from the above theorem that : Two triangles are similar ⇔ They are equiangular.*

COROLLARY *(AA Similarity) If two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar.*

PROOF Let ΔABC and ΔDEF be two triangles such that $\angle A = \angle D$ and $\angle B = \angle E$.

In triangles ABC and DEF, we have

$$\angle A + \angle B + \angle C = 180° \text{ and } \angle D + \angle E + \angle F = 180°$$

$$\Rightarrow \quad \angle A + \angle B + \angle C = \angle D + \angle E + \angle F$$

$$\Rightarrow \quad \angle D + \angle E + \angle C = \angle D + \angle E + \angle F \qquad [\because \angle A = \angle D \text{ and } \angle B = \angle E]$$

$$\Rightarrow \quad \angle C = \angle F.$$

$$\therefore \quad \angle A = \angle D, \angle B = \angle E \text{ and } \angle C = \angle F.$$

Thus, the two triangles are equiangular and hence they are similar.

Q.E.D.

In the above discussion we have seen that if three angles of one triangle are respectively equal to three angles of another triangle, their corresponding sides are proportional and hence the triangles are similar. Now a natural question arises. Is the converse of this statement true? In other words, if the sides of a triangle are respectively proportional to the sides of another triangle, is it true that their corresponding angles are equal? For this, let us perform the following activity:

ACTIVITY Draw two triangles ABC and DEF such that $AB = 4.5$ cm, $BC = 9$ cm, $CA = 12$ cm, $DE = 3$ cm, $EF = 6$ cm and $FD = 8$ cm as shown in Fig. 7.80 (i) and (ii).

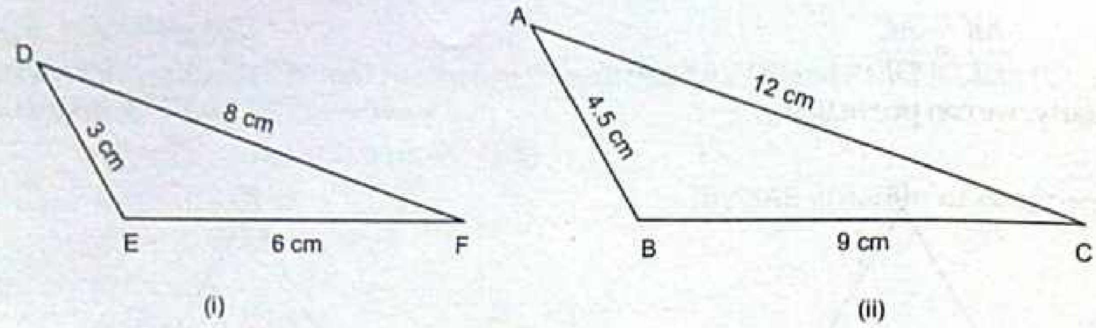


Fig. 7.80

We have,

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = \frac{3}{2}$$

That is the corresponding sides of triangles ABC and DEF are proportional.

Now, measure $\angle A, \angle B, \angle C, \angle D, \angle E$ and $\angle F$. You will observe that $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$ i.e., the corresponding angles of two triangles are equal and hence they are similar. Let us now prove this result as a criterion of similarity of two triangles as a theorem.

THEOREM 2 *(SSS Similarity Criterion) If the corresponding sides of two triangles are proportional, then they are similar.* **[NCERT]**

GIVEN Two triangles ABC and DEF such that $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

TO PROVE $\Delta ABC \sim \Delta DEF$

CONSTRUCTION Let P and Q be points on DE and DF respectively such that $DP = AB$ and $DQ = AC$. Join PQ.

PROOF We have,

$$\frac{AB}{DE} = \frac{AC}{DF}$$

$\Rightarrow \quad \frac{DP}{DE} = \frac{DQ}{DF}$ $\quad$ [$\because AB = DP$ and $AC = DQ$]

$\Rightarrow \quad PQ \parallel EF$ $\quad$ [By the converse of Thale's Theorem]

$\Rightarrow \quad \angle DPQ = \angle E$ and $\angle DQP = \angle F$ $\quad$ [Corresponding angles]

Thus, in triangles DPQ and DEF, we have

$$\angle DPQ = \angle E \text{ and } \angle DQP = \angle F$$

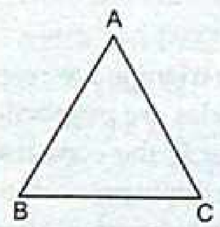


Fig. 7.81

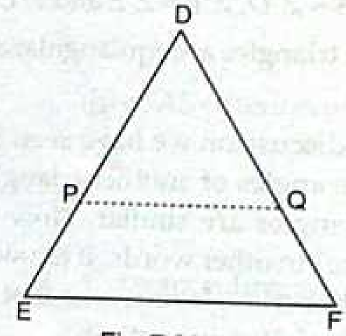


Fig. 7.82

Therefore, by AA-criterion of similarity, we have

$$\Delta DPQ \sim \Delta DEF \quad \text{...(i)}$$

$\Rightarrow \quad \frac{DP}{DE} = \frac{PQ}{EF}$ [By def. of similarity]

$\Rightarrow \quad \frac{AB}{DE} = \frac{PQ}{EF}$ $[\because DP = AB]$

But, $\quad \frac{AB}{DE} = \frac{BC}{EF}$

$\therefore \quad \frac{PQ}{EF} = \frac{BC}{EF}$

$\Rightarrow \quad PQ = BC$

Thus, in triangles ABC and DPQ, we have

$AB = DP, AC = DQ$ and $BC = PQ$

Therefore, by SSS criterion of congruence, we have

$\Delta ABC \cong \Delta DPQ$ ...(ii)

From (i) and (ii), we have

$\Delta ABC \cong \Delta DPQ$ and $\Delta DPQ \sim \Delta DEF$

$\Rightarrow \quad \Delta ABC \sim \Delta DPQ$ and $\Delta DPQ \sim \Delta DEF$ $[\because \Delta ABC \cong \Delta DPQ \Leftrightarrow \Delta ABC \sim \Delta DPQ]$

$\Rightarrow \quad \Delta ABC \sim \Delta DEF$

Q.E.D.

In view of the above two theorems, we can also give the following definitions of the similarity of two triangles.

DEFINITION 1 *Two triangles are similar if their corresponding angles are equal i.e. they are equiangular.*

DEFINITION 2 *Two triangles are similar if their corresponding sides are proportional.*

In class IX, we have learnt about various criteria for congruency of two triangles. We observe that corresponding to SSS congruence criterion there is SSS similarity criterion. This suggests us to look for a similarity criterion corresponding to SAS congruency criterion of triangles. To check the existence of such criterion, let us perform the following activity:

ACTIVITY Draw two triangles ABC and DEF such that $AB = 6$ cm, $\angle A = 60°$, $AC = 12$ cm, $DE = 4$ cm, $\angle D = 60°$ and $DF = 8$ cm as shown in Fig. 7.83.

We observe that $\frac{AB}{DE} = \frac{AC}{DF}$ (each equal to 3/2 and $\angle A$ (included between the sides AB and AC) is equal to $\angle D$ (included between the sides DE and DF). That is, one angle of a triangle is equal to one angle of another triangle and sides including those angles are in the same ratio.

Now, measure $\angle B$, $\angle C$, $\angle E$ and $\angle F$. You will find that $\angle B = \angle E$ and $\angle C = \angle F$. So, by AAA similarity criterion, we obtain $\Delta ABC \sim \Delta DEF$.

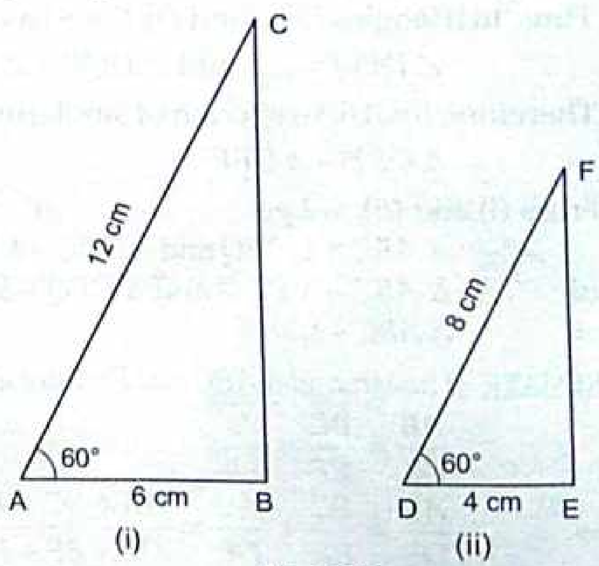


Fig. 7.83

It follows from the above activity that, if one angle of a triangle is equal to one angle of the other triangle and the sides including these angles are proportional, then the two triangles are similar. We prove the above observation as a theorem given below.

THEOREM 3 *(SAS Similarity Criterion) If in two triangles, one pair of corresponding sides are proportional and the included angles are equal then the two triangles are similar.* **[NCERT]**

GIVEN Two triangles ABC and DEF such that $\angle A = \angle D$ and $\frac{AB}{DE} = \frac{AC}{DF}$

TO PROVE $\Delta ABC \sim \Delta DEF$

CONSTRUCTION Mark points P and Q on DE and DF respectively such that $DP = AB$ and $DQ = AC$. Join PQ.

PROOF In triangles ABC and DPQ, we have

$$AB = DP, \angle A = \angle D \text{ and } AC = DQ$$

Therefore, by SAS Criterion of Congruence, we have

$$\Delta ABC \cong \Delta DPQ \quad \ldots(i)$$

Now, $\frac{AB}{DE} = \frac{AC}{DF}$

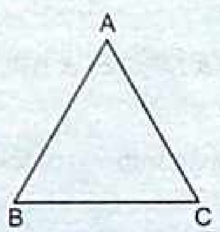


Fig. 7.84

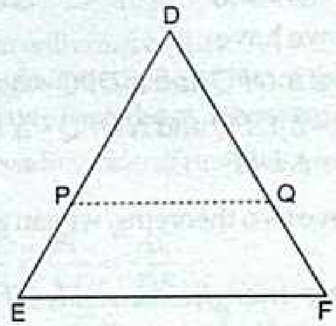


Fig. 7.85

$\Rightarrow \quad \frac{DP}{DE} = \frac{DQ}{DF}$ $\quad$ [$\because AB = DP$ and $AC = DQ$]

$\Rightarrow \quad PQ \parallel EF$ $\quad$ [By the converse of Thale's Theorem]

$\Rightarrow \quad \angle DPQ = \angle E$ and $\angle DQP = \angle F$[Corresponding angles]

Thus, in triangles DPQ and DEF, we have

$$\angle DPQ = \angle E \text{ and } \angle DQP = \angle F$$

Therefore, by AAA-criterion of similarity, we have

$$\Delta DPQ \sim \Delta DEF \quad \ldots(ii)$$

From (i) and (ii), we get

$\Delta ABC \cong \Delta DPQ$ and $\Delta DPQ \sim \Delta DEF$

$\Rightarrow \quad \Delta ABC \sim \Delta DPQ$ and $\Delta DPQ \sim \Delta DEF$

$\Rightarrow \quad \Delta ABC \sim \Delta DEF$

Q.E.D.

REMARK *If two triangles ABC and DEF are similar, then*

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

$\Rightarrow \quad \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = \frac{AB + BC + AC}{DE + EF + DF}$ $\quad$ [Using ratio and proportion]

$\Rightarrow \quad \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = \frac{\text{Perimeter of } \Delta ABC}{\text{Perimeter of } \Delta DEF}$

Thus, if two triangles are similar, then their corresponding sides are proportional and they are proportional to the corresponding perimerers.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Examine each pair of triangles in Fig. 7.86 and state which pair of triangles are similar. Also, state the similarity criterion used by you for answering the question and write the similarity relation in symbolic form.* **[NCERT]**

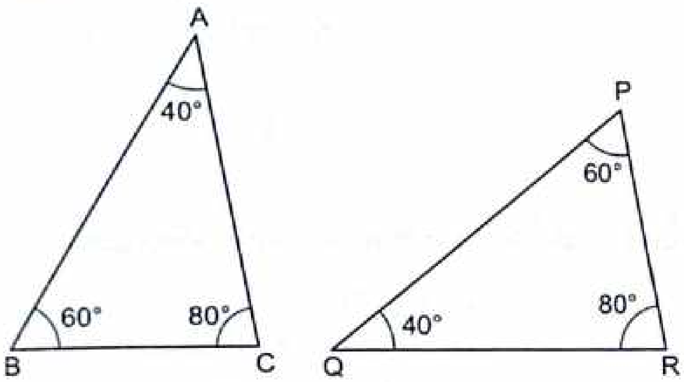


Fig. 7.86 (i)

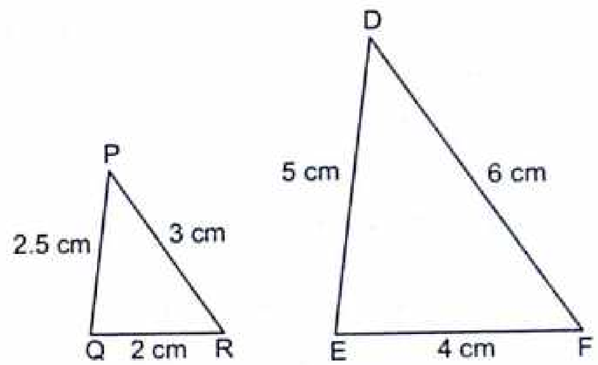


Fig. 7.86 (ii)

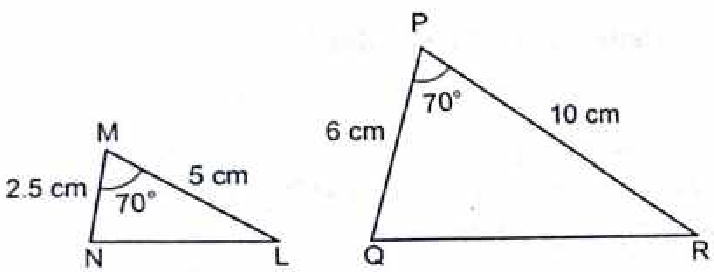


Fig. 7.86 (iii)

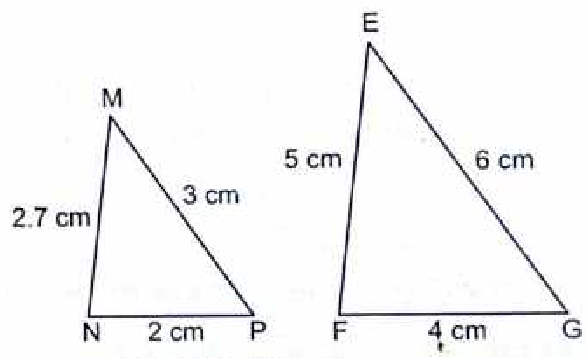


Fig. 7.86 (iv)

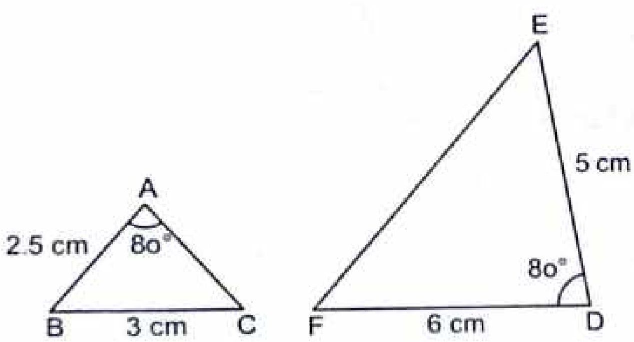


Fig. 7.86 (v)

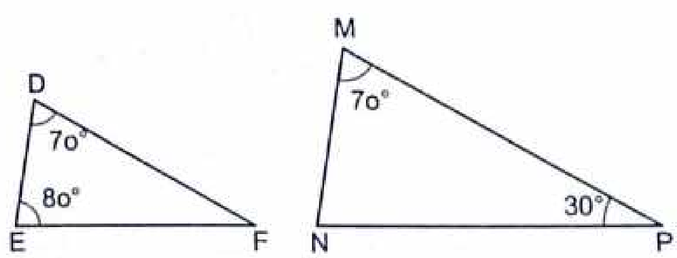


Fig. 7.86 (vi)

SOLUTION (i) In triangles ABC and PQR, we observe that

$$\angle A = \angle Q = 40^\circ, \angle B = \angle P = 60^\circ \text{ and } \angle C = \angle R = 80^\circ$$

Therefore, by AAA-criterion of similarity

$$\Delta ABC \sim \Delta QPR \text{ or, } \Delta PQR \sim \Delta BAC \text{ or, } \Delta ACB \sim \Delta QRP$$

(ii) In triangle PQR and DEF, we observe that

$$\frac{PQ}{DE} = \frac{QR}{EF} = \frac{PR}{DF} = \frac{1}{2}$$

Therefore, by *SSS*-criterion of similarity, we have

$$\Delta PQR \sim \Delta DEF$$

(iii) In triangles *LMN* and *PQR*, we have

$$\angle M = \angle P = 70^\circ$$

But, $\dfrac{MN}{PQ} \neq \dfrac{ML}{PR}$

Therefore, these two triangles are not similar as they do not satisfy SAS criterion of similarity.

(iv) In Δ's *MNP* and *EFG*, we observe that

$$\frac{NP}{FG} = \frac{MP}{EG} \neq \frac{MN}{EF}$$

Therefore, these two triangles are not similarly as they do not satisfy *SSS*-criterion of similarity.

(v) In Δ's *ABC* and *DEF*, we have

$$\angle A = \angle D = 80^\circ$$

But, $\dfrac{AB}{DE} \neq \dfrac{AC}{DF}$ [$\because AC$ is not given]

So, by *SAS*-criterion of similarity these two triangles are not similar.

(vi) In Δ's *DEF* and *MNP*, we have

$$\angle D = \angle M = 70^\circ$$

$$\angle E = \angle N = 80^\circ \qquad [\because \angle N = 180^\circ - \angle M - \angle P = 180^\circ - 70^\circ - 30^\circ = 80^\circ]$$

So, by *AA*-criterion of similarity, we obtain $\Delta DEF \sim \Delta MNP$.

EXAMPLE 2 *In Fig. 7.87, find ∠ F.*

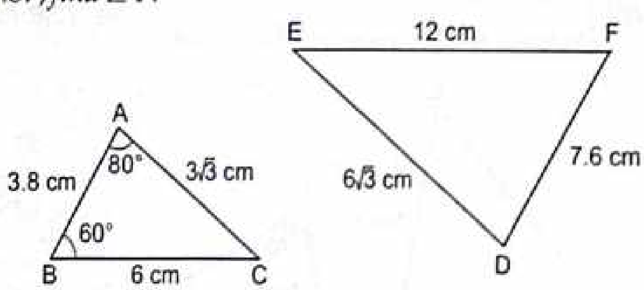


Fig. 7.87

SOLUTION In triangles *ABC* and *DEF*, we have

$$\frac{AB}{DF} = \frac{BC}{FE} = \frac{CA}{ED} = \frac{1}{2}$$

Therefore, by *SSS*-criterion of similarity, we have

$$\Delta ABC \sim \Delta DFE$$

$\Rightarrow \quad \angle A = \angle D, \angle B = \angle F$ and $\angle C = \angle E$

$\Rightarrow \quad \angle D = 80^\circ, \angle F = 60^\circ$

Hence, $\angle F = 60^\circ$.

EXAMPLE 3 *In Fig. 7.88, $\Delta ACB \sim \Delta APQ$. If $BC = 8$ cm, $PQ = 4$ cm, $BA = 6.5$ cm, $AP = 2.8$ cm, find CA and AQ.*

SOLUTION We have,

$\Delta ACB \sim \Delta APQ$

$\Rightarrow \quad \frac{AC}{AP} = \frac{CB}{PQ} = \frac{AB}{AQ}$

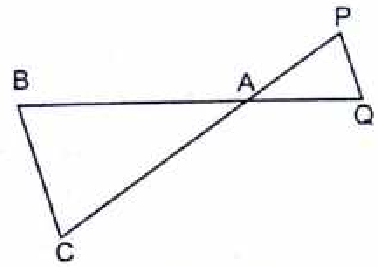


Fig. 7.88

$\Rightarrow \quad \frac{AC}{AP} = \frac{CB}{PQ}$ and $\frac{CB}{PQ} = \frac{AB}{AQ}$

$\Rightarrow \quad \frac{AC}{2.8} = \frac{8}{4}$ and $\frac{8}{4} = \frac{6.5}{AQ}$

$\Rightarrow \quad \frac{AC}{2.8} = 2$ and $\frac{6.5}{AQ} = 2 \Rightarrow AC = (2 \times 2.8)\text{ cm} = 5.6\text{ cm}$ and $AQ = \frac{6.5}{2}\text{ cm} = 3.25\text{ cm}$

EXAMPLE 4 *In Fig. 7.89, if $\Delta EDC \sim \Delta EBA$, $\angle BEC = 115°$ and $\angle EDC = 70°$. Find $\angle DEC$, $\angle DCE$, $\angle EAB$, $\angle AEB$ and $\angle EBA$.* **[NCERT]**

SOLUTION Since BD is a line and EC is a ray on it.

$\therefore \quad \angle DEC + \angle BEC = 180°$

$\Rightarrow \quad \angle DEC + 115° = 180°$

$\Rightarrow \quad \angle DEC = 180° - 115° = 65°$

But, $\quad \angle AEB = \angle DEC$ [Vertically opposite angles]

$\therefore \quad \angle AEB = 65°$

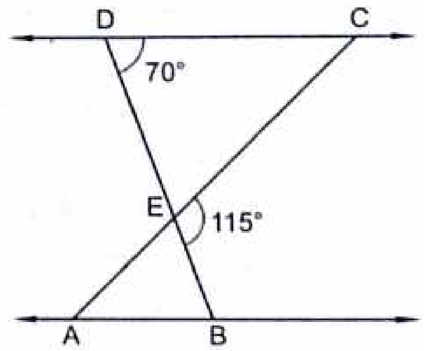


Fig. 7.89

In ΔCDE, we have

$\angle CDE + \angle DEC + \angle DCE = 180°$

$\Rightarrow \quad 70° + 65° + \angle DCE = 180°$

$\Rightarrow \quad \angle DCE = 180° - 135° = 45°$

It is given that $\Delta EDC \sim \Delta EBA$

$\therefore \quad \angle EBA = \angle EDC, \angle EAB = \angle ECD$

$\Rightarrow \quad \angle EBA = 70°$ and $\angle EAB = 45°$ $\quad [\because \angle ECD = \angle DCE = 45°]$

Hence, $\angle DEC = 65°$, $\angle DCE = 45°$, $\angle EAB = 45°$, $\angle AEB = 65°$ and $\angle EBA = 70°$.

EXAMPLE 5 *In Fig. 7.90, if $\Delta POS \sim \Delta ROQ$, prove that $PS \parallel QR$.* **[NCERT]**

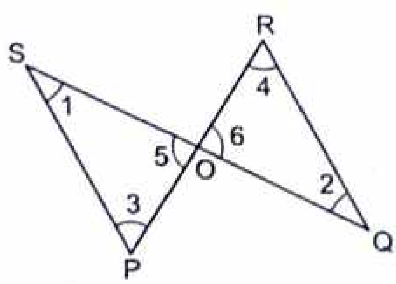


Fig. 7.90

SOLUTION We have,

$\Delta POS \sim \Delta ROQ$

$\Rightarrow \quad \angle 3 = \angle 4$ and $\angle 1 = \angle 2$

Thus, PS and QR are two lines and the transversal PR cuts them in such a way that $\angle 3 = \angle 4$ i.e., alternate angles are equal. Hence, $PS \parallel QR$.

EXAMPLE 6 *In Fig. 7.90, if $PS \parallel QR$, prove that $\Delta POS \sim \Delta ROQ$.*

SOLUTION It is given that $PS \parallel QR$ and transversal PR cuts them at P and R.

$\therefore \quad \angle 3 = \angle 4$

Again, $PS \parallel QR$ and transversal SQ cuts them at S and Q

$\therefore \quad \angle 1 = \angle 2$

Also, $\angle 5 = \angle 6$ [Vertically opposite angles]

Thus, in Δ POS and QOR, we have

$\angle 1 = \angle 2$ i. e., $\angle S = \angle Q$

$\angle 3 = \angle 4$ i.e., $\angle P = \angle R$

and, $\angle 5 = \angle 6$ i.e., $\angle POS = \angle QOR$

Therefore, by AAA-criterion of similarity, we obtain $\Delta POS \sim \Delta ROQ$.

EXAMPLE 7 *In Fig. 7.91, QA and PB are perpendiculars to AB. If AO = 10 cm, BO = 6 cm and PB = 9 cm. Find AQ.*

SOLUTION In triangles AOQ and BOP, we have

$\angle OAQ = \angle OBP$ [Each equal to 90°]

$\angle AOQ = \angle BOP$ [Vertically opposite angles]

Therefore, by AA- criterion of similarity, we obtain

$\Delta AOQ \sim \Delta BOP$

$$\Rightarrow \quad \frac{AO}{BO} = \frac{OQ}{OP} = \frac{AQ}{BP}$$

$$\Rightarrow \quad \frac{AO}{BO} = \frac{AQ}{BP}$$

$$\Rightarrow \quad \frac{10}{6} = \frac{AQ}{9}$$

$$\Rightarrow \quad AQ = \frac{10 \times 9}{6} = 15 \text{ cm}$$

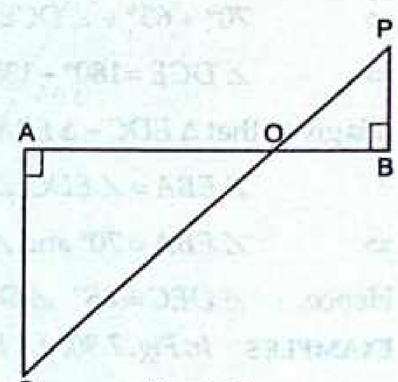


Fig. 7.91

EXAMPLE 8 *In Fig. 7.92, if $\angle ADE = \angle B$ show that $\Delta ADE \sim \Delta ABC$. If $AD = 3.8$ cm, $AE = 3.6$ cm, $BE = 2.1$ cm and $BC = 4.2$ cm, find DE.*

SOLUTION In triangles ADE and ABC, we have

$\angle ADE = \angle B$ (Given) and $\angle A = \angle A$ (Common)

So, by AA-criterion of similarity, we have

$$\Delta ADE \sim \Delta ABC$$

$$\Rightarrow \quad \frac{AD}{AB} = \frac{DE}{BC}$$

$$\Rightarrow \quad \frac{AD}{AE + EB} = \frac{DE}{BC}$$

$$\Rightarrow \quad \frac{3.8}{3.6 + 2.1} = \frac{DE}{4.2}$$

$$\Rightarrow \quad DE = \frac{3.8 \times 4.2}{3.6 + 2.1} \text{ cm} = 2.8 \text{ cm}$$

Hence, $DE = 2.8$ cm

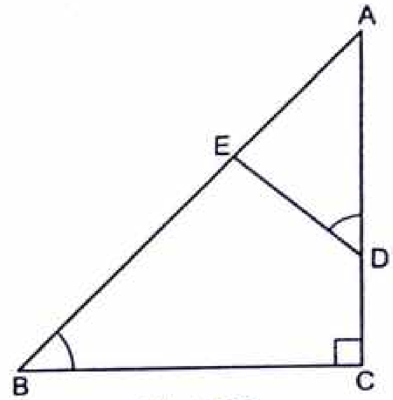


Fig. 7.92

EXAMPLE 9 *In Fig. 7.93, $\frac{AO}{OC} = \frac{BO}{OD} = \frac{1}{2}$ and $AB = 5$ cm. Find the value of DC.*

SOLUTION In $\Delta\ AOB$ and $\Delta\ COD$, we have

$$\angle AOB = \angle COD \quad \text{[Vertically opposite angles]}$$

$$\frac{AO}{OC} = \frac{OB}{OD} \quad \text{[Given]}$$

So, by SAS-criterion of similarity, we have

$$\Delta AOB \sim \Delta COD$$

$$\Rightarrow \quad \frac{AO}{OC} = \frac{BO}{OD} = \frac{AB}{DC}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{5}{DC} \quad [\because AB = 5 \text{ cm}]$$

$$\Rightarrow \quad DC = 10 \text{ cm}$$

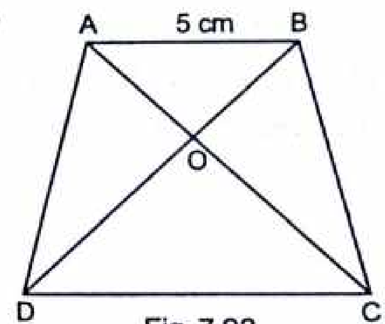


Fig. 7.93

EXAMPLE 10 *In Fig. 7.94, if $\angle A = \angle C$, then prove that $\Delta\ AOB \sim \Delta\ COD$.*

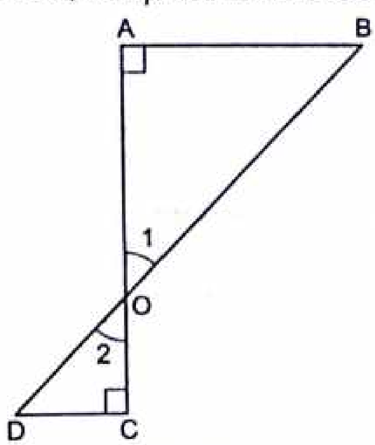


Fig. 7.94

SOLUTION In triangles AOB and COD, we obtain

$\angle A = \angle C$ (Given)

and, $\angle 1 = \angle 2$ [Vertically opposite angles]

Therefore, by AA-criterion of similarity, we obtain

$$\Delta AOB \sim \Delta COD$$

EXAMPLE 11 *In Fig.7.95, if $AB \perp BC$ and $DE \perp AC$. Prove that $\Delta ABC \sim \Delta AED$.* **[CBSE 2009]**

SOLUTION In Δ's *ABC* and *AED*, we have

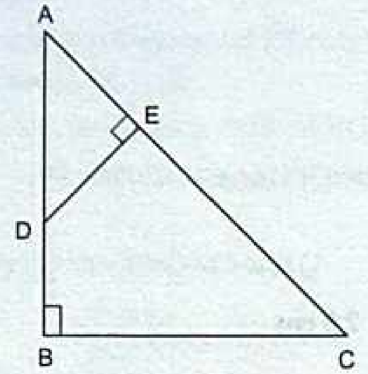


Fig. 7.95

$\angle ABC = \angle AED = 90°$

$\angle BAC = \angle EAD$ [Each equal to $\angle A$]

Therefore, by *AA*-criterion of similarity, we obtain $\Delta ABC \sim \Delta AED$.

EXAMPLE 12 *In Fig. 7.96, if $\angle P = \angle RTS$, prove that $\Delta RPQ \sim \Delta RTS$.*

SOLUTION In triangles *RPQ* and *RTS*, we have

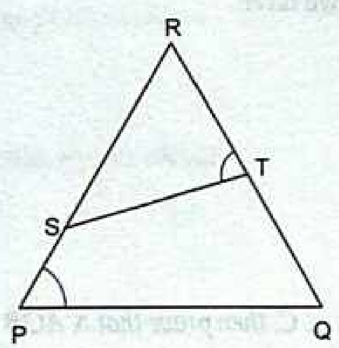


Fig. 7.96

$\angle RPQ = \angle RTS$ [Given]

$\angle PRQ = \angle TRS$ [Each equal to *R*]

Therefore, by *AA*-criterion of similarity, we obtain $\Delta RPQ \sim \Delta RTS$.

EXAMPLE 13 *In Fig. 7.97, if $\frac{QT}{PR} = \frac{QR}{QS}$ and $\angle 1 = \angle 2$. Prove that $\Delta PQS \sim \Delta TQR$.* **[NCERT]**

SOLUTION We have,

$$\frac{QT}{PR} = \frac{QR}{QS}$$ [Given]

$$\Rightarrow \quad \frac{QT}{QR} = \frac{PR}{QS}$$ ...(i)

We also have,

$\angle 1 = \angle 2$ [Given]

$\Rightarrow \quad PR = PQ$ $\left[\begin{array}{l}\text{Sides opposite to equal} \\ \text{angles are equal}\end{array}\right]$ ...(ii)

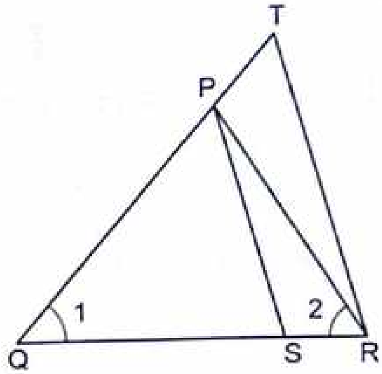


Fig. 7.97

From (i) and (ii), we get

$$\frac{QT}{QR} = \frac{PQ}{QS}$$

$$\Rightarrow \quad \frac{PQ}{QT} = \frac{QS}{QR} \quad \text{...(iii)}$$

Thus, in triangles PQS and TQR, we have

$$\frac{PQ}{QT} = \frac{QS}{QR} \text{ and } \angle PQS = \angle TQR = \angle Q$$

So, by SAS-criterion of similarity, we obtain $\Delta PQS \sim \Delta TQR$.

EXAMPLE 14 *In Fig. 7.98, AD and CE are two altitudes of ΔABC. Prove that*

(i) $\Delta AEF \sim \Delta CDF$ (ii) $\Delta ABD \sim \Delta CBE$

(iii) $\Delta AEF \sim \Delta ADB$ (iv) $\Delta FDC \sim \Delta BEC$ **[NCERT]**

SOLUTION (i) In triangles AEF and CDF, we have

$\angle AEF = \angle CDF = 90°$ [$\because CE \perp AB$ and $AD \perp BC$]

$\angle AFE = \angle CFD$ [Vertically opposite angles]

Thus, by AA-criterion of similarity, we have

$\Delta AEF \sim \Delta CDF$

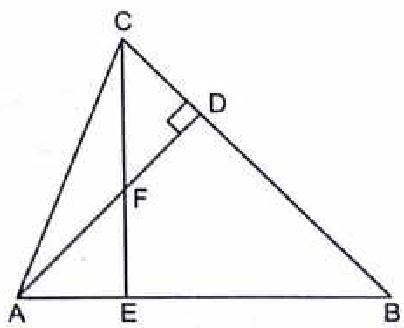


Fig. 7.98

(ii) In Δ's ABD and CBE, we have

$\angle ABD = \angle CBE = \angle B$ [Common angle]

$\angle ADB = \angle CEB = 90°$ [$\because AD \perp BC$ and $CE \perp AB$]

Thus, by AA-criterion of similarity, we have

$\Delta ABD \sim \Delta CBE$

(iii) In Δ's AEF and ADB, we have

$\angle AEF = \angle ADB = 90°$ [$\because AD \perp BC$ and $CE \perp AB$]

$\angle FAE = \angle DAB$ [Common angle]

Thus, by AA-criterion of similarity, we have

$\Delta AEF \sim \Delta ADB$

(iv) In Δ's FDC and BEC, we have

$\angle FDC = \angle BEC = 90°$ [$\because AD \perp BC$ and $CE \perp AB$]

$\angle FCD = \angle ECB$ [Common angle]

Thus, by *AA*-criterion of similarity, we obtain $\Delta FDC \sim \Delta BEC$.

EXAMPLE 15 *In Fig. 7.99(i) and (ii), if CD and GH (D and H lie on AB and FE) are respectively bisectors of $\angle ACB$ and $\angle EGF$ and $\Delta ABC \sim \Delta FEG$, prove that* **[NCERT]**

(i) $\Delta DCA \sim \Delta HGF$ (ii) $\frac{CD}{GH} = \frac{AC}{FG}$ (iii) $\Delta DCB \sim \Delta HGE$

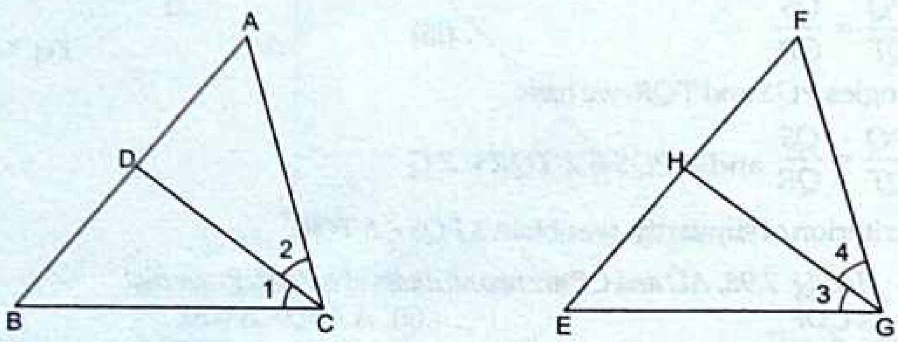


Fig. 7.99 (i) Fig. 7.99 (ii)

SOLUTION (i) We have,

$\Delta ABC \sim \Delta FEG$

$\Rightarrow \quad \angle A = \angle F$

and, $\quad \angle C = \angle G$ ...(i)

$\Rightarrow \quad \frac{1}{2}\angle C = \frac{1}{2}\angle G$

$\Rightarrow \quad \angle 1 = \angle 3$ and $\angle 2 = \angle 4$ $\left[\because CD \text{ and } GH \text{ are bisector of } \angle C \text{ and } \angle G \text{ respectively}\right]$ ...(ii)

Thus, in Δ's *ACD* and *FGH*, we have

$\angle A = \angle F$ [From (i)]

$\angle 2 = \angle 4$ [From (ii)]

Therefore, by *AA*-criterion of similarity, we obtain

$\Delta ACD \sim \Delta FGH$ or, $\Delta DCA \sim \Delta HGF$

(ii) We have,

$$\Delta ACD \sim \Delta FGH \Rightarrow \frac{AC}{FG} = \frac{CD}{GH}$$

(iii) In Δ's *DCB* and *HGE*, we have

$\angle 1 = \angle 3$ [From (ii)]

$\angle B = \angle E$ [$\because \Delta ABC \sim \Delta FEG$]

Thus, by *AA*-criterion of similarity, we obtain $\Delta DCB \sim \Delta HGE$.

