10 Trigonometric Ratios

TRIGONOMETRY It is that branch of mathematics which deals with the measurement of angles and the problems allied with angles.

TRIGONOMETRIC RATIOS (T-RATIOS) OF AN ACUTE ANGLE OF A RIGHT TRIANGLE

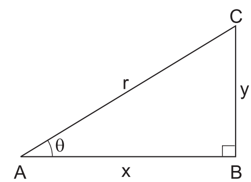
Let $\angle BAC = \theta$ be an acute angle of a right-angled $\triangle ABC$.

In right-angled $\triangle ABC$, let

base $= AB = x$ units,

perpendicular $= BC = y$ units

and hypotenuse $= AC = r$ units.



We define the following ratios, known as Trigonometric Ratios for θ.

(i) sine $\theta = \dfrac{\text{perpendicular}}{\text{hypotenuse}} = \dfrac{y}{r}$, and is written as $\sin\theta$.

(ii) cosine $\theta = \dfrac{\text{base}}{\text{hypotenuse}} = \dfrac{x}{r}$, and is written as $\cos\theta$.

(iii) tangent $\theta = \dfrac{\text{perpendicular}}{\text{base}} = \dfrac{y}{x}$, and is written as $\tan\theta$.

(iv) cosecant $\theta = \dfrac{\text{hypotenuse}}{\text{perpendicular}} = \dfrac{r}{y}$, and is written as $\operatorname{cosec}\theta$.

(v) secant $\theta = \dfrac{\text{hypotenuse}}{\text{base}} = \dfrac{r}{x}$, and is written as $\sec\theta$.

(vi) cotangent $\theta = \dfrac{\text{base}}{\text{perpendicular}} = \dfrac{x}{y}$, and is written as $\cot\theta$.

RECIPROCAL RELATION

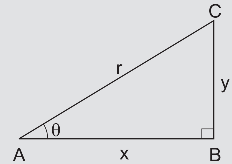
Clearly, we have

(i) $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ (ii) $\sec\theta = \dfrac{1}{\cos\theta}$ (iii) $\cot\theta = \dfrac{1}{\tan\theta}$

SUMMARY

Consider a $\triangle ABC$ in which $\angle B = 90^\circ$ and $\angle A = \theta$.

Let $AB = x$, $BC = y$ and $AC = r$. Then,



(i) $\sin\theta = \dfrac{y}{r}$ (ii) $\cos\theta = \dfrac{x}{r}$ (iii) $\tan\theta = \dfrac{y}{x}$

(iv) $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta} = \dfrac{r}{y}$ (v) $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{r}{x}$

(vi) $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{x}{y}$

Also, we have

(i) $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ (ii) $\sec\theta = \dfrac{1}{\cos\theta}$ (iii) $\cot\theta = \dfrac{1}{\tan\theta}$

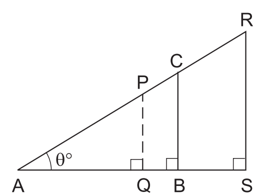
T-RATIOS OF AN ANGLE ARE WELL-DEFINED

THEOREM *Show that the value of each of the trigonometric ratios of an angle does not depend on the size of the triangle. It only depends on the angle.*

PROOF Consider a $\triangle ABC$ in which $\angle B = 90^\circ$ and $\angle A = \theta^\circ$.

Take a point P on AC and draw $PQ \perp AB$.