EXAMPLE 16 *In Fig. 7.100, CD and GH are respectively the medians of ΔABC and ΔEFG. If $\Delta ABC \sim \Delta FEG$, prove that*

(i) $\Delta ADC \sim \Delta FHG$ (ii) $\frac{CD}{GH} = \frac{AB}{FE}$ (iii) $\Delta CDB \sim \Delta GHE$

[NCERT]

SOLUTION It is given that CD and GD are medians of Δ's ABC and EFG respectively.

$$\therefore \quad 2AD = AB \text{ and } 2FH = FE \qquad \ldots(i)$$

It is also given that $\Delta\, ABC \sim \Delta\, FEG$

$$\therefore \quad \frac{AB}{FE} = \frac{AC}{FG} = \frac{BC}{EG} \text{ and, } \angle A = \angle F, \angle B = \angle E, \angle C = \angle G \qquad \ldots(ii)$$

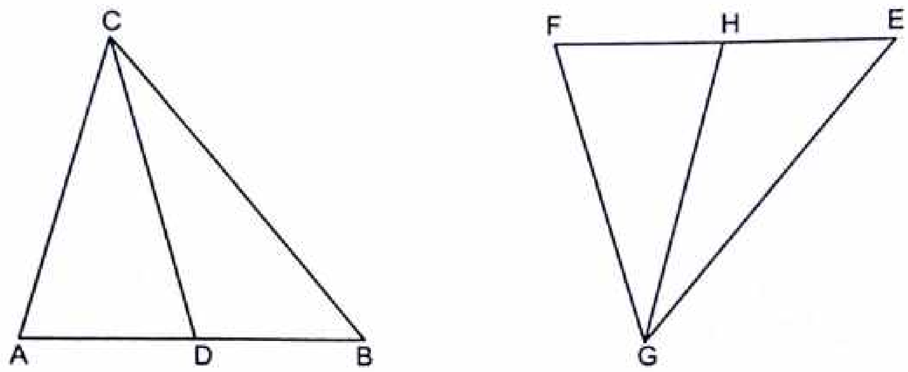


Fig. 7.100

Now, $\dfrac{AB}{FE} = \dfrac{AC}{FG} = \dfrac{BC}{EG}$

$\Rightarrow \quad \dfrac{2AD}{2FH} = \dfrac{AC}{FG} = \dfrac{BC}{EG}$ [Using (i)]

$\Rightarrow \quad \dfrac{AD}{FH} = \dfrac{AC}{FG} = \dfrac{BC}{EG}$ ...(iii)

(i) In Δ's ADC and FHG, we have

$\dfrac{AD}{FH} = \dfrac{AC}{FG}$ [From (iii)]

and, $\angle A = \angle F$

So, by SAS criterion of similarity, we obtain $\Delta\, ADC \sim \Delta\, FHG$.

(ii) We have,

$\Delta\, ADC \sim \Delta\, FHG$ [Proved above]

$\Rightarrow \quad \dfrac{DC}{HG} = \dfrac{AD}{FH}$

$\Rightarrow \quad \dfrac{CD}{GH} = \dfrac{2AD}{2FH}$

$\Rightarrow \quad \dfrac{CD}{GH} = \dfrac{AB}{FE}$ [$\because AB = 2\,AD$ and $FE = 2\,FH$]

(iii) We have,

$\dfrac{AB}{FE} = \dfrac{AC}{FG} = \dfrac{BC}{EG}$ [From (i)]

Also, $\dfrac{CD}{GH} = \dfrac{AB}{FE}$ [As proved above]

$\therefore \quad \dfrac{CD}{GH} = \dfrac{BC}{EG}$ ...(iv)

Again, $\frac{AB}{FE} = \frac{AC}{FG} = \frac{BC}{EG}$

$\Rightarrow \quad \frac{2DB}{2HE} = \frac{BC}{EG}$ [$\because$ D and H are mid-points of AB and FE respectively]

$\Rightarrow \quad \frac{DB}{HE} = \frac{BC}{EG}$ ...(v)

From (iv) and (v), we have

$$\frac{CD}{GH} = \frac{BC}{EG} = \frac{DB}{HE}$$

$\Rightarrow \quad \frac{CD}{GH} = \frac{DB}{HE} = \frac{BC}{EG}$

$\Rightarrow \quad \Delta CDB \sim \Delta GHE$ [By *SSS* criterion of similarity]

EXAMPLE 17 *In Fig. 7.101, if $BD \perp AC$ and $CE \perp AB$, prove that*

(i) $\Delta AEC \sim \Delta ADB$ (ii) $\frac{CA}{AB} = \frac{CE}{DB}$ **[NCERT]**

SOLUTION (i) In Δ's AEC and ADB, we have

$\angle AEC = \angle ADB = 90°$ [$\because CE \perp AB$ and $BD \perp AC$]

and, $\angle EAC = \angle DAB$ [Each equal to $\angle A$]

Therefore, by *AA*-criterion of similarity, we obtain

$$\Delta AEC \sim \Delta ADB$$

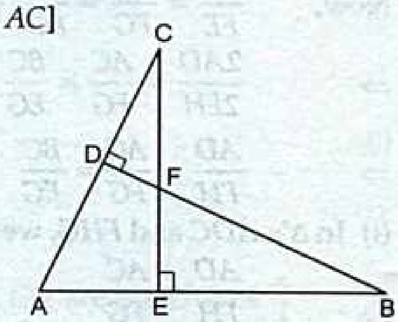


Fig. 7.101

(ii) We have,

$\Delta AEC \sim \Delta ADB$ [As proved above]

$\Rightarrow \quad \frac{CA}{BA} = \frac{EC}{DB}$

$\Rightarrow \quad \frac{CA}{AB} = \frac{CE}{DB}$

EXAMPLE 18 *D is a point on the side BC of ΔABC such that $\angle ADC = \angle BAC$. Prove that $\frac{CA}{CD} = \frac{CB}{CA}$ or, $CA^2 = CB \times CD$.* **[NCERT, CBSE 2004]**

SOLUTION In ΔABC and ΔDAC, we have

$\angle ADC = \angle BAC$ and $\angle C = \angle C$

Therefore, by *AA*-criterion of similarity, we obtain

$$\Delta ABC \sim \Delta DAC$$

$\Rightarrow \quad \frac{AB}{DA} = \frac{BC}{AC} = \frac{AC}{DC}$

$\Rightarrow \quad \frac{CB}{CA} = \frac{CA}{CD}$

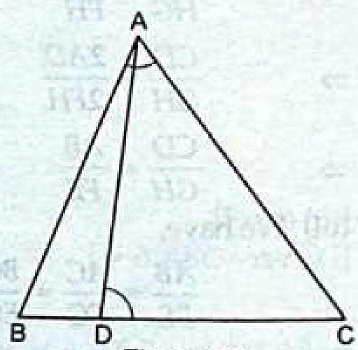


Fig. 7.102

EXAMPLE 19 *In Fig. 7.103, considering triangles BEP and CPD, prove that $BP \times PD = EP \times PC$.*

GIVEN A ΔABC in which $BD \perp AC$ and $CE \perp AB$ and BD and CE intersect at P.

TO PROVE $BP \times PD = EP \times PC$

PROOF In $\Delta\, EPB$ and $\Delta\, DPC$, we have

$\angle PEB = \angle PDC$ [Each equal to 90°]

$\angle EPB = \angle DPC$ [Vertically opposite angles]

Thus, by *AA*-criterion of similarity, we obtain

$$\Delta\, EPB \sim \Delta\, DPC$$

$$\frac{EP}{DP} = \frac{PB}{PC}$$

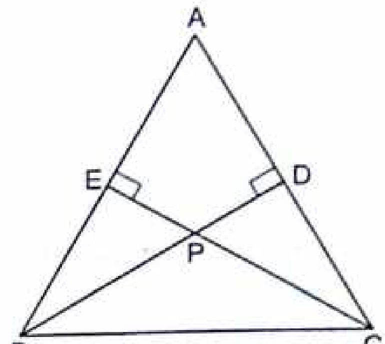


Fig. 7.103

$$\Rightarrow \quad BP \times PD = EP \times PC$$

EXAMPLE 20 *P and Q are points on sides AB and AC respectively of $\Delta\, ABC$. If $AP = 3$ cm, $PB = 6$ cm, $AQ = 5$ cm and $QC = 10$ cm, show that $BC = 3\,PQ$.*

SOLUTION We have,

$AB = AP + PB = (3 + 6)$ cm $= 9$ cm and, $AC = AQ + QC = (5 + 10)$ cm $= 15$ cm.

$$\therefore \quad \frac{AP}{AB} = \frac{3}{9} = \frac{1}{3} \text{ and } \frac{AQ}{AC} = \frac{5}{15} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{AP}{AB} = \frac{AQ}{AC}$$

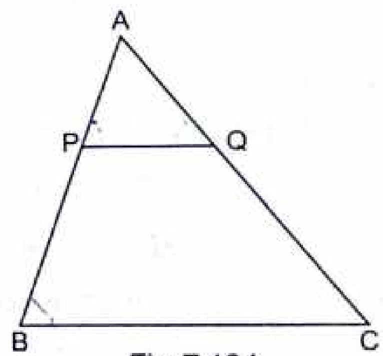


Fig. 7.104

Thus, in triangles APQ and ABC, we have

$$\frac{AP}{AB} = \frac{AQ}{AC} \text{ and } \angle A = \angle A \quad \text{[Common]}$$

Therefore, by *SAS*-criterion of similarity, we have

$$\Delta\, APQ \sim \Delta\, ABC$$

$$\Rightarrow \quad \frac{AP}{AB} = \frac{PQ}{BC} = \frac{AQ}{AC}$$

$$\Rightarrow \quad \frac{PQ}{BC} = \frac{AQ}{AC}$$

$$\Rightarrow \quad \frac{PQ}{BC} = \frac{5}{15} \Rightarrow \frac{PQ}{BC} = \frac{1}{3} \Rightarrow BC = 3PQ$$

EXAMPLE 21 *In Fig. 7.105, express x in terms of a, b and c.*

SOLUTION In $\Delta\, KPN$ and $\Delta\, KLM$, we have

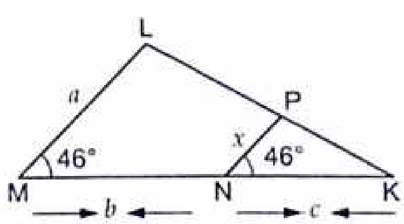


Fig. 7.105

$\angle KNP = \angle KML = 46°$ [Given]

$\angle K = \angle K$ [Common]

$\therefore \quad \Delta\, KNP \sim \Delta\, KML$ [By *AA*-criterion of similarity]

$\Rightarrow \quad \frac{KN}{KM} = \frac{NP}{ML}$ [∵ Corresponding sides of similar triangles are proportional]

$\Rightarrow \quad \frac{c}{b+c} = \frac{x}{a} \Rightarrow x = \frac{ac}{b+c}$

EXAMPLE 22 *The diagonal BD of a parallelogram ABCD intersects the segment AE at the point F, where E is any point on the side BC. Prove that DF × EF = FB × FA*

SOLUTION In ΔAFD and ΔBFE, we have

$\angle 1 = \angle 2$ [Vertically opposite angles]

$\angle 3 = \angle 4$ [Alternate angles]

So, by *AA*-criterion of similarity, we have

$\Delta FBE \sim \Delta FDA$

$\Rightarrow \quad \frac{FB}{FD} = \frac{FE}{FA}$

$\Rightarrow \quad \frac{FB}{DF} = \frac{EF}{FA}$

$\Rightarrow \quad DF \times EF = FB \times FA$

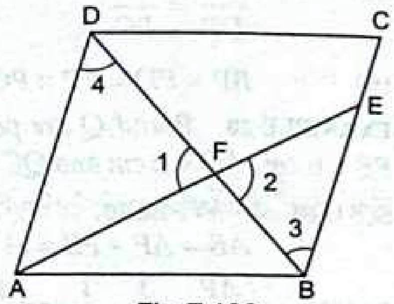


Fig. 7.106

EXAMPLE 23 *In a ΔABC, BD and CE are the altitudes. Prove that ΔADB and ΔAEC are similar. Is $\Delta CDB \sim \Delta BEC$?*

SOLUTION In ΔABD and ΔAEC, we obtain

$\angle ADB = \angle AEC$ [Each equal to 90°]

$\angle BAD = \angle EAC$ [Common]

So, by *AA*-criterion of similarity, we have

$\Delta BDA \sim \Delta CEA$ or, $\Delta ADB \sim \Delta AEC$.

Clearly, ΔCDB is not similar to ΔBEC, because they are not equiangular.

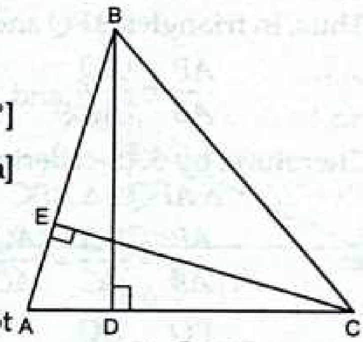


Fig. 7.107

EXAMPLE 24 *E is a point on side AD produced of a parallelogram ABCD and BE intersects CD at F. Prove that $\Delta ABE \sim \Delta CFB$.* **[NCERT, CBSE 2008]**

SOLUTION In Δ's *ABE* and *CFB*, we have

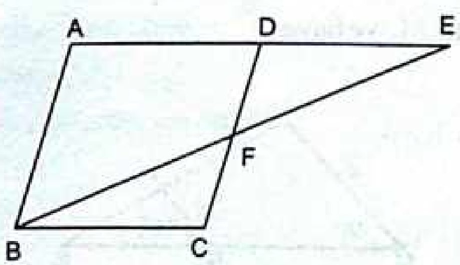


Fig. 7.108

$\angle AEB = \angle CBF$ [Alternate angles]

$\angle A = \angle C$ [Opposite angles of a parallelogram]

Thus, by *AA*-criterion of similarity, we have

$\Delta ABE \sim \Delta CFB$.

EXAMPLE 25 *In Fig. 7.109, AD and BE are respectively perpendiculars to BC and AC. Show that*

(i) $\Delta ADC \sim \Delta BEC$ (ii) $CA \times CE = CB \times CD$

(iii) $\Delta ABC \sim \Delta DEC$ (iv) $CD \times AB = CA \times DE$

SOLUTION (i) in Δ's ADC and BEC, we have

$\angle ADC = \angle BEC = 90°$ [Given]

$\angle ACD = \angle BCE$ [Common]

So, by AA-criterion of similarity, we obtain

$\Delta ADC \sim \Delta BEC$

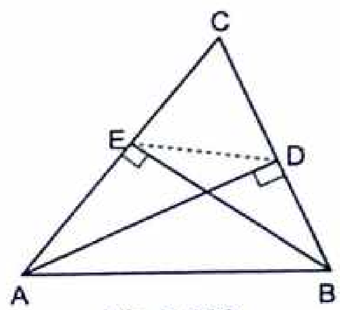


Fig. 7.109

(ii) We have,

$\Delta ADC \sim \Delta BEC$ [As proved above]

$\Rightarrow \quad \dfrac{AC}{BC} = \dfrac{DC}{EC}$ ...(i)

$\Rightarrow \quad CA \times CE = CB \times CD$

(iii) In Δ's ABC and DEC, we have

$\dfrac{AC}{BC} = \dfrac{DC}{EC}$ [From(i)]

$\Rightarrow \quad \dfrac{AC}{DC} = \dfrac{BC}{EC}$

Also, $\angle ACB = \angle DCE$

So, by SAS-criterion of similarity, we obtain

$\Delta ABC \sim \Delta DEC$ [Common]

(iv) We have,

$\Delta ABC \sim \Delta DEC$ [As proved above]

$\Rightarrow \quad \dfrac{AB}{DE} = \dfrac{AC}{DC} \Rightarrow AB \times DC = AC \times DE \Rightarrow CD \times AB = CA \times DE$

EXAMPLE 26 *In Fig. 7.110, E is a point on side CB produced of an isosceles triangle ABC with AB = AC. If AD ⊥ BC and EF ⊥ AC, prove that (i) Δ ABD ~ Δ ECF (ii) AB × EF = AD × EC.*

[NCERT, CBSE 2010]

SOLUTION It is given that ΔABC is isosceles with

$AB = AC$

$\therefore \quad \angle B = \angle C$

Now, in Δ's ABD and ECF, we have

$\angle ABD = \angle ECF$ [$\because \angle B = \angle C$]

$\angle ADB = \angle EFC = 90°$ [$\because AD \perp BC$ and $EF \perp AC$]

So, by AA-criterion of similarity, we have

$\Delta ABD \sim \Delta ECF$

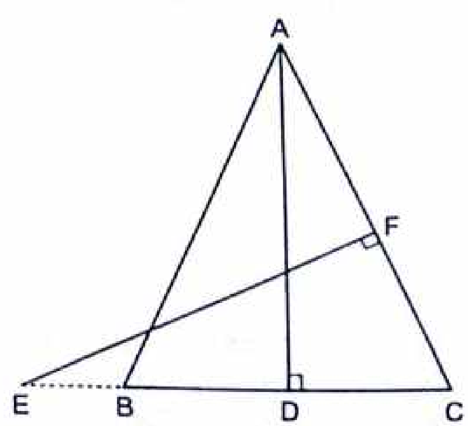


Fig. 7.110

$\Rightarrow \quad \dfrac{AB}{EC} = \dfrac{AD}{EF}$

$\Rightarrow \quad AB \times EF = AD \times EC$

EXAMPLE 27 *In Fig. 7.111, $\Delta FEC \cong \Delta GBD$ and $\angle 1 = \angle 2$. Prove that $\Delta ADE \sim \Delta ABC$.*

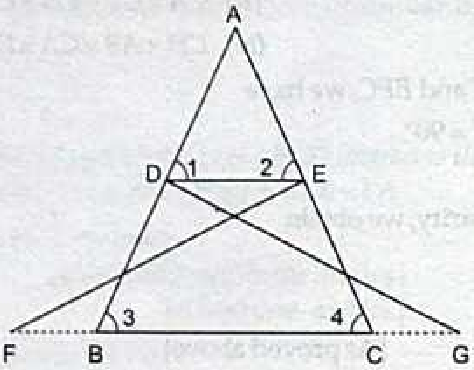


Fig. 7.111

SOLUTION We have,

$\Delta FEC \cong \Delta GBD$

$\Rightarrow \quad EC = BD$ ...(i)

It is given that

$\angle 1 = \angle 2$

$\Rightarrow \quad AD = AE$ [Sides opposite to equal angles are equal] ...(ii)

From (i) and (ii), we have

$$\frac{AE}{EC} = \frac{AD}{BD}$$

$\Rightarrow \quad DE \parallel BC$ [By the converse of basic proportionality theorem]

$\Rightarrow \quad \angle 1 = \angle 3$ and $\angle 2 = \angle 4$

Thus, in Δ's ADE and ABC, we have

$\angle A = \angle A$

$\angle 1 = \angle 3$

$\angle 2 = \angle 4$

So, by *AAA*-criterion of similarity, we have

$\Delta ADE \sim \Delta ABC$

EXAMPLE 28 *In Fig. 7.112, $\frac{OA}{OC} = \frac{OD}{OB}$. Prove that $\angle A = \angle C$ and $\angle B = \angle D$.* **[NCERT]**

SOLUTION in Δ's AOD and COB, we have

$$\frac{OA}{OC} = \frac{OD}{OB}$$

$$\Rightarrow \quad \frac{OA}{OD} = \frac{OC}{OB}$$

Also, $\angle 1 = \angle 2$ [Vertically opposite angles]

So, by *SAS*-criterion of similarity, we obtain

$\Delta AOD \sim \Delta COB$

$\Rightarrow \quad \angle A = \angle C$ and $\angle B = \angle D$

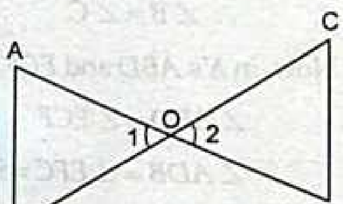


Fig. 7.112

EXAMPLE 29 *If a perpendicular is drawn from the vertex containing the right angle of a right triangle to the hypotenuse then prove that the triangle on each side of the perpendicular are similar to each other and to the original triangle. Also, prove that the square of the perpendicular is equal to the product of the lengths of the two parts of the hypotenuse.*

GIVEN A right triangle ABC right angled at B. $BD \perp AC$.

TO PROVE (i) $\Delta ADB \sim \Delta BDC$ (ii) $\Delta ADB \sim \Delta ABC$ (iii) $\Delta BDC \sim \Delta ABC$

(iv) $BD^2 = AD \times DC$ (v) $AB^2 = AD \times AC$ (vi) $BC^2 = CD \times AC$

[CBSE 2009]

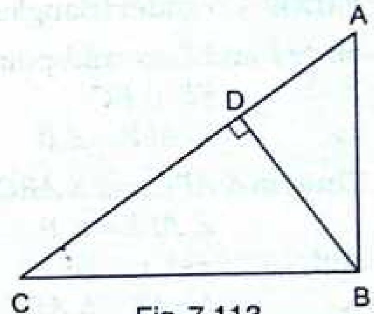


Fig. 7.113

PROOF (i) We have,

$\angle ABD + \angle DBC = 90°$

Also, $\angle C + \angle DBC + \angle BDC = 180°$

$\Rightarrow \angle C + \angle DBC + 90° = 180°$

$\Rightarrow \angle C + \angle DBC = 90°$

But, $\angle ABD + \angle DBC = 90°$

$\therefore \angle ABD + \angle DBC = \angle C + \angle DBC$

$\Rightarrow \angle ABD = \angle C$ ...(i)

Thus, in ΔADB and ΔBDC, we have

$\angle ABD = \angle C$ [From (i)]

and, $\angle ADB = \angle BDC$ [Each equal to 90°]

So, by AA-similarity criterion, we obtain $\Delta ADB \sim \Delta BDC$.

(ii) In ΔADB and ΔABC, we have

$\angle ADB = \angle ABC$ [Each equal to 90°]

and, $\angle A = \angle A$ [Common]

So, by AA-similarity criterion, we obtain $\Delta ADB \sim \Delta ABC$.

(iii) In ΔBDC and ΔABC, we have

$\angle BDC = \angle ABC$ [Each equal to 90°]

$\angle C = \angle C$ [Common]

So, by AA-similarity criterion, we obtain $\Delta BDC \sim \Delta ABC$.

(iv) From (i), we have

$\Delta ADB \sim \Delta BDC$

$\Rightarrow \frac{AD}{BD} = \frac{BD}{DC}$

$\Rightarrow BD^2 = AD \times DC$

(v) From (ii), we have

$\Delta ADB \sim \Delta ABC$

$\Rightarrow \frac{AD}{AB} = \frac{AB}{AC}$

$\Rightarrow AB^2 = AD \times AC$

(vi) From (iii), we have

$\Delta BDC \sim \Delta ABC$

$\Rightarrow \frac{BC}{AC} = \frac{DC}{BC} \Rightarrow BC^2 = CD \times AC$

EXAMPLE 30 *Prove that the line segments joining the mid-points of the sides of a triangle form four triangles, each of which is similar to the original triangle.*

SOLUTION

GIVEN ΔABC in which D, E, F are the mid-points of sides BC, CA and AB respectively.

TO PROVE Each of the triangles AFE, FBD, EDC and DEF is similar to ΔABC.

PROOF Consider triangles AFE and ABC.

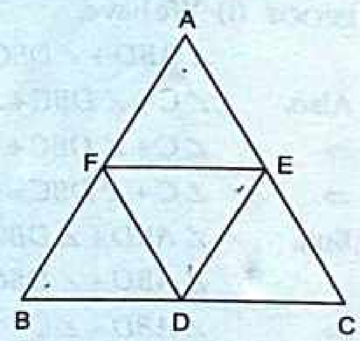


Fig. 7.114

Since F and E are mid-points of AB and AC respectively.

$\therefore \quad FE \parallel BC$

$\Rightarrow \quad \angle AFE = \angle B$ [Corresponding angles]

Thus, in ΔAFE and ΔABC, we have

$\angle AFE = \angle B$

and, $\angle A = \angle A$ [Common]

$\therefore \quad \Delta AFE \sim \Delta ABC.$

Similarly, we have

$\Delta FBD \sim \Delta ABC$ and $\Delta EDC \sim \Delta ABC$.

Now, we shall show that $\Delta DEF \sim \Delta ABC$.

Clearly, $ED \parallel AF$ and $DF \parallel EA$.

$\therefore \quad AFDE$ is a parallelogram.

$\Rightarrow \quad \angle EDF = \angle A$ [$\because$ Opposite angles of a parallelogram are equal]

Similarly, $BDEF$ is a parallelogram.

$\therefore \quad \angle DEF = \angle B$ [$\because$ Opposite angles of a parallelogram are equal]

Thus, in triangles DEF and ABC, we have

$\angle EDF = \angle A$ and $\angle DEF = \angle B$

So, by AA-criterion of similarity, we have

$\Delta DEF \sim \Delta ABC.$

Thus, each one of the triangles AFE, FBD, EDC and DEF is similar to ΔABC.

EXAMPLE 31 *In ΔABC, DE is parallel to base BC, with D on AB and E on AC. If $\frac{AD}{DB} = \frac{2}{3}$, find $\frac{BC}{DE}$.*

[CBSE 2002 C]

SOLUTION In ΔABC, we have

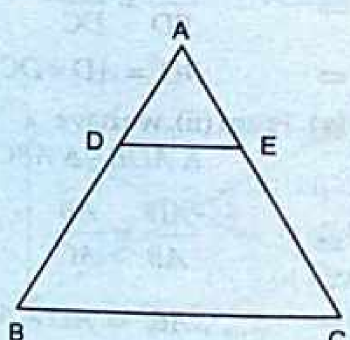


Fig. 7.115

$DE \parallel BC$

$\Rightarrow \quad \frac{AB}{AD} = \frac{AC}{AE}$

Thus, in triangles ABC and ADE, we have

$\frac{AB}{AD} = \frac{AC}{AE}$

and, $\angle A = \angle A$

Therefore, by SAS-criterion of similarity, we have

$\Delta ABC \sim \Delta ADE$

$\Rightarrow \quad \frac{AB}{AD} = \frac{BC}{DE}$ ...(i)

It is given that

$$\frac{AD}{DB}=\frac{2}{3}$$

$$\Rightarrow \quad \frac{DB}{AD}=\frac{3}{2}$$

$$\Rightarrow \quad \frac{DB}{AD}+1=\frac{3}{2}+1 \Rightarrow \frac{DB+AD}{AD}=\frac{5}{2} \Rightarrow \frac{AB}{AD}=\frac{5}{2} \quad \ldots(ii)$$

From (i) and (ii), we get $\frac{BC}{DE}=\frac{5}{2}$.

EXAMPLE 32 *In Fig. 7.116, if $\Delta ABE \cong \Delta ACD$, prove that $\Delta ADE \sim \Delta ABC$.* **[NCERT]**

SOLUTION It is given that $\Delta ABE \cong \Delta ACD$.

$\therefore \quad AB=AC$ [$\because$ Corresponding parts of congruent triangles are equal]

and, $\quad AE=AD$

$$\Rightarrow \quad \frac{AB}{AD}=\frac{AC}{AE}$$

$$\Rightarrow \quad \frac{AB}{AC}=\frac{AD}{AE} \quad \ldots(i)$$

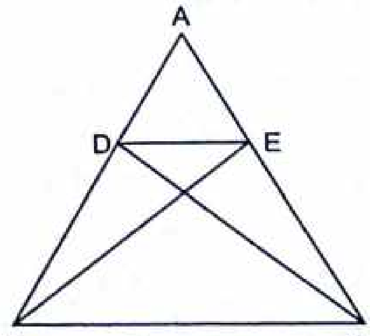


Fig. 7.116

Thus, in triangles ADE and ABC, we obtain

$$\frac{AB}{AC}=\frac{AD}{AE}$$

and, $\quad \angle BAC=\angle DAE$ [Common]

Hence, by SAS-criterion of similarity, we obtain $\Delta ADE \sim \Delta ABC$.

LEVEL-2

EXAMPLE 33 *A vertical stick 12 m long casts a shadow 8 m long on the ground. At the same time a tower casts the shadow 40 m long on the ground. Determine the height of the tower.*

SOLUTION Let AB be the vertical stick and AC be its shadow. Also, let DE be the vertical tower and DF be its shadow. Join BC and EF. Let $DE=x$ metres.

We have,

$$AB=12 \text{ m}, AC=8 \text{ m}, \text{ and } DF=40 \text{ m}.$$

In ΔABC and ΔDEF, we have

$\angle A=\angle D=90°$ and $\angle C=\angle F$ [Angular elevation of the sum]

Therefore, by AA- criterion of similarity, we obtain $\Delta ABC \sim \Delta DEF$

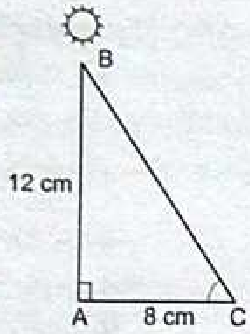


Fig. 7.117

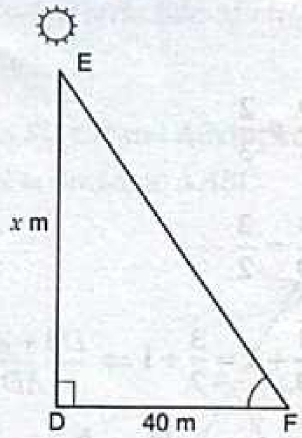


Fig. 7.118

$$\Rightarrow \quad \frac{AB}{DE} = \frac{AC}{DF}$$

$$\Rightarrow \quad \frac{12}{x} = \frac{8}{40} \Rightarrow \frac{12}{x} = \frac{1}{5} \Rightarrow x = 60 \text{ metres}$$

EXAMPLE 34 *In Fig. 7.119, $\angle CAB = 90°$ and $AD \perp BC$. If $AC = 75$ cm, $AB = 1$ m and $BD = 1.25$ m, find AD.*

SOLUTION We have,

$$AB = 1 \text{ m} = 100 \text{ cm}, AC = 75 \text{ cm and } BD = 125 \text{ cm}.$$

In ΔBAC and ΔBDA, we have

$\angle BAC = \angle BDA$ [Each equal to 90°]

and, $\angle B = \angle B$

So, by AA-criterion of similarity, we obtain

$$\Delta BAC \sim \Delta BDA$$

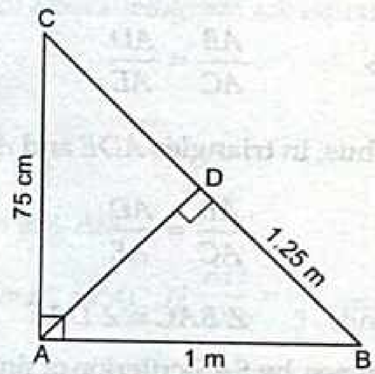


Fig. 7.119

$$\Rightarrow \quad \frac{BA}{BD} = \frac{AC}{AD}$$

$$\Rightarrow \quad \frac{100}{125} = \frac{75}{AD}$$

$$\Rightarrow \quad AD = \frac{125 \times 75}{100} \text{ cm} = 93.75 \text{ cm}$$

EXAMPLE 35 *The perimeters of two similar triangles are 30 cm and 20 cm respectively. If one side of the first triangle is 12 cm, determine the corresponding side of the second triangle.*

SOLUTION Let ΔABC and ΔDEF be two similar triangles of perimeters P_1 and P_2 respectively. Also, let $AB = 12$ cm, $P_1 = 30$ cm and $P_2 = 20$ cm. Then,

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = \frac{P_1}{P_2}$$ [∵ Ratio of corresponding sides of similar triangles is equal to the ratio of their perimeters]

$$\Rightarrow \quad \frac{AB}{DE} = \frac{P_1}{P_2}$$

$$\Rightarrow \quad \frac{12}{DE} = \frac{30}{20}$$

$$\Rightarrow \quad DE = \frac{12 \times 20}{30} \text{ cm} = 8 \text{ cm}$$

Hence, the corresponding side of the second triangle is 8 cm.

EXAMPLE 36 *The perimeters of two similar triangles ABC and PQR are respectively 36 cm and 24 cm. If PQ = 10 cm, find AB.*

SOLUTION Since the ratio of the corresponding sides of similar triangles is same as the ratio of their perimeters.

$$\therefore \quad \Delta ABC \sim \Delta PQR$$

$$\Rightarrow \quad \frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR} = \frac{36}{24}$$

$$\Rightarrow \quad \frac{AB}{PQ} = \frac{36}{24}$$

$$\Rightarrow \quad \frac{AB}{10} = \frac{36}{24}$$

$$\Rightarrow \quad AB = \frac{36 \times 10}{24} \text{ cm} = 15 \text{ cm}$$

EXAMPLE 37 *Two triangles BAC and BDC, right angled at A and D respectively, are drawn on the same base BC and on the same side of BC. If AC and DB intersect at P, prove that $AP \times PC = DP \times PB$.* **[CBSE 2000C]**

SOLUTION In ΔAPB and ΔDPC, we have

$\angle A = \angle D = 90°$

and, $\angle APB = \angle DPC$ [Vertically opposite angles]

Thus, by *AA*-criterion of similarity, we obtain

$\Delta APB \sim \Delta DPC$

$$\Rightarrow \quad \frac{AP}{DP} = \frac{PB}{PC}$$

$$\Rightarrow \quad AP \times PC = DP \times PB$$

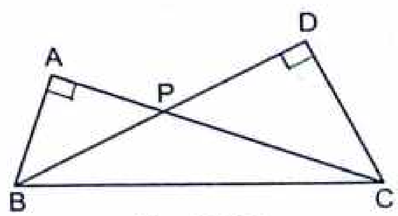


Fig. 7.120

EXAMPLE 38 *In Fig. 7.121, $\angle BAC = 90°$ and segment $AD \perp BC$. Prove that $AD^2 = BD \times DC$.*

SOLUTION In ΔABD and ΔACD, we have

$\angle ADB = \angle ADC$ [Each equal to 90°]

and, $\angle DBA = \angle DAC$ $\left[\begin{array}{l}\text{Each equal to complement of} \\ \angle BAD \text{ i.e. } 90° - \angle BAD\end{array}\right]$

Therefore, by AA-criterion of similarity, we have

$\Delta DBA \sim \Delta DAC$ $\left[\begin{array}{l}\therefore \angle D \leftrightarrow \angle D, \angle B \leftrightarrow \angle DAC \\ \text{and } \angle BAD \leftrightarrow \angle DCA\end{array}\right]$

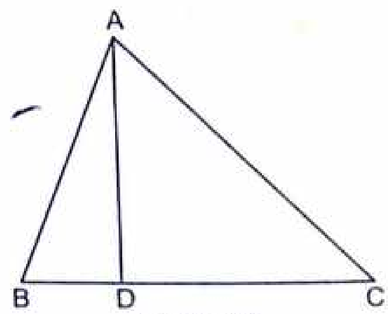


Fig. 7.121

$$\Rightarrow \quad \frac{DB}{DA} = \frac{DA}{DC} \quad \text{[In similar triangles corresponding sides are proportional]}$$

$$\Rightarrow \quad \frac{BD}{AD} = \frac{AD}{DC}$$

$$\Rightarrow \quad AD^2 = BD \times DC$$

EXAMPLE 39 *In ΔABC, if $AD \perp BC$ and $AD^2 = BD \times DC$, prove that $\angle BAC = 90°$.*

SOLUTION We have,

$$AD^2 = BD \times DC$$

$$\Rightarrow \quad AD \times AD = BD \times DC$$

$$\Rightarrow \quad \frac{AD}{DC} = \frac{BD}{AD}$$

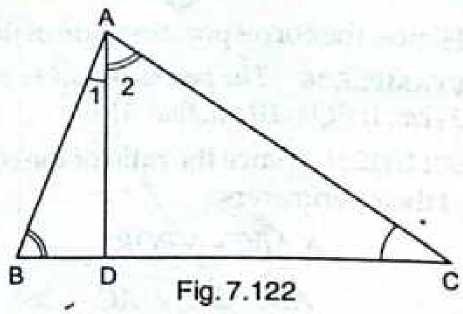


Fig. 7.122

Thus, in ΔABD and ΔACD, we have

$$\frac{AD}{DC} = \frac{BD}{AD}$$

and, $\angle BDA = \angle CDA$ [Each equal to 90°]

So, by *SAS*-criterion of similarity, we get

$$\Delta DBA \sim \Delta DAC$$

$\Rightarrow$ ΔDBA and ΔDAC are equiangular

$$\Rightarrow \quad \angle 1 = \angle C \text{ and } \angle 2 = \angle B$$

$$\Rightarrow \quad \angle 1 + \angle 2 = \angle B + \angle C$$

$$\Rightarrow \quad \angle A = \angle B + \angle C \quad [\because \angle 1 + \angle 2 = \angle A]$$

But, $\angle A + \angle B + \angle C = 180°$

$$\therefore \quad \angle A + \angle A = 180° \quad [\because \angle B + \angle C = \angle A]$$

$$\Rightarrow \quad 2\angle A = 180° \Rightarrow \angle A = 90°$$

Hence, $\angle BAC = 90°$.

EXAMPLE 40 *In Fig. 7.123, ABCD is a trapezium with $AB \parallel DC$. If ΔAED is similar to ΔBEC, prove that $AD = BC$.*

SOLUTION In ΔEDC and ΔEBA, we have

$\angle 1 = \angle 2$ [Alternate angles]

$\angle 3 = \angle 4$ [Alternate angles]

and, $\angle CED = \angle AEB$ [Vertically opposite angles]

$$\therefore \quad \Delta EDC \sim \Delta EBA$$

$$\Rightarrow \quad \frac{ED}{EB} = \frac{EC}{EA}$$

$$\Rightarrow \quad \frac{ED}{EC} = \frac{EB}{EA} \quad \text{...(i)}$$

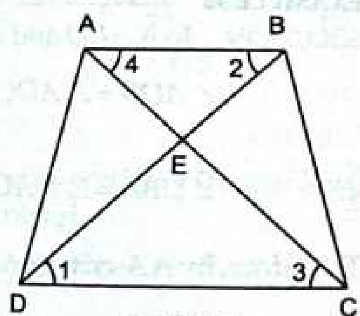


Fig. 7.123

It is given that $\Delta AED \sim \Delta BEC$

$$\therefore \quad \frac{ED}{EC} = \frac{EA}{EB} = \frac{AD}{BC} \quad \text{...(ii)}$$

From (i) and (ii), we get

$$\frac{EB}{EA} = \frac{EA}{EB}$$

$$\Rightarrow \quad (EB)^2 = (EA)^2$$

$$\Rightarrow \quad EB = EA$$

Substituting $EB = EA$ in (ii), we get

$$\frac{EA}{EA} = \frac{AD}{BC}$$

$\Rightarrow \quad \dfrac{AD}{BC} = 1$

$\Rightarrow \quad AD = BC$

EXAMPLE 41 *Through the mid-point M of the side CD of a parallelogram ABCD, the line BM is drawn intersecting AC in L and AD produced in E. Prove that EL = 2 BL.* **[CBSE 2009]**

SOLUTION In $\Delta\, BMC$ and $\Delta\, EMD$, we have

$MC = MD$ [$\because$ M is the mid-point of CD]

$\angle\, CMB = \angle\, EMD$ [Vertically opposite angles]

and, $\angle\, MBC = \angle\, MED$ [Alternate angles]

So, by AAS-criterion of congruence, we have

$\therefore \quad \Delta\, BMC \cong \Delta\, EMD$

$\Rightarrow \quad BC = DE$ ...(i)

Also, $AD = BC$ [$\because$ $ABCD$ is a parallelogram] ...(ii)

$AD + DE = BC + BC$

$\Rightarrow \quad AE = 2\,BC$ ...(iii)

Now, in $\Delta\, AEL$ and $\Delta\, CBL$, we have

$\angle\, ALE = \angle\, CLB$ [Vertically opposite angles]

$\angle\, EAL = \angle\, BCL$ [Alternate angles]

So, by AA-criterion of similarity of triangles, we have

$\Delta\, AEL \sim \Delta\, CBL$

$\Rightarrow \quad \dfrac{EL}{BL} = \dfrac{AE}{CB}$

$\Rightarrow \quad \dfrac{EL}{BL} = \dfrac{2BC}{BC}$ [Using equations (iii)]

$\Rightarrow \quad \dfrac{EL}{BL} = 2$

$\Rightarrow \quad EL = 2\,BL$

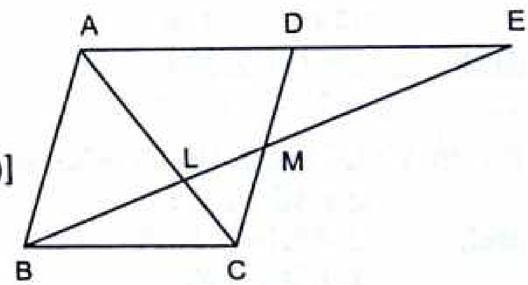


Fig. 7.124

EXAMPLE 42 *In a $\Delta\, ABC$, let P and Q be points on AB and AC respectively such that PQ || BC. Prove that the median AD bisects PQ.*

SOLUTION Suppose the median AD intersects PQ at E.

Now, $PQ \parallel BC$

$\Rightarrow \quad \angle\, APE = \angle\, B$ and $\angle\, AQE = \angle\, C$ [Corresponding angles]

So, in Δ's APE and ABD, we have

$\angle\, APE = \angle\, ABD$

and, $\angle\, PAE = \angle\, BAD$ [Common]

$\therefore \quad \Delta\, APE \sim \Delta\, ABD$

$\Rightarrow \quad \dfrac{PE}{BD} = \dfrac{AE}{AD}$ ...(i)

Similarly, we have

$\Delta\, AQE \sim \Delta\, ACD$

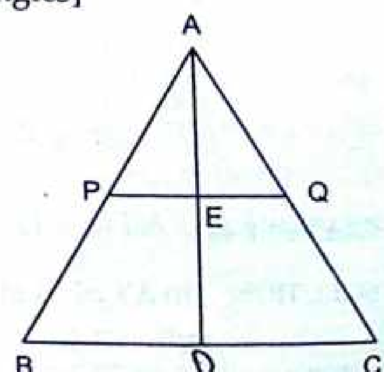


Fig. 7.125

$\therefore \quad \frac{QE}{CD} = \frac{AE}{AD}$ ...(ii)

From (i) and (ii), we get

$\frac{PE}{BD} = \frac{QE}{CD}$

$\Rightarrow \quad \frac{PE}{BD} = \frac{QE}{BD}$ [$\because$ AD is the median $\therefore BD = CD$]

$\Rightarrow \quad PE = QE$

Hence, AD bisects PQ.

EXAMPLE 43 *In Fig. 7.126, DEFG is a square and $\angle BAC = 90°$. Prove that*

(i) $\Delta AGF \sim \Delta DBG$ (ii) $\Delta AGF \sim \Delta EFC$ (iii) $\Delta DBG \sim \Delta EFC$ (iv) $DE^2 = BD \times EC$

[CBSE 2009]

SOLUTION (i) In ΔAGF and ΔDBG, we have

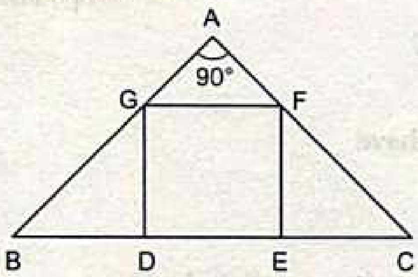


Fig. 7.126

$\angle GAF = \angle BDG$ [Each equal to 90°]

and, $\angle AGF = \angle DBG$ [Corresponding angles]

$\therefore \quad \Delta AGF \sim \Delta DBG$ [By AA-criterion of similarity]

(ii) In ΔAGF and ΔEFC, we have

$\angle FAG = \angle CEF$ [Each equal to 90°]

and, $\angle AFG = \angle ECF$ [Corresponding angles]

$\therefore \quad \Delta AGF \sim \Delta EFC$ [By AA-criterion of similarity]

(iii) Since $\Delta AGF \sim \Delta DBG$ and $\Delta AGF \sim \Delta EFC$

$\therefore \quad \Delta DBG \sim \Delta EFC$

(iv) We have,

$\Delta DBG \sim \Delta EFC$ [Using (iii)]

$\therefore \quad \frac{BD}{EF} = \frac{DG}{EC}$

$\Rightarrow \quad \frac{BD}{DE} = \frac{DE}{EC}$ [$\because$ $DEFG$ is a square $\therefore EF = DE, DG = DE$]

$\Rightarrow \quad DE^2 = BD \times EC$

EXAMPLE 44 *In Fig. 7.127, if $AD \perp BC$ and $\frac{BD}{DA} = \frac{DA}{DC}$, prove that ΔABC is a right triangle.*

SOLUTION In Δ's BDA and ADC, we have

$\frac{DB}{DA} = \frac{DA}{DC}$ [Given]

and, $\angle BDA = \angle ADC$ [Each equal to 90°]

So, by SAS-criterion of similarity, we have

$$\Delta BDA \sim \Delta ADC$$

$\Rightarrow \quad \angle ABD = \angle CAD$ and $\angle BAD = \angle ACD$

$\Rightarrow \quad \angle ABD + \angle ACD = \angle CAD + \angle BAD$

$\Rightarrow \quad \angle B + \angle C = \angle A$

$\Rightarrow \quad \angle A + \angle B + \angle C = 2\angle A$ [Adding $\angle A$ on both sides]

$\Rightarrow \quad 2\angle A = 180°$

$\Rightarrow \quad \angle A = 90°$

$\Rightarrow \quad \Delta ABC$ is a right triangle.

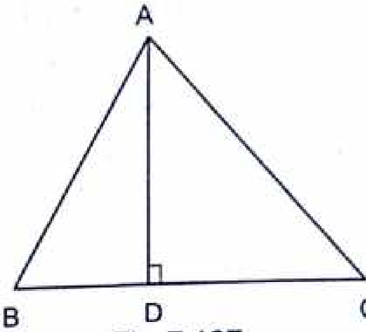


Fig. 7.127

EXAMPLE 45 *In Fig. 7.128, $\angle ACB = 90°$ and $CD \perp AB$. Prove that $\frac{CB^2}{CA^2} = \frac{BD}{AD}$.*

SOLUTION In triangles ACD and ABC, we have

$\angle ADC = \angle ACB$ [Each equal to 90°]

and, $\angle DAC = \angle BAC$ [Common]

So, by AA-criterion of similarity, we obtain

$$\Delta ACD \sim \Delta ABC$$

$\Rightarrow \quad \frac{AC}{AB} = \frac{AD}{AC}$

$\Rightarrow \quad AC^2 = AB \times AD$ ...(i)

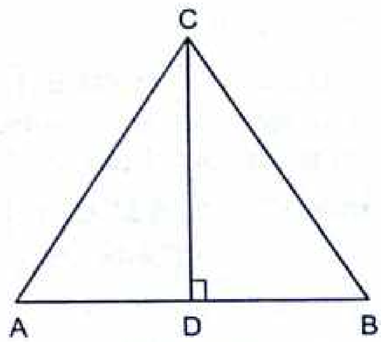


Fig. 7.128

In Δ's BCD and BAC, we have

$\angle BDC = \angle BCA$

and, $\angle DBC = \angle ABC$

So, by AA-criterion of similarity, we obtain

$$BCD \sim BAC$$

$\Rightarrow \quad \frac{BC}{BD} = \frac{BA}{BC}$

$\Rightarrow \quad BC^2 = AB \times BD$ ...(ii)

Dividing (ii) by (i), we get

$$\frac{BC^2}{AC^2} = \frac{AB \times BD}{AB \times AD} \Rightarrow \frac{BC^2}{AC^2} = \frac{BD}{AD}$$

EXAMPLE 46 *Through the mid-point M of the side CD of a parallelogram $ABCD$, the line BM is drawn intersecting AC at L and AD produced at E. Prove that $EL = 2BL$.*

SOLUTION In D's BMC and EMD, we have

$\angle BMC = \angle EMD$ [Vertically opposite angles]

$MC = MD$ [$\because M$ is the mid-point of CD]

$\angle MCB = \angle MDE$ [Alternate angles]

So, by AAS-congruence criterion, we have

$$\Delta BMC \cong \Delta EMD$$

$\Rightarrow \quad BC = ED$ [$\because$ Corresponding parts of congruent triangle are equal]

In Δ's AEL and CBL, we have

$\angle ALE = \angle CLB$ [Vertically opposite angles]

$\angle EAL = \angle BCL$ [Alternate angles]

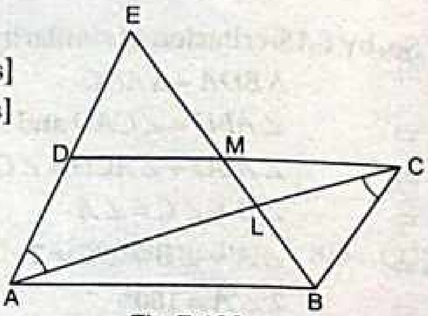


Fig. 7.129

So, by AA-criterion of similarity, we have

$\Delta AEL \sim \Delta CBL$

$\Rightarrow \quad \frac{AE}{BC} = \frac{EL}{BL} = \frac{AL}{CL}$

$\Rightarrow \quad \frac{EL}{BL} = \frac{AE}{BC}$

$\Rightarrow \quad \frac{EL}{BL} = \frac{AD + DE}{BC} = \frac{BC + DE}{BC} = \frac{2BC}{BC} = 2$ $[\because AD = BC \text{ and } DE = BC]$

$\Rightarrow \quad EL = 2\,BL.$

LEVEL-3

EXAMPLE 47 *Two poles of height a metres and b metres are p metres apart. Prove that the height of the point of intersection of the lines joining the top of each pole to the foot of the opposite pole is given by* $\frac{ab}{a+b}$ *metres.*

SOLUTION Let AB and CD be two poles of heights a metres and b metres respectively such that the poles are p metres apart i.e. $AC = p$ metres. Suppose the lines AD and BC meet at O such that $OL = h$ metres. Let $CL = x$ and $LA = y$. Then, $x + y = p$.

In ΔABC and ΔLOC, we have

$\angle CAB = \angle CLO$ [Each equal to 90°]

$\angle C = \angle C$ [Common]

$\therefore \quad \Delta CAB \sim \Delta CLO$ [By AA-criterion of similarity]

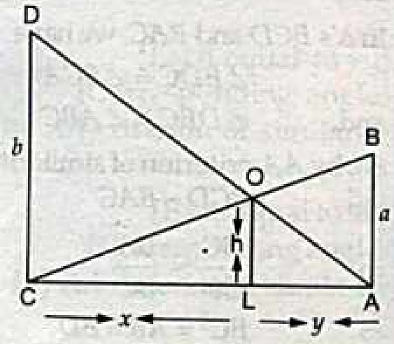


Fig. 7.130

$\Rightarrow \quad \frac{CA}{CL} = \frac{AB}{LO}$

$\Rightarrow \quad \frac{p}{x} = \frac{a}{h}$

$\Rightarrow \quad x = \frac{ph}{a}$ ...(i)

In ΔALO and ΔACD, we have

$\angle ALO = \angle ACD$ [Each equal to 90°]

$\angle A = \angle A$ [Common]

$\therefore \quad \Delta ALO \sim \Delta ACD$ [By AA-criterion of similarity]

$\Rightarrow \quad \frac{AL}{AC} = \frac{OL}{DC}$

$\Rightarrow \quad \frac{y}{p} = \frac{h}{b}$

$\Rightarrow \quad y = \frac{ph}{b}$ $[\because AC = x + y = p]$ ...(ii)

From (i) and (ii), we have

$$x + y = \frac{ph}{a} + \frac{ph}{b}$$

$$\Rightarrow \quad p = ph\left(\frac{1}{a} + \frac{1}{b}\right) \qquad [\because x + y = p]$$

$$\Rightarrow \quad 1 = h\left(\frac{a+b}{ab}\right) \Rightarrow h = \frac{ab}{a+b} \text{ metres}$$

Hence, the height of the intersection of the lines joining the top of each pole to the foot of the opposite pole is $\frac{ab}{a+b}$ metres.

EXAMPLE 48 *ABC is a triangle in which AB = AC and D is a point on AC such that $BC^2 = AC \times CD$. Prove that BD = BC.*

GIVEN ΔABC in which $AB = AC$ and D is a point on the side AC such that

$$BC^2 = AC \times CD$$

TO PROVE $BD = BC$

CONSTRUCTION Join BD

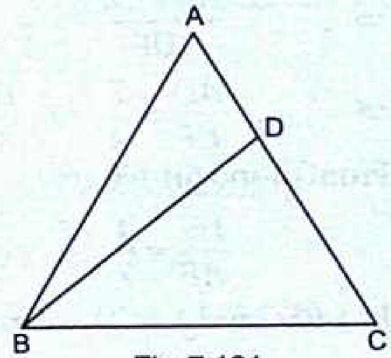


Fig. 7.131

PROOF We have,

$$BC^2 = AC \times CD$$

$$\Rightarrow \quad \frac{BC}{CD} = \frac{AC}{BC} \qquad \ldots(i)$$

Thus, in ΔABC and ΔBDC, we have

$$\frac{AC}{BC} = \frac{BC}{CD} \qquad [\text{From (i)}]$$

and, $\angle C = \angle C$ [Common]

$\therefore$ $\Delta ABC \sim \Delta BDC$ [By SAS criterion of similarity]

$$\Rightarrow \quad \frac{AB}{BD} = \frac{BC}{DC}$$

$$\Rightarrow \quad \frac{AC}{BD} = \frac{BC}{CD} \qquad [\because AB = AC]$$

$$\Rightarrow \quad \frac{AC}{BC} = \frac{BD}{CD} \qquad \ldots(ii)$$

From (i) and (ii), we get

$$\frac{BC}{CD} = \frac{BD}{CD} \Rightarrow BD = BC$$

EXAMPLE 49 *In trapezium ABCD, AB || DC and DC = 2AB. EF drawn parallel to AB cuts AD in F and BC in E such that $\frac{BE}{EC} = \frac{3}{4}$. Diagonal DB intersects EF at G. Prove that 7 FE = 10 AB.*

SOLUTION In ΔDFG and ΔDAB, we have

$\angle 1 = \angle 2$ $[\because AB \parallel DC \parallel EF \therefore \angle 1 \text{ and } \angle 2 \text{ are corresponding angles}]$

$\angle FDG = \angle ADB$ [Common]

So, by *AA*-criterion of similarity, we have

$\therefore \quad \Delta DFG \sim \Delta DAB$

$\Rightarrow \quad \frac{DF}{DA} = \frac{FG}{AB}$ ...(i)

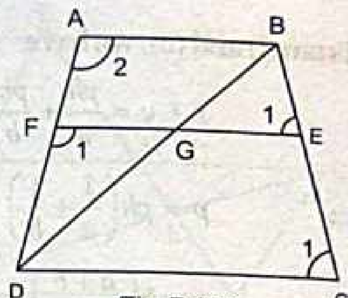


Fig. 7.132

In trapezium *ABCD*, we have

$EF \parallel AB \parallel DC$

$\therefore \quad \frac{AF}{DF} = \frac{BE}{EC}$

$\Rightarrow \quad \frac{AF}{DF} = \frac{3}{4}$ $\left[\because \frac{BE}{EC} = \frac{3}{4} \text{(Given)}\right]$

$\Rightarrow \quad \frac{AF}{DF} + 1 = \frac{3}{4} + 1$ [Adding 1 on both sides]

$\Rightarrow \quad \frac{AF + DF}{DF} = \frac{7}{4}$

$\Rightarrow \quad \frac{AD}{DF} = \frac{7}{4} \Rightarrow \frac{DF}{AD} = \frac{4}{7}$ ...(ii)

From (i) and (ii), we get

$\frac{FG}{AB} = \frac{4}{7} \Rightarrow FG = \frac{4}{7} AB$ ...(iii)

In ΔBEG and ΔBCD, we have

$\angle BEG = \angle BCD$ [Corresponding angles]

$\angle B = \angle B$ [Common]

$\therefore \quad \Delta BEG \sim \Delta BCD$ [By *AA*-criterion of similarity]

$\Rightarrow \quad \frac{BE}{BC} = \frac{EG}{CD}$

$\Rightarrow \quad \frac{3}{7} = \frac{EG}{CD}$ $\left[\because \frac{BE}{EC} = \frac{3}{4} \Rightarrow \frac{EC}{BE} = \frac{4}{3} \Rightarrow \frac{EC}{BE} + 1 = \frac{4}{3} + 1 \Rightarrow \frac{BC}{BE} = \frac{7}{3}\right]$

$\Rightarrow \quad EG = \frac{3}{7} CD$

$\Rightarrow \quad EG = \frac{3}{7} \times 2\, AB$ $[\because CD = 2\, AB \text{ (given)}]$

$\Rightarrow \quad EG = \frac{6}{7} AB$ ...(iv)

Adding (iii) and (iv), we get

$FG + EG = \frac{4}{7} AB + \frac{6}{7} AB \Rightarrow EF = \frac{10}{7} AB \Rightarrow 7\, EF = 10\, AB$

EXAMPLE 50 *Through the vertex D of a parallelogram ABCD, a line is drawn to intersect the sides BA and BC produced at E and F respectively. Prove that*

$$\frac{DA}{AE} = \frac{FB}{BE} = \frac{FC}{CD}$$

SOLUTION In Δ's EAD and DCF, we have

$\angle 1 = \angle 2$ [$\because AB \parallel DC$ $\therefore$ Corresponding angles are equal]

$\angle 3 = \angle 4$ [$\because AD \parallel BC$ $\therefore$ Corresponding angles are equal]

Therefore, by *AA*-criterion of similarity, we have

$$\Delta EAD \sim \Delta DCF$$

$$\Rightarrow \quad \frac{EA}{DC} = \frac{AD}{CF} = \frac{DE}{FD}$$

$$\Rightarrow \quad \frac{EA}{DC} = \frac{AD}{CF}$$

$$\Rightarrow \quad \frac{AD}{AE} = \frac{CF}{CD} \quad \ldots(i)$$

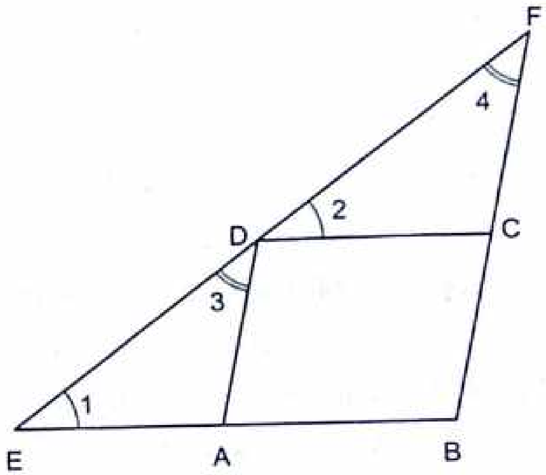


Fig. 7.133

Now, in Δ's EAD and EBF, we have

$\angle 1 = \angle 1$ [Common angle]

$\angle 3 = \angle 4$

So, by *AA*-criterion of similarity, we have

$$\Delta EAD \sim \Delta EBF$$

$$\Rightarrow \quad \frac{EA}{EB} = \frac{AD}{BF} = \frac{ED}{EF}$$

$$\Rightarrow \quad \frac{EA}{EB} = \frac{AD}{BF}$$

$$\Rightarrow \quad \frac{AD}{AE} = \frac{FB}{BE} \quad \ldots(ii)$$

From (i) and (ii), we obtain: $\dfrac{AD}{AE} = \dfrac{FB}{BE} = \dfrac{CF}{CD}$

EXAMPLE 51 *In Fig. 7.134, ABC is a right triangle right angled at B and D is the foot of the perpendicular drawn from B on AC. If $DM \perp BC$ and $DN \perp AB$, prove that*

(i) $DM^2 = DN \times MC$ (ii) $DN^2 = DM \times AN$ **[NCERT]**

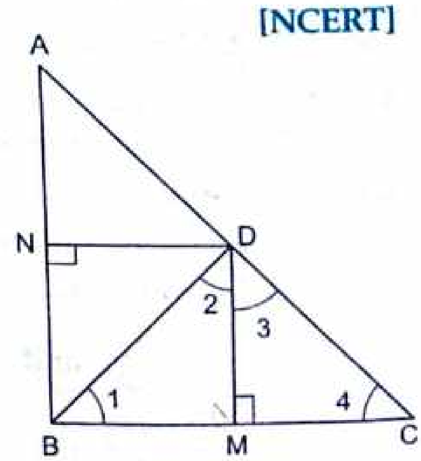


Fig. 7.134

SOLUTION We have,

$AB \perp BC$ and $DM \perp BC$

$\Rightarrow \quad AB \parallel DM$

Similarly, we have

$CB \perp AB$ and $DN \perp AB$

$\Rightarrow \quad CB \parallel DN$

Hence, quadrilateral $BMDN$ is a rectangle.

$\therefore \quad BM = ND$

(i) In ΔBMD, we have

$$\angle 1 + \angle BMD + \angle 2 = 180°$$
$$\Rightarrow \quad \angle 1 + 90° + \angle 2 = 180°$$
$$\Rightarrow \quad \angle 1 + \angle 2 = 90°$$

Similarly, in ΔDMC, we have

$$\angle 3 + \angle 4 = 90°$$

Since $BD \perp AC$. Therefore,

$$\angle 2 + \angle 3 = 90°$$

Now, $\angle 1 + \angle 2 = 90°$ and $\angle 2 + \angle 3 = 90°$

$$\Rightarrow \quad \angle 1 + \angle 2 = \angle 2 + \angle 3$$
$$\Rightarrow \quad \angle 1 = \angle 3$$

Also, $\angle 3 + \angle 4 = 90°$ and $\angle 2 + \angle 3 = 90°$

$$\Rightarrow \quad \angle 3 + \angle 4 = \angle 2 + \angle 3 \Rightarrow \angle 2 = \angle 4$$

Thus, in Δ's BMD and DMC, we have

$$\angle 1 = \angle 3 \text{ and } \angle 2 = \angle 4$$

So, by AA-criterion of similarity, we obtain

$$\Delta BMD \sim \Delta DMC$$
$$\Rightarrow \quad \frac{BM}{DM} = \frac{MD}{MC}$$
$$\Rightarrow \quad \frac{DN}{DM} = \frac{DM}{MC} \qquad [\because BM = ND]$$
$$\Rightarrow \quad DM^2 = DN \times MC$$

(ii) Proceeding as in (i), we can prove that

$$\Delta BND \sim \Delta DNA$$
$$\Rightarrow \quad \frac{BN}{DN} = \frac{ND}{NA}$$
$$\Rightarrow \quad \frac{DM}{DN} = \frac{DN}{AN} \qquad [\because BN = DM]$$
$$\Rightarrow \quad DN^2 = DM \times AN$$

EXAMPLE 52 *ABC is an isosceles triangle with $AB = AC$ and D is a point on AC such that $BC^2 = AC \times CD$. Prove that $BD = BC$.*

SOLUTION We have,

$$BC^2 = AC \times CD \text{ and } AB = AC$$
$$\Rightarrow \quad BC \times BC = AC \times CD \text{ and } \angle B = \angle C$$
$$\Rightarrow \quad \frac{BC}{AC} = \frac{CD}{BC} \text{ and } \angle B = \angle C$$
$$\Rightarrow \quad \frac{BC}{CA} = \frac{DC}{CB} \text{ and } \angle B = \angle C$$

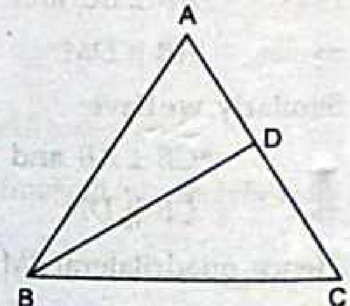


Fig. 7.135

So, by *SAS*-criterion of similarity, we obtain

$$\Delta BCA \sim \Delta DCB$$

$$\Rightarrow \quad \frac{BC}{DC} = \frac{CA}{CB} = \frac{BA}{DB}$$

$$\Rightarrow \quad \frac{CA}{CB} = \frac{BA}{DB}$$

$$\Rightarrow \quad \frac{BA}{CA} = \frac{DB}{CB}$$

$$\Rightarrow \quad 1 = \frac{DB}{CB} \qquad [\because AB = AC]$$

$$\Rightarrow \quad DB = CB \Rightarrow BD = BC$$

EXERCISE 7.5

LEVEL-1

1. In Fig. 7.136, $\Delta ACB \sim \Delta APQ$. If $BC = 8$ cm, $PQ = 4$ cm, $BA = 6.5$ cm and $AP = 2.8$ cm, find CA and AQ.

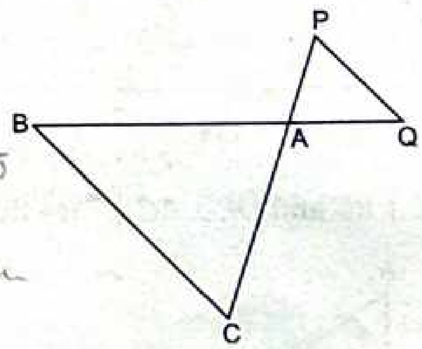


Fig. 7.136

2. In Fig. 7. 137, $AB \parallel QR$. Find the length of PB.

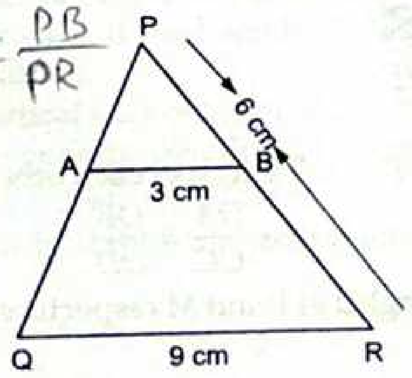


Fig. 7.137

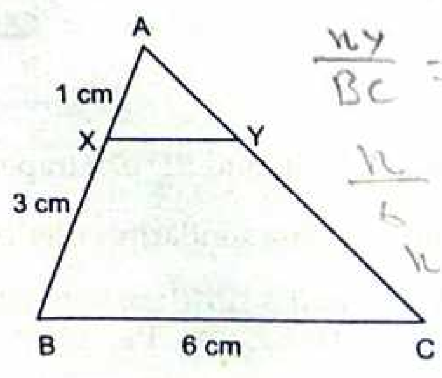


Fig. 7.138

3. In Fig. 7.138, $XY \parallel BC$. Find the length of XY.
4. In a right angled triangle with sides a and b and hypotenuse c, the altitude drawn on the hypotenuse is x. Prove that $ab = cx$.

5. In Fig. 7.139, $\angle ABC = 90^\circ$ and $BD \perp AC$. If $BD = 8$ cm and $AD = 4$ cm, find CD.

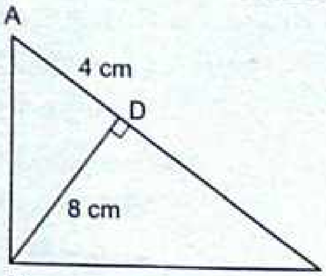


Fig. 7.139

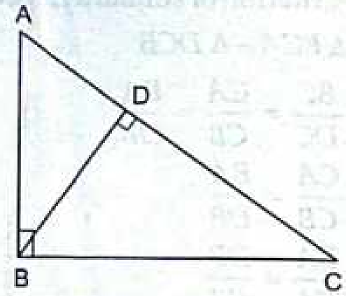


Fig. 7.140

6. In Fig. 7.140, $\angle ABC = 90^\circ$ and $BD \perp AC$. If $AB = 5.7$ cm, $BD = 3.8$ cm and $CD = 5.4$ cm, find BC.

7. In Fig. 7.141, $DE \parallel BC$ such that $AE = (1/4)\,AC$. If $AB = 6$ cm, find AD.

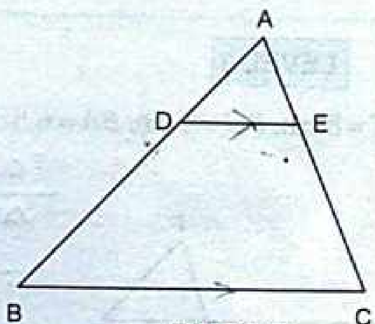


Fig. 7.141

8. In Fig. 7.142, if $AB \perp BC$, $DC \perp BC$ and $DE \perp AC$, prove that $\Delta\, CED \sim \Delta\, ABC$.

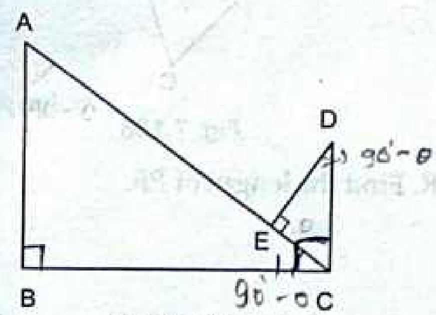


Fig. 7.142

9. Diagonals AC and BD of a trapezium $ABCD$ with $AB \parallel DC$ intersect each other at the point O. Using similarity criterion for two triangles, show that $\dfrac{OA}{OC} = \dfrac{OB}{OD}$.

10. If $\Delta\, ABC$ and $\Delta\, AMP$ are two right triangles, right angled at B and M respectively such that $\angle MAP = \angle BAC$. Prove that

(i) $\Delta\, ABC \sim \Delta\, AMP$ (ii) $\dfrac{CA}{PA} = \dfrac{BC}{MP}$

LEVEL-2

11. A vertical stick 10 cm long casts a shadow 8 cm long. At the same time a tower casts a shadow 30 m long. Determine the height of the tower.

12. In Fig. 7.143, $\angle A = \angle CED$, prove that $\Delta CAB \sim \Delta CED$. Also, find the value of x.

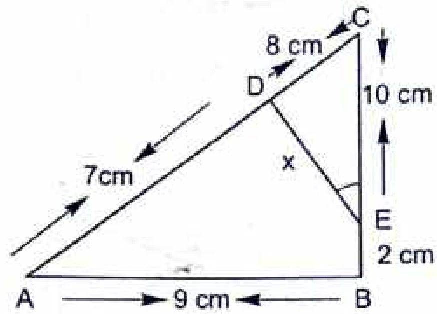


Fig. 7.143

13. The perimeters of two similar triangles are 25 cm and 15 cm respectively. If one side of first triangle is 9 cm, what is the corresponding side of the other triangle? **[CBSE 2002 C]**

14. In ΔABC and ΔDEF, it is being given that: $AB = 5$ cm, $BC = 4$ cm and $CA = 4.2$ cm; $DE = 10$ cm, $EF = 8$ cm and $FD = 8.4$ cm. If $AL \perp BC$ and $DM \perp EF$, find $AL : DM$.

15. D and E are the points on the sides AB and AC respectively of a ΔABC such that: $AD = 8$ cm, $DB = 12$ cm, $AE = 6$ cm and $CE = 9$ cm. Prove that $BC = 5/2\ DE$.

16. D is the mid-point of side BC of a ΔABC. AD is bisected at the point E and BE produced cuts AC at the point X. Prove that $BE : EX = 3 : 1$

17. $ABCD$ is a parallelogram and APQ is a straight line meeting BC at P and DC produced at Q. prove that the rectangle obtained by BP and DQ is equal to the rectangle contained by AB and BC.

18. In ΔABC, AL and CM are the perpendiculars from the vertices A and C to BC and AB respectively. If AL and CM intersect at O, prove that:

    (i) $\Delta OMA \sim \Delta OLC$ (ii) $\dfrac{OA}{OC} = \dfrac{OM}{OL}$

19. $ABCD$ is a quadrilateral in which $AD = BC$. If P, Q, R, S be the mid-points of AB, AC, CD and BD respectively, show that $PQRS$ is a rhombus.

20. In an isosceles ΔABC, the base AB is produced both the ways to P and Q such that $AP \times BQ = AC^2$. Prove that $\Delta APC \sim \Delta BCQ$.

21. A girl of heigh 90 cm is walking away from the base of a lamp-post at a speed of 1.2 m/sec. If the lamp is 3.6 m above the ground, find the length of her shadow after 4 seconds.

22. A vertical stick of length 6 m casts a shadow 4 m long on the ground and at the same time a tower casts a shadow 28 m long. Find the height of the tower. **[NCERT]**

23. In Fig. 7.144, ΔABC is right angled at C and $DE \perp AB$. Prove that $\Delta ABC \sim \Delta ADE$ and hence find the lengths of AE and DE.

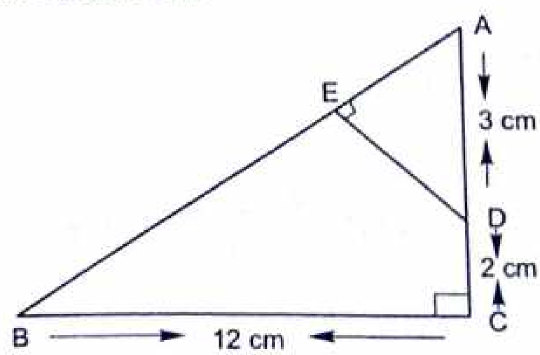


Fig. 7.144

LEVEL-3

24. In Fig. 7.145, PA, QB and RC are each perpendicular to AC. Prove that $\frac{1}{x}+\frac{1}{z}=\frac{1}{y}$.

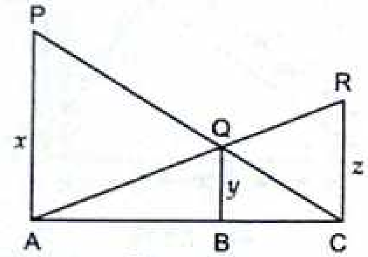


Fig. 7.145

25. In Fig. 7.146, we have $AB \parallel CD \parallel EF$. If $AB = 6$ cm, $CD = x$ cm, $EF = 10$ cm, $BD = 4$ cm and $DE = y$ cm, calculate the values of x and y.

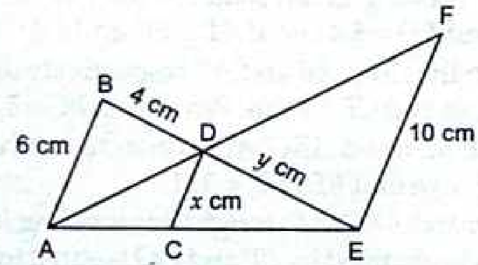


Fig. 7.146

ANSWERS

1. $CA = 5.6$ cm, $AQ = 3.25$ cm
2. 2 cm
3. 1.5 cm
5. 16 cm
6. 8.1 cm
7. 1.5 cm
11. 37.5 m
12. 6
13. 5.4 cm
14. 1 : 2
21. 1.6 m
22. 42 m
23. $DE = \frac{36}{13}$ cm and $AE = \frac{15}{13}$ cm
25. $x = 3.75$ cm; $y = 6.67$ cm

HINT TO SELECTED PROBLEMS

2. Use: $\Delta PAB \sim \Delta PQR$
5. In ΔDBA and ΔDCB, we have

$\angle DBA = \angle DCB$ [Each equal to $90° - \angle DBC$]

and, $\angle D = \angle D = 90°$ [Common]

$\therefore \quad \Delta DBA \sim \Delta DCB$

$$\Rightarrow \frac{BD}{CD} = \frac{AD}{BD} \Rightarrow CD = \frac{BD^2}{AD}$$

6. Clearly, $\Delta ABC \sim \Delta BDC$

$$\Rightarrow \frac{AB}{BD} = \frac{BC}{DC} \Rightarrow \frac{5.7}{3.8} = \frac{BC}{5.4} \Rightarrow BC = \frac{5.7 \times 5.4}{3.8} = 8.1 \text{ cm}$$

7. Clearly, $\Delta ADE \sim \Delta ABC$

$$\Rightarrow \frac{AD}{AB} = \frac{AE}{AC} \Rightarrow \frac{AD}{6} = \frac{1}{4} \Rightarrow AD = \frac{6}{4} = 1.5 \text{ cm}.$$

12. In $\Delta\, CAB$ and $\Delta\, CED$, we have

$\angle A = \angle CED$ and $\angle C = \angle C$ [Common]

$\therefore \quad \Delta\, CAB \sim \Delta\, CED$

$$\Rightarrow \quad \frac{CA}{CE} = \frac{AB}{DE} = \frac{CB}{CD} \Rightarrow \frac{AB}{DE} = \frac{CB}{CD} \Rightarrow \frac{9}{x} = \frac{10+2}{8} \Rightarrow x = 6 \text{ cm}$$

13. Ratio of the corresponding sides = Ratio of perimeters.

$$\Rightarrow \quad \frac{2}{x} = \frac{25}{15} \Rightarrow x = \frac{27}{5} = 5.4 \text{ cm}$$

14. Clearly, $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF} = \dfrac{1}{2}$

$\therefore \quad \Delta\, ABC \sim \Delta\, DEF$

Now, use the result that in similar triangles the ratio of corresponding altitudes is same as the ratio of corresponding sides.

15. Clearly, $\dfrac{AD}{AB} = \dfrac{AE}{AC} = \dfrac{2}{3}$ and $\angle A$ is common in $\Delta\, ABC$ and $\Delta\, ADE$

$\therefore \quad \Delta\, ADE \sim \Delta\, ABC \Rightarrow \dfrac{BC}{DE} = \dfrac{AB}{AD}$

17. Use the result $\Delta\, ABP \sim \Delta\, QDA$ to prove that $AB \times BC = BP \times DQ$.

21. Let AB be the lamp-post and CD be the girl after walking for 4 seconds. Let DE be the length of her shadow such that $DE = x$ metres, $BD = 1.2 \times 4 = 4.8$ m.