Then, $\triangle AQP$ is similar to $\triangle ABC$.



$$\therefore \quad \frac{AQ}{AB} = \frac{AP}{AC} = \frac{PQ}{CB}$$

$$\Rightarrow \quad \frac{PQ}{AP} = \frac{CB}{AC} = \sin\theta.$$

Similarly, $\dfrac{AQ}{AP} = \dfrac{AB}{AC} = \cos\theta$

and $\dfrac{PQ}{AQ} = \dfrac{CB}{AB} = \tan\theta.$

Similarly, if we produce AC to R and draw $RS \perp AB$ produced then $\triangle ASR$ is similar to $\triangle ABC$.

$$\therefore \quad \frac{AS}{AB} = \frac{AR}{AC} = \frac{RS}{CB}$$

$$\Rightarrow \quad \frac{RS}{AR} = \frac{CB}{AC} = \sin\theta.$$

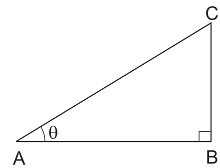
Similarly, $\dfrac{AS}{AR} = \dfrac{AB}{AC} = \cos\theta$

and $\dfrac{RS}{AS} = \dfrac{CB}{AB} = \tan\theta.$

Hence, the trigonometric ratios of an angle do not depend on the size of the triangle. They only depend on the angle.

REMARK Consider a $\triangle ABC$ in which $\angle B = 90^\circ$ and $\angle A = \theta^\circ$.

We know that in a right triangle, the hypotenuse is the longest side.



$\therefore \quad \sin\theta = \frac{BC}{AC} < 1$ and $\cos\theta = \frac{AB}{AC} < 1$.

Thus, $\sin\theta < 1$ and $\cos\theta < 1$.

POWER OF T-RATIOS We write

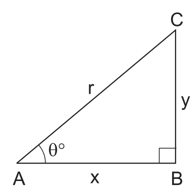
$(\sin\theta)^2 = \sin^2\theta$; $\quad (\sin\theta)^3 = \sin^3\theta$; $\quad (\cos\theta)^3 = \cos^3\theta$; etc.

QUOTIENT RELATION OF T-RATIOS

THEOREM 1 *For any acute angle θ, prove that*

(i) $\tan\theta = \frac{\sin\theta}{\cos\theta}$; *(ii)* $\cot\theta = \frac{\cos\theta}{\sin\theta}$; *(iii)* $\tan\theta \cdot \cot\theta = 1$.

PROOF Consider a right-angled $\triangle ABC$ in which $\angle B = 90^\circ$ and $\angle A = \theta^\circ$. Let $AB = x$ units, $BC = y$ units and $AC = r$ units. Then,



(i) $\tan\theta = \frac{y}{x} = \frac{(y/r)}{(x/r)}$

[dividing num. and denom. by r]

$= \frac{\sin\theta}{\cos\theta}$.

$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta}$.

(ii) $\cot\theta = \frac{x}{y} = \frac{(x/r)}{(y/r)}$ [dividing num. and denom. by r]

$= \frac{\cos\theta}{\sin\theta}$.

$\therefore \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$.

(iii) $\tan\theta \cdot \cot\theta = \frac{\sin\theta}{\cos\theta} \cdot \frac{\cos\theta}{\sin\theta} = 1$.

SUMMARY

(i) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ (ii) $\cot\theta = \frac{\cos\theta}{\sin\theta}$ (iii) $\tan\theta \cdot \cot\theta = 1$

SQUARE RELATION

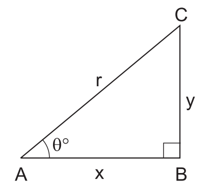
THEOREM 2 *For any acute angle θ, prove that*

(i) $\sin^2\theta + \cos^2\theta = 1;$

(ii) $1 + \tan^2\theta = \sec^2\theta;$

(iii) $1 + \cot^2\theta = \text{cosec}^2\theta.$

PROOF Consider a right-angled ΔABC in which $\angle B = 90^\circ$ and $\angle A = \theta^\circ$. Let $AB = x$ units, $BC = y$ units and $AC = r$ units.