In Δ's ABE and CDE, we have

$\angle B = \angle D$ and $\angle E = \angle E$

So, by AA-similarity criterion, we obtain $\Delta\, ABE \sim \Delta\, CDE$

$$\therefore \quad \frac{BE}{DE} = \frac{AB}{CD} \Rightarrow \frac{4.8 + x}{x} = \frac{3.6}{0.9} \Rightarrow x = 1.6 \text{ m}$$

23. In triangles ABC and ADE, we have

$$\angle ACB = \angle AED = 90° \quad \text{and,} \quad \angle BAC = \angle DAE$$

So, by AA similarity criterion, we obtain

$$\Delta\, ABC \sim \Delta ADE$$

$$\Rightarrow \quad \frac{AB}{AD} = \frac{BC}{DE} = \frac{AC}{AE}$$

$$\Rightarrow \quad \frac{13}{3} = \frac{12}{DE} = \frac{5}{AE} \qquad [\because AB^2 = AC^2 + BC^2 = 5^2 + 12^2]$$

$\Rightarrow \quad DE = \frac{36}{13}$ and $AE = \frac{15}{13}$

24. In ΔPAC, we have

$$BQ \parallel AP \Rightarrow \frac{BQ}{AP} = \frac{CB}{CA} \Rightarrow \frac{y}{x} = \frac{CB}{CA} \quad \ldots(i)$$

In ΔACR, we have

$$BQ \parallel CR \Rightarrow \frac{BQ}{CR} = \frac{AB}{AC} \Rightarrow \frac{y}{z} = \frac{AB}{AC} \quad \ldots(ii)$$

Adding (i) and (ii), we get

$$\frac{y}{x} + \frac{y}{z} = \frac{CB}{AC} + \frac{AB}{AC}$$

$$\Rightarrow \quad \frac{y}{x} + \frac{y}{z} = \frac{AB + BC}{AC} \Rightarrow \frac{y}{x} + \frac{y}{z} = \frac{AC}{AC} \Rightarrow \frac{y}{x} + \frac{y}{z} = 1 \Rightarrow \frac{1}{x} + \frac{1}{z} = \frac{1}{y}$$

7.8 MORE ON CHARACTERISTIC PROPERTIES

In the previous section, we have learnt about characteristic properties of similar triangles and their applications. In this section, we shall discuss some more results as theorems derived from the characteristic properties of similar triangles.

THEOREM 1 *If two triangles are equiangular, prove that the ratio of the corresponding sides is same as the ratio of the corresponding medians.* **[NCERT]**

GIVEN Two triangles ABC and DEF in which $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$, AP and DQ are their medians.

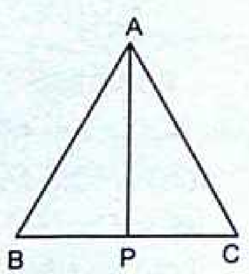


Fig. 7.147

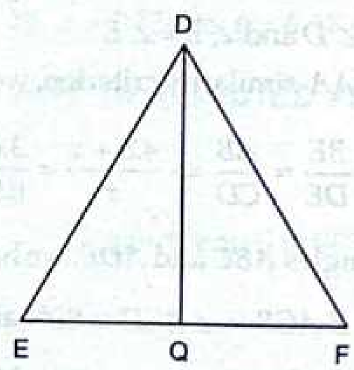


Fig. 7.148

TO PROVE $\frac{BC}{EF} = \frac{AP}{DQ}$

PROOF Since equiangular triangles are similar.

$\therefore \quad \Delta ABC \sim \Delta DEF$

$$\Rightarrow \quad \frac{AB}{DE} = \frac{BC}{EF} \quad \ldots(i)$$

$\Rightarrow \quad \frac{AB}{DE} = \frac{2BP}{2EQ}$ $\quad \left[\begin{array}{l}\because P \text{ and } Q \text{ are mid-points of } BC \text{ and } EF \text{ respectively} \\ \therefore BC = 2BP \text{ and } EF = 2\,EQ\end{array}\right]$

$\Rightarrow \quad \frac{AB}{DE} = \frac{BP}{EQ}$ ...(ii)

Now, in Δ ABP and Δ DFQ, we have

$\frac{AB}{DE} = \frac{BP}{EQ}$ [From (ii)]

and, $\angle B = \angle E$ [Given]

So, by SAS-criterion of similarity, we have

$\Delta ABP \sim \Delta DEQ$

$\Rightarrow \quad \frac{AB}{DE} = \frac{AP}{DQ}$ ...(iii)

From (i) and (iii), we get : $\frac{BC}{EF} = \frac{AP}{DQ}$

Hence, the ratio of the corresponding sides is same as the ratio of corresponding medians.

Q.E.D.

THEOREM 2 *If two triangles are equiangular, prove that the ratio of the corresponding sides is same as the ratio of the corresponding angle bisector segments.*

GIVEN Two triangles ABC and DEF in which $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$; and AX, DY are the bisectors of $\angle A$ and $\angle D$ respectively.

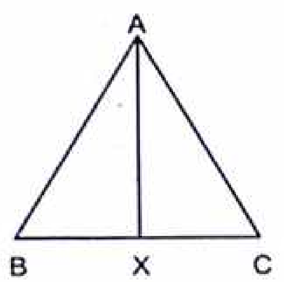


Fig. 7.149

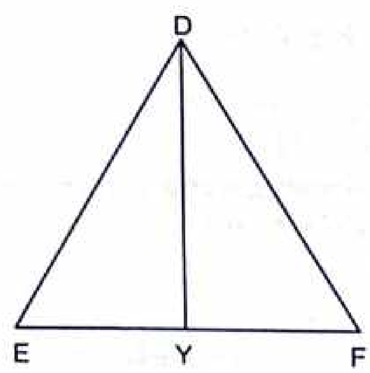


Fig. 7.150

TO PROVE $\frac{BC}{EF} = \frac{AX}{DY}$

PROOF Since equiangular triangles are similar.

$\Delta ABC \sim \Delta DEF$

$\Rightarrow \quad \frac{AB}{DE} = \frac{BC}{EF}$ ...(i)

In Δ ABX and DEY, we have

$\angle B = \angle E$ [Given]

and, $\angle BAX = \angle EDY$ $\left[\because \angle A = \angle D \Rightarrow \frac{1}{2}\angle A = \frac{1}{2}\angle D \Rightarrow \angle BAX = \angle EDY\right]$

So, by *AA*-criterion of similarity, we have

$$\Delta ABX \sim \Delta DEY$$

$$\Rightarrow \quad \frac{AB}{DE} = \frac{AX}{DY} \qquad \text{...(ii)}$$

From (i) and (ii), we get : $\frac{BC}{EF} = \frac{AX}{DY}$

Q.E.D.

THEOREM 3 *If two triangles are equiangular, prove that the ratio of the corresponding sides is same as the ratio of the corresponding altitudes.*

GIVEN Two triangles *ABC* and *DEF* in which

$\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$ and $AL \perp BC, DM \perp EF$

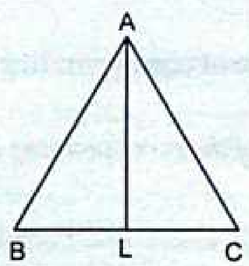


Fig. 7.151

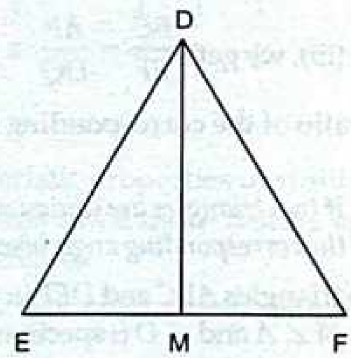


Fig. 7.152

TO PROVE $\frac{BC}{EF} = \frac{AL}{DM}$

PROOF Since equiangular triangles are similar.

$$\therefore \quad \Delta ABC \sim \Delta DEF$$

$$\Rightarrow \quad \frac{AB}{DE} = \frac{BC}{EF} \qquad \text{...(I)}$$

In triangle *ALB* and *DME*, we have

$\angle ALB = \angle DME$ [Each equal to 90°]

$\angle B = \angle E$ [Given]

So, by *AA*-criterion of similarity, we have

$$\Delta ALB \sim \Delta DME$$

$$\Rightarrow \quad \frac{AB}{DE} = \frac{AL}{DM} \qquad \text{...(ii)}$$

From (i) and (ii), we get : $\frac{BC}{EF} = \frac{AL}{DM}$

Q.E.D.

THEOREM 4 *If one angle of a triangle is equal to one angle of another triangle and the bisectors of these equal angles divide the opposite side in the same ratio, prove that the triangles are similar.*

GIVEN Two triangles ABC and DEF in which $\angle A = \angle D$. The bisectors AP and DQ or $\angle A$ and $\angle D$ intersect BC and EF in P and Q respectively such that $\frac{BP}{PC} = \frac{EQ}{QF}$.

TO PROVE $\Delta ABC \sim \Delta DEF$

PROOF We know that the bisectors of an angle of a triangle intersects the opposite side in the ratio of the sides containing the angle.

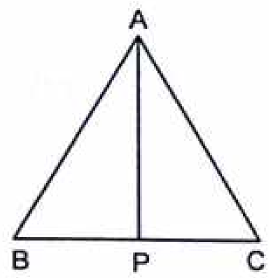


Fig. 7.153

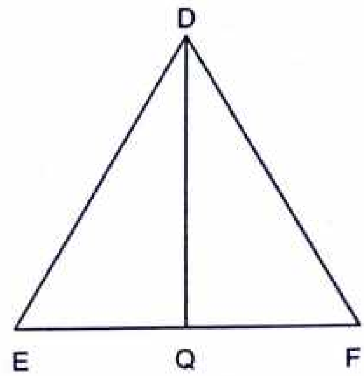


Fig. 7.154

$\therefore$ AP is the bisector of $\angle A$

$\Rightarrow \quad \frac{BP}{PC} = \frac{AB}{AC}$ ...(i)

DQ is the bisector of $\angle D$

$\Rightarrow \quad \frac{EQ}{QF} = \frac{DE}{DF}$ ...(ii)

But, $\quad \frac{BP}{PC} = \frac{EQ}{QF}$ [Given]

Therefore, from (i) and (ii), we get

$$\frac{AB}{AC} = \frac{DE}{DF}$$

Thus, in triangles ABC and DEF, we have

$$\frac{AB}{AC} = \frac{DE}{DF}$$

and, $\quad \angle A = \angle D$ [Given]

So, by SAS-criterion of similarity, we obtain : $\Delta ABC \sim \Delta DEF$.

Q.E.D.

THEOREM 5 *If two sides and a median bisecting one of these sides of a triangle are respectively proportional to the two sides and the corresponding median of another triangle, then the triangles are similar.* **[NCERT]**

GIVEN ΔABC and ΔDEF in which AP and DQ are the medians such that

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AP}{DQ}$$

TO PROVE $\Delta ABC \sim \Delta DEF$

PROOF We have,

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AP}{DQ}$$

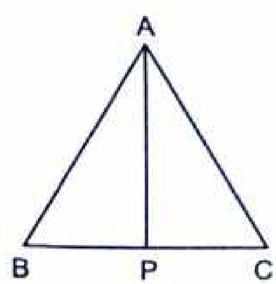


Fig. 7.155

$\Rightarrow \quad \dfrac{AB}{DE} = \dfrac{\frac{1}{2}BC}{\frac{1}{2}EF} = \dfrac{AP}{DQ}$

$\Rightarrow \quad \dfrac{AB}{DE} = \dfrac{BP}{EQ} = \dfrac{AP}{DQ}$

$\Rightarrow \quad \Delta ABP \sim \Delta DEQ$ [By SSS-similarity]

$\Rightarrow \quad \angle B = \angle E$

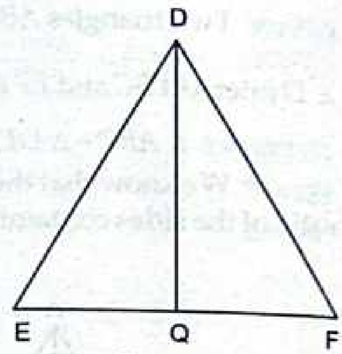


Fig. 7.156

Now, in ΔABC and ΔDEF, we have

$\dfrac{AB}{DE} = \dfrac{BC}{EF}$ [Given]

and, $\angle B = \angle E$

So, by SAS-criterion of similarity, we obtain $\Delta ABC \sim \Delta DEF$

Q.E.D.

THEOREM 6 *If two sides and a median bisecting the third side of a triangle are respectively proportional to the corresponding sides and the median of another triangle, then the two triangles are similar.*

GIVEN Two triangles ABC and DEF in which AP and DQ are the medians such that $\dfrac{AB}{DE} = \dfrac{AC}{DF} = \dfrac{AP}{DQ}$

TO PROVE $\Delta ABC \sim \Delta DEF$

CONSTRUCTION Produce AP to G so that $PG = AP$. Join CG. Also, produce DQ to H so that $QH = DQ$. Join FH.

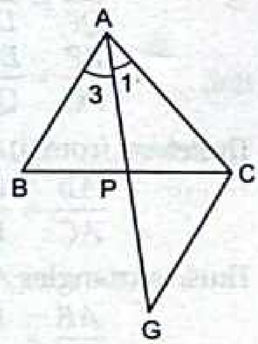


Fig. 7.157

PROOF In ΔAPB and ΔGPC, we have

$BP = CP$ [$\because AP$ is the median]

$AP = GP$ [By construction]

and, $\angle APB = \angle CPG$ [Vertically opposite angles]

So, by SAS-criterion of congruence, we have

$\Delta APB \cong \Delta GPC$

$\Rightarrow \quad AB = GC$ ...(i)

Again, In ΔDQE and ΔHQF, we have

$EQ = FQ$ [$\because DQ$ is the median]

$DQ = HQ$ [By construction]

and, $\angle DQE = \angle HQF$ [Vertically opposite angles]

So, by SAS-criterion of congruence, we have

$\Delta DQE \cong \Delta HQF$

$\Rightarrow \quad DE = HF$ ...(ii)

Now, $\dfrac{AB}{DE} = \dfrac{AC}{DF} = \dfrac{AP}{DQ}$ [Given]

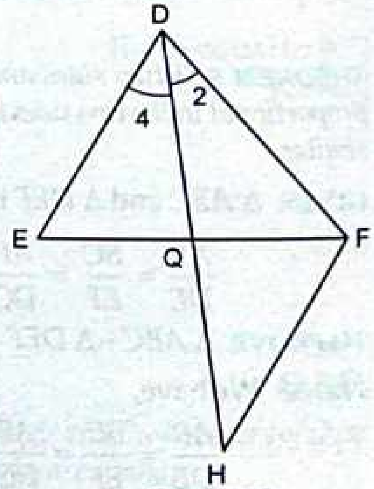


Fig. 7.158

$$\Rightarrow \quad \frac{GC}{HF} = \frac{AC}{DF} = \frac{AP}{DQ} \qquad \left[\begin{array}{l} \because AB = GC \text{ and } DE = HF \\ \text{(From (i) and (ii))} \end{array}\right]$$

$$\Rightarrow \quad \frac{GC}{HF} = \frac{AC}{DF} = \frac{2AP}{2DQ}$$

$$\Rightarrow \quad \frac{GC}{HF} = \frac{AC}{DF} = \frac{AG}{DH} \qquad [\because 2AP = AG \text{ and } 2DQ = DH]$$

$\Rightarrow \quad \Delta AGC \sim \Delta DHF$ [By *SSS*-criterion of similarity]

$\Rightarrow \quad \angle 1 = \angle 2$

Similarly, we have

$$\angle 3 = \angle 4 \Rightarrow \angle 1 + \angle 3 = \angle 2 + \angle 4 \Rightarrow \angle A = \angle D \qquad \ldots\text{(iii)}$$

Thus, in ΔABC and ΔDEF, we have

$\angle A = \angle D$ [From (iii)]

and, $\dfrac{AB}{DE} = \dfrac{AC}{DF}$ [Given]

So, by *SAS*-criterion of similarity, we obtain $\Delta ABC \sim \Delta DEF$

Q.E.D.

7.9 AREAS OF TWO SIMILAR TRIANGLES

In this section, we will discuss some theorems concerning the ratio of areas of similar triangles.

THEOREM 1 *The ratio of the areas of two similar triangles are equal to the ratio of the squares of any two corresponding sides.* **[NCERT, CBSE 2000, 2002, 2004, 2006C, 2008, 2010]**

GIVEN Two triangles *ABC* and *DEF* such that $\Delta ABC \sim \Delta DEF$.

TO PROVE $\dfrac{\text{Area }(\Delta ABC)}{\text{Area }(\Delta DEF)} = \dfrac{AB^2}{DE^2} = \dfrac{BC^2}{EF^2} = \dfrac{AC^2}{DF^2}$

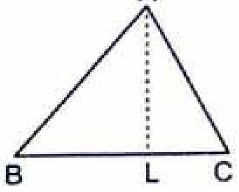


Fig. 7.159

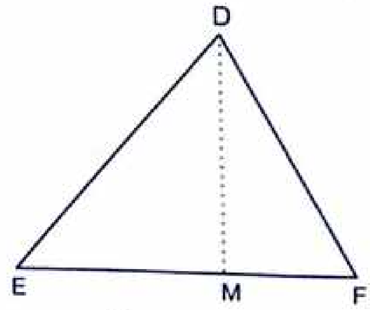


Fig. 7.160

CONSTRUCTION Draw $AL \perp BC$ and $DM \perp EF$.

PROOF Since similar triangles are equiangular and their corresponding sides are proportional. Therefore,

$\Delta ABC \sim \Delta DEF$

$\Rightarrow \quad \angle A = \angle D, \angle B = \angle E, \angle C = \angle F$ and $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$ ...(i)

Thus, in $\Delta\, ALB$ and $\Delta\, DME$, we have

$\Rightarrow \quad \angle ALB = \angle DME$ [Each equal to 90°]

and, $\quad \angle B = \angle E$ [From (i)]

So, by *AA*-criterion of similarity, we have

$$\Delta\, ALB \sim \Delta\, DME$$

$$\Rightarrow \quad \frac{AL}{DM} = \frac{AB}{DE} \quad \text{...(ii)}$$

From (i) and (ii), we get

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = \frac{AL}{DM} \quad \text{...(iii)}$$

Now,

$$\frac{\text{Area}\,(\Delta\, ABC)}{\text{Area}\,(\Delta\, DEF)} = \frac{\frac{1}{2}(BC \times AL)}{\frac{1}{2}(EF \times DM)}$$

$$\Rightarrow \quad \frac{\text{Area}\,(\Delta\, ABC)}{\text{Area}\,(\Delta\, DEF)} = \frac{BC}{EF} \times \frac{AL}{DM}$$

$$\Rightarrow \quad \frac{\text{Area}\,(\Delta\, ABC)}{\text{Area}\,(\Delta\, DEF)} = \frac{BC}{EF} \times \frac{BC}{EF} \quad \left[\text{From (iii)}, \frac{BC}{EF} = \frac{AL}{DM}\right]$$

$$\Rightarrow \quad \frac{\text{Area}\,(\Delta\, ABC)}{\text{Area}\,(\Delta\, DEF)} = \frac{BC^2}{EF^2}$$

But, $\quad \dfrac{BC}{EF} = \dfrac{AB}{DE} = \dfrac{AC}{DF}$

$$\Rightarrow \quad \frac{BC^2}{EF^2} = \frac{AB^2}{DE^2} = \frac{AC^2}{DF^2}$$

Hence, $\quad \dfrac{\text{Area}\,(\Delta\, ABC)}{\text{Area}\,(\Delta\, DEF)} = \dfrac{AB^2}{DE^2} = \dfrac{BC^2}{EF^2} = \dfrac{AC^2}{DF^2}$

Q.E.D.

THEOREM 2 *The areas of two similar triangles are in the ratio of the squares of the corresponding altitudes.*

GIVEN Two triangles *ABC* and *DEF* such that $\Delta\, ABC \sim \Delta\, DEF$ and $AL \perp BC$, $DM \perp EF$.

TO PROVE $\dfrac{\text{Area}\,(\Delta\, ABC)}{\text{Area}\,(\Delta\, DEF)} = \dfrac{AL^2}{DM^2}$

PROOF Since the ratio of the areas of two similar triangles is equal to the ratio of the squares of any two corresponding sides.

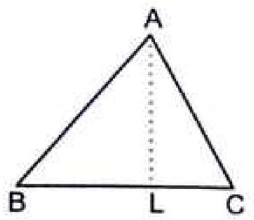


Fig. 7.161

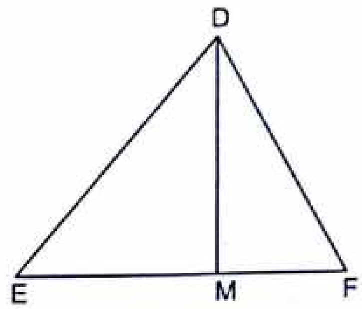


Fig. 7.162

$$\therefore \quad \frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \frac{AB^2}{DE^2} \quad \ldots\text{(i)}$$

Now, in ΔALB and ΔDME, we have

$\angle ALB = \angle DME$ [Each equal to 90°]

and, $\angle B = \angle E$ [$\because \Delta ABC \sim \Delta DEF \therefore \angle A = \angle D, \angle B = \angle E, \angle C = \angle F$]

So, by *AA*-criterion of similarity, we have

$$\Delta ALB \sim \Delta DME$$

$$\Rightarrow \quad \frac{AB}{DE} = \frac{AL}{DM}$$

$$\Rightarrow \quad \frac{AB^2}{DE^2} = \frac{AL^2}{DM^2} \quad \ldots\text{(ii)}$$

From (i) and (ii), we get

$$\frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \frac{AL^2}{DM^2}$$

Q.E.D.

THEOREM 3 *The areas of two similar triangles are in the ratio of the squares of the corresponding medians.* **[NCERT]**

GIVEN Two triangles *ABC* and *DEF* such that $\Delta ABC \sim \Delta DEF$ and *AP*, *DQ* are their medians.

TO PROVE $\frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \frac{AP^2}{DQ^2}$

PROOF Since the ratio of the areas of two similar triangles is equal to the ratio of the squares of any two corresponding sides.

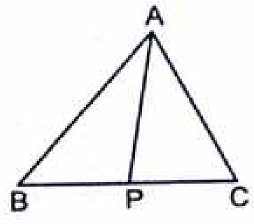


Fig. 7.163

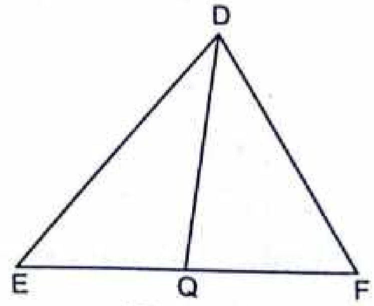


Fig. 7.164

$$\therefore \quad \frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \frac{AB^2}{DE^2} \quad \ldots\text{(i)}$$

Now, $\Delta ABC \sim \Delta DEF$

$$\Rightarrow \quad \frac{AB}{DE} = \frac{BC}{EF}$$

$$\Rightarrow \quad \frac{AB}{DE} = \frac{2BP}{2EQ} = \frac{BP}{EQ} \qquad \text{...(ii)}$$

Thus, in triangles APB and DQE, we have

$$\frac{AB}{DE} = \frac{BP}{EQ} \text{ and } \angle B = \angle E \qquad [\because \Delta ABC \sim \Delta DEF]$$

So, by SAS-criterion of similarity, we have

$$\Delta APB \sim \Delta DQE$$

$$\Rightarrow \quad \frac{BP}{EQ} = \frac{AP}{DQ} \qquad \text{...(iii)}$$

From (ii) and (iii), we get

$$\frac{AB}{DE} = \frac{AP}{DQ}$$

$$\Rightarrow \quad \frac{AB^2}{DE^2} = \frac{AP^2}{DQ^2} \qquad \text{...(iv)}$$

From (i) and (iv), we get

$$\frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \frac{AP^2}{DQ^2}$$

Q.E.D.

THEOREM 4 *The areas of two similar triangles are in the ratio of the squares of the corresponding angle bisector segments.*

GIVEN $\Delta ABC \sim \Delta DEF$ and AX and DY are bisector of $\angle A$ and $\angle D$ respectively.

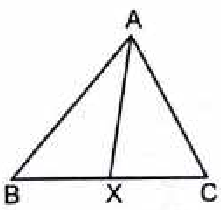


Fig. 7.165

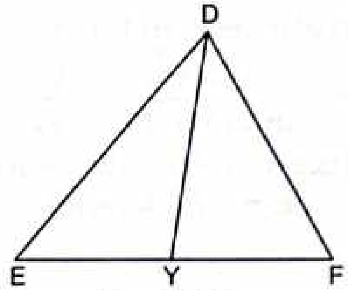


Fig. 7.166

TO PROVE $\dfrac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \dfrac{AX^2}{DY^2}$

PROOF Since the ratio of the areas of two similar triangles are equal to the ratio of the squares of any two corresponding sides.

$$\therefore \quad \frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \frac{AB^2}{DE^2} \qquad \text{...(i)}$$

Now, $\quad \Delta ABC \sim \Delta DEF$

$$\Rightarrow \quad \angle A = \angle D$$

$$\Rightarrow \quad \frac{1}{2}\angle A = \frac{1}{2}\angle D$$

$$\Rightarrow \quad \angle BAX = \angle EDY$$

Thus, in triangles ABX and DEY, we have

$$\angle BAX = \angle EDY \text{ and } \angle B = \angle E \qquad [\because \Delta ABC \sim \Delta DEF]$$

So, by AA-similarity criterion, we have

$$\Delta ABX \sim \Delta DEY$$

$$\Rightarrow \quad \frac{AB}{DE} = \frac{AX}{DY}$$

$$\Rightarrow \quad \frac{AB^2}{DE^2} = \frac{AX^2}{DY^2} \qquad \ldots(ii)$$

From (i) and (ii), we get

$$\frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \frac{AP^2}{DY^2} \qquad \text{Q.E.D.}$$

THEOREM 5 *If the areas of two similar triangles are equal, then the triangles are congruent i.e. equal and similar triangles are congruent.* **[NCERT, CBSE 2002C, 2018]**

GIVEN Two triangles ABC and DEF such that $\Delta ABC \sim \Delta DEF$ and Area (ΔABC) = Area (ΔDEF).

TO PROVE $\Delta ABC \cong \Delta DEF$

PROOF We have,

$$\Delta ABC \sim \Delta DEF \Rightarrow \angle A = \angle D, \angle B = \angle E, \angle C = \angle F \text{ and } \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

In order to prove that $\Delta ABC \cong \Delta DEF$, it is sufficient to show that $AB = DE$, $BC = EF$ and $AC = DF$.

It is given that : Area (ΔABC) = Area (ΔDEF)

$$\Rightarrow \quad \frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = 1$$

$$\Rightarrow \quad \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{AC^2}{DF^2} = 1 \qquad \left[\because \frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{AC^2}{DF^2}\right]$$

$$\Rightarrow \quad AB^2 = DE^2,\ BC^2 = EF^2 \text{ and } AC^2 = DF^2$$

$$\Rightarrow \quad AB = DE, BC = EF \text{ and } AC = DF$$

Hence, $\quad \Delta ABC \cong \Delta DEF$. **Q.E.D.**

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *If $\Delta ABC \sim \Delta DEF$ such that $AB = 1.2$ cm and $DE = 1.4$ cm. Find the ratio of areas of ΔABC and ΔDEF.*

SOLUTION We know that the ratio of areas of two similar triangles is equal to the ratio of the squares of any two corresponding sides.

$$\therefore \quad \frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \frac{AB^2}{DE^2} \Rightarrow \frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \frac{(1.2)^2}{(1.4)^2} = \left(\frac{12}{14}\right)^2 = \frac{36}{49}$$

EXAMPLE 2 *In two similar triangles ABC and PQR, if their corresponding altitudes AD and PS are in the ratio 4 : 9, find the ratio of the areas of Δ ABC and Δ PQR.*

SOLUTION Since the areas of two similar triangles are in the ratio of the squares of the corresponding altitudes.

$$\therefore \quad \frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta PQR)} = \frac{AD^2}{PS^2} \Rightarrow \frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta PQR)} = \left(\frac{4}{9}\right)^2 = \frac{16}{81} \qquad [\because AD : PS = 4:9]$$

Hence, Area (ΔABC) : Area $(\Delta PQR) = 16 : 81$

EXAMPLE 3 *If Δ ABC is similar to Δ DEF such that BC = 3 cm, EF = 4 cm and area of Δ ABC = 54 cm^2. Determine the area of Δ DEF.*

SOLUTION Since the ratio of areas of two similar triangles is equal to the ratio of the squares of any two corresponding sides.

$$\therefore \quad \frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \frac{BC^2}{EF^2}$$

$$\Rightarrow \quad \frac{54}{\text{Area}(\Delta DEF)} = \frac{3^2}{4^2} \Rightarrow \text{Area}(\Delta DEF) = \frac{54 \times 16}{9} = 96 \text{ cm}^2$$

EXAMPLE 4 *If $\Delta ABC \sim \Delta DEF$ such that area of Δ ABC is 9 cm^2 and the area of Δ DEF is 16 cm^2 and BC = 2.1 cm. Find the length of EF.*

SOLUTION We have,

$$\frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \frac{BC^2}{EF^2}$$

$$\Rightarrow \quad \frac{9}{16} = \frac{(2.1)^2}{EF^2} \Rightarrow \frac{3}{4} = \frac{2.1}{EF} \Rightarrow EF = \frac{4 \times 2.1}{3} \text{ cm} = 2.8 \text{ cm}$$

EXAMPLE 5 *In Fig. 7.166, PB and QA are perpendiculars to segment AB. If PO = 5 cm, QO = 7 cm and Area Δ POB = 150 cm^2 find the area of Δ QOA.*

SOLUTION In ΔOAQ and ΔOBP, we have

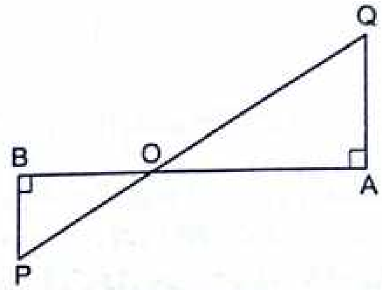
$\angle A = \angle B$ [Each equal to 90°]

$\angle AOQ = \angle BOP$

So, by *AA*-criterion of similarity, we have

$\Delta AOQ \sim \Delta BOP$

$$\Rightarrow \quad \frac{\text{Area}(\Delta AOQ)}{\text{Area}(\Delta BOP)} = \frac{OQ^2}{OP^2}$$



$$\Rightarrow \quad \frac{\text{Area}(\Delta AOQ)}{150} = \frac{7^2}{5^2}$$

$$\Rightarrow \quad \text{Area}(\Delta AOQ) = \frac{49}{25} \times 150 \text{ cm}^2 = 294 \text{ cm}^2$$

Fig. 7.167

EXAMPLE 6 *In Fig. 7.168, ABCD is a trapezium in which AB || DC and AB = 2 DC. Determine the ratio of the areas of ΔAOB and ΔCOD.* **[NCERT]**

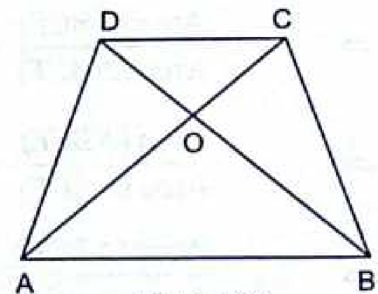
SOLUTION In triangle AOB and COD, we have

$\angle AOB = \angle COD$ [Vertically opposite angles]

and, $\angle OAB = \angle OCD$ [Alternate angles]

So, by AA-criterion of similarity, we have

$\Delta AOB \sim \Delta COD$



$$\Rightarrow \quad \frac{\text{Area}(\Delta AOB)}{\text{Area}(\Delta COD)} = \frac{AB^2}{DC^2}$$

$$\Rightarrow \quad \frac{\text{Area}(\Delta AOB)}{\text{Area}(\Delta COD)} = \frac{(2DC)^2}{(DC)^2} = \frac{4}{1}$$

Fig. 7.168

Hence, Area (ΔAOB) : Area $(\Delta COD) = 4:1$.

EXAMPLE 7 *In the trapezium ABCD, AB || CD and AB = 2 CD. If the area of $\Delta AOB = 84 \text{ cm}^2$, find the area of ΔCOD.* **[CBSE 2005]**

SOLUTION From example 6, we have

$$\frac{\text{Area}(\Delta AOB)}{\text{Area}(\Delta COD)} = \frac{4}{1}$$

$$\Rightarrow \quad \frac{84}{\text{Area}(\Delta COD)} = \frac{4}{1} \Rightarrow \text{Area}(\Delta COD) = 21 \text{ cm}^2$$

LEVEL-2

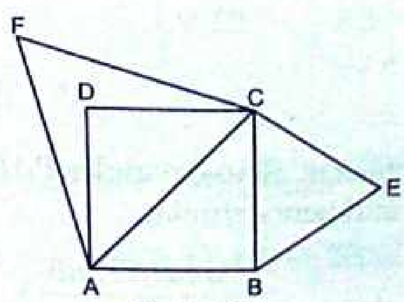
EXAMPLE 8 *Prove that the area of the triangle BCE described on one side BC of a square ABCD as base is one half the area of the similar triangle ACF described on the diagonal AC as base.*

SOLUTION $ABCD$ is a square. ΔBCE is described on side BC is similar to ΔACF described on diagonal AC.

Since $ABCD$ is a square. Therefore,

$AB = BC = CD = DA$ and, $AC = \sqrt{2}\, BC$ [$\because$ Diagonal = $\sqrt{2}$ Side)]

Now, $\Delta BCE \sim \Delta ACF$



$$\Rightarrow \quad \frac{\text{Area}(\Delta BCE)}{\text{Area}(\Delta ACF)} = \frac{BC^2}{AC^2}$$

Fig. 7.169

$$\Rightarrow \quad \frac{\text{Area}(\Delta BCE)}{\text{Area}(\Delta ACF)} = \frac{BC^2}{(\sqrt{2}BC)^2} = \frac{1}{2}$$

$$\Rightarrow \quad \text{Area}(\Delta BCE) = \frac{1}{2}\text{Area}(\Delta ACF)$$

EXAMPLE 9 *Prove that the area of the equilateral triangle described on the side of a square is half the area of the equilateral triangle described on its diagonal.* **[NCERT, CBSE 2018]**

GIVEN A square $ABCD$. Equilateral triangles $\Delta\ BCE$ and $\Delta\ ACF$ have been described on side BC and diagonal AC respectively.

TO PROVE $\text{Area}(\Delta\ BCE) = \frac{1}{2}.\text{Area}(\Delta\ ACF)$

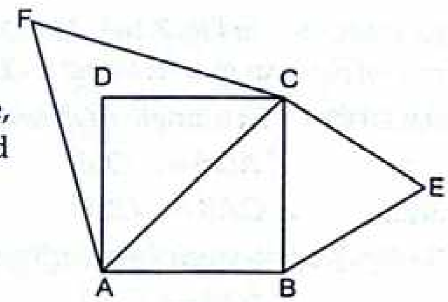


Fig. 7.170

PROOF Since $\Delta\ BCE$ and $\Delta\ ACF$ are equilateral. Therefore, they are equiangular (each angle being equal to 60°) and hence

$$\Delta\ BCE \sim \Delta\ ACF$$

$$\Rightarrow \quad \frac{\text{Area}(\Delta BCE)}{\text{Area}(\Delta ACF)} = \frac{BC^2}{AC^2}$$

$$\Rightarrow \quad \frac{\text{Area}(\Delta BCE)}{\text{Area}(\Delta ACF)} = \frac{BC^2}{(\sqrt{2}BC)^2} \qquad \left[\begin{array}{l} \because ABCD \text{ is a square} \\ \therefore \text{diagonal} = \sqrt{2}(\text{side}) \Rightarrow AC = \sqrt{2}BC \end{array}\right]$$

$$\Rightarrow \quad \frac{\text{Area}(\Delta BCE)}{\text{Area}(\Delta ACF)} = \frac{1}{2}$$

EXAMPLE 10 *Equilateral triangles are drawn on the sides of a right triangle. Show that the area of the triangle on the hypotenuse is equal to the sum of the areas of triangles on the other two sides.*

[CBSE 2002]

GIVEN A right angled triangle ABC with right angle at B. Equilateral triangles PAB, QBC and RAC are described on sides AB, BC and CA respectively.

TO PROVE $\text{Area}(\Delta\ PAB) + \text{Area}(\Delta\ QBC) = \text{Area}(\Delta\ RAC)$

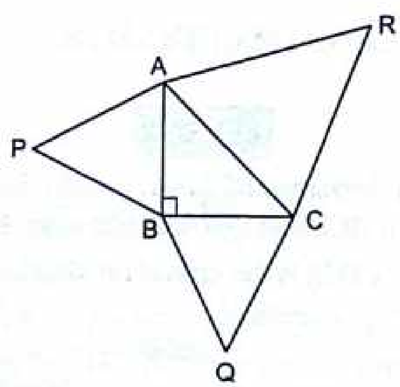


Fig. 7.171

PROOF Since triangles PAB, QBC and RAC are equilateral. Therefore, they are equiangular and hence similar.

$$\therefore \quad \frac{\text{Area}(\Delta PAB)}{\text{Area}(\Delta RAC)} + \frac{\text{Area}(\Delta QBC)}{\text{Area}(\Delta RAC)} = \frac{AB^2}{AC^2} + \frac{BC^2}{AC^2}$$

$$\Rightarrow \quad \frac{\text{Area}(\Delta PAB)}{\text{Area}(\Delta RAC)} + \frac{\text{Area}(\Delta QBC)}{\text{Area}(\Delta RAC)} = \frac{AB^2 + BC^2}{AC^2}$$

$$\Rightarrow \quad \frac{\text{Area}(\Delta PAB)}{\text{Area}(\Delta RAC)} + \frac{\text{Area}(\Delta QBC)}{\text{Area}(\Delta RAC)} = \frac{AC^2}{AC^2} = 1 \quad \left[\begin{array}{l}\because \Delta ABC \text{ is a right angled triangle} \\ \text{with } \angle B = 90^\circ \therefore AC^2 = AB^2 + BC^2\end{array}\right]$$

$$\Rightarrow \quad \frac{\text{Area}(\Delta PAB) + \text{Area}(\Delta QBC)}{\text{Area}(\Delta RAC)} = 1$$

$$\Rightarrow \quad \text{Area}(\Delta PAB) + \text{Area}(\Delta QBC) = \text{Area}(\Delta RAC)$$

EXAMPLE 11 *D, E, F are the mid-points of the sides BC, CA and AB respectively of a ΔABC. Determine the ratio of the areas of ΔDEF and ΔABC.* **[NCERT]**

SOLUTION Since D and E are the mid-points of the sides BC and AB respectively of ΔABC.

$\therefore \quad DE \parallel BA \Rightarrow DE \parallel FA$ ...(i)

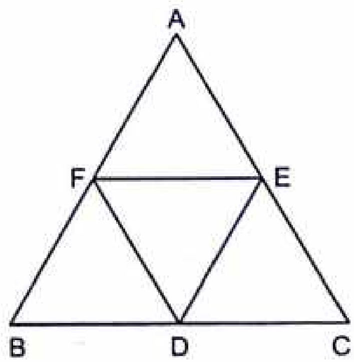


Fig. 7.172

Since D and F are mid-points of the sides BC and AB respectively of ΔABC. Therefore,

$DF \parallel CA \Rightarrow DF \parallel AE$ ...(ii)

From (i), and (ii), we conclude that $AFDE$ is a parallelogram.

Similarly, $BDEF$ is a parallelogram.

In ΔDEF and ΔABC, we have

$\angle FDE = \angle A$ [Opposite angles of parallelogram $AFDE$]

and, $\angle DEF = \angle B$ [Opposite angles of parallelogram $BDEF$]

So, by AA-similarity criterion, we have

$\Delta DEF \sim \Delta ABC$

$$\Rightarrow \quad \frac{\text{Area}(\Delta DEF)}{\text{Area}(\Delta ABC)} = \frac{DE^2}{AB^2} = \frac{(1/2AB)^2}{AB^2} = \frac{1}{4} \qquad \left[\because DE = \frac{1}{2}AB\right]$$

Hence, $\text{Area}(\Delta DEF) : \text{Area}(\Delta ABC) = 1 : 4$

EXAMPLE 12 *D and E are points on the sides AB and AC respectively of a ΔABC such that $DE \parallel BC$ and divides ΔABC into two parts, equal in area, Find $\frac{BD}{AB}$.* **[NCERT, CBSE 2000]**

SOLUTION We have,

Area (ΔADE) = Area (trapezium $BCED$)

$\Rightarrow$ Area (ΔADE) + Area (ΔADE) = Area (trapezium $BCED$) + Area (ΔADE)

$\Rightarrow$ 2 Area (ΔADE) = Area (ΔABC) ...(i)

In ΔADE and ΔABC, we have

$\angle ADE = \angle B$ [$\because DE \parallel BC \therefore \angle ADE = \angle B$ (Corresponding angles]

and, $\angle A = \angle A$ [Common]

$\therefore$ $\Delta ADE \sim \Delta ABC$

$$\Rightarrow \frac{\text{Area }(\Delta ADE)}{\text{Area }(\Delta ABC)} = \frac{AD^2}{AB^2}$$

$$\Rightarrow \frac{\text{Area }(\Delta ADE)}{2\text{ Area }(\Delta ADE)} = \frac{AD^2}{AB^2}$$

$$\Rightarrow \frac{1}{2} = \left(\frac{AD}{AB}\right)^2$$

$$\Rightarrow \frac{AD}{AB} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow AB = \sqrt{2}AD$$

$$\Rightarrow AB = \sqrt{2}(AB - BD)$$

$$\Rightarrow (\sqrt{2}-1)AB = \sqrt{2}BD \Rightarrow \frac{BD}{AB} = \frac{\sqrt{2}-1}{\sqrt{2}} = \frac{2-\sqrt{2}}{2}$$

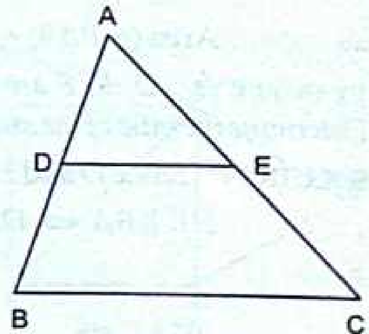


Fig. 7.173

EXAMPLE 13 *Two isosceles triangles have equal vertical angles and their areas are in the ratio 16 : 25. Find the ratio of their corresponding heights.* **[CBSE 2000]**

SOLUTION Let ΔABC and ΔDEF be the given triangles such that $AB = AC$ and $DE = DF$, $\angle A = \angle D$

and, $\frac{\text{Area }(\Delta ABC)}{\text{Area }(\Delta DEF)} = \frac{16}{25}$ ...(i)

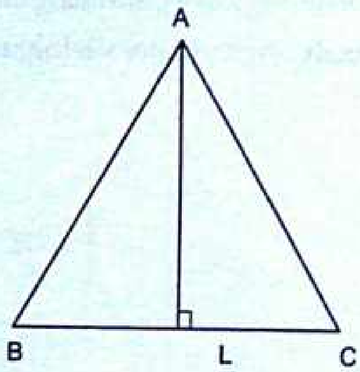


Fig. 7.174

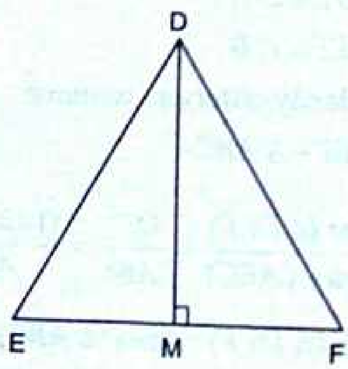


Fig. 7.175

Draw $AL \perp BC$ and $DM \perp EF$.

Now, $AB = AC, DE = DF$

$\Rightarrow \quad \frac{AB}{AC} = 1 \text{ and } \frac{DE}{DF} = 1$

$\Rightarrow \quad \frac{AB}{AC} = \frac{DE}{DF}$

$\Rightarrow \quad \frac{AB}{DE} = \frac{AC}{DF}$

Thus, in triangles ABC and DEF, we have

$\frac{AB}{DE} = \frac{AC}{DF}$ and $\angle A = \angle D$ [Given]

So, by *SAS*-similarity criterion, we have

$\Delta ABC \sim \Delta DEF$

$\Rightarrow \quad \frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \frac{AL^2}{DM^2}$

$\Rightarrow \quad \frac{16}{25} = \frac{AL^2}{DM^2}$ [Using (i)]

$\Rightarrow \quad \frac{AL}{DM} = \frac{4}{5}$

Hence, $AL : DM = 4 : 5$.

EXAMPLE 14 *In Fig 7.176, $DE \parallel BC$ and $AD : DB = 5 : 4$. Find* $\frac{\text{Area}(\Delta DEF)}{\text{Area}(\Delta CFB)}$. **[CBSE 2000]**

SOLUTION In ΔABC, we have

$DE \parallel BC$

$\Rightarrow \quad \angle ADE = \angle ABC$ and $\angle AED = \angle ACB$ [Corresponding angles]

Thus, in triangles ADE and ABC, we have

$\angle A = \angle A$ [Common]

$\angle ADE = \angle ABC$

and, $\angle AED = \angle ACB$

$\therefore \quad \Delta AED \sim \Delta ABC$ [By AAA similarity]

$\Rightarrow \quad \frac{AD}{AB} = \frac{DE}{BC}$

We have,

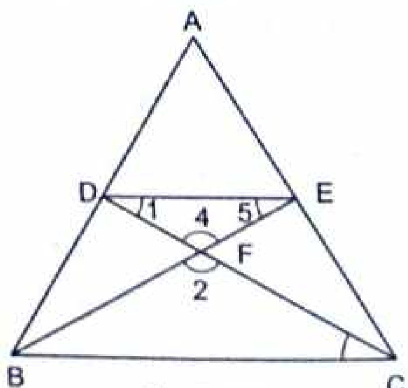


Fig. 7.176

$\frac{AD}{DB} = \frac{5}{4}$

$\Rightarrow \quad \frac{DB}{AD} = \frac{4}{5}$

$\Rightarrow \quad \frac{DB}{AD} + 1 = \frac{4}{5} + 1$

$\Rightarrow \quad \frac{DB + AD}{AD} = \frac{9}{5} \Rightarrow \frac{AB}{AD} = \frac{9}{5} \Rightarrow \frac{AD}{AB} = \frac{5}{9}$

$\therefore \quad \frac{DE}{BC} = \frac{5}{9}$

In $\Delta\ DFE$ and $\Delta\ CFB$, we have

$\angle 1 = \angle 3$ [Alternate interior angles]

$\angle 2 = \angle 4$ [Vertically opposite angles]

Therefore, by *AA*-similarity criterion, we have

$\Delta\ DFE \sim \Delta\ CFB$

$\Rightarrow \quad \frac{\text{Area}(\Delta\ DFE)}{\text{Area}(\Delta\ CFB)} = \frac{DE^2}{BC^2}$

$\Rightarrow \quad \frac{\text{Area}(\Delta\ DFE)}{\text{Area}(\Delta\ CFB)} = \left(\frac{5}{9}\right)^2 = \frac{25}{81}$ [Using (i)]

EXAMPLE 15 *In Fig. 7.177, XY || AC and XY divides triangular region ABC into two parts equal in area. Determine* $\frac{AX}{AB}$. **[CBSE 2008]**

SOLUTION We have,

$XY \parallel AC$

and, $\text{Area}(\Delta\ BXY) = \text{Area}(\text{quad. } XYCA)$

$\Rightarrow \quad \text{Area}(\Delta\ ABC) = 2\ \text{Area}(\Delta\ BXY)$ ...(i)

Now, $XY \parallel AC$ and BA is a transversal.

$\Rightarrow \quad \angle BXY = \angle BAC$ ...(ii)

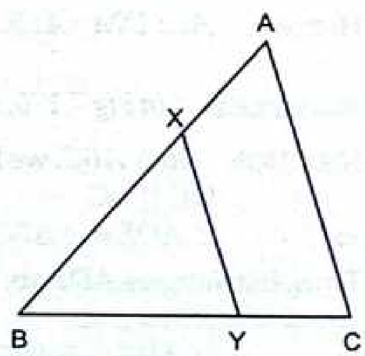


Fig. 7.177

Thus, in Δ's BAC and BXY, we have

$\angle XBY = \angle ABC$ [Common]

$\angle BXY = \angle BAC$ [From (ii)]

Therefore, *AA*-criterion of similarity, we have

$\Delta\ BAC \sim \Delta\ BXY$

$\Rightarrow \quad \frac{\text{Area}(\Delta\ BAC)}{\text{Area}(\Delta\ BXY)} = \frac{BA^2}{BX^2}$

$\Rightarrow \quad 2 = \frac{BA^2}{BX^2}$ [Using (i)]

$\Rightarrow \quad BA = \sqrt{2}BX$

$\Rightarrow \quad BA = \sqrt{2}(BA - AX) \Rightarrow (\sqrt{2}-1)BA = \sqrt{2}\,AX \Rightarrow \frac{AX}{AB} = \frac{\sqrt{2}-1}{\sqrt{2}}$

EXERCISE 7.6

LEVEL-1

1. Triangles ABC and DEF are similar.
   (i) If area $(\Delta\ ABC) = 16\ \text{cm}^2$, area $(\Delta\ DEF) = 25\ \text{cm}^2$ and $BC = 2.3$ cm, find EF.
   (ii) If area $(\Delta\ ABC) = 9\ \text{cm}^2$, area $(\Delta\ DEF) = 64\ \text{cm}^2$ and $DE = 5.1$ cm, find AB.

(iii) If $AC = 19$ cm and $DF = 8$ cm, find the ratio of the area of two triangles.

(iv) If area $(\Delta ABC) = 36\text{ cm}^2$, area $(\Delta DEF) = 64\text{ cm}^2$ and $DE = 6.2$ cm, find AB.

(v) If $AB = 1.2$ cm and $DE = 1.4$ cm, find the ratio of the areas of ΔABC and ΔDEF.

2. In Fig. 7.178, $\Delta ACB \sim \Delta APQ$. If $BC = 10$ cm, $PQ = 5$ cm, $BA = 6.5$ cm and $AP = 2.8$ cm, find CA and AQ. Also, find the area (ΔACB) : area (ΔAPQ).

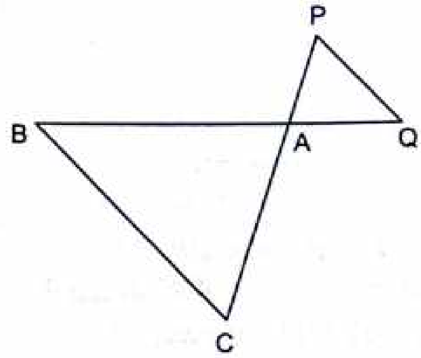


Fig. 7.178

3. The areas of two similar triangles are 81 cm^2 and 49 cm^2 respectively. Find the ratio of their corresponding heights. What is the ratio of their corresponding medians?
4. The areas of two similar triangles are 169 cm^2 and 121 cm^2 respectively. If the longest side of the larger triangle is 26 cm, find the longest side of the smaller triangle.
5. The areas of two similar triangles are 25 cm^2 and 36 cm^2 respectively. If the altitude of the first triangle is 2.4 cm, find the corresponding altitude of the other.
6. The corresponding altitudes of two similar triangles are 6 cm and 9 cm respectively. Find the ratio of their areas.
7. ABC is a triangle in which $\angle A = 90°$, $AN \perp BC$, $BC = 12$ cm and $AC = 5$ cm. Find the ratio of the areas of ΔANC and ΔABC.
8. In Fig. 7.179, $DE \parallel BC$

   (i) If $DE = 4$ cm, $BC = 6$ cm and Area $(\Delta ADE) = 16\text{ cm}^2$, find the area of ΔABC.

   (ii) If $DE = 4$ cm, $BC = 8$ cm and Area $(\Delta ADE) = 25\text{ cm}^2$, find the area of ΔABC.

   (iii) If $DE : BC = 3 : 5$. Calculate the ratio of the areas of ΔADE and the trapezium $BCED$.

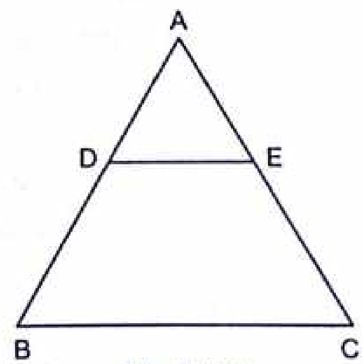


Fig. 7.179

9. In ΔABC, D and E are the mid-points of AB and AC respectively. Find the ratio of the areas of ΔADE and ΔABC.
10. The areas of two similar triangles are 100 cm^2 and 49 cm^2 respectively. If the altitude of the bigger triangle is 5 cm, find the corresponding altitude of the other. **[CBSE 2002]**

11. The areas of two similar triangles are 121 cm^2 and 64 cm^2 respectively. If the median of the first triangle is 12.1 cm, find the corresponding median of the other. **[CBSE 2001]**
12. If $\Delta ABC \sim \Delta DEF$ such that $AB = 5$ cm, area $(\Delta ABC) = 20\ cm^2$ and area $(\Delta DEF) = 45\ cm^2$, determine DE.
13. In ΔABC, PQ is a line segment intersecting AB at P and AC at Q such that $PQ \parallel BC$ and PQ divides ΔABC into two parts equal in area. Find $\frac{BP}{AB}$.
14. The areas of two similar triangles ABC and PQR are in the ratio $9:16$. If $BC = 4.5$ cm, find the length of QR. **[CBSE 2004]**
15. ABC is a triangle and PQ is a straight line meeting AB in P and AC in Q. If $AP = 1$ cm, $PB = 3$ cm, $AQ = 1.5$ cm, $QC = 4.5$ m, prove that area of ΔAPQ is one-sixteenth of the area of ΔABC. **[CBSE 2005]**
16. If D is a point on the side AB of ΔABC such that $AD : DB = 3.2$ and E is a point on BC such that $DE \parallel AC$. Find the ratio of areas of ΔABC and ΔBDE. **[CBSE 2006 C]**
17. If ΔABC and ΔBDE are equilateral triangles, where D is the mid point of BC, find the ratio of areas of ΔABC and ΔBDE. **[CBSE 2010]**

LEVEL-2

18. Two isosceles triangles have equal vertical angles and their areas are in the ratio 36 : 25. Find the ratio of their corresponding heights.
19. In Fig. 7.180, ΔABC and ΔDBC are on the same base BC. If AD and BC intersect at O, prove that **[NCERT, CBSE 2000, 2005]**

$$\frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DBC)} = \frac{AO}{DO}$$

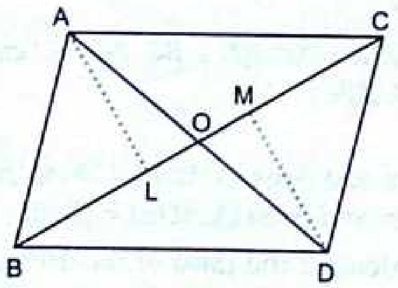


Fig. 7.180

20. $ABCD$ is a trapezium in which $AB \parallel CD$. The diagonals AC and BD intersect at O. Prove that: (i) $\Delta AOB \sim \Delta COD$ (ii) If $OA = 6$ cm, $OC = 8$ cm, Find:

    (a) $\frac{\text{Area}(\Delta AOB)}{\text{Area}(\Delta COD)}$ (b) $\frac{\text{Area}(\Delta AOD)}{\text{Area}(\Delta COD)}$
21. In ΔABC, P divides the side AB such that $AP : PB = 1 : 2$. Q is a point in AC such that $PQ \parallel BC$. Find the ratio of the areas of ΔAPQ and trapezium $BPQC$.
22. AD is an altitude of an equilateral triangle ABC. On AD as base, another equilateral triangle ADE is constructed. Prove that Area (ΔADE) : Area $(\Delta ABC) = 3 : 4$. **[CBSE 2010]**

ANSWERS

1. (i) 2.875 cm (ii) 1.9125 cm (iii) 361 : 64 (iv) 4.65 (v) 36 : 49
2. 5.6 cm, 3.25 cm, 4 : 1
3. 9 : 7 ; 9 : 7
4. 22 cm

5. 2.88 cm
6. $4:9$
7. $25:144$
8. (i) 36 cm^2, (ii) 100 cm^2, (iii) $9:16$
9. $1:4$
10. 3.5 cm
11. 8.8 cm
12. 7.5 cm
13. $\dfrac{\sqrt{2}-1}{\sqrt{2}}$
14. 6 cm
16. $25:4$
17. 4:1
18. $6:5$
20. (a) $\dfrac{9}{16}$ (b) $\dfrac{3}{4}$
21. $1:8$

HINT TO SELECTED PROBLEMS

3. Since the ratio of the areas of two similar triangles is equal to the ratio of the squares of their corresponding altitudes and is also equal to the ratio of the squares of their corresponding medians. Hence,

ratio of altitudes $= 9:7 =$ ratio of medians.

4. We have,

$$\frac{169}{121} = \frac{26^2}{(\text{Side of the smaller triangle})^2}$$

$$\Rightarrow \frac{13}{11} = \frac{26}{\text{Side of the smaller triangle}} \Rightarrow \text{Side} = 22\text{ cm}$$

7. We have,

$$\Delta ANC \sim \Delta ABC \Rightarrow \frac{\text{Area}(\Delta ANC)}{\text{Area}(\Delta ABC)} = \frac{AC^2}{BC^2} = \frac{25}{144}$$

8. (i) In ΔADE and ΔABC, we have

$\angle ADE = \angle B$ [Corresponding angles ($\because DE \parallel BC$)]

$\angle A = \angle A$ [Common]

$$\therefore \quad \Delta ADE \sim \Delta ABC \Rightarrow \frac{\text{Area}(\Delta ADE)}{\text{Area}(\Delta ABC)} = \frac{DE^2}{BC^2}$$

(iii) $$\frac{\text{Area}(\Delta ADE)}{\text{Area}(\Delta ABC)} = \frac{DE^2}{BC^2} = \frac{3^2}{5^2} = \frac{9}{25}$$

Let Area $(\Delta ADE) = 9x$ sq. units and Area $(\Delta ABC) = 25x$ sq. units.

$\therefore$ Area (trap. $BCED$) $=$ Area $(\Delta ABC) -$ Area $(\Delta ADE) = 25x - 9x = 16x$

9. Since D and E are the mid-points of AB and AC respectively. Therefore, $DE \parallel BC$. Consequently, we have

$$\Delta ADE \sim \Delta ABC \Rightarrow \frac{\text{Area}(\Delta ADE)}{\text{Area}(\Delta ABC)} = \frac{AD^2}{AB^2} = \frac{AD^2}{(2AD)^2} = \frac{1}{4}$$

19. Draw $AL \perp BC$ and $DM \perp BC$. In ΔALO and ΔDMO, we have

$\angle ALO = \angle DMO = 90^\circ$ and, $\angle AOL = \angle DOM$

$\therefore \quad \Delta ALO \sim \Delta DMO$

$$\Rightarrow \quad \frac{AL}{DM} = \frac{AO}{DO} \quad \text{...(i)}$$

$$\therefore \quad \frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DBC)} = \frac{(1/2)\,BC \times AL}{(1/2)\,BC \times DM} = \frac{AL}{DM} = \frac{AO}{DO} \quad \text{[Using (i)]}$$

7.10 PYTHAGORAS THEOREM

In this section, we shall prove an important theorem known as Pythagoras Theorem. This Theorem is also known as Baudhayan Theorem.

THEOREM 1 *In a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.* **[NCERT, CBSE 2001, 2002, 2004, 2005, 2006C, 2009, 2010, 2018]**

GIVEN A right-angled triangle ABC in which $\angle B = 90°$.

TO PROVE $(\text{Hypotenuse})^2 = (\text{Base})^2 + (\text{Perpendicular})^2$ i.e. $AC^2 = AB^2 + BC^2$.

CONSTRUCTION From B draw $BD \perp AC$.

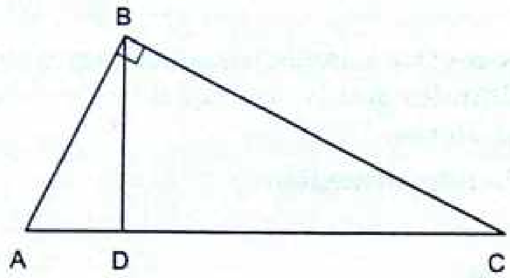


Fig. 7.181

PROOF In triangles ADB and ABC, we have

$\angle ADB = \angle ABC$ [Each equal to 90°]

and, $\angle A = \angle A$ [Common]

So, by AA-similarity criterion, we have

$\Delta ADB \sim \Delta ABC$

$\Rightarrow \quad \frac{AD}{AB} = \frac{AB}{AC}$ [$\because$ In similar triangles corresponding sides are proportional]

$\Rightarrow \quad AB^2 = AD \times AC$ ...(i)

In triangles BDC and ABC, we have

$\angle CDB = \angle ABC$ [Each equal to 90°]

and, $\angle C = \angle C$ [Common]

So, by AA-similarity criterion, we have

$\Delta BDC \sim \Delta ABC$

$\Rightarrow \quad \frac{DC}{BC} = \frac{BC}{AC}$ [$\because$ In similar triangles corresponding sides are proportional]

$\Rightarrow \quad BC^2 = AC \times DC$ ...(ii)

Adding equations (i) and (ii), we get

$AB^2 + BC^2 = AD \times AC + AC \times DC$

$\Rightarrow \quad AB^2 + BC^2 = AC(AD + DC)$

$\Rightarrow \quad AB^2 + BC^2 = AC \times AC$

$\Rightarrow \quad AB^2 + BC^2 = AC^2$

Hence, $AC^2 = AB^2 + BC^2$

Q.E.D.

The converse of the above theorem is also true as proved below.

THEOREM 2 *(Converse of Pythagoras Theorem) In a triangle, if the square of one side is equal to the sum of the squares of the other two sides, then the angle opposite to the side is a right angle.*

[NCERT, CBSE 2000C, 2006C, 2009, 2010]

GIVEN A triangle ABC such that $AC^2 = AB^2 + BC^2$.

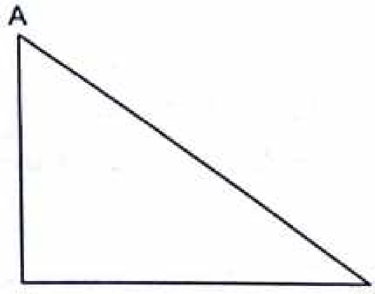


Fig. 7.182

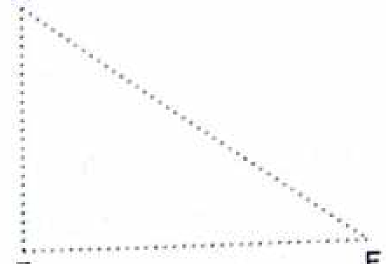


Fig. 7.183

CONSTRUCTION Construct a triangle DEF such that $DE = AB$, $EF = BC$ and $\angle E = 90°$.

PROOF In order to prove that $\angle B = 90°$, it is sufficient to show that $\Delta ABC \sim \Delta DEF$. For this we proceed as follows:

Since ΔDEF is a right angled triangle with right angle at E. Therefore, by Pythagoras theorem, we have

$$DF^2 = DE^2 + EF^2$$

$\Rightarrow \quad DF^2 = AB^2 + BC^2$ [$\because DE = AB$ and $EF = BC$ (By construction)]

$\Rightarrow \quad DF^2 = AC^2$ [$\because AB^2 + BC^2 = AC^2$ (Given)]

$\Rightarrow \quad DF = AC$ ...(i)

Thus, in ΔABC and ΔDEF, we have

$AB = DE, BC = EF$ [By construction]

and, $AC = DF$ [From equation (i)]

$\therefore \quad \Delta ABC \cong \Delta DEF$

$\Rightarrow \quad \angle B = \angle E = 90°$

Hence, ΔABC is a right triangle right-angled at B. **Q.E.D.**

7.10.1 SOME IMPORTANT RESULTS BASED UPON PYTHAGORAS THEOREM

THEOREM 1 *(Result on obtuse triangle) In Fig. 7.184, ΔABC is an obtuse triangle, obtuse-angled at B. If $AD \perp CB$, prove that $AC^2 = AB^2 + BC^2 + 2\,BC \times BD$.* **[NCERT]**

GIVEN An obtuse triangle ABC, obtuse-angled at B and AD is perpendicular to CB produced.

TO PROVE $AC^2 = AB^2 + BC^2 + 2\,BC \times BD$.

PROOF Since ΔADB is a right triangle right-angled at D. Therefore, by Pythagoras theorem, we have

$$AB^2 = AD^2 + DB^2 \quad \text{...(i)}$$

Again, ΔADC is a right triangle right-angled at D.

Therefore, by Pythagoras theorem, we have

$$AC^2 = AD^2 + DC^2$$

$\Rightarrow \quad AC^2 = AD^2 + (DB + BC)^2$

$\Rightarrow \quad AC^2 = AD^2 + DB^2 + BC^2 + 2\,BC \cdot BD$

$\Rightarrow \quad AC^2 = (AD^2 + DB^2) + BC^2 + 2\,BC \cdot BD$

$\Rightarrow \quad AC^2 = AB^2 + BC^2 + 2\,BC \cdot BD$ [Using (i)]

Hence, $AC^2 = AB^2 + BC^2 + 2\,BC \cdot BD$

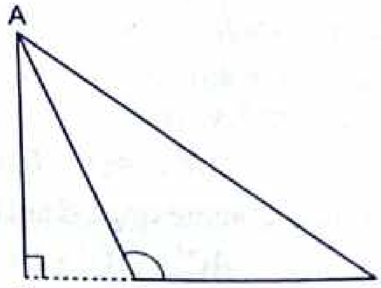


Fig. 7.184

REMARK *In the above theorem BD is known as the projection of AB on BC and the theorem can also be stated as:*

In an obtuse triangle, the square of the side opposite to obtuse angle is equal to the sum of the squares of other two sides plus twice the product of one side and the projection of other on first.

THEOREM 2 (Result on acute triangle) *In Fig. 7.185, $\angle B$ of ΔABC is an acute angle and $AD \perp BC$, prove that $AC^2 = AB^2 + BC^2 - 2\,BC \times BD$* **[NCERT]**

GIVEN A ΔABC in which $\angle B$ is an acute angle and $AD \perp BC$

TO PROVE $AC^2 = AB^2 + BC^2 - 2BC \times BD$.

PROOF Since ΔADB is a right triangle right-angled at D. So, by Pythagoras theorem, we have

$$AB^2 = AD^2 + BD^2 \quad \ldots(i)$$

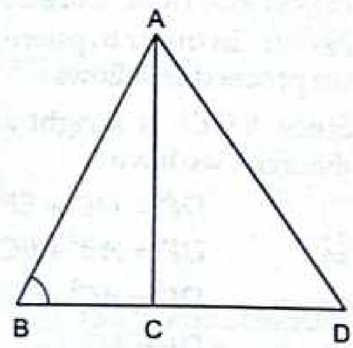


Fig. 7.185

Again, ΔADC is a right triangle right-angled at D. Applying Pythagoras theorem, we obtain

$$AC^2 = AD^2 + DC^2$$

$\Rightarrow$ $AC^2 = AD^2 + (BC - BD)^2$

$\Rightarrow$ $AC^2 = AD^2 + (BC^2 + BD^2 - 2\,BC \cdot BD)$

$\Rightarrow$ $AC^2 = (AD^2 + BD^2) + BC^2 - 2\,BC \cdot BD$

$\Rightarrow$ $AC^2 = AB^2 + BC^2 - 2\,BC \cdot BD$ [Using (i)]

Hence, $AC^2 = AB^2 + BC^2 - 2\,BC \cdot BD$

Q.E.D.

REMARK *In the above theorem BD is known as the projection of AB on BC and the theorem can also be stated as:*

In an acute triangle, the square of the side opposite to an acute angle is equal to the sum of the squares of other two sides minus twice the product of one side and the projection of other on first.

THEOREM 3 *Prove that in any triangle, the sum of the squares of any two sides is equal to twice the square of half of the third side together with twice the square of the median which bisects the third side.*

GIVEN A ΔABC in which AD is a median.

TO PROVE $AB^2 + AC^2 = 2AD^2 + 2\left(\frac{1}{2}BC\right)^2$ or, $AB^2 + AC^2 = 2(AD^2 + BD^2)$

CONSTRUCTION Draw $AE \perp BC$.

PROOF Since $\angle AED = 90°$. Therefore, in ΔADE, we have

$$\angle ADE < 90° \Rightarrow \angle ADB > 90°$$

Thus, ΔADB is an obtuse-angled triangle and ΔADC is an acute-angled triangle.

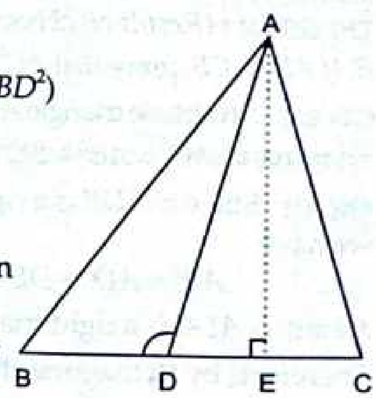


Fig. 7.186

ΔABD is obtuse-angled at D and $AE \perp BD$ produced. Therefore, by theorem 1, we have

$$AB^2 = AD^2 + BD^2 + 2\,BD \times DE \quad \ldots(i)$$

ΔACD is acute-angled at D and $AE \perp CD$. Therefore, by theorem 2, we have

$$AC^2 = AD^2 + DC^2 - 2\,DC \times DE$$

$\Rightarrow$ $AC^2 = AD^2 + BD^2 - 2\,BD \times DE$ [$\because CD = BD$] ...(ii)

Adding equations (i) and (ii), we get

$$AB^2 + AC^2 = 2(AD^2 + BD^2)$$

$$\Rightarrow \quad AB^2 + AC^2 = 2\left\{AD^2 + \left(\frac{BC}{2}\right)^2\right\}$$

$$\Rightarrow \quad AB^2 + AC^2 = 2\,AD^2 + 2\left(\frac{1}{2}BC\right)^2$$

$$\Rightarrow \quad AB^2 + AC^2 = 2\,AD^2 + 2\,BD^2$$

$$\Rightarrow \quad AB^2 + AC^2 = 2\,(AD^2 + BD^2)$$

Q.E.D.

THEOREM 4 *Prove that three times the sum of the squares of the sides of a triangle is equal to four times the sum of the squares of the medians of the triangle.*

GIVEN A $\Delta\, ABC$ in which AD, BE and CF are three medians.

TO PROVE $3\,(AB^2 + BC^2 + CA^2) = 4\,(AD^2 + BE^2 + CF^2)$

PROOF Since in any triangle, the sum of the squares of any two sides is equal to twice the square of half of the third side together with twice the square of the median bisecting it.

Therefore, taking AD as the median bisecting side BC, we have

$$AB^2 + AC^2 = 2\,(AD^2 + BD^2)$$

$$\Rightarrow \quad AB^2 + AC^2 = 2\left\{AD^2 + \left(\frac{BC}{2}\right)^2\right\}$$

$$\Rightarrow \quad AB^2 + AC^2 = 2\left\{AD^2 + \frac{BC^2}{4}\right\}$$

$$\Rightarrow \quad 2\,(AB^2 + AC^2) = (4\,AD^2 + BC^2) \qquad \text{...(i)}$$

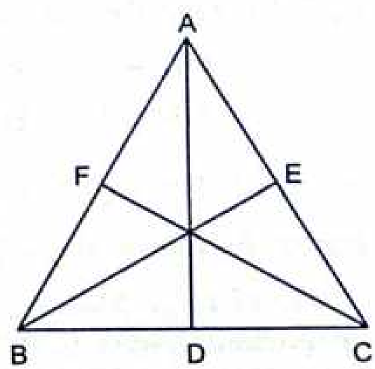


Fig. 7.187

Similarly, by taking BE and CF respectively as the medians, we get

$$2\,(AB^2 + BC^2) = (4\,BE^2 + AC^2) \qquad \text{...(ii)}$$

and, $$2\,(AC^2 + BC^2) = (4\,CF^2 + AB^2) \qquad \text{...(iii)}$$

Adding (i), (ii) and (iii), we get

$$4\,(AB^2 + BC^2 + AC^2) = 4\,(AD^2 + BE^2 + CF^2) + (BC^2 + AC^2 + AB^2)$$

$$\Rightarrow \quad 3\,(AB^2 + BC^2 + AC^2) = 4\,(AD^2 + BE^2 + CF^2)$$

Hence, $$3\,(AB^2 + BC^2 + AC^2) = 4\,(AD^2 + BE^2 + CF^2)$$

Q.E.D.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *A right triangle has hypotenuse of length p cm and one side of length q cm. If* $p - q = 1$, *find the length of the third side of the triangle.*

SOLUTION Let the third side be x cm. Then, by Pythagoras theorem, we have

$$p^2 = q^2 + x^2$$

$$\Rightarrow \quad x^2 = p^2 - q^2 = (p-q)\,(p+q) = p + q \qquad [\because p - q = 1]$$

$$\Rightarrow \quad x = \sqrt{p+q} = \sqrt{2q+1} \qquad [\because p - q = 1 \therefore p = q + 1]$$

Hence, the length of the third side is $\sqrt{2q+1}$ cm.

EXAMPLE 2 *The sides of certain triangles are given below. Determine which of them are right triangles:*

(i) $a = 6\ cm, b = 8\ cm$ *and* $c = 10\ cm$ (ii) $a = 5\ cm, b = 8\ cm$ *and* $c = 11\ cm$.

SOLUTION (i) We have,

$a = 6$ cm, $b = 8$ cm and $c = 10$ cm

Here, the larger side is $c = 10$ cm.

We have, $a^2 + b^2 = 6^2 + 8^2 = 36 + 64 = 100 = c^2$

So, the triangle with the given sides is a right triangle.

(ii) Here, the larger side is $c = 11$ cm.

Clearly, $a^2 + b^2 = 25 + 64 = 89 \neq c^2$

So, the triangle with the given sides is not a right triangle.

EXAMPLE 3 *A man goes 10 m due east and then 24 m due north. Find the distance from the starting point.*

SOLUTION Let the initial position of the man be O and his final position be B. Since the man goes 10 m due east and then 24 m due north. Therefore, $\Delta\, AOB$ is a right triangle right-angled at A such that $OA = 10$ m and $AB = 24$ m.

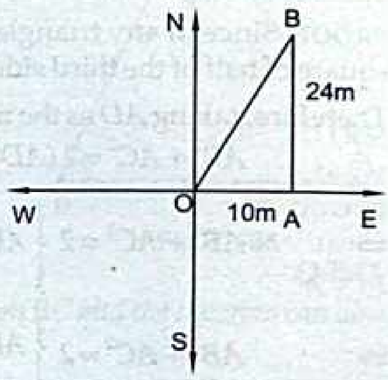


Fig. 7.188

By Pythagoras theorem, we have

$$OB^2 = OA^2 + AB^2$$

$$\Rightarrow \quad OB^2 = 10^2 + 24^2 = 100 + 576 = 676$$

$$\Rightarrow \quad OB = \sqrt{676} = 26 \text{ m}$$

Hence, the man is at a distance of 26 m from the starting point.

EXAMPLE 4 *A ladder is placed in such a way that its foot is at a distance of 5 m from a wall and its tip reaches a window 12 m above the ground. Determine the length of the ladder.*

SOLUTION Let AB be the ladder and B be the window. Then,

$$BC = 12 \text{ m and } AC = 5 \text{ m}$$

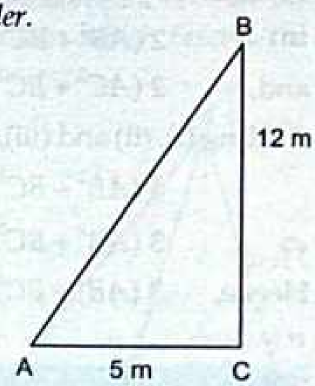


Fig. 7.189

Since $\Delta\, ABC$ is a right triangle right-angled at C.

$$\therefore \quad AB^2 = AC^2 + BC^2$$

$$\Rightarrow \quad AB^2 = 5^2 + 12^2 = 169$$

$$\Rightarrow \quad AB^2 = 13 \text{ m}$$

Hence, the length of the ladder is 13 m.

EXAMPLE 5 *A ladder 25 m long reaches a window of a building 20 m above the ground. Determine the distance of the foot of the ladder from the building.*

SOLUTION Suppose that AB is the ladder, B is the window and CB is the building. Then, triangle ABC is a right triangle with right-angle at C.

$$\therefore \quad AB^2 = AC^2 + BC^2$$

$$\Rightarrow \quad 25^2 = AC^2 + 20^2$$

$$\Rightarrow \quad AC^2 = 625 - 400 = 225$$

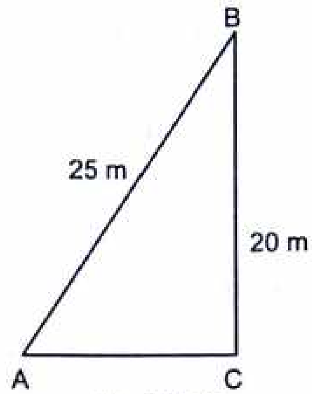


Fig. 7.190

$\Rightarrow \quad AC = \sqrt{225}\text{ m} = 15\text{ m}$

Hence, the foot of the ladder is at a distance 15 m from the building.

EXAMPLE 6 *A ladder 15 m long reaches a window which is 9 m above the ground on one side of a street. Keeping its foot at the same point, the ladder is turned to other side of the street to reach a window 12 m high. Find the width of the street.*

SOLUTION Let AB be the width of the street and C be the foot of the ladder. Let D and E be the windows at heights of 9 m and 12 m respectively from the ground. Then, CD and EF are the two positions of the ladder.

Clearly, $AD = 9$ m, $BE = 12$ m, $CD = CE = 15$ m.