Then, by Pythagoras' theorem, we have

$$x^2 + y^2 = r^2.$$

(i) $\sin^2\theta + \cos^2\theta = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \left(\frac{y^2}{r^2} + \frac{x^2}{r^2}\right)$

$= \frac{(x^2 + y^2)}{r^2} = \frac{r^2}{r^2}$ $\quad [\because\ x^2 + y^2 = r^2]$

$= 1.$

$\therefore \quad \sin^2\theta + \cos^2\theta = 1.$

(ii) $1 + \tan^2\theta = 1 + \left(\frac{y}{x}\right)^2 = 1 + \frac{y^2}{x^2} = \frac{y^2 + x^2}{x^2} = \frac{r^2}{x^2}$ $\quad [\because\ x^2 + y^2 = r^2]$

$= \left(\frac{r}{x}\right)^2 = \sec^2\theta.$

$\therefore \quad 1 + \tan^2\theta = \sec^2\theta.$

(iii) $1 + \cot^2\theta = 1 + \left(\frac{x}{y}\right)^2 = 1 + \frac{x^2}{y^2} = \frac{x^2 + y^2}{y^2} = \frac{r^2}{y^2}$ $\quad [\because\ x^2 + y^2 = r^2]$

$= \left(\frac{r}{y}\right)^2 = \text{cosec}^2\theta.$

$\therefore \quad 1 + \cot^2\theta = \text{cosec}^2\theta.$

SUMMARY

(i) $\sin^2\theta + \cos^2\theta = 1$ (ii) $1 + \tan^2\theta = \sec^2\theta$ (iii) $1 + \cot^2\theta = \text{cosec}^2\theta$

SOLVED EXAMPLES

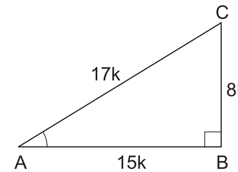
EXAMPLE 1 *If* $\sin A = \frac{8}{17}$, *find other trigonometric ratios of* $\angle A$.

SOLUTION Let us draw a ΔABC in which $\angle B = 90^\circ$.

Then, $\sin A = \frac{BC}{AC} = \frac{8}{17}$.

Let $BC = 8k$ and $AC = 17k$, where k is positive.

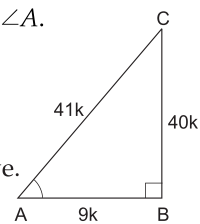
By Pythagoras' theorem, we have



$$AC^2 = AB^2 + BC^2$$
$$\Rightarrow \quad AB^2 = AC^2 - BC^2$$
$$\Rightarrow \quad AB^2 = (17k)^2 - (8k)^2 = 289k^2 - 64k^2$$
$$= 225k^2$$
$$\Rightarrow \quad AB = \sqrt{225k^2} = 15k.$$
$$\therefore \quad \sin A = \frac{BC}{AC} = \frac{8k}{17k} = \frac{8}{17}; \ \cos A = \frac{AB}{AC} = \frac{15k}{17k} = \frac{15}{17};$$
$$\tan A = \frac{\sin A}{\cos A} = \left(\frac{8}{17} \times \frac{17}{15}\right) = \frac{8}{15};$$
$$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{17}{8}; \ \sec A = \frac{1}{\cos A} = \frac{17}{15}$$
and $\cot A = \dfrac{1}{\tan A} = \dfrac{15}{8}$.

EXAMPLE 2 *If* $\cos A = \frac{9}{41}$, *find other trigonometric ratios of* $\angle A$.

SOLUTION Let us draw a $\triangle ABC$ in which $\angle B = 90°$.



Then, $\cos A = \dfrac{AB}{AC} = \dfrac{9}{41}$.

Let $AB = 9k$ and $AC = 41k$, where k is positive.

By Pythagoras' theorem, we have

$$AC^2 = AB^2 + BC^2$$
$$\Rightarrow \quad BC^2 = AC^2 - AB^2$$
$$\Rightarrow \quad BC^2 = (41k)^2 - (9k)^2 = 1681k^2 - 81k^2 = 1600k^2$$
$$\Rightarrow \quad BC = \sqrt{1600k^2} = 40k.$$
$$\therefore \quad \sin A = \frac{BC}{AC} = \frac{40k}{41k} = \frac{40}{41}; \ \cos A = \frac{9}{41} \text{ (given)};$$
$$\tan A = \frac{\sin A}{\cos A} = \left(\frac{40}{41} \times \frac{41}{9}\right) = \frac{40}{9};$$
$$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{41}{40}; \ \sec A = \frac{1}{\cos A} = \frac{41}{9}$$
and $\cot A = \dfrac{1}{\tan A} = \dfrac{9}{40}$.