In $\Delta\, ACD$, we have

$$CD^2 = AC^2 + AD^2$$

$\Rightarrow \quad 15^2 = AC^2 + 9^2$

$\Rightarrow \quad AC^2 = 225 - 81 = 144$

$\Rightarrow \quad AC = 12$ m

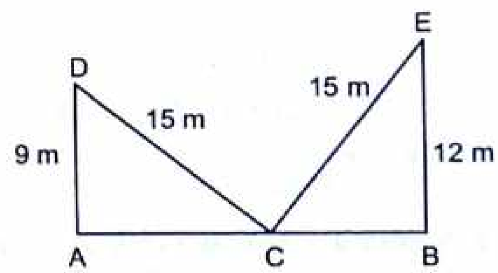


Fig. 7.191

In $\Delta\, BCE$, we have

$$CE^2 = BC^2 + BE^2$$

$\Rightarrow \quad 15^2 = BC^2 + 12^2$

$\Rightarrow \quad BC^2 = 225 - 144 = 81$

$\Rightarrow \quad BC = 9$ m

Hence, width of the street $= AB = AC + CB = (12 + 9)$ m $= 21$ m.

EXAMPLE 7 *The hypotenuse of a right triangle is 6 m more than the twice of the shortest side. If the third side is 2 m less than the hypotenuse, find the sides of the triangle.*

SOLUTION Let the shortest side be x metres in length. Then,

Hypotenuse $= (2x + 6)$ m and, Third side $= (2x + 4)$ m

By Pythagoras theorem, we have

$$(2x + 6)^2 = x^2 + (2x + 4)^2$$

$\Rightarrow \quad 4x^2 + 24x + 36 = x^2 + 4x^2 + 16x + 16$

$\Rightarrow \quad x^2 - 8x - 20 = 0$

$\Rightarrow \quad (x - 10)(x + 2) = 0$

$\Rightarrow \quad x = 10$ or, $x = -2$

$\Rightarrow \quad x = 10$ $\quad$ [$\because x$ cannot be negative]

Hence, the sides of the triangle are 10 m, 26 m and 24 m.

EXAMPLE 8 *P and Q are the mid-points of the sides CA and CB respectively of a Δ ABC, right angled at C. Prove that :*

(i) $4AQ^2 = 4AC^2 + BC^2$ **[CBSE 2010]**

(ii) $4BP^2 = 4BC^2 + AC^2$ **[CBSE 2001]**

(iii) $4(AQ^2 + BP^2) = 5AB^2$ **[NCERT, CBSE 2001, 2006C]**

SOLUTION (i) Since $\Delta\, AQC$ is a right triangle right-angled at C.

$\therefore \quad AQ^2 = AC^2 + QC^2$

$\Rightarrow \quad 4AQ^2 = 4AC^2 + 4QC^2$ [Multiplying both sides by 4]

$\Rightarrow \quad 4AQ^2 = 4AC^2 + (2QC)^2$

$\Rightarrow \quad 4AQ^2 = 4AC^2 + BC^2 \quad [\because BC = 2QC]$

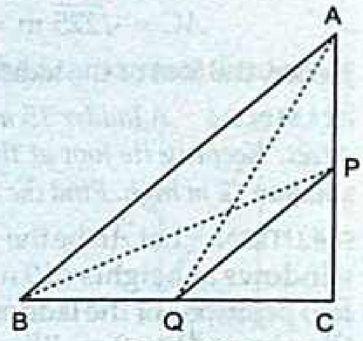


Fig. 7.192

(ii) Since $\Delta\, BPC$ is a right triangle right-angled at C.

$\therefore \quad BP^2 = BC^2 + CP^2$

$\Rightarrow \quad 4BP^2 = 4BC^2 + 4CP^2$ [Multiplying both sides by 4]

$\Rightarrow \quad 4BP^2 = 4BC^2 + (2CP)^2$

$\Rightarrow \quad 4BP^2 = 4BC^2 + AC^2 \quad [\because AC = 2CP]$

(iii) From (i) and (ii), we have

$4AQ^2 = 4AC^2 + BC^2$ and, $4BP^2 = 4BC^2 + AC^2$

$\therefore \quad 4AQ^2 + 4BP^2 = (4AC^2 + BC^2) + (4BC^2 + AC^2)$

$\Rightarrow \quad 4(AQ^2 + BP^2) = 5(AC^2 + BC^2)$

$\Rightarrow \quad 4(AQ^2 + BP^2) = 5AB^2 \quad \left[\text{In } \Delta ABC, \text{ we have } AB^2 = AC^2 + BC^2\right]$

EXAMPLE 9 *ABC is a right triangle right-angled at B. Let D and E be any points on AB and BC respectively. Prove that $AE^2 + CD^2 = AC^2 + DE^2$* **[CBSE 2002C, 2007]**

SOLUTION Since $\Delta\, ABE$ is right triangle, right-angled at B.

$\therefore \quad AE^2 = AB^2 + BE^2 \quad \ldots$(i)

Again, $\Delta\, DBC$ is right triangle right-angled at B.

$\therefore \quad CD^2 = BD^2 + BC^2 \quad \ldots$(ii)

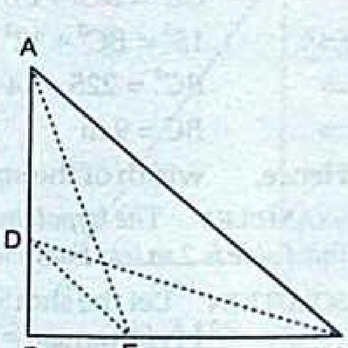


Fig. 7.193

Adding (i) and (ii), we get

$AE^2 + CD^2 = (AB^2 + BE^2) + (BD^2 + BC^2)$

$\Rightarrow \quad AE^2 + CD^2 = (AB^2 + BC^2) + (BE^2 + BD^2)$

Using Pythagoras theorem in $\Delta\, ABC$ and $\Delta\, DBE$, we have

$AC^2 = AB^2 + BC^2$ and $DE^2 = BE^2 + BD^2$

$\therefore \quad AE^2 + CD^2 = AC^2 + DE^2$

Hence, $\quad AE^2 + CD^2 = AC^2 + DE^2$

EXAMPLE 10 *Prove that three times the square of any side of an equilateral-triangle is equal to four times the square of the altitude.* **[CBSE 2002]**

SOLUTION Let ABC be an equilateral triangle and let $AD \perp BC$.

In $\Delta\, ADB$ and $\Delta\, ADC$, we have

$AB = AC$ [Given]

$\angle B = \angle C$ [Each equal to 60°]

and, $\angle ADB = \angle ADC$ [Each equal to 90°]

$\therefore \quad \Delta ADB \cong \Delta ADC$

$\Rightarrow \quad BD = DC$

$\Rightarrow \quad BD = DC = \frac{1}{2} BC$

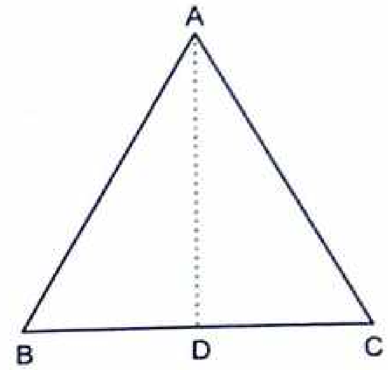


Fig. 7.194

Since ΔADB is a right triangle right-angled at D.

$\therefore \quad AB^2 = AD^2 + BD^2$

$\Rightarrow \quad AB^2 = AD^2 + \left(\frac{1}{2} BC\right)^2$

$\Rightarrow \quad AB^2 = AD^2 + \frac{BC^2}{4}$

$\Rightarrow \quad AB^2 = AD^2 + \frac{AB^2}{4}$ $[\because BC = AB]$

$\Rightarrow \quad \frac{3}{4} AB^2 = AD^2 \Rightarrow 3AB^2 = 4AD^2$

EXAMPLE 11 *In an equilateral triangle with side a, prove that*

(i) Altitude $= \frac{a\sqrt{3}}{2}$ **[CBSE 2001C]** (ii) Area $= \frac{\sqrt{3}}{4} a^2$ **[NCERT, CBSE 2002]**

SOLUTION Let ABC be an equilateral triangle the length of whose each side is a units. Draw $AD \perp BC$. Then, D is the mid-point of BC.

$\Rightarrow \quad AB = a, BD = \frac{1}{2} BC = \frac{a}{2}$

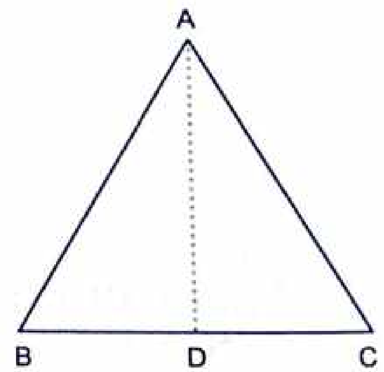


Fig. 7.195

Since ΔABD is a right triangle right-angled at D.

$\therefore \quad AB^2 = AD^2 + BD^2$

$\Rightarrow \quad a^2 = AD^2 + \left(\frac{a}{2}\right)^2$

$\Rightarrow \quad AD^2 = a^2 - \frac{a^2}{4} = \frac{3a^2}{4}$

$\Rightarrow \quad AD = \frac{\sqrt{3}a}{2}$

$\therefore \quad$ Altitude $\frac{\sqrt{3}}{2} a$

Now,

Area of $\Delta ABC = (1/2)$ (Base × Height)

$\Rightarrow \quad$ Area of $\Delta ABC = \frac{1}{2}(BC \times AD) = \frac{1}{2} \times a \times \frac{\sqrt{3}}{2} a = \frac{\sqrt{3}}{4} a^2$

EXAMPLE 12 *ABC is an isosceles right triangle right-angled at C. Prove that $AB^2 = 2AC^2$.* **[NCERT]**

SOLUTION Since ΔABC is a right triangle right-angled at C.

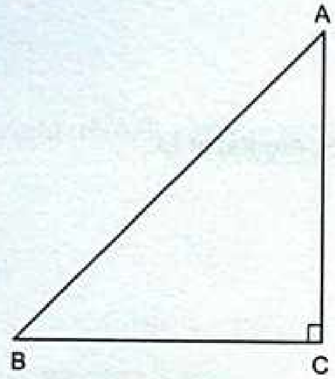


Fig. 7.196

$\therefore \quad AB^2 = AC^2 + BC^2$

$\Rightarrow \quad AB^2 = AC^2 + AC^2 \qquad [\because BC = AC]$

$\Rightarrow \quad AB^2 = 2AC^2$

EXAMPLE 13 *In ΔABC, AD is perpendicular to BC. Prove that:*

(i) $AB^2 + CD^2 = AC^2 + BD^2$ **[CBSE 2008, 2009]** (ii) $AB^2 - BD^2 = AC^2 - CD^2$

SOLUTION Since triangles ABD and ACD are right triangles right-angled at D.

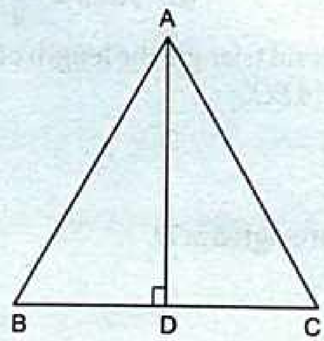


Fig. 7.197

$\therefore \quad AB^2 = AD^2 + BD^2 \qquad ...(i)$

and, $\quad AC^2 = AD^2 + CD^2 \qquad ...(ii)$

Subtracting (ii) from (i), we get

$AB^2 - AC^2 = BD^2 - CD^2$

$\Rightarrow \quad AB^2 + CD^2 = AC^2 + BD^2$ and $AB^2 - BD^2 = AC^2 - CD^2$

EXAMPLE 14 *P and Q are points on the sides CA and CB respectively of ΔABC right angled at C. Prove that $AQ^2 + BP^2 = AB^2 + PQ^2$* **[NCERT, CBSE 2002]**

SOLUTION In right-angled triangles ACQ and PCB, we have

$AQ^2 = AC^2 + CQ^2$ and $PB^2 = PC^2 + CB^2$

$\Rightarrow \quad AQ^2 + BP^2 = (AC^2 + CQ^2) + (PC^2 + CB^2)$

$\Rightarrow \quad AQ^2 + BP^2 = (AC^2 + BC^2) + (PC^2 + QC^2)$

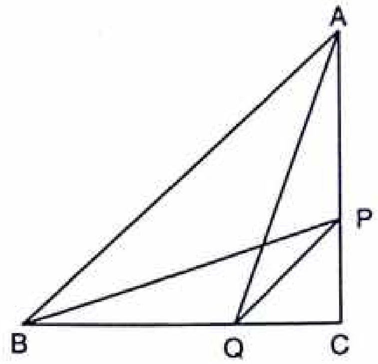


Fig. 7.198

$\Rightarrow \quad AQ^2 + BP^2 = AB^2 + PQ^2$ $\left[\begin{array}{l}\text{By Pythagoras theorem, we obtain} \\ AC^2 + BC^2 = AB^2 \text{ and } PC^2 + QC^2 = PQ^2\end{array}\right]$

EXAMPLE 15 *ABC is an isosceles triangle with $AC = BC$. If $AB^2 = 2AC^2$, prove that $\Delta\, ABC$ is right triangle.* **[NCERT, CBSE 2000]**

SOLUTION We have,

$AC = BC$ and $AB^2 = 2\,AC^2$

Now, $AB^2 = 2\,AC^2$

$\Rightarrow \quad AB^2 = AC^2 + AC^2$

$\Rightarrow \quad AB^2 = AC^2 + BC^2$ $[\because AC = BC \text{ (Given)}]$

$\Rightarrow \quad \Delta\, ABC$ is a right triangle right-angled at C.

LEVEL-2

EXAMPLE 16 *In Fig. 7.199, ABC is a right triangle right-angled at B. AD and CE are the two medians drawn from A and C respectively. If AC = 5 cm and $AD = \frac{3\sqrt{5}}{2}$ cm, find the length of CE.*

SOLUTION Since $\Delta\, ABD$ is a right triangle right-angled at B. Therefore,

$$AD^2 = AB^2 + BD^2$$

$$\Rightarrow \quad AD^2 = AB^2 + \left(\frac{BC}{2}\right)^2 \qquad [\because BD = DC]$$

$$\Rightarrow \quad AD^2 = AB^2 + \frac{1}{4}\cdot BC^2 \qquad \text{...(i)}$$

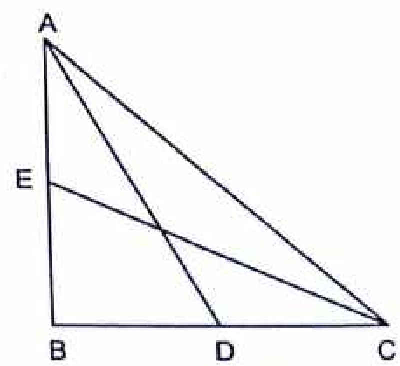


Fig. 7.199

Again, $\Delta\, BCE$ is a right triangle right angled at B.

$$\therefore \quad CE^2 = BC^2 + BE^2$$

$$\Rightarrow \quad CE^2 = BC^2 + \left(\frac{AB}{2}\right)^2 \qquad [\because BE = EA]$$

$$\Rightarrow \quad CE^2 = BC^2 + \frac{1}{4}\cdot AB^2 \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$AD^2 + CE^2 = AB^2 + \frac{1}{4}\,BC^2 + BC^2 + \frac{1}{4}\,AB^2$$

$\Rightarrow \quad AD^2 + CE^2 = \frac{5}{4}(AB^2 + BC^2)$

$\Rightarrow \quad AD^2 + CE^2 = \frac{5}{4} AC^2$ $\quad [\because \Delta ABC$ is right triangle $\therefore AC^2 = AB^2 + BC^2]$

$\Rightarrow \quad \left(\frac{3\sqrt{5}}{2}\right)^2 + CE^2 = \frac{5}{4} \times 25$

$\Rightarrow \quad CE^2 = \frac{125}{4} - \frac{45}{4} = 20$

$\Rightarrow \quad CE = \sqrt{20}$ cm $= 2\sqrt{5}$ cm

EXAMPLE 17 *The perpendicular AD on the base BC of a ΔABC intersects BC at D so that $DB = 3CD$. Prove that $2AB^2 = 2AC^2 + BC^2$.* **[NCERT, CBSE 2005, 2009]**

SOLUTION We have,

$DB = 3CD$

$\therefore \quad BC = BD + DC$

$\Rightarrow \quad BC = 3CD + CD$

$\Rightarrow \quad BC = 4CD$

$\Rightarrow \quad CD = \frac{1}{4} BC$

$\therefore \quad CD = \frac{1}{4} BC$ and $BD = 3CD = \frac{3}{4} BC$ $\quad$ ...(i)

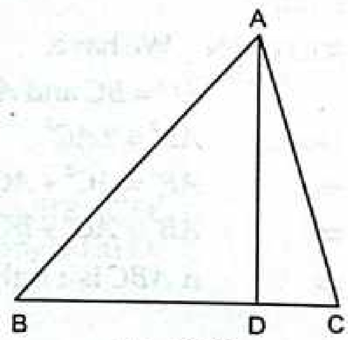


Fig. 7.200

Since ΔABD is a right triangle right-angled at D.

$\therefore \quad AB^2 = AD^2 + BD^2$ $\quad$ ...(ii)

Similarly, ΔACD is a right triangle right angled at D.

$\therefore \quad AC^2 = AD^2 + CD^2$ $\quad$ ...(iii)

Subtracting equation (iii) from equation (ii) we get

$AB^2 - AC^2 = BD^2 - CD^2$

$\Rightarrow \quad AB^2 - AC^2 = \left(\frac{3}{4} BC\right)^2 - \left(\frac{1}{4} BC\right)^2$ $\quad \left[\text{From (i) } CD = \frac{1}{4} BC, BD = \frac{3}{4} BC\right]$

$\Rightarrow \quad AB^2 - AC^2 = \frac{9}{16} BC^2 - \frac{1}{16} BC^2$

$\Rightarrow \quad AB^2 - AC^2 = \frac{1}{2} BC^2$

$\Rightarrow \quad 2(AB^2 - AC^2) = BC^2 \Rightarrow 2AB^2 = 2AC^2 + BC^2$

EXAMPLE 18 *ABC is a right triangle right-angled at C. Let $BC = a$, $CA = b$, $AB = c$ and let p be the length of perpendicular from C on AB, prove that*

(i) $cp = ab$ **[CBSE 2002]** $\quad$ (ii) $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$

SOLUTION Let $CD \perp AB$. Then, $CD = p$.

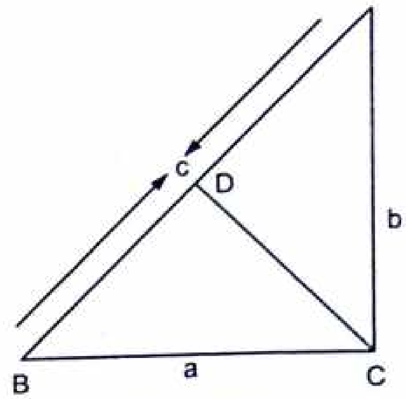


Fig. 7.201

$\therefore$ Area of $\Delta\ ABC = \frac{1}{2}$(Base× Height)

$\Rightarrow$ Area of $\Delta\ ABC = \frac{1}{2}(AB \times CD) = \frac{1}{2}cp$

Also,

Area of $\Delta\ ABC = \frac{1}{2}(BC \times AC) = \frac{1}{2}ab$

$\therefore \quad \frac{1}{2}cp = \frac{1}{2}ab$

$\Rightarrow \quad cp = ab$

(ii) Since $\Delta\ ABC$ is a right triangle right-angled at C.

$\therefore \quad AB^2 = BC^2 + AC^2$

$\Rightarrow \quad c^2 = a^2 + b^2$

$\Rightarrow \quad \left(\frac{ab}{p}\right)^2 = a^2 + b^2 \qquad \left[\because cp = ab \therefore c = \frac{ab}{p}\right]$

$\Rightarrow \quad \frac{a^2b^2}{p^2} = a^2 + b^2$

$\Rightarrow \quad \frac{1}{p^2} = \frac{a^2 + b^2}{a^2b^2} \Rightarrow \frac{1}{p^2} = \frac{1}{b^2} + \frac{1}{a^2} \Rightarrow \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$

EXAMPLE 19 *In an isosceles triangle ABC with AB = AC, BD is perpendicular from B to the side AC. Prove that* $BD^2 - CD^2 = 2CD \cdot AD$

SOLUTION Since $\Delta\ ADB$ is a right triangle right-angled at D.

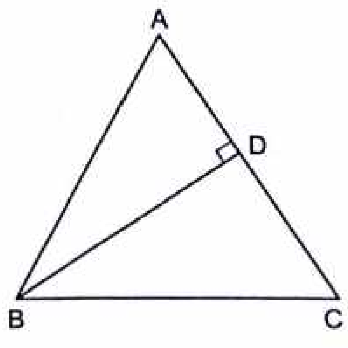


Fig. 7.202

$\therefore \quad AB^2 = AD^2 + BD^2$

$\Rightarrow \quad AC^2 = AD^2 + BD^2 \qquad [\because AB = AC]$

$\Rightarrow \quad (AD + CD)^2 = AD^2 + BD^2$

$\Rightarrow \quad AD^2 + CD^2 + 2\,AD \cdot CD = AD^2 + BD^2$

$\Rightarrow \quad BD^2 - CD^2 = 2\,CD \cdot AD$

EXAMPLE 20 *ABC is a triangle in which $AB = AC$ and D is any point in BC. Prove that $AB^2 - AD^2 = BD \cdot CD$* **[CBSE 2005]**

SOLUTION Draw $AE \perp BC$

In ΔAEB and ΔAEC, we have

$AB = AC$,

$AE = AE$ [Common]

and, $\angle B = \angle C$ [$\because AB = AC$]

$\therefore$ $\Delta AEB \cong \Delta AEC$

$\Rightarrow$ $BE = CE$

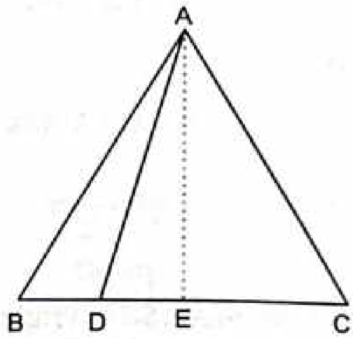


Fig. 7.203

Since ΔAED and ΔABE are right triangles right-angled at E. Therefore,

$AD^2 = AE^2 + DE^2$ and $AB^2 = AE^2 + BE^2$

$\Rightarrow$ $AB^2 - AD^2 = BE^2 - DE^2$

$\Rightarrow$ $AB^2 - AD^2 = (BE + DE)(BE - DE)$

$\Rightarrow$ $AB^2 - AD^2 = (CE + DE)(BE - DE)$ [$\because BE = CE$]

$\Rightarrow$ $AB^2 - AD^2 = CD \cdot BD$

Hence, $AB^2 - AD^2 = BD \cdot CD$

EXAMPLE 21 *From a point O in the interior of a ΔABC, perpendiculars OD, OE and OF are drawn to the sides BC, CA and AB respectively. Prove that:*

(i) $AF^2 + BD^2 + CE^2 = OA^2 + OB^2 + OC^2 - OD^2 - OE^2 - OF^2$

(ii) $AF^2 + BD^2 + CE^2 = AE^2 + CD^2 + BF^2$ **[NCERT]**

SOLUTION Let O be a point in the interior of ΔABC and let $OD \perp BC$, $OE \perp CA$ and $OF \perp AB$.

(i) In right triangles ΔOFA, ΔODB and ΔOEC, we have

$OA^2 = AF^2 + OF^2$

$OB^2 = BD^2 + OD^2$

and, $OC^2 = CE^2 + OE^2$

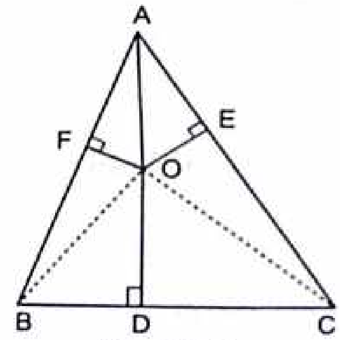


Fig. 7.204

Adding all these results, we get

$OA^2 + OB^2 + OC^2 = AF^2 + BD^2 + CE^2 + OF^2 + OD^2 + OE^2$

$\Rightarrow$ $AF^2 + BD^2 + CE^2 = OA^2 + OB^2 + OC^2 - OD^2 - OE^2 - OF^2$

(ii) In right triangles ΔODB and ΔODC, we have

$OB^2 = OD^2 + BD^2$

and, $OC^2 = OD^2 + CD^2$

$\therefore$ $OB^2 - OC^2 = (OD^2 + BD^2) - (OD^2 + CD^2)$

$\Rightarrow$ $OB^2 - OC^2 = BD^2 - CD^2$ ...(i)

Similarly, we have

$OC^2 - OA^2 = CE^2 - AE^2$ ...(ii)

and, $OA^2 - OB^2 = AF^2 - BF^2$ ...(iii)

Adding (i), (ii) and (iii), we get

$$(OB^2 - OC^2) + (OC^2 - OA^2) + (OA^2 - OB^2) = (BD^2 - CD^2) + (CE^2 - AE^2) + (AF^2 - BF^2)$$

$\Rightarrow \quad (BD^2 + CE^2 + AF^2) - (AE^2 + CD^2 + BF^2) = 0$

$\Rightarrow \quad AF^2 + BD^2 + CE^2 = AE^2 + BF^2 + CD^2$

EXAMPLE 22 *A point O in the interior of a rectangle ABCD is joined with each of the vertices A, B, C and D. Prove that* $OB^2 + OD^2 = OC^2 + OA^2$ **[NCERT, CBSE 2006C]**

SOLUTION Let $ABCD$ be the given rectangle and let O be a point within it. Join OA, OB, OC and OD.

Through O, draw $EOF \parallel AB$. Then, $ABFE$ is a rectangle.

In right triangles $\Delta\, OEA$ and $\Delta\, OFC$, we have

$OA^2 = OE^2 + AE^2$ and $OC^2 = OF^2 + CF^2$

$\Rightarrow \quad OA^2 + OC^2 = (OE^2 + AE^2) + (OF^2 + CF^2)$

$\Rightarrow \quad OA^2 + OC^2 = OE^2 + OF^2 + AE^2 + CF^2 \quad \ldots(i)$

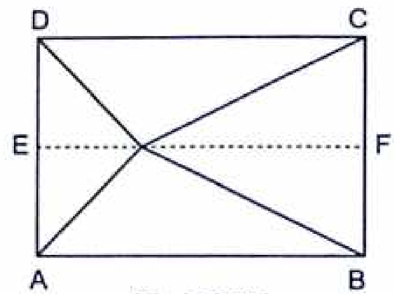


Fig. 7.205

Now, in right triangles $\Delta\, OFB$ and $\Delta\, ODE$, we have

$OB^2 = OF^2 + FB^2$ and $OD^2 = OE^2 + DE^2$

$\Rightarrow \quad OB^2 + OD^2 = (OF^2 + FB^2) + (OE^2 + DE^2)$

$\Rightarrow \quad OB^2 + OD^2 = OE^2 + OF^2 + DE^2 + BF^2$

$\Rightarrow \quad OB^2 + OD^2 = OE^2 + OF^2 + CF^2 + AE^2 \quad [\because DE = CF \text{ and } AE = BF] \quad \ldots(ii)$

From (i) and (ii), we get

$OA^2 + OC^2 = OB^2 + OD^2$

EXAMPLE 23 *ABCD is a rhombus. Prove that* $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$

[NCERT, CBSE 2005]

SOLUTION Let the diagonals AC and BD of rhombus $ABCD$ intersect at O. Since the diagonals of a rhombus bisect each other at right angles.

$\therefore \quad \angle AOB = \angle BOC = \angle COD = \angle DOA = 90^\circ$ and $AO = CO$, $BO = OD$.

Since $\Delta\, AOB$ is a right triangle right-angled at O.

$\therefore \quad AB^2 = OA^2 + OB^2$

$\Rightarrow \quad AB^2 = \left(\frac{1}{2}AC\right)^2 + \left(\frac{1}{2}BD\right)^2 \quad [\because OA = OC \text{ and } OB = OD]$

$\Rightarrow \quad 4\,AB^2 = AC^2 + BD^2 \quad \ldots(i)$

Similarly, we have

$4\,BC^2 = AC^2 + BD^2 \quad \ldots(ii)$

$4\,CD^2 = AC^2 + BD^2 \quad \ldots(iii)$

and, $\quad 4\,AD^2 = AC^2 + BD^2 \quad \ldots(iv)$

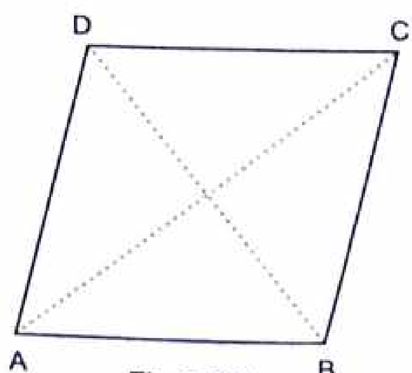


Fig. 7.206

Adding all these results, we get

$4\,(AB^2 + BC^2 + CD^2 + AD^2) = 4\,(AC^2 + BD^2)$

$\Rightarrow \quad AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$

EXAMPLE 24 *In a triangle ABC, AC > AB, D is the mid-point of BC and AE ⊥ BC. Prove that :*

(i) $AC^2 = AD^2 + BC \cdot DE + \frac{1}{4} BC^2$ **[NCERT]**

(ii) $AB^2 = AD^2 - BC \cdot DE + \frac{1}{4} BC^2$ **[CBSE 2006C]**

(iii) $AB^2 + AC^2 = 2\,AD^2 + \frac{1}{2} BC^2$

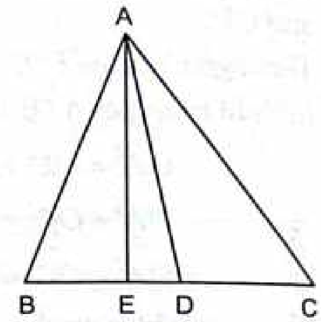


Fig. 7.207

SOLUTION We have, $\angle AED = 90°$,

$\therefore$ $\angle ADE < 90°$ and $\angle ADC > 90°$.

i.e., $\angle ADE$ is acute and $\angle ADC$ is obtuse.

(i) In $\Delta\, ADC$, $\angle ADC$ is an obtuse angle.

$\therefore$ $AC^2 = AD^2 + DC^2 + 2\,DC \cdot DE$

$\Rightarrow$ $AC^2 = AD^2 + \left(\frac{1}{2} BC\right)^2 + 2 \cdot \frac{1}{2} BC \cdot DE$

$\Rightarrow$ $AC^2 = AD^2 + \frac{1}{4} BC^2 + BC \cdot DE$

$\Rightarrow$ $AC^2 = AD^2 + BC \cdot DE + \frac{1}{4} BC^2$ ...(i)

(ii) In $\Delta\, ABD$, $\angle ADE$ is an acute angle.

$\therefore$ $AB^2 = AD^2 + BD^2 - 2\,BD \cdot DE$

$\Rightarrow$ $AB^2 = AD^2 + \left(\frac{1}{2} BC\right)^2 + 2 \cdot \frac{1}{2} BC \cdot DE$

$\Rightarrow$ $AB^2 = AD^2 + \frac{1}{4} BC^2 - BC \cdot DE$

$\Rightarrow$ $AB^2 = AD^2 - BC \cdot DE + \frac{1}{4} BC^2$ ...(ii)

(iii) From (i) and (ii), we get

$$AB^2 + AC^2 = 2AD^2 + \frac{1}{2} BC^2$$

EXAMPLE 25 *In an equilateral triangle ABC the side BC is trisected at D. Prove that* $9\,AD^2 = 7\,AB^2$

SOLUTION Let ABC be an equilateral triangle and let D be a point on BC such that $BD = \frac{1}{3} BC$. Draw $AE \perp BC$. Join AD. **[NCERT, CBSE 2018]**

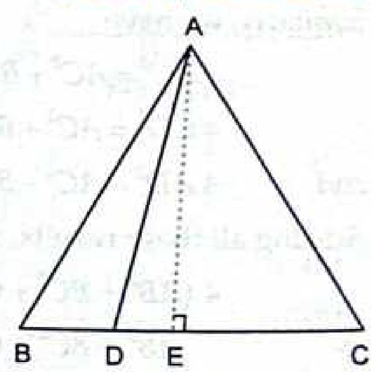


Fig. 7.208

In $\Delta\, AEB$, and $\Delta\, AEC$, we have $AB = AC$,

$\angle AEB = \angle AEC = 90°$

and, $AE = AE$

So, by RHS-criterion of similarity, we have

$\Delta\, AEB \sim \Delta\, AEC$

$\Rightarrow$ $BE = EC$

Thus, we have

$$BD = \frac{1}{3}BC, DC = \frac{2}{3}BC \text{ and } BE = EC = \frac{1}{2}BC \quad \text{...(i)}$$

Since $\angle C = 60°$. Therefore, ΔADC is an acute triangle.

$\therefore \quad AD^2 = AC^2 + DC^2 - 2\,DC \times EC$

$\Rightarrow \quad AD^2 = AC^2 + \left(\frac{2}{3}BC\right)^2 - 2 \times \frac{2}{3}BC \times \frac{1}{2}BC$ [Using (i)]

$\Rightarrow \quad AD^2 = AC^2 + \frac{4}{9}BC^2 - \frac{2}{3}BC^2$

$\Rightarrow \quad AD^2 = AB^2 + \frac{4}{9}AB^2 - \frac{2}{3}AB^2$ $[\because AB = BC = AC]$

$\Rightarrow \quad AD^2 = \frac{9AB^2 + 4AB^2 - 6AB^2}{9} = \frac{7}{9}AB^2$

$\Rightarrow \quad 9\,AD^2 = 7\,AB^2$

ALITER Draw $AE \perp BC$. Triangle ABC is equilateral. Therefore, E is the mid-point of BC.

$\therefore \quad BE = CE = \frac{1}{2}BC.$

Applying Pythagoras theorem in right triangles AEB and AED, we obtain

$AB^2 = AE^2 + BE^2$ and $AD^2 = AE^2 + DE^2$

$\Rightarrow \quad AB^2 - AD^2 = (AE^2 + BE^2) - (AE^2 + DE^2)$

$\Rightarrow \quad AB^2 - AD^2 = BE^2 - DE^2$

$\Rightarrow \quad AB^2 - AD^2 = \left(\frac{1}{2}AB\right)^2 - \left(\frac{1}{6}AB\right)^2 \quad \left[\because DE = BE - BD = \frac{1}{2}AB - \frac{1}{3}AB = \frac{1}{6}AB\right]$

$\Rightarrow \quad AB^2 - AD^2 = \frac{2}{9}AB^2 \Rightarrow \frac{7}{9}AB^2 = AD^2 \Rightarrow 9AD^2 = 7AB^2$

EXAMPLE 26 *In a ΔABC, $AD \perp BC$ and $AD^2 = BD \times CD$. Prove that ΔABC is a right triangle.*

[CBSE 2006C]

SOLUTION In right triangles ADB and ADC, we have

$AB^2 = AD^2 + BD^2$ ...(i)

and, $AC^2 = AD^2 + DC^2$ ...(ii)

Adding (i) and (ii), we get

$AB^2 + AC^2 = 2\,AD^2 + BD^2 + DC^2$

$\Rightarrow \quad AB^2 + AC^2 = 2\,BD \times CD + BD^2 + CD^2$ $[\because AD^2 = BD \times CD$ (Given)$]$

$\Rightarrow \quad AB^2 + AC^2 = (BD + CD)^2 = BC^2$

Thus, in ΔABC, we obtain

$AB^2 + AC^2 = BC^2$

Hence, ΔABC is a right triangle right-angled at A.

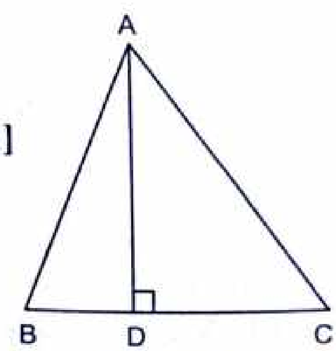


Fig. 7.209

EXAMPLE 27 *In Fig. 7.210, ABC is a right triangle right angled at B and points D and E trisect BC. Prove that* $8AE^2 = 3AC^2 + 5AD^2$. **[CBSE 2006 C]**

SOLUTION Since D and E are the points of trisection of BC. Therefore,

$$BD = DE = CE.$$

Let $BD = DE = CE = x$. Then, $BE = 2x$ and $BC = 3x$.

In right triangles ABD, ABE and ABC, we have

$\Rightarrow \quad AD^2 = AB^2 + BD^2$

$\Rightarrow \quad AD^2 = AB^2 + x^2 \quad$ ...(i)

$AE^2 = AB^2 + BE^2$

$\Rightarrow \quad AE^2 = AB^2 + 4x^2 \quad$ ...(ii)

and, $\quad AC^2 = AB^2 + BC^2$

$\Rightarrow \quad AC^2 = AB^2 + 9x^2 \quad$ ...(iii)

Now, $\quad 8AE^2 - 3AC^2 - 5AD^2 = 8(AB^2 + 4x^2) - 3(AB^2 + 9x^2) - 5(AB^2 + x^2)$

$\Rightarrow \quad 8AE^2 - 3AC^2 - 5AD^2 = 0$

$\Rightarrow \quad 8AE^2 = 3AC^2 + 5AD^2$

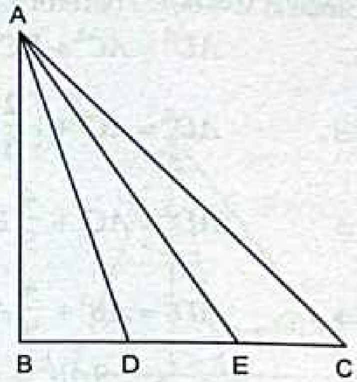


Fig. 7.210

EXAMPLE 28 *ABC is a right triangle right-angled at C and* $AC = \sqrt{3}\,BC$. *Prove that* $\angle ABC = 60°$.

SOLUTION Let D be the mid-point of AB. Join CD. Since ABC is a right triangle right-angled at C.

$\therefore \quad AB^2 = AC^2 + BC^2$

$\Rightarrow \quad AB^2 = (\sqrt{3}BC)^2 + BC^2 \quad [\because AC = \sqrt{3}\,BC \text{ (Given)}]$

$\Rightarrow \quad AB^2 = 4BC^2$

$\Rightarrow \quad AB = 2BC$

But, $\quad BD = \frac{1}{2}AB$ or, $AB = 2BD$

$\therefore \quad BD = BC$

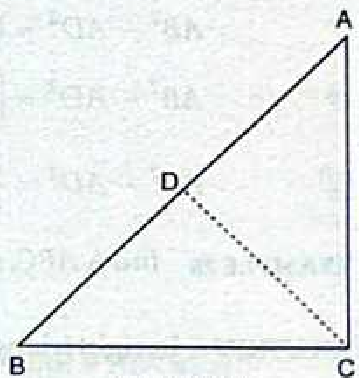


Fig. 7.211

We know that the mid-point of the hypotenuse of a right triangle is equidistant from the vertices.

$\therefore \quad CD = AD = BD$

$\Rightarrow \quad CD = BC \quad [\because BD = BC]$

Thus, in ΔBCD, we have

$BD = CD = BC$

$\Rightarrow \quad \Delta BCD$ is equilateral

$\Rightarrow \quad \angle ABC = 60°$

EXAMPLE 29 *In a right triangle if a perpendicular is drawn from the right angle to the hypotenuse, prove that the square of the perpendicular is equal to the rectangle contained by the two segments of the hypotenuse.*

GIVEN A right triangle ABC right-angled at A, $AD \perp BC$.

TO PROVE $AD^2 = BD \times CD$

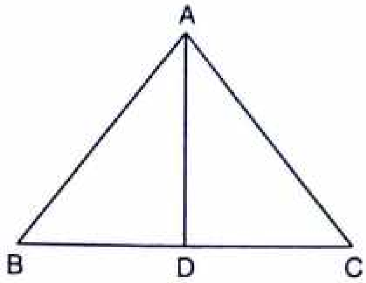


Fig. 7.212

PROOF Since $\Delta\, ABD$ and $\Delta\, ACD$ are right triangles.

$\therefore \quad AB^2 = AD^2 + BD^2 \quad \ldots(i)$

and, $\quad AC^2 = AD^2 + CD^2 \quad \ldots(ii)$

Adding equations (i) and (ii), we get

$$AB^2 + AC^2 = 2\,AD^2 + BD^2 + CD^2$$

$\Rightarrow \quad BC^2 = 2\,AD^2 + BD^2 + CD^2 \quad \left[\because \Delta ABC \text{ is right-angled at } A \therefore AB^2 + AC^2 = BC^2\right]$

$\Rightarrow \quad (BD + CD)^2 = 2\,AD^2 + BD^2 + CD^2$

$\Rightarrow \quad BD^2 + CD^2 + 2\,BD \times CD = 2\,AD^2 + BD^2 + CD^2$

$\Rightarrow \quad 2\,BD \times CD = 2\,AD^2$

$\Rightarrow \quad AD^2 = BD \times CD$

Hence, $\quad AD^2 = BD \times CD.$

EXAMPLE 30 *ABC is an isosceles triangle right-angled at B. Similar triangles ACD and ABE are constructed on sides AC and AB. Find the ratio between the areas of Δ ABE and Δ ACD.*

[CBSE 2001, 2002]

SOLUTION Let $AB = BC = x$.

It is given that $\Delta\, ABC$ is right-angled at B.

$\therefore \quad AC^2 = AB^2 + BC^2$

$\Rightarrow \quad AC^2 = x^2 + x^2$

$\Rightarrow \quad AC = \sqrt{2}\, x$

It is given that

$$\Delta\, ABE \sim \Delta\, ACD$$

$\Rightarrow \quad \dfrac{\text{Area}\,(\Delta\, ABE)}{\text{Area}\,(\Delta\, ACD)} = \dfrac{AB^2}{AC^2}$

$\Rightarrow \quad \dfrac{\text{Area}\,(\Delta\, ABE)}{\text{Area}\,(\Delta\, ACD)} = \dfrac{x^2}{(\sqrt{2}x)^2}$

$\Rightarrow \quad \dfrac{\text{Area}\,(\Delta\, ABE)}{\text{Area}\,(\Delta\, ACD)} = \dfrac{1}{2}$

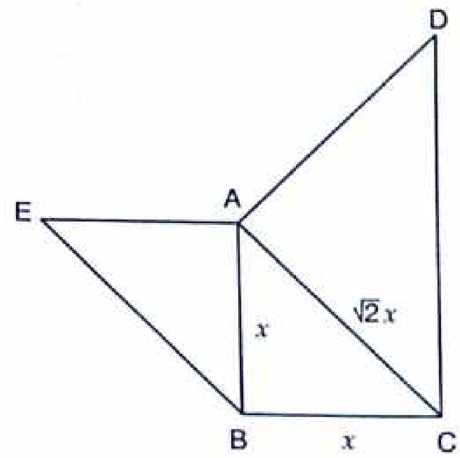


Fig. 7.213

EXAMPLE 31 *ABC is a right-angled triangle right angled at A. A circle is inscribed in it the lengths of the two sides containing the right angle are 6 cm and 8 cm. Find the radius of the circle.*

[CBSE 2002]

SOLUTION Using Pythagoras theorem in $\Delta\, BAC$, we have

$$BC^2 = AB^2 + AC^2$$

$\Rightarrow \quad BC^2 = 6^2 + 8^2 = 100$

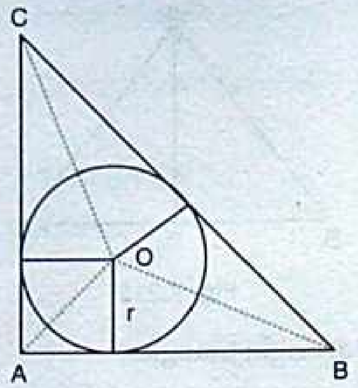


Fig. 7.214

$\Rightarrow \quad BC = 10$ cm

Now,

Area of $\Delta\, ABC$ = Area of $\Delta\, OAB$ + Area of $\Delta\, OBC$ + Area of $\Delta\, OCA$

$\Rightarrow \quad \frac{1}{2} AB \times AC = \frac{1}{2} AB \times r + \frac{1}{2} BC \times r + \frac{1}{2} CA \times r$

$\Rightarrow \quad \frac{1}{2} \times 6 \times 8 = \frac{1}{2}(6 \times r) + \frac{1}{2}(10 \times r) + \frac{1}{2}(8 \times r)$

$\Rightarrow \quad 48 = 24\, r \Rightarrow r = 2$ cm

EXAMPLE 32 *In $\Delta\, PQR$, $QM \perp PR$ and $PR^2 - PQ^2 = QR^2$. Prove that $QM^2 = PM \times MR$* **[NCERT]**

SOLUTION In $\Delta\, PQR$, we have

$PR^2 - PQ^2 = QR^2$

$\therefore \quad PR^2 = PQ^2 + QR^2$

$\Rightarrow \quad \Delta\, PQR$ is a right triangle right-angled at Q.

$\Rightarrow \quad \angle 2 + \angle 3 = 90°$

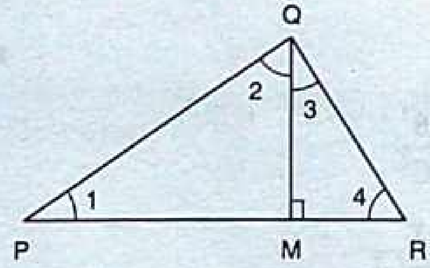


Fig. 7.215

Also, $\quad \angle 1 + \angle 2 = 90° \quad [\because \angle PMQ = 90°]$

$\therefore \quad \angle 1 = \angle 3$

Similarly, we have

$\angle 2 = \angle 4$

Thus, in Δ's PMQ and QMR, we have

$\angle 1 = \angle 3$ and $\angle 2 = \angle 4$

So, by AA-criterion for similarity, we obtain

$$\Delta PMQ \sim \Delta QMR$$

$$\Rightarrow \quad \frac{PM}{QM} = \frac{MQ}{MR}$$

$$\Rightarrow \quad QM^2 = PM \times MR.$$

EXAMPLE 33 *Prove that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.* **[NCERT]**

SOLUTION We know that if AD is a median of $\Delta\, ABC$, then

$$AB^2 + AC^2 = 2\,AD^2 + \frac{1}{2}\,BC^2 \qquad \text{[See Example 24 (iii)]}$$

Since diagonals of a parallelogram bisect each other. Therefore, BO and DO are medians of triangles ABC and ADC respectively.

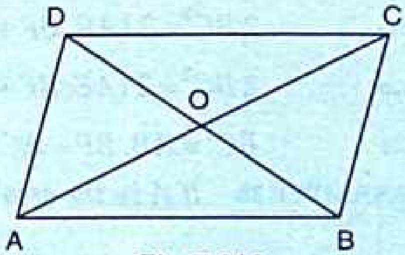


Fig. 7.216

$$\therefore \quad AB^2 + BC^2 = 2\,BO^2 + \frac{1}{2}\,AC^2 \qquad \ldots(i)$$

$$\text{and,} \quad AD^2 + CD^2 = 2\,DO^2 + \frac{1}{2}\,AC^2 \qquad \ldots(ii)$$

Adding (i) and (ii), we have

$$AB^2 + BC^2 + CD^2 + AD^2 = 2\,(BO^2 + DO^2) + AC^2$$

$$\Rightarrow \quad AB^2 + BC^2 + CD^2 + AD^2 = 2\left(\frac{1}{4}\,BD^2 + \frac{1}{4}\,BD^2\right) + AC^2 \qquad \left[\because DO = \frac{1}{2}\,BD\right]$$

$$\Rightarrow \quad AB^2 + BC^2 + CD^2 + AD^2 = AC^2 + BD^2$$

EXAMPLE 34 *In a right triangle ABC right-aangled at C, P and Q are the points on the sides CA and CB respectively, which divide these sides in the ratio 2 : 1. Prove that*

(i) $9\,AQ^2 = 9\,AC^2 + 4\,BC^2$ (ii) $9\,BP^2 = 9\,BC^2 + 4\,AC^2$ (iii) $9\,(AQ^2 + BP^2) = 13\,AB^2$

SOLUTION It is given that P divides CA in the ratio 2 : 1. Therefore,

$$CP = \frac{2}{3}\,AC \qquad \ldots(i)$$

Also, Q divides CB in the ratio 2 :1.

$$\therefore \quad QC = \frac{2}{3}\,BC \qquad \ldots(ii)$$

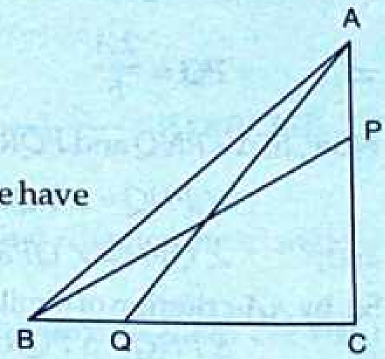


Fig. 7.217

(i) Applying Pythagoras theorem in right-angled triangle ACQ, we have

$$AQ^2 = QC^2 + AC^2$$

$$\Rightarrow \quad AQ^2 = \frac{4}{9}\,BC^2 + AC^2 \qquad \text{[Using (ii)]}$$

$$\Rightarrow \quad 9\,AQ^2 = 4\,BC^2 + 9\,AC^2 \qquad \ldots(iii)$$

(ii) Applying Pythagoras theorem in right triangle BCP, we have

$$BP^2 = BC^2 + CP^2$$

$$\Rightarrow \quad BP^2 = BC^2 + \frac{4}{9}\,AC^2 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad 9\,BP^2 = 9\,BC^2 + 4\,AC^2 \qquad \ldots(iv)$$

Adding (iii) and (iv), we get

$$9(AQ^2 + BP^2) = 13(BC^2 + AC^2)$$

$$\Rightarrow \quad 9(AQ^2 + BP^2) = 13AB^2 \qquad [\because BC^2 = AC^2 + AB^2]$$

LEVEL-3

EXAMPLE 35 *In a $\Delta\, ABC$, the angles at B and C are acute. If BE and CF be drawn perpendiculars on AC and AB respectively, prove that $BC^2 = AB \times BF + AC \times CE$.*

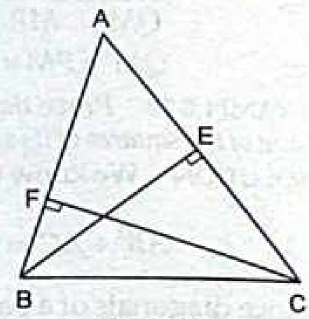


Fig. 7.218

SOLUTION In $\Delta\, ABC$, $\angle B$ is acute and $CF \perp AB$.

$\therefore \quad AC^2 = AB^2 + BC^2 - 2\, AB.\, BF$ ...(i)

Similarly in $\Delta\, ABC$, $\angle B$ is acute and $BE \perp AC$.

$\therefore \quad AB^2 = BC^2 + AC^2 - 2\, AC.\, CE$ ...(ii)

Adding (i) and (ii), we get

$$AC^2 + AB^2 = AB^2 + BC^2 - 2\, AB.\, BF + BC^2 + AC^2 - 2\, AC.\, CE$$

$\Rightarrow \quad 2\, BC^2 - 2\,(AB.\, BF + AC.CE) = 0$

$\Rightarrow \quad 2\, BC^2 = 2\,(AB.\, BF + AC.CE)$

$\Rightarrow \quad BC^2 = AB.\, BF + AC.CE$

EXAMPLE 36 *If A be the area of a right triangle and b one of the sides containing the right angle, prove that the length of the altitude on the hypotenuse is $\dfrac{2Ab}{\sqrt{b^4 + 4A^2}}$.*

SOLUTION Let PQR be a right triangle right-angled at Q such that $QR = b$ and $A =$ Area of $\Delta\, PQR$. Draw QN perpendicular to PR.

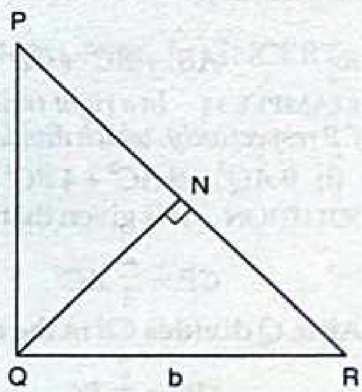


Fig. 7.219

Now,

$A =$ Area of $\Delta\, PQR$

$\Rightarrow \quad A = \dfrac{1}{2}\,(QR \times PQ)$

$\Rightarrow \quad A = \dfrac{1}{2}\,(b \times PQ)$

$\Rightarrow \quad PQ = \dfrac{2A}{b}$ ...(i)

Now, in Δ's PNQ and PQR, we have

$\angle PNQ = \angle PQR$ [Each equal to 90°]

and, $\quad \angle QPN = \angle QPR$ [Common]

So, by AA-criterion of similarity, we obtain

$\Delta\, PNQ \sim \Delta\, PQR$

$\Rightarrow \quad \dfrac{PQ}{PR} = \dfrac{NQ}{QR}$ ...(ii)

Applying Pythagoras theorem in $\Delta\, PQR$, we obtain

$PQ^2 + QR^2 = PR^2$

$\Rightarrow \quad \dfrac{4A^2}{b^2} + b^2 = PR^2$

$\Rightarrow \quad PR = \sqrt{\dfrac{4A^2 + b^4}{b^2}} = \dfrac{\sqrt{4A^2 + b^4}}{b}$

From (i) and (ii), we have

$$\frac{2A}{b \times PR} = \frac{NQ}{b}$$

$$\Rightarrow \quad NQ = \frac{2A}{PR} \Rightarrow NQ = \frac{2Ab}{\sqrt{4A^2 + b^4}} \qquad \left[\because PR = \frac{\sqrt{4A^2 + b^4}}{b}\right]$$

EXAMPLE 37 *The perimeter of a right triangle is 60 cm. Its hypotenuse is 25 cm. Find the area of the triangle.* **[CBSE 2016]**

SOLUTION Let ABC be a right triangle right angled at B. It is given that $AC = 25$ cm. Let $AB = x$ and $BC = y$.

Using Pythagoras theorem, we obtain

$$AB^2 + BC^2 = AC^2$$

$$\Rightarrow \quad x^2 + y^2 = 25^2 \qquad ...(i)$$

It is given that the perimeter of triangle ABC is 60 cm.

$$\therefore \quad AB + BC + CA = 60$$

$$\Rightarrow \quad x + y + 25 = 60$$

$$\Rightarrow \quad x + y = 35$$

$$\Rightarrow \quad (x + y)^2 = 35^2 \qquad \text{[On squaring both sides]}$$

$$\Rightarrow \quad x^2 + y^2 + 2xy = 35^2$$

$$\Rightarrow \quad 25^2 + 2xy = 35^2 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad 2xy = 35^2 - 25^2$$

$$\Rightarrow \quad 2xy = (35 + 25)(35 - 25)$$

$$\Rightarrow \quad 2xy = 60 \times 10$$

$$\Rightarrow \quad xy = 300$$

$$\Rightarrow \quad \frac{1}{2}xy = 150$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = 150 \text{ cm}^2$$

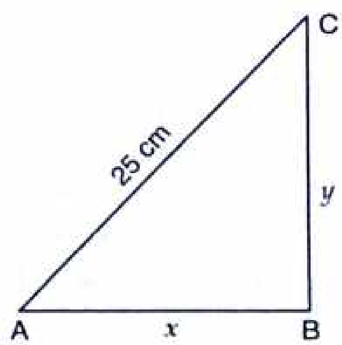


Fig. 7.220

EXERCISE 7.7

LEVEL-1

1. If the sides of a triangle are 3 cm, 4 cm and 6 cm long, determine whether the triangle is a right-angled triangle.
2. The sides of certain triangles are given below. Determine which of them are right triangles.
   (i) $a = 7$ cm, $b = 24$ cm and $c = 25$ cm
   (ii) $a = 9$ cm, $b = 16$ cm and $c = 18$ cm
   (iii) $a = 1.6$ cm, $b = 3.8$ cm and $c = 4$ cm
   (iv) $a = 8$ cm, $b = 10$ cm and $c = 6$ cm
3. A man goes 15 metres due west and then 8 metres due north. How far is he from the starting point?
4. A ladder 17 m long reaches a window of a building 15 m above the ground. Find the distance of the foot of the ladder from the building.
5. Two poles of heights 6 m and 11 m stand on a plane ground. If the distance between their feet is 12 m, find the distance between their tops. [NCERT, CBSE 2002C]

6. In an isosceles triangle ABC, $AB = AC = 25$ cm, $BC = 14$ cm. Calculate the altitude from A on BC.
7. The foot of a ladder is 6 m away from a wall and its top reaches a window 8 m above the ground. If the ladder is shifted in such a way that its foot is 8 m away from the wall, to what height does its tip reach?
8. Two poles of height 9 m and 14 m stand on a plane ground. If the distance between their feet is 12 m, find the distance between their tops.
9. Using Pythagoras theorem determine the length of AD in terms of b and c shown in Fig. 7. 221.

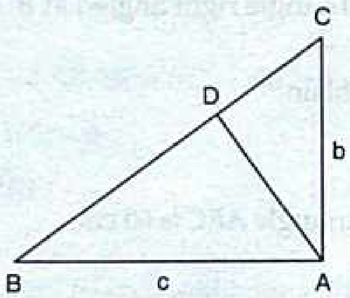


Fig. 7.221

10. A triangle has sides 5 cm, 12 cm and 13 cm. Find the length to one decimal place, of the perpendicular from the opposite vertex to the side whose length is 13 cm.
11. $ABCD$ is a square. F is the mid-point of AB. BE is one third of BC. If the area of $\Delta FBE = 108\ \text{cm}^2$, find the length of AC.
12. In an isosceles triangle ABC, if $AB = AC = 13$ cm and the altitude from A on BC is 5 cm, find BC.
13. In a ΔABC, $AB = BC = CA = 2a$ and $AD \perp BC$. Prove that
    (i) $AD = a\sqrt{3}$ (ii) Area $(\Delta ABC) = \sqrt{3}\, a^2$
14. The lengths of the diagonals of a rhombus are 24 cm and 10 cm. Find each side of the rhombus.
15. Each side of a rhombus is 10 cm. If one of its diagonals is 16 cm find the length of the other diagonal.
16. Calculate the height of an equilateral triangle each of whose sides measures 12 cm.
17. In Fig. 7.222, $\angle B < 90°$ and segment $AD \perp BC$, show that
    (i) $b^2 = h^2 + a^2 + x^2 - 2ax$ (ii) $b^2 = a^2 + c^2 - 2ax$

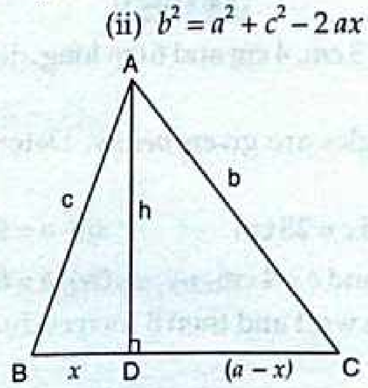


Fig. 7.222

18. In an equilateral ΔABC, $AD \perp BC$, prove that $AD^2 = 3\,BD^2$ **[CBSE 2002C]**

19. ΔABD is a right triangle right-angled at A and $AC \perp BD$. Show that

(i) $AB^2 = BC \cdot BD$ (ii) $AC^2 = BC \cdot DC$ (iii) $AD^2 = BD \cdot CD$ (iv) $\frac{AB^2}{AC^2} = \frac{BD}{DC}$ [NCERT]

20. A guy wire attached to a vertical pole of height 18 m is 24 m long and has a stake attached to the other end. How far from the base of the pole should the stake be driven so that the wire will be taut? [NCERT]

21. Determine whether the triangle having sides $(a-1)$ cm, $2\sqrt{a}$ cm and $(a+1)$ cm is a right angled triangle. [CBSE 2010]

LEVEL-2

22. In an acute-angled triangle, express a median in terms of its sides.

23. In right-angled triangle ABC in which $\angle C = 90°$, if D is the mid-point of BC, prove that $AB^2 = 4AD^2 - 3AC^2$. [CBSE 2010]

24. In Fig. 7.223, D is the mid-point of side BC and $AE \perp BC$. If $BC = a$ $AC = b$, $AB = c$, $ED = x$, $AD = p$ and $AE = h$, prove that:

(i) $b^2 = p^2 + ax + \frac{a^2}{4}$ (ii) $c^2 = p^2 - ax + \frac{a^2}{4}$ (iii) $b^2 + c^2 = 2p^2 + \frac{a^2}{2}$

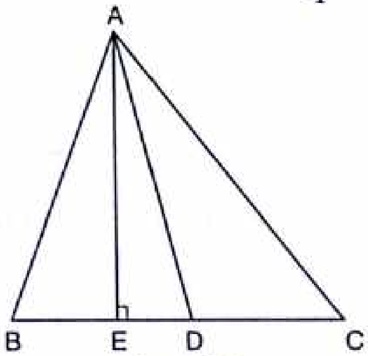


Fig. 7.223

25. In ΔABC, $\angle A$ is obtuse, $PB \perp AC$ and $QC \perp AB$. Prove that:

(i) $AB \times AQ = AC \times AP$ (ii) $BC^2 = (AC \times CP + AB \times BQ)$

26. In a right ΔABC right-angled at C, if D is the mid-point of BC, prove that $BC^2 = 4(AD^2 - AC^2)$.

27. In a quadrilateral $ABCD$, $\angle B = 90°$, $AD^2 = AB^2 + BC^2 + CD^2$, prove that $\angle ACD = 90°$

28. An aeroplane leaves an airport and flies due north at a speed of 1000 km/hr. At the same time, another aeroplane leaves the same airport and flies due west at a speed of 1200 km/hr. How far apart will be the two planes after $1\frac{1}{2}$ hours? [NCERT]

ANSWERS

1. No
2. (i), (iv)
3. 17 m
4. 8 m
5. 13 m
6. 24 m
7. 6 m
8. 13 m
9. $\frac{bc}{\sqrt{b^2 + c^2}}$
10. 4.6 cm
11. 50.904 cm
12. 24 cm
14. 13 cm
15. 12 cm
16. 10.39 cm
20. $6\sqrt{7}$ m
21. yes
22. $\frac{2AB^2 + 2AC^2 - BC^2}{4}$
28. 2343 km (Approx)

HINT TO SELECTED PROBLEMS

5. Find the hypotenuse of a right triangle having two sides $(11-6)$ m $=5$ m and 12 m.
6. Let D be the foot of the perpendicular from A on BC. Then,

$$\Delta\, ABD \cong ACD \Rightarrow BD = CD = 7 \text{ cm}.$$

Now, apply Pythagoras theorem in $\Delta\, ABD$.

9. Area of $\Delta\, ABC = \frac{1}{2}(AB \times AC) = \frac{1}{2}bc$.

Also, Area of $\Delta\, ABC = \frac{1}{2}(BC \times AD) = \frac{1}{2}\sqrt{b^2+c^2} \times AD$

$$\therefore \quad \frac{1}{2}\sqrt{b^2+c^2} \times AD = \frac{1}{2}bc \Rightarrow AD = \frac{bc}{\sqrt{b^2+c^2}}$$

10. Let $AB = 5$ cm, $BC = 12$ cm and $AC = 13$ cm. Then, $AC^2 = AB^2 + BC^2$. This proves that $\Delta\, ABC$ is a right triangle, right-angled at B. Let BD be the length of perpendicular from B on AC.

Now,

Area $\Delta\, ABC = \frac{1}{2}(BC \times BA) = \frac{1}{2}(12 \times 5) = 30 \text{ cm}^2$

Also, Area of $\Delta\, ABC = \frac{1}{2}\, AC \times BD = \frac{1}{2}(13 \times BD) \Rightarrow (13 \times BD) = 30 \Rightarrow BD = \frac{60}{13}$ cm.

13. First prove that $\Delta\, ABD \cong \Delta\, ACD$ and then use Phythagoras theorem in $\Delta\, ABD$ to find AD.
14. Let $ABCD$ be a rhombus in which $AC = 24$ cm and $BD = 10$ cm. Suppose the diagonals intersect at O. Since the diagonals of a rhombus bisect each other at right angles. Therefore, $\Delta\, OAB$ is a right triangle, right-angled at O such that

$$OA = \frac{1}{2}\, AC = 12 \text{ cm and } OB = \frac{1}{2}\, BD = 5 \text{ cm}.$$

Using Pythagoras theorem, we obtain

$$AB^2 = OA^2 + OB^2 = 12^2 + 5^2 = 169 \Rightarrow AB = 13 \text{ cm}.$$

22. Let ABC be an acute angled triangle and let AD be a median. Then,

$$AB^2 + AC^2 = 2(AD^2 + BD^2)$$ [See Theorem 3 on page 7.100]

$$\Rightarrow \quad AD^2 = \frac{2AB^2 + 2AC^2 - BC^2}{4}$$

23. In right triangles ABC and ADC, we have

$$AB^2 = AC^2 + BC^2 \quad \text{...(i)}$$

and, $AD^2 = AC^2 + CD^2$ ...(ii)

Now,

$$AB^2 = AC^2 + BC^2$$

$\Rightarrow \quad AB^2 = AC^2 + 4CD^2 \qquad \left[\because CD = BD = \frac{1}{2} BC\right]$

$\Rightarrow \quad AB^2 = AC^2 + 4(AD^2 - AC^2) \qquad$ [Using (ii)]

$\Rightarrow \quad AB^2 = 4\,AD^2 - 3AC^2$

REVISION EXERCISE

1. In each of the figures [7.224 (i)- (iv)] given below, a line segment is drawn parallel to one side of the triangle and the lengths of certain line-segments are marked. Find the value of x in each of the following :

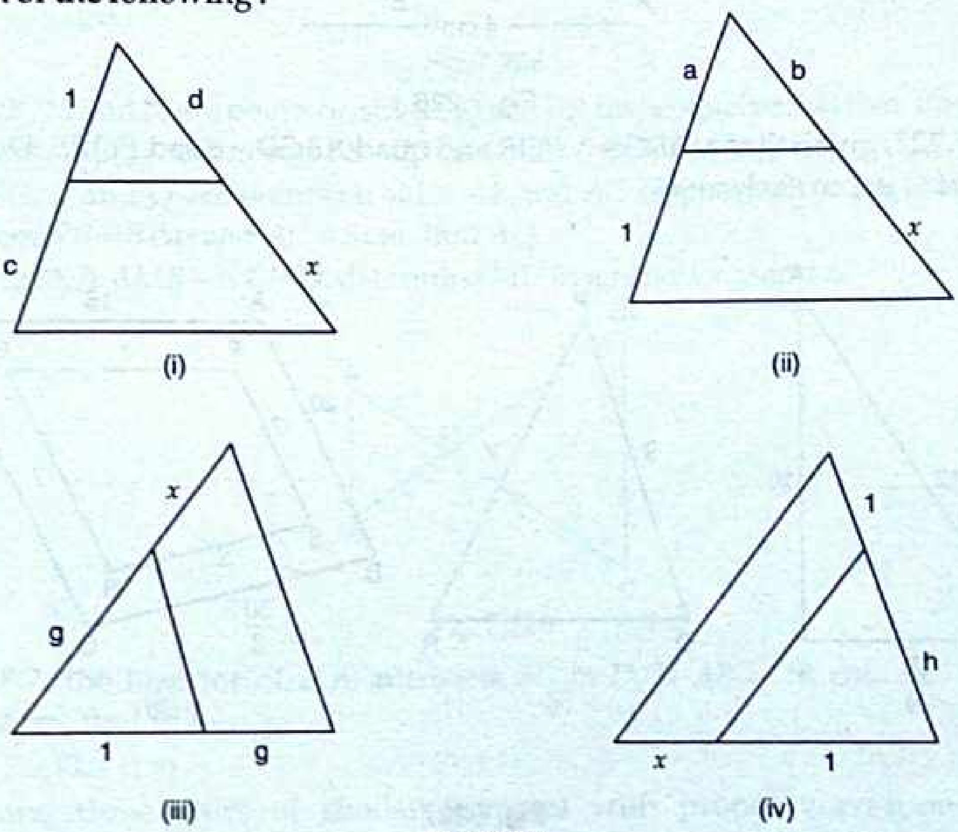


Fig. 7.224

2. What values of x will make $DE \parallel AB$ in the Fig. 7.225?

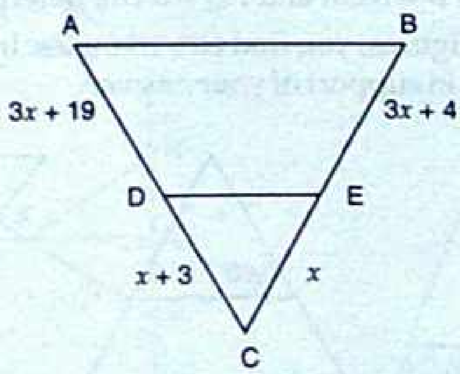


Fig. 7.225

3. In $\Delta\, ABC$, points P and Q are on CA and CB, respectively such that $CA = 16$ cm, $CP = 10$ cm, $CB = 30$ cm and $CQ = 25$ cm. Is $PQ \parallel AB$?

4. In Fig. 7.226, $DE \parallel CB$. Determine AC and AE.

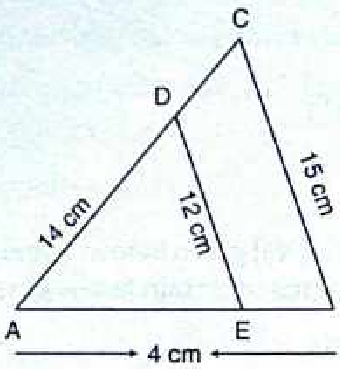


Fig. 7.226

5. In Fig. 7.227, given that $\Delta ABC \sim \Delta PQR$ and quad $ABCD \sim$ quad $PQRS$. Determine the values of x, y, z in each case.

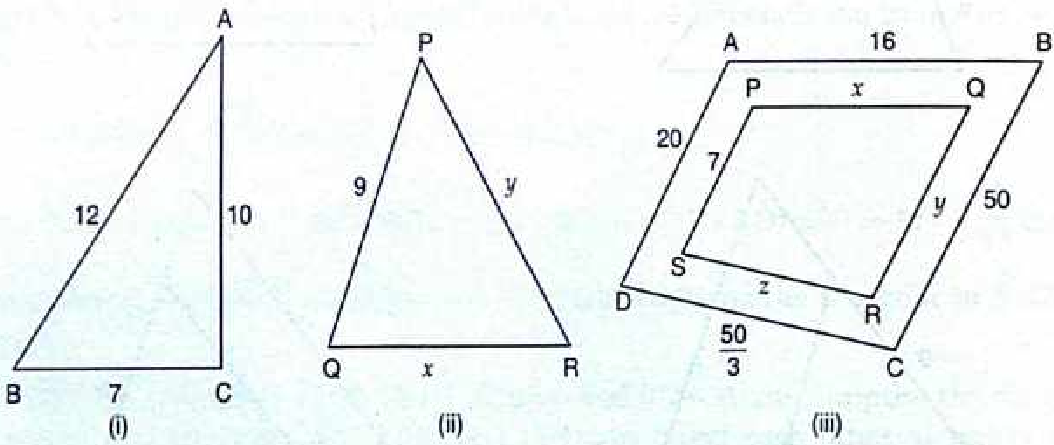
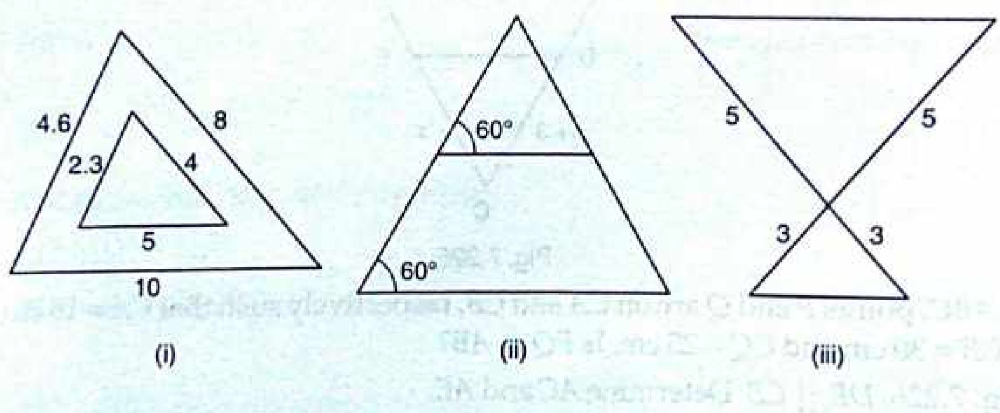
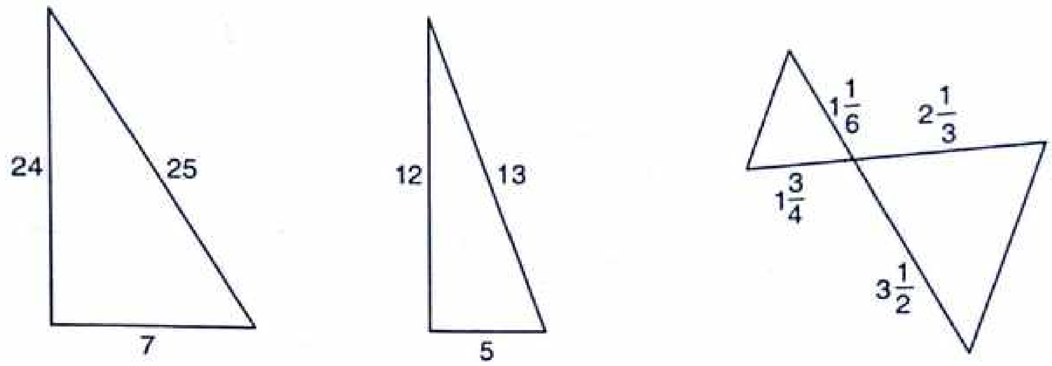


Fig. 7.227

6. In ΔABC, P and Q are points on sides AB and AC respectively such that $PQ \parallel BC$. If $AP = 4$ cm, $PB = 6$ cm and $PQ = 3$ cm, determine BC.

7. In each of the following figures, you find two triangles. Indicate whether the triangles are similar. Give reasons in support of your answer.





(iv) (v) (vi)

Fig. 7.228

8. In $\Delta\ PQR$, M and N are points on sides PQ and PR respectively such that $PM = 15$ cm and $NR = 8$ cm. If $PQ = 25$ cm and $PR = 20$ cm state whether $MN \parallel QR$.
9. In $\Delta\ ABC$, P and Q are points on sides AB and AC respectively such that $PQ \parallel BC$. If $AP = 3$ cm, $PB = 5$ cm and $AC = 8$ cm, find AQ.
10. In Fig.7.229, $\Delta\ AMB \sim \Delta\ CMD$; determine MD in terms of x, y and z.

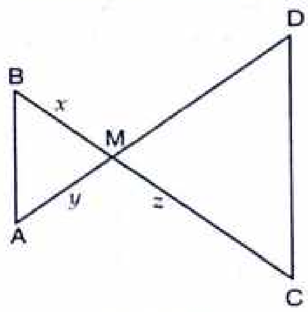


Fig. 7.229

11. In $\Delta\ ABC$, the bisector of $\angle A$ intersects BC in D. If $AB = 18$ cm, $AC = 15$ cm and $BC = 22$ cm, find BD
12. In Fig. 7.230, $l \parallel m$
    (i) Name three pairs of similar triangles with proper correspondence; write similarities.
    (ii) Prove that $\frac{AB}{PQ} = \frac{AC}{PR} = \frac{BC}{RQ}$

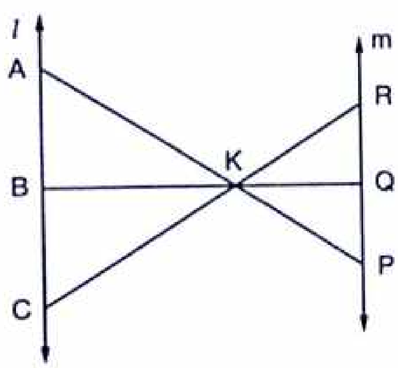


Fig.7.230

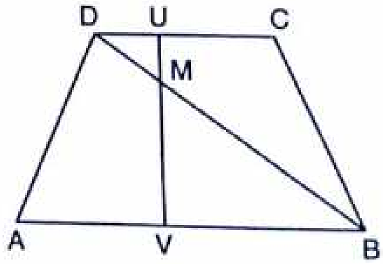


Fig. 7.231

13. In Fig. 7.231, $AB \parallel DC$ Prove that
    (i) $\Delta\ DMU \sim \Delta\ BMV$ (ii) $DM \times BV = BM \times DU$

14. $ABCD$ is a trapezium in which $AB \parallel DC$. P and Q are points on sides AD and BC such that $PQ \parallel AB$. If $PD = 18$, $BQ = 35$ and $QC = 15$, find AD.

15. In ΔABC, D and E are points on sides AB and AC respectively such that
$AD \times EC = AE \times DB$. Prove that $DE \parallel BC$.