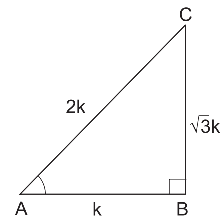
EXAMPLE 3 *If* $\tan A = \sqrt{3}$, *find other trigonometric ratios of* $\angle A$.

SOLUTION Let us draw a $\triangle ABC$ in which $\angle B = 90°$.

Then, $\tan A = \dfrac{BC}{AB} = \dfrac{\sqrt{3}}{1}$.

Let $BC = \sqrt{3}\,k$ and $AB = k$, where k is positive.

By Pythagoras' theorem, we have



$$AC^2 = AB^2 + BC^2 = k^2 + (\sqrt{3}k)^2$$
$$= k^2 + 3k^2 = 4k^2.$$
$$\therefore \quad AC = \sqrt{4k^2} = 2k.$$
$$\therefore \quad \sin A = \frac{BC}{AC} = \frac{\sqrt{3}k}{2k} = \frac{\sqrt{3}}{2};$$
$$\cos A = \frac{AB}{AC} = \frac{k}{2k} = \frac{1}{2};$$
$$\tan A = \sqrt{3} \text{ (given)}; \ \operatorname{cosec} A = \frac{1}{\sin A} = \frac{2}{\sqrt{3}};$$
$$\sec A = \frac{1}{\cos A} = 2 \text{ and } \cot A = \frac{1}{\tan A} = \frac{1}{\sqrt{3}}.$$

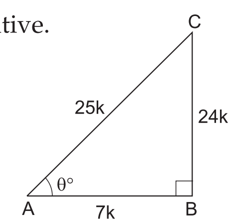
EXAMPLE 4 *If* $\sec\theta = \frac{25}{7}$, *find all trigonometric ratios of* θ.

SOLUTION Let us draw a $\triangle ABC$ in which $\angle B = 90°$. Let $\angle A = \theta°$.

Then, $\sec\theta = \frac{AC}{AB} = \frac{25}{7}$.

Let $AC = 25k$ and $AB = 7k$, where k is positive.

By Pythagoras' theorem, we have



$$AC^2 = AB^2 + BC^2$$
$$\Rightarrow \quad BC^2 = AC^2 - AB^2 = (25k)^2 - (7k)^2$$
$$= 625k^2 - 49k^2 = 576k^2$$
$$\Rightarrow \quad BC = \sqrt{576k^2} = 24k.$$
$$\therefore \quad \sin\theta = \frac{BC}{AC} = \frac{24k}{25k} = \frac{24}{25}; \ \cos\theta = \frac{1}{\sec\theta} = \frac{7}{25};$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \left(\frac{24}{25} \times \frac{25}{7}\right) = \frac{24}{7}; \ \operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{25}{24};$$
$$\sec\theta = \frac{25}{7} \text{ (given) and } \cot\theta = \frac{1}{\tan\theta} = \frac{7}{24}.$$

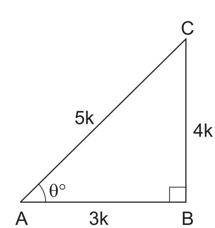
EXAMPLE 5 *If* $\cos\theta = \frac{3}{5}$, *find the value of* $\left(\frac{5\operatorname{cosec}\theta - 4\tan\theta}{\sec\theta + \cot\theta}\right)$.

SOLUTION Let us draw a $\triangle ABC$ in which $\angle B = 90°$.

Let $\angle A = \theta°$.

Then, $\cos\theta = \frac{AB}{AC} = \frac{3}{5}$.

Let $AB = 3k$ and $AC = 5k$, where k is positive.



By Pythagoras' theorem, we have

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow \quad BC^2 = AC^2 - AB^2 = (5k)^2 - (3k)^2 = 25k^2 - 9k^2 = 16k^2$$

$$\Rightarrow \quad BC = \sqrt{16k^2} = 4k.$$

$$\therefore \quad \sec\theta = \frac{1}{\cos\theta} = \frac{5}{3};\ \tan\theta = \frac{BC}{AB} = \frac{4k}{3k} = \frac{4}{3};$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{3}{4};\ \text{and}\ \operatorname{cosec}\theta = \frac{AC}{BC} = \frac{5k}{4k} = \frac{5}{4}.$$

$$\therefore \quad \left(\frac{5\operatorname{cosec}\theta - 4\tan\theta}{\sec\theta + \cot\theta}\right) = \frac{\left(5 \times \frac{5}{4} - 4 \times \frac{4}{3}\right)}{\left(\frac{5}{3} + \frac{3}{4}\right)} = \frac{\left(\frac{25}{4} - \frac{16}{3}\right)}{\left(\frac{5}{3} + \frac{3}{4}\right)}$$

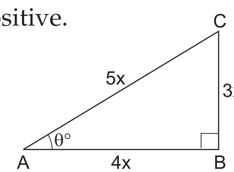
$$= \frac{\frac{75-64}{12}}{\frac{20+9}{12}} = \left(\frac{11}{12} \times \frac{12}{29}\right) = \frac{11}{29}.$$

EXAMPLE 6 *If* $\sec\theta = \frac{5}{4}$, *show that* $\frac{(2\cos\theta - \sin\theta)}{(\cot\theta - \tan\theta)} = \frac{12}{7}$.

SOLUTION Consider a $\triangle ABC$ in which $\angle A = \theta$ and $\angle B = 90^\circ$.

$$\sec\theta = \frac{\text{hypotenuse}}{\text{base}} = \frac{AC}{AB} = \frac{5}{4} = \frac{5x}{4x}\ \text{(say)}.$$

$\therefore$ $AC = 5x$ and $AB = 4x$, where x is positive.



By Pythagoras' theorem we have

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow \quad BC^2 = AC^2 - AB^2 = (5x)^2 - (4x)^2 = 9x^2$$

$$\Rightarrow \quad BC = 3x.$$

$$\therefore \quad \cos\theta = \frac{AB}{AC} = \frac{4x}{5x} = \frac{4}{5};\ \sin\theta = \frac{BC}{AC} = \frac{3x}{5x} = \frac{3}{5};$$

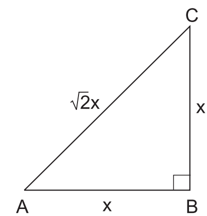
$$\cot\theta = \frac{AB}{BC} = \frac{4x}{3x} = \frac{4}{3};\ \text{and}\ \tan\theta = \frac{BC}{AB} = \frac{3x}{4x} = \frac{3}{4}.$$

$$\therefore \quad \frac{(2\cos\theta - \sin\theta)}{(\cot\theta - \tan\theta)} = \frac{\left(2 \times \frac{4}{5} - \frac{3}{5}\right)}{\left(\frac{4}{3} - \frac{3}{4}\right)} = \frac{\left(\frac{8}{5} - \frac{3}{5}\right)}{\left(\frac{4}{3} - \frac{3}{4}\right)} = \frac{\left(\frac{5}{5}\right)}{\left(\frac{7}{12}\right)} = \frac{1}{\left(\frac{7}{12}\right)} = \frac{12}{7}.$$

EXAMPLE 7 *In a* $\triangle ABC$ *it is given that* $\angle B = 90^\circ$ *and* $AB : AC = 1 : \sqrt{2}$. *Find the value of* $\left(\frac{2\tan A}{1 + \tan^2 A}\right)$.

SOLUTION Consider a $\triangle ABC$ in which $\angle B = 90^\circ$ and $AB : AC = 1 : \sqrt{2}$.