16. $ABCD$ is a trapezium having $AB \parallel DC$. Prove that O, the point of intersection of diagonals, divides the two diagonals in the same ratio. Also prove that

$$\frac{ar\,(\Delta OCD)}{ar\,(\Delta OAB)} = \frac{1}{9}, \text{ if } AB = 3\,CD.$$

17. Corresponding sides of two triangles are in the ratio 2 : 3. If the area of the smaller triangle is 48 cm^2, determine the area of the larger triangle.

18. The areas of two similar triangles are 36 cm^2 and 100 cm^2. If the length of a side of the smaller triangle in 3 cm, find the length of the corresponding side of the larger triangle.

19. Corresponding sides of two similar triangles are in the ratio 1 : 3. If the area of the smaller triangle in 40 cm^2, find the area of the larger triangle.

20. In Fig. 7.232, each of PA, QB, RC and SD is perpendicular to l. If $AB = 6$ cm, $BC = 9$ cm, $CD = 12$ cm and $PS = 36$ cm, then determine PQ, QR and RS.

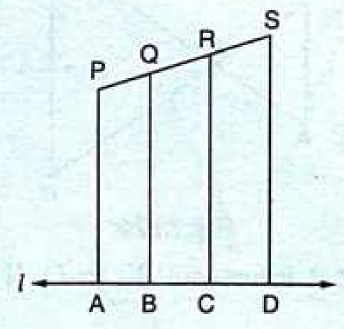


Fig. 7.232

21. In each of the figures given below, an altitude is drawn to the hypotenuse by a right-angled triangle. The length of different line-segments are marked in each figure. Determine x, y, z in each case

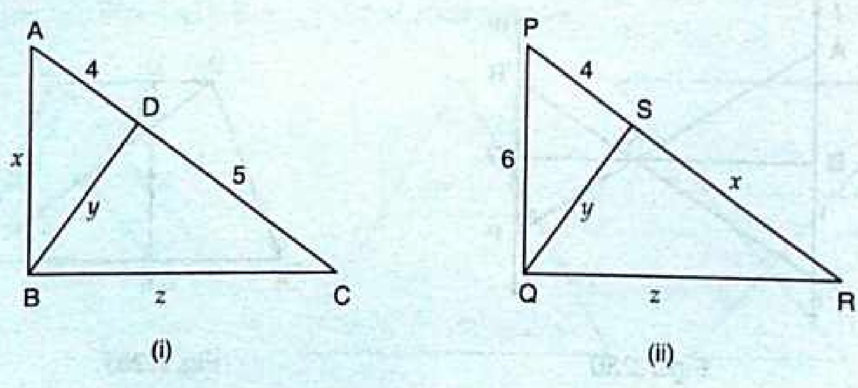


Fig. 7.233

22. Prove that in an equilateral triangle, three times the square of a side is equal to four times the square of its altitudes.

LEVEL-2

23. In ΔABC, AD and BE are altitudes. Prove that $\frac{ar(\Delta DEC)}{ar(\Delta ABC)} = \frac{DC^2}{AC^2}$.

24. The diagonals of quadrilateral $ABCD$ intersect at O. Prove that $\frac{ar(\Delta ACB)}{ar(\Delta ACD)} = \frac{BO}{DO}$.

25. In ΔABC, ray AD bisects $\angle A$ and intersects BC in D. If $BC = a$, $AC = b$ and $AB = c$, prove that

(i) $BD = \frac{ac}{b+c}$ (ii) $DC = \frac{ab}{b+c}$

26. There is a staircase as shown in Fig. 7.234, connecting points A and B. Measurements of steps are marked in the figure. Find the straight line distance between A and B.

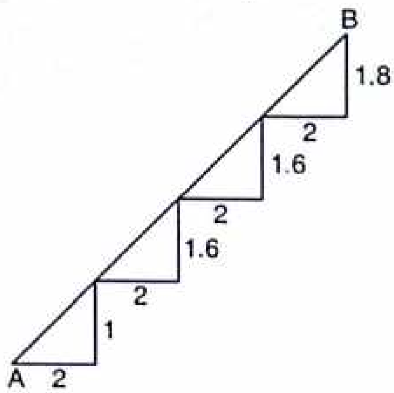


Fig. 7.234

27. In ΔABC, $\angle A = 60°$. Prove that $BC^2 = AB^2 + AC^2 - AB \cdot AC$.

28. In ΔABC, $\angle C$ is an obtuse angle. $AD \perp BC$ and $AB^2 = AC^2 + 3\,BC^2$. Prove that $BC = CD$.

29. A point D is on the side BC of an equilateral triangle ABC such that $DC = \frac{1}{4}\,BC$. Prove that $AD^2 = 13\,CD^2$.

30. In ΔABC, if $BD \perp AC$ and $BC^2 = 2\,AC \cdot CD$, then prove that $AB = AC$.

31. In a quadrilateral $ABCD$, given that $\angle A + \angle D = 90°$. Prove that $AC^2 + BD^2 = AD^2 + BC^2$.

32. In ΔABC, given that $AB = AC$ and $BD \perp AC$. Prove that $BC^2 = 2\,AC \cdot CD$

33. $ABCD$ is a rectangle. Points M and N are on BD such that $AM \perp BD$ and $CN \perp BD$. Prove that $BM^2 + BN^2 = DM^2 + DN^2$.

34. In ΔABC, AD is a median. Prove that $AB^2 + AC^2 = 2\,AD^2 + 2\,DC^2$.

35. In ΔABC, $\angle ABC = 135°$. Prove that $AC^2 = AB^2 + BC^2 + 4\,ar(\Delta ABC)$

36. In a quadrilateral $ABCD$, $\angle B = 90°$. If $AD^2 = AB^2 + BC^2 + CD^2$ then prove that $\angle ACD = 90°$

37. In a triangle ABC, N is a point on AC such that $BN \perp AC$. If $BN^2 = AN \cdot NC$, prove that $\angle B = 90°$.

38. Nazima is fly fishing in a stream. The tip of her fishing rod is 1.8 m above the surface of the water and the fly at the end of the string rests on the water 3.6 m away and 2.4 m from a point directly under the tip of the rod. Assuming that her string (from the tip of her rod to the fly) is taut, how much string does she have out (See Fig. 7.235)? If she pulls the string at the rate of 5 cm per second, what will the horizontal distance of the fly from her after 12 seconds.

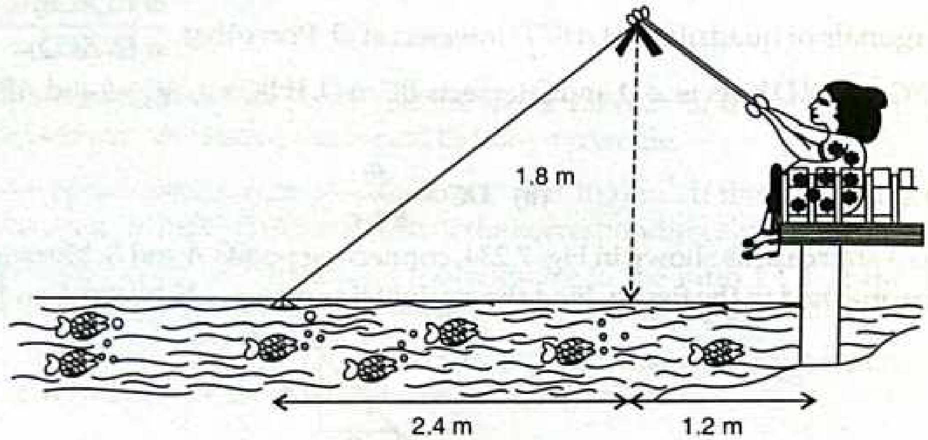


Fig. 7.235

ANSWERS

1. (i) cd (ii) $\frac{b}{a}$ (iii) g^2 (iv) $\frac{1}{h}$ 2. 2 3. No

4. $\frac{35}{2}, \frac{16}{5}$ 5. (i) $x = \frac{21}{4}, y = \frac{15}{2}$ (ii) $x = \frac{28}{5}, y = \frac{35}{2}, z = \frac{35}{6}$

6. 7.5 cm 7. (i) Yes (ii) Yes (iii) Yes (iv) Yes (v) Yes (vi) No

8. Yes 9. 3 cm 10. $\frac{xz}{y}$ 11. 12 14. 60 17. 108 cm^2

18. 5 cm 19. 360 cm^2 20. 8 cm, 12 cm, 16 cm

21. (i) $x = 6, y = 2\sqrt{5}, z = 3\sqrt{5}$, (ii) $x = 5, y = 2\sqrt{5}, z = 3\sqrt{5}$ 26. 10.

HINT TO SELECTED PROBLEM

31. Extend AB and CD to intersect at O.

Now, $\angle AOD = 90°$

$\Rightarrow AC^2 = OA^2 + OC^2$ and $BD^2 = OB^2 + OD^2$

$\Rightarrow AC^2 + BD^2 = (OA^2 + OD^2) + (OB^2 + OC^2) = AD^2 + BC^2$

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

Answer each of the following questions either in one word or one sentence or as per requirement of the questions:

1. State basic proportionality theorem and its converse.
2. In Fig. 7.236, find AC.

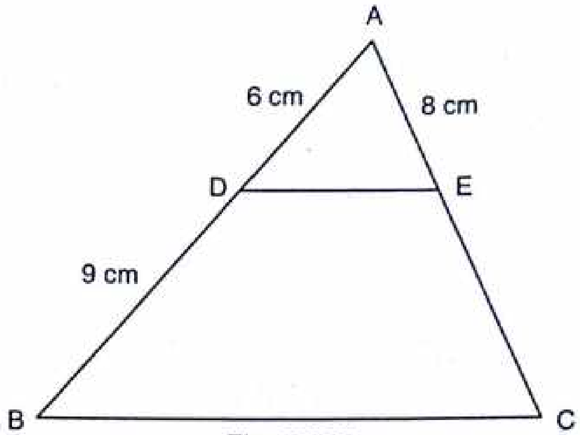


Fig. 7.236

3. In Fig. 7.237, if AD is the bisector of $\angle A$, what is AC?

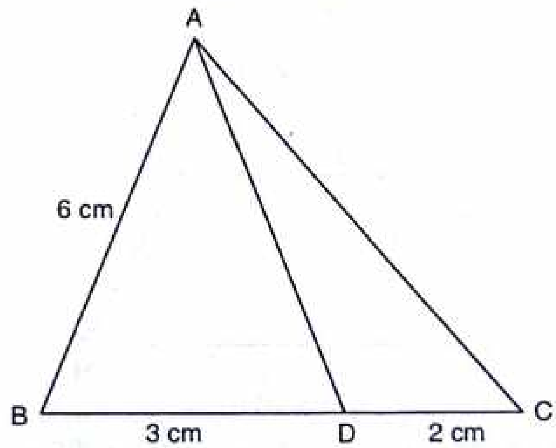


Fig. 7.237

4. Given $\Delta ABC \sim \Delta PQR$, if $\frac{AB}{PQ} = \frac{1}{3}$, then find $\frac{\text{Area }(\Delta ABC)}{\text{Area }(\Delta PQR)}$. [CBSE 2018]
5. State SSS similarity criterion.
6. State SAS similarity criterion.
7. In the adjoining figure, DE is parallel to BC and $AD = 1$ cm, $BD = 2$ cm. What is the ratio of the area of ΔABC to the area of ΔADE?
8. In the figure given below $DE \parallel BC$. If $AD = 2.4$ cm, $DB = 3.6$ cm and $AC = 5$ cm. Find AE.
9. If the areas of two similar triangles ABC and PQR are in the ratio 9 : 16 and $BC = 4.5$ cm, what is the length of QR?
10. The areas of two similar triangles are 169 cm^2 and 121 cm^2 respectively. If the longest side of the larger triangle is 26 cm, what is the length of the longest side of the smaller triangle?
11. If ABC and DEF are similar triangles such that $\angle A = 57°$ and $\angle E = 73°$, what is the measure of $\angle C$?
12. If the altitude of two similar triangles are in the ratio 2 : 3, what is the ratio of their areas?
13. If ΔABC and ΔDEF are two triangles such that $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = \frac{3}{4}$, then write Area ($\Delta ABC$): Area ($\Delta DEF$).
14. If ΔABC and ΔDEF are similar triangles such that $AB = 3$ cm, $BC = 2$ cm $CA = 2.5$ cm and $EF = 4$ cm, write the perimeter of ΔDEF.
15. State Pythagoras theorem and its converse.
16. The lengths of the diagonals of a rhombus are 30 cm and 40 cm. Find the side of the rhombus. [CBSE 2008]

17. In Fig. 7.238, $PQ \parallel BC$ and $AP : PB = 1 : 2$. Find $\frac{\text{area}(\Delta APQ)}{\text{area}(\Delta ABC)}$. **[CBSE 2008]**

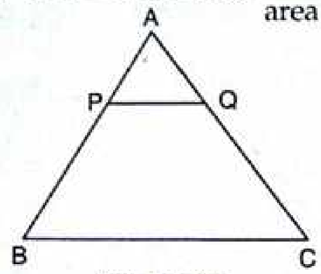


Fig. 7.238

18. In Fig. 7.239, S and T are points on the sides PQ and PR respectively of ΔPQR such that $PT = 2$ cm, $TR = 4$ cm and ST is parallel to QR. Find the ratio of the areas of ΔPST and ΔPQR. **[CBSE 2010]**

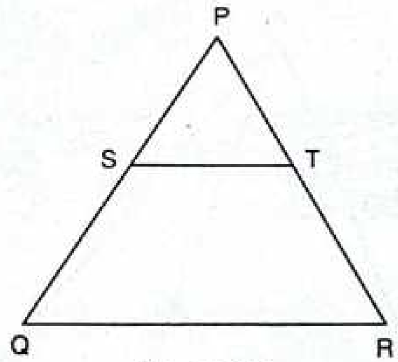


Fig. 7.239

19. In Fig. 7.240, ΔAHK is similar to ΔABC. If $AK = 10$ cm, $BC = 3.5$ cm and $HK = 7$ cm, find AC. **[CBSE 2010]**

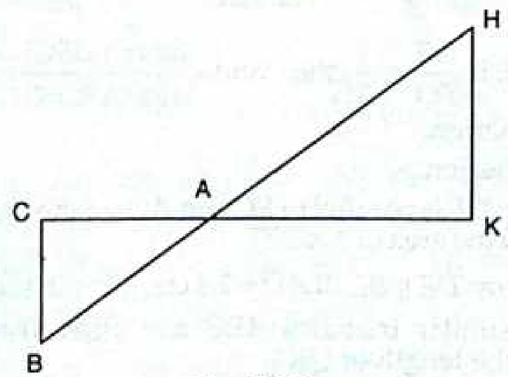


Fig. 7.240

20. In Fig. 7.241, $DE \parallel BC$ in ΔABC such that $BC = 8$ cm, $AB = 6$ cm and $DA = 1.5$ cm. Find DE.

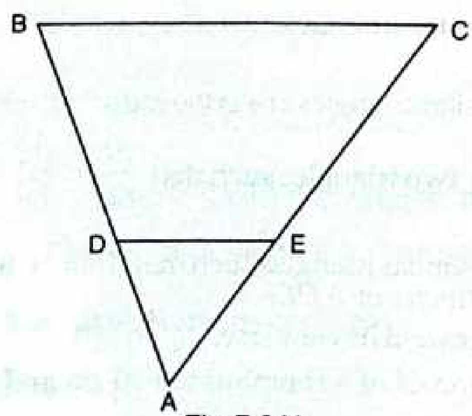


Fig. 7.241

21. In Fig. 7.242, $DE \parallel BC$ and $AD = \frac{1}{2} BD$. If $BC = 4.5$ cm, find DE. [CBSE 2010]

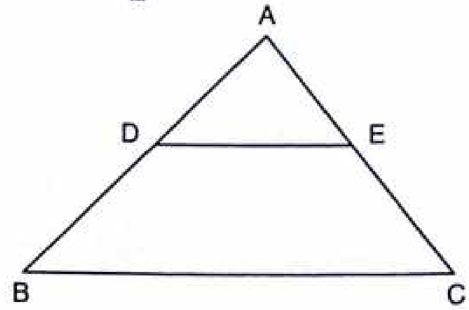


Fig. 7.242

LEVEL-2

22. In Fig. 7.243, $LM = LN = 46°$. Express x in terms of a, b and c where a, b, c are lengths of LM, MN and and NK respectively.

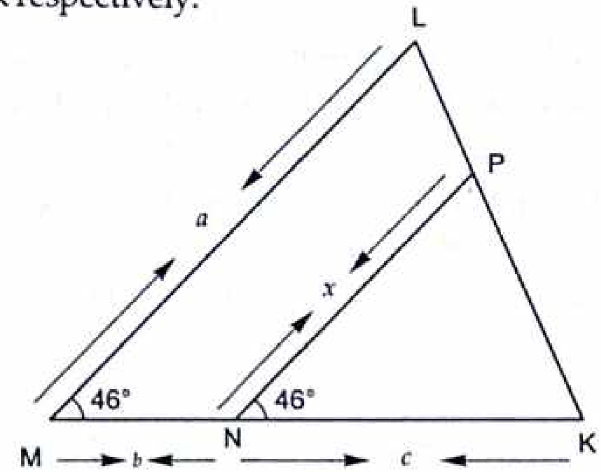


Fig. 7.243

ANSWERS

1. 20 cm	2. 4 cm	4. $\frac{1}{9}$	7. 9 : 1	8. 2 cm	9. 6 cm
10. 22 cm	11. 50°	12. 4 : 9	13. 9 : 16	14. 15 cm	16. 25 cm
17. 1 : 4	18. 1 : 9	19. 5 cm	20. 2 cm	21. 1.5 cm	22. $\frac{ac}{b+c}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

LEVEL-1

1. Sides of two similar triangles are in the ratio 4 : 9. Areas of these triangles are in the ratio.
   (a) 2 : 3 (b) 4 : 9 (c) 81 : 16 (d) 16 : 81
2. The areas of two similar triangles are in respectively 9 cm^2 and 16 cm^2. The ratio of their corresponding sides is
   (a) 3 : 4 (b) 4 : 3 (c) 2 : 3 (d) 4 : 5
3. The areas of two similar triangles ΔABC and ΔDEF are 144 cm^2 and 81 cm^2 respectively. If the longest side of larger ΔABC be 36 cm, then. the longest side of the smaller triangle ΔDEF is

(a) 20 cm (b) 26 cm (c) 27 cm (d) 30 cm

4. ΔABC and ΔBDE are two equilateral triangles such that D is the mid-point of BC. The ratio of the areas of triangles ABC and BDE is

(a) 2 : 1 (b) 1 : 2 (c) 4 : 1 (d) 1 : 4

5. If ΔABC and ΔDEF are similar such that $2\,AB = DE$ and $BC = 8$ cm, then $EF =$

(a) 16 cm (b) 12 cm (c) 8 cm (d) 4 cm.

6. If ΔABC and ΔDEF are two triangles such that $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = \frac{2}{5}$, then Area ($\Delta ABC$) : Area ($\Delta DEF$) =

(a) 2 : 5 (b) 4 : 25 (c) 4 : 15 (d) 8 : 125

7. XY is drawn parallel to the base BC of a ΔABC cutting AB at X and AC at Y. If $AB = 4\,BX$ and $YC = 2$ cm, then $AY =$

(a) 2 cm (b) 4 cm (c) 6 cm (d) 8 cm.

8. Two poles of height 6 m and 11 m stand vertically upright on a plane ground. If the distance between their foot is 12 m, the distance between their tops is

(a) 12 m (b) 14 m (c) 13 m. (d) 11 m

9. In ΔABC, D and E are points on side AB and AC respectively such that $DE \parallel BC$ and $AD : DB = 3 : 1$. If $EA = 3.3$ cm, then $AC =$

(a) 1.1 cm (b) 4 cm (c) 4.4 cm (d) 5.5 cm

10. In triangles ABC and DEF, $\angle A = \angle E = 40°$, $AB : ED = AC : EF$ and $\angle F = 65°$, then $\angle B =$

(a) 35° (b) 65° (c) 75° (d) 85°

11. If ABC and DEF are similar triangles such that $\angle A = 47°$ and $\angle E = 83°$, then $\angle C =$

(a) 50° (b) 60° (c) 70° (d) 80°

12. If D, E, F are the mid-points of sides BC, CA and AB respectively of ΔABC, then the ratio of the areas of triangles DEF and ABC is

(a) 1 : 4 (b) 1 : 2 (c) 2 : 3 (d) 4 : 5

13. In an equilateral triangle ABC, if $AD \perp BC$, then

(a) $2\,AB^2 = 3\,AD^2$ (b) $4\,AB^2 = 3\,AD^2$ (c) $3\,AB^2 = 4\,AD^2$ (d) $3\,AB^2 = 2\,AD^2$

14. If ΔABC is an equilateral triangle such that $AD \perp BC$, then $AD^2 =$

(a) $\frac{3}{2}DC^2$ (b) $2\,DC^2$ (c) $3\,CD^2$ (d) $4\,DC^2$

15. In a ΔABC, AD is the bisector of $\angle BAC$. If $AB = 6$ cm, $AC = 5$ cm and $BD = 3$ cm, then $DC =$

(a) 11.3 cm (b) 2.5 cm (c) 3 : 5 cm (d) None of these.

16. In a ΔABC, AD is the bisector of $\angle BAC$. If $AB = 8$ cm, $BD = 6$ cm and $DC = 3$ cm. Find AC

(a) 4 cm (b) 6 cm (c) 3 cm (d) 8 cm

17. $ABCD$ is a trapezium such that $BC \parallel AD$ and $AB = 4$ cm. If the diagonals AC and BD intersect at O such that $\frac{AO}{OC} = \frac{DO}{OB} = \frac{1}{2}$, then $BC =$

(a) 7 cm (b) 8 cm (c) 9 cm (d) 6 cm

18. If ABC is a right triangle right-angled at B and M, N are the mid-points of AB and BC respectively, then $4(AN^2 + CM^2) =$

(a) $4\,AC^2$ (b) $5\,AC^2$ (c) $\frac{5}{4}\,AC^2$ (d) $6\,AC^2$

19. If in ΔABC and ΔDEF, $\frac{AB}{DE} = \frac{BC}{FD}$, then $\Delta ABC \sim \Delta DEF$ *when*

(a) $\angle A = \angle F$ (b) $\angle A = \angle D$ (c) $\angle B = \angle D$ (d) $\angle B = \angle E$

20. If in two triangles ABC and DEF, $\frac{AB}{DE} = \frac{BC}{FE} = \frac{CA}{FD}$, then

(a) $\Delta FDE \sim \Delta CAB$ (b) $\Delta FDE \sim \Delta ABC$ (c) $\Delta CBA \sim \Delta FDE$ (d) $\Delta BCA \sim \Delta FDE$

21. $\Delta ABC \sim \Delta DEF$, ar $(\Delta ABC) = 9\text{ cm}^2$, ar $(\Delta DEF) = 16\text{ cm}^2$. If $BC = 2.1$ cm, then the measure of EF is

(a) 2.8 cm (b) 4.2 cm (c) 2.5 cm (d) 4.1 cm

22. The length of the hypotenuse of an isosceles right triangle whose one side is $4\sqrt{2}$ cm is

(a) 12 cm (b) 8 cm (c) $8\sqrt{2}$ cm (d) $12\sqrt{2}$ cm

23. A man goes 24 m due west and then 7 m due north. How far is he from the starting point?

(a) 31 m (b) 17 m (c) 25 m (d) 26 m

24. $\Delta ABC \sim \Delta DEF$. If $BC = 3$ cm, $EF = 4$ cm and ar $(\Delta ABC) = 54\text{ cm}^2$, then ar $(\Delta DEF) =$

(a) 108 cm^2 (b) 96 cm^2 (c) 48 cm^2 (d) 100 cm^2

25. $\Delta ABC \sim \Delta PQR$ such that ar $(\Delta ABC) = 4$ ar (ΔPQR). If $BC = 12$ cm, then $QR =$

(a) 9 cm (b) 10 cm (c) 6 cm (d) 8 cm

26. The areas of two similar triangles are 121 cm^2 and 64 cm^2 respectively. If the median of the first triangle is 12.1 cm, then the corresponding median of the other triangle is

(a) 11 cm (b) 8.8 cm (c) 11.1 cm (d) 8.1 cm

27. In an equilateral triangle ABC if $AD \perp BC$, then $AD^2 =$

(a) CD^2 (b) $2CD^2$ (c) $3CD^2$ (d) $4CD^2$

28. In an equilateral triangle ABC if $AD \perp BC$, then

(a) $5AB^2 = 4AD^2$ (b) $3AB^2 = 4AD^2$ (c) $4AB^2 = 3AD^2$ (d) $2AB^2 = 3AD^2$

29. In an isosceles triangle ABC if $AC = BC$ and $AB^2 = 2AC^2$, then $\angle C =$

(a) 30° (b) 45° (c) 90° (d) 60°

30. ΔABC is an isosceles triangle in which $\angle C = 90°$. If $AC = 6$ cm, then $AB =$

(a) $6\sqrt{2}$ cm (b) 6 cm (c) $2\sqrt{6}$ cm (d) $4\sqrt{2}$ cm

31. If in two triangles ABC and DEF, $\angle A = \angle E$, $\angle B = \angle F$, then which of the following is not true?

(a) $\frac{BC}{DF} = \frac{AC}{DE}$ (b) $\frac{AB}{DE} = \frac{BC}{DF}$ (c) $\frac{AB}{EF} = \frac{AC}{DE}$ (d) $\frac{BC}{DF} = \frac{AB}{EF}$

32. In Fig. 7.244 the measures of $\angle D$ and $\angle F$ are respectively

(a) $50°, 40°$ (b) $20°, 30°$ (c) $40°, 50°$ (d) $30°, 20°$

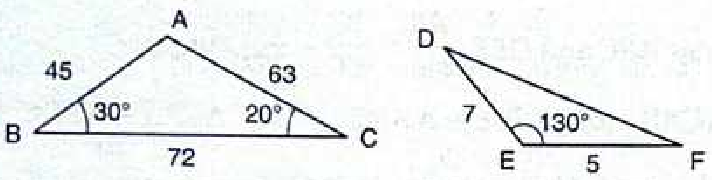


Fig. 7.244

33. In Fig. 7.245, the value of x for which $DE \parallel AB$ is

(a) 4 (b) 1 (c) 3 (d) 2

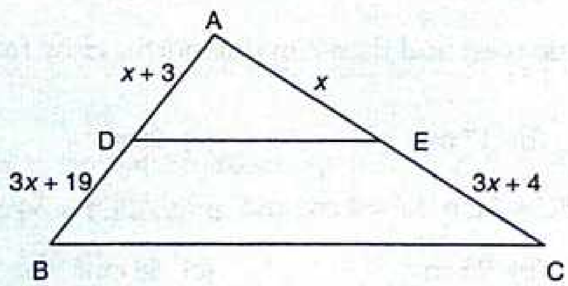


Fig. 7.245

34. In Fig. 7.246, if $\angle ADE = \angle ABC$, then $CE =$

(a) 2 (b) 5 (c) 9/2 (d) 3

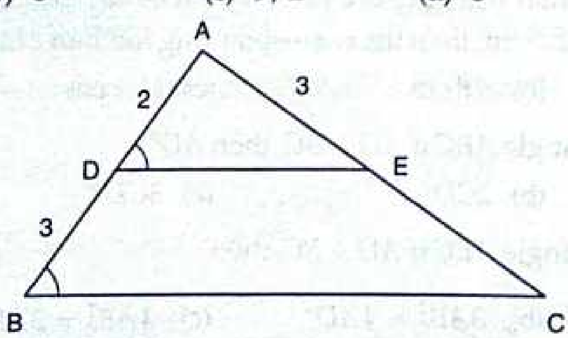


Fig. 7.246

35. In Fig. 7.247, $RS \parallel DB \parallel PQ$. If $CP = PD = 11$ cm and $DR = RA = 3$ cm. Then the values of x and y are respectively

(a) 12, 10 (b) 14, 6 (c) 10, 7 (d) 16, 8

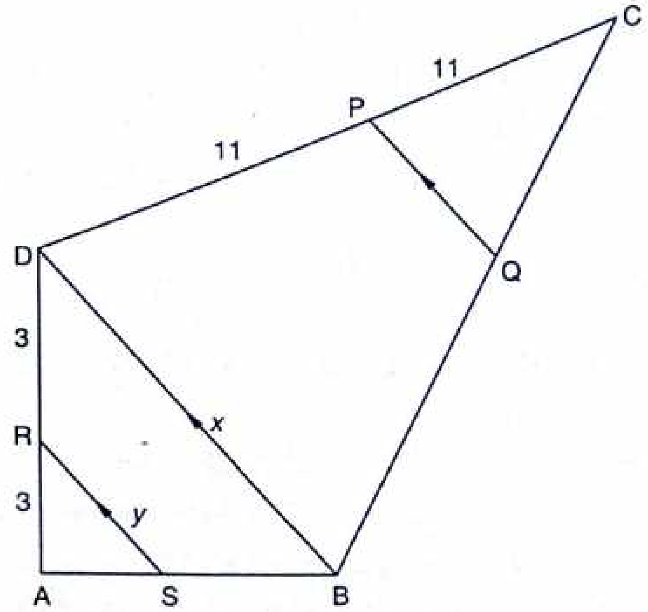


Fig. 7.247

36. In Fig. 7.248, if $PB \parallel CF$ and $DP \parallel EF$, then $\frac{AD}{DE} =$

(a) $\frac{3}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{2}{3}$

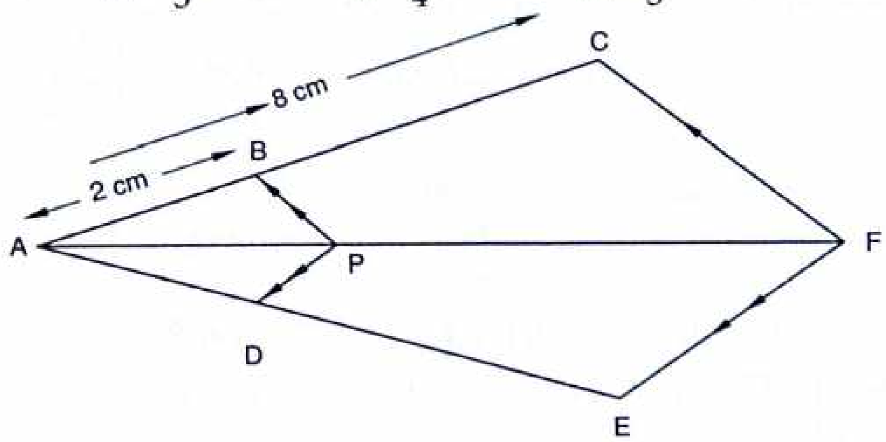


Fig. 7.248

37. A chord of a circle of radius 10 cm subtends a right angle at the centre. The length of the chord (in cm) is

(a) $5\sqrt{2}$ (b) $10\sqrt{2}$ (c) $\frac{5}{\sqrt{2}}$ (d) $10\sqrt{3}$ **[CBSE 2014]**

LEVEL-2

38. A vertical stick 20 m long casts a shadow 10 m long on the ground. At the same time, a tower casts a shadow 50 m long on the ground. The height of the tower is

(a) 100 m (b) 120 m (c) 25 m (d) 200 m

39. Two isosceles triangles have equal angles and their areas are in the ratio 16 : 25. The ratio of their corresponding heights is

(a) 4 : 5 (b) 5 : 4 (c) 3 : 2 (d) 5 : 7

40. ΔABC is such that $AB = 3$ cm, $BC = 2$ cm and $CA = 2.5$ cm. If $\Delta DEF \sim \Delta ABC$ and $EF = 4$ cm, then perimeter of ΔDEF is

(a) 7.5 cm (b) 15 cm (c) 22.5 cm (d) 30 cm.

41. In ΔABC, a line XY parallel to BC cuts AB at X and AC at Y. If BY bisects $\angle XYC$, then

(a) $BC = CY$ (b) $BC = BY$ (c) $BC \neq CY$ (d) $BC \neq BY$

42. In a ΔABC, $\angle A = 90°$, $AB = 5$ cm and $AC = 12$ cm. If $AD \perp BC$, then $AD =$

(a) $\frac{13}{2}$ cm (b) $\frac{60}{13}$ cm (c) $\frac{13}{60}$ cm (d) $\frac{2\sqrt{15}}{13}$ cm

43. In a ΔABC, perpendicular AD from A on BC meets BC at D. If $BD = 8$ cm, $DC = 2$ cm and $AD = 4$ cm, then

(a) ΔABC is isosceles (b) ΔABC is equilateral

(c) $AC = 2AB$ (d) ΔABC is right-angled at A.

44. In a ΔABC, point D is on side AB and point E is on side AC, such that $BCED$ is a trapezium. If $DE : BC = 3 : 5$, then Area (ΔADE) : Area ($\square BCED$) =

(a) $3:4$ (b) $9:16$ (c) $3:5$ (d) $9:25$

45. If ABC is an isosceles triangle and D is a point on BC such that $AD \perp BC$, then

(a) $AB^2 - AD^2 = BD.DC$ (b) $AB^2 - AD^2 = BD^2 - DC^2$

(c) $AB^2 + AD^2 = BD.DC$ (d) $AB^2 + AD^2 = BD^2 - DC^2$

46. ΔABC is a right triangle right-angled at A and $AD \perp BC$. Then, $\frac{BD}{DC} =$

(a) $\left(\frac{AB}{AC}\right)^2$ (b) $\frac{AB}{AC}$ (c) $\left(\frac{AB}{AD}\right)^2$ (d) $\frac{AB}{AD}$

47. If E is a point on side CA of an equilateral triangle ABC such that $BE \perp CA$, then $AB^2 + BC^2 + CA^2 =$

(a) $2\,BE^2$ (b) $3\,BE^2$ (c) $4\,BE^2$ (d) $6\,BE^2$

48. In a right triangle ABC right-angled at B, if P and Q are points on the sides AB and AC respectively, then

(a) $AQ^2 + CP^2 = 2(AC^2 + PQ^2)$ (b) $2(AQ^2 + CP^2) = AC^2 + PQ^2$

(c) $AQ^2 + CP^2 = AC^2 + PQ^2$ (d) $AQ + CP = \frac{1}{2}(AC + PQ)$.

49. If $\Delta ABC \sim \Delta DEF$ such that $DE = 3$ cm, $EF = 2$ cm, $DF = 2.5$ cm, $BC = 4$ cm, then perimeter of ΔABC is

(a) 18 cm (b) 20 cm (c) 12 cm (d) 15 cm

50. If $\Delta ABC \sim \Delta DEF$ such that $AB = 9.1$ cm and $DE = 6.5$ cm. If the perimeter of ΔDEF is 25 cm, then the perimeter of ΔABC is

(a) 36 cm (b) 30 cm (c) 34 cm (d) 35 cm

51. In an isosceles triangle ABC, if $AB = AC = 25$ cm and $BC = 14$ cm, then the measure of altitude from A on BC is

(a) 20 cm (b) 22 cm (c) 18 cm (d) 24 cm

ANSWERS

1. (d)	2. (a)	3. (c)	4. (c)	5. (d)	6. (b)
7. (c)	8. (c)	9. (c)	10. (c)	11. (a)	12. (a)
13. (c)	14. (c)	15. (b)	16. (a)	17. (b)	18. (b)
19. (c)	20. (a)	21. (a)	22. (b)	23. (c)	24. (b)
25. (c)	26. (b)	27. (c)	28. (b)	29. (c)	30. (a)
31. (b)	32. (b)	33. (d)	34. (c)	35. (d)	36. (b)
37. (b)	38. (a)	39. (a)	40. (b)	41. (a)	42. (b)
43. (d)	44. (d)	45. (a)	46. (b)	47. (c)	48. (c)
49. (d)	50. (d)	51. (d)			

SUMMARY

1. Two figures having the same shape but not necessarily the same size are called similar figures.
2. All congruent figures are similar but the converse is not true.
3. Two polygons having the same number of sides are similar, if
   (i) their corresponding angles are equal and
   (ii) their corresponding sides are proportional (i.e., in the same ratio).
4. If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, then the other two sides are divided in the same ratio.
5. If a line divides any two sides of a triangle in the same ratio, then the line is parallel to the third side of the triangle.
6. The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.
7. If a line through one vertex of a triangle divides the opposite side in the ratio of other two sides, then the line bisects the angle at the vertex.
8. The external bisector of an angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle.
9. The line drawn from the mid-point of one side of a triangle is parallel of another side bisects the third side.
10. The line joining the mid-points of two sides of a triangle is parallel to the third side.
11. The diagonals of a trapezium divide each other proportionally.
12. If the diagonals of a quadrilateral divide each other proportionally, then it is a trapezium.
13. Any line parallel to the parallel sides of a trapezium divides the non-parallel sides proportionally.
14. If three or more parallel lines are intersected by two transversals, then the intercepts made by them on the transversals are proportional.
15. *AAA Similarity criterion:* If in two triangles, corresponding angles are equal, then the triangles are similar.
16. *AA Similarity criterion:* If in two triangles, two angles of one triangle are respectively equal to the two angles of the other triangle, then the two triangles are similar.
17. *SSS Similarity criterion:* If in two triangles, corresponding sides are in the same ratio, then the two triangles are similar.

18. If one angle of a triangles is equal to one angle of another triangle and the sides including these angles are in the same ratio, then the triangles are similar.
19. If two triangles are equiangular, then
    (i) the ratio of the corresponding sides is same as the ratio of corresponding medians.
    (ii) the ratio of the corresponding sides is same as the ratio of the corresponding angle bisector segments.
    (iii) the ratio of the corresponding sides is same as the ratio of the corresponding altitudes.
20. If one angle of a triangle is equal to one angle of another triangle and the bisectors of these equal angles divide the opposite side in the same ratio, then the triangles are similar.
21. If two sides and a median bisecting one of these sides of a triangle are respectively proportional to the two sides and the corresponding median of another triangle, then the triangles are similar.
22. If two sides and a median bisecting the third side of a triangle are respectively proportional to the two sides and the corresponding median of another triangle, then the triangles are similar.
23. The ratio of the areas of two similar triangles is equal to the ratio of
    (i) the squares of any two corresponding sides.
    (ii) the squares of the corresponding altitudes.
    (iii) the squares of the corresponding medians.
    (iv) the squares of the corresponding angle bisector segments.
24. If the areas of two similar triangles are equal, then the triangles are congruent i.e., equal and similar triangles congruent.
25. If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse, then the triangles on both sides of the perpendicular are similar to the whole. triangle and also to each other.
26. *Pythagoras Theorem:* In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.
27. *Converse of Pythagoras Theorem:* If in a triangle, square of one side is equal to the sum of the squares of the other two sides, then the angle opposite to first side is a right angle.
28. In any triangle, the sum of the squares of any two sides is equal to twice the square of half of the third side together with the twice of the square of the median which bisects the third side.
29. Three times the sum of the squares of the sides of a triangle is equal to four times the sum of the squares of the medians of the triangle.
30. Three times the square of any side of an equilateral triangle is equal to four times the square of the altitude.