Let $AB = x$. Then, $AC = \sqrt{2}\,x$.



By Pythagoras' theorem, we have

$$AC^2 = AB^2 + BC^2 \Rightarrow BC^2 = AC^2 - AB^2$$

$$\Rightarrow \quad BC^2 = (\sqrt{2}\,x)^2 - (x)^2 = 2x^2 - x^2 = x^2$$

$$\Rightarrow \quad BC = x.$$

$$\therefore \quad \tan A = \frac{BC}{AB} = \frac{x}{x} = 1.$$

So, the given expression $= \left(\dfrac{2\tan A}{1+\tan^2 A}\right) = \left(\dfrac{2\times 1}{1+1}\right) = \dfrac{2}{2} = 1.$

EXAMPLE 8 *If* $3\tan\theta = 4$, *evaluate* $\dfrac{3\sin\theta + 2\cos\theta}{3\sin\theta - 2\cos\theta}$.

SOLUTION $3\tan\theta = 4 \Rightarrow \tan\theta = \dfrac{4}{3}$.

Given expression

$$= \frac{3\sin\theta + 2\cos\theta}{3\sin\theta - 2\cos\theta}$$

$$= \frac{3\tan\theta + 2}{3\tan\theta - 2} \qquad \text{[dividing num. and denom. by } \cos\theta]$$

$$= \frac{\left(3\times\frac{4}{3}+2\right)}{\left(3\times\frac{4}{3}-2\right)} = \frac{6}{2} = 3. \qquad \left[\because \ \tan\theta = \frac{4}{3}\right]$$

EXAMPLE 9 *If* $5\cot\theta = 3$, *find the value of* $\left(\dfrac{5\sin\theta - 3\cos\theta}{4\sin\theta + 3\cos\theta}\right)$.

SOLUTION $5\cot\theta = 3 \Rightarrow \cot\theta = \dfrac{3}{5}$.

Given expression $= \left(\dfrac{5\sin\theta - 3\cos\theta}{4\sin\theta + 3\cos\theta}\right) = \dfrac{(5 - 3\cot\theta)}{(4 + 3\cot\theta)}$

[dividing num. and denom. by $\sin\theta$]

$$= \frac{\left(5 - 3\times\frac{3}{5}\right)}{\left(4 + 3\times\frac{3}{5}\right)} = \frac{\left(5 - \frac{9}{5}\right)}{\left(4 + \frac{9}{5}\right)} = \left(\frac{16}{5}\times\frac{5}{29}\right) = \frac{16}{29}.$$

EXAMPLE 10 *If* $7\sin^2\theta + 3\cos^2\theta = 4$, *show that* $\tan\theta = \dfrac{1}{\sqrt{3}}$. [CBSE 2008]

SOLUTION $7\sin^2\theta + 3\cos^2\theta = 4$

$$\Rightarrow \quad 4\sin^2\theta + 3\sin^2\theta + 3\cos^2\theta = 4$$

$$\Rightarrow \quad 4\sin^2\theta + 3(\sin^2\theta + \cos^2\theta) = 4$$

$\Rightarrow \quad 4\sin^2\theta + 3 \times 1 = 4$ $\quad [\because \; \sin^2\theta + \cos^2\theta = 1]$

$\Rightarrow \quad 4\sin^2\theta = 1 \Rightarrow \sin^2\theta = \frac{1}{4}$.

$\therefore \quad \cos^2\theta = (1 - \sin^2\theta) = \left(1 - \frac{1}{4}\right) = \frac{3}{4}$.

$\therefore \quad \tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \left(\frac{1}{4} \times \frac{4}{3}\right) = \frac{1}{3}$.

Hence, $\tan\theta = \frac{1}{\sqrt{3}}$.

EXAMPLE 11 *If* $\cot\theta = \frac{15}{8}$ *then evaluate* $\frac{(2 + 2\sin\theta)(1 - \sin\theta)}{(1 + \cos\theta)(2 - 2\cos\theta)}$. [CBSE 2009]

SOLUTION Given expression $= \frac{(2 + 2\sin\theta)(1 - \sin\theta)}{(1 + \cos\theta)(2 - 2\cos\theta)}$

$$= \frac{2(1 + \sin\theta)(1 - \sin\theta)}{2(1 + \cos\theta)(1 - \cos\theta)}$$

$$= \frac{(1 - \sin^2\theta)}{(1 - \cos^2\theta)} = \frac{\cos^2\theta}{\sin^2\theta} = \cot^2\theta$$

$$= (\cot\theta)^2 = \left(\frac{15}{8}\right)^2 = \frac{225}{64}.$$

Hence, the value of the given expression is $\frac{225}{64}$.

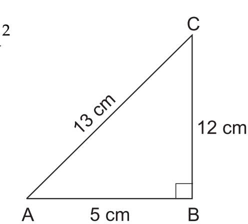
EXAMPLE 12 *In* $\triangle ABC$, *right-angled at B,* $AB = 5$ cm *and* $BC = 12$ cm. *Find the values of* $\sin A$, $\sec A$, $\sin C$ *and* $\sec C$.

SOLUTION In $\triangle ABC$, $\angle B = 90^\circ$, $AB = 5$ cm and $BC = 12$ cm.

By Pythagoras' theorem, we have

$AC^2 = (AB^2 + BC^2) = \{(5)^2 + (12)^2\}\text{ cm}^2$

$= (25 + 144)\text{ cm}^2 = 169\text{ cm}^2$.



$\therefore \quad AC = \sqrt{169\text{ cm}^2} = 13$ cm.

For T-ratios of $\angle A$, we have

base $= AB = 5$ cm,

perpendicular $= BC = 12$ cm

and hypotenuse $= AC = 13$ cm.

$\therefore \quad \sin A = \frac{BC}{AC} = \frac{12}{13}$ and $\sec A = \frac{AC}{AB} = \frac{13}{5}$.

For T-ratios of $\angle C$, we have

base $= BC = 12$ cm,

perpendicular = $AB = 5$ cm

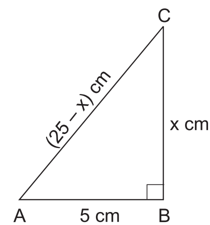
and hypotenuse = $AC = 13$ cm.

$\therefore \quad \sin C = \frac{AB}{AC} = \frac{5}{13}$ and $\sec C = \frac{AC}{BC} = \frac{13}{12}$.

EXAMPLE 13 *In a* $\triangle ABC$, $\angle B = 90°$, $AB = 5$ cm *and* $(BC + AC) = 25$ cm. *Find the values of* $\sin A$, $\cos A$, $\text{cosec}\, C$ *and* $\sec C$.

SOLUTION Let $BC = x$ cm. Then, $AC = (25 - x)$ cm.

By Pythagoras' theorem, we have



$AB^2 + BC^2 = AC^2$

$\Rightarrow \quad 5^2 + x^2 = (25 - x)^2$

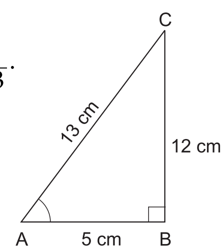
$\Rightarrow \quad 25 + x^2 = 625 - 50x + x^2$

$\Rightarrow \quad 50x = 600$

$\Rightarrow \quad x = 12.$

$\therefore \quad BC = 12$ cm, $AC = 13$ cm and $AB = 5$ cm.

For T-ratios of $\angle A$, we have



$\sin A = \frac{BC}{AC} = \frac{12}{13}$ and $\cos A = \frac{AB}{AC} = \frac{5}{13}$.

For T-ratios of $\angle C$, we have

base, $BC = 12$ cm,

perpendicular, $AB = 5$ cm

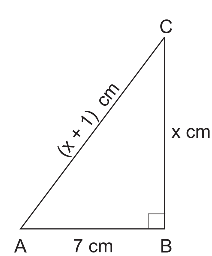
and hypotenuse, $AC = 13$ cm.

$\therefore \quad \text{cosec}\, C = \frac{AC}{AB} = \frac{13}{5}$ and $\sec C = \frac{AC}{BC} = \frac{13}{12}$.

EXAMPLE 14 *In a* $\triangle ABC$, $\angle B = 90°$, $AB = 7$ cm and $(AC - BC) = 1$ cm. *Find the values of* $\sin A$, $\cos A$, $\sin C$ *and* $\cos C$.

SOLUTION Let $BC = x$ cm. Then, $AC = (x + 1)$ cm.

By Pythagoras' theorem, we have



$AB^2 + BC^2 = AC^2$

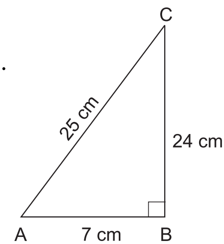
$\Rightarrow \quad 7^2 + x^2 = (x + 1)^2$

$\Rightarrow \quad 49 + x^2 = x^2 + 2x + 1$

$\Rightarrow \quad 2x = 48$

$\Rightarrow \quad x = 24.$

$\therefore \quad BC = 24$ cm, $AC = 25$ cm and $AB = 7$ cm.



For T-ratios of $\angle A$, we have

$$\sin A = \frac{BC}{AC} = \frac{24}{25} \text{ and } \cos A = \frac{AB}{AC} = \frac{7}{25}.$$

For T-ratios of $\angle C$, we have

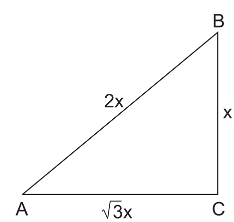
$$\sin C = \frac{AB}{AC} = \frac{7}{25} \text{ and } \cos C = \frac{BC}{AC} = \frac{24}{25}.$$

EXAMPLE 15 *In a* $\triangle ABC$, $\angle C = 90^\circ$ *and* $\tan A = \frac{1}{\sqrt{3}}$. *Find the values of:*

(i) $(\sin A \cdot \cos B + \cos A \cdot \sin B)$ *(ii)* $(\cos A \cdot \cos B - \sin A \cdot \sin B)$

[CBSE 2008]

SOLUTION Consider a $\triangle ABC$ in which $\angle C = 90^\circ$ and $\tan A = \frac{1}{\sqrt{3}}$.



Then, $\tan A = \frac{1}{\sqrt{3}} \Rightarrow \frac{BC}{AC} = \frac{1}{\sqrt{3}}$.

Let $BC = x$. Then, $AC = \sqrt{3}x$.

By Pythagoras' theorem, we have

$$AB^2 = AC^2 + BC^2$$
$$= (\sqrt{3}x)^2 + x^2 = (3x^2 + x^2) = 4x^2$$

$\Rightarrow \quad AB = \sqrt{4x^2} = 2x.$

For T-ratios of $\angle A$, we have

base $= AC = \sqrt{3}x$, perpendicular $= BC = x$ and

hypotenuse $= AB = 2x$.

$\therefore \quad \sin A = \frac{BC}{AB} = \frac{x}{2x} = \frac{1}{2}$ and $\cos A = \frac{AC}{AB} = \frac{\sqrt{3}x}{2x} = \frac{\sqrt{3}}{2}$.

For T-ratios of $\angle B$, we have

base $= BC = x$, perpendicular $= AC = \sqrt{3}x$ and

hypotenuse $= AB = 2x$.

$\therefore \quad \sin B = \frac{AC}{AB} = \frac{\sqrt{3}x}{2x} = \frac{\sqrt{3}}{2}$ and $\cos B = \frac{BC}{AB} = \frac{x}{2x} = \frac{1}{2}$.

(i) $(\sin A \cdot \cos B + \cos A \cdot \sin B) = \left(\frac{1}{2} \times \frac{1}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}\right)$

$$= \left(\frac{1}{4} + \frac{3}{4}\right) = 1.$$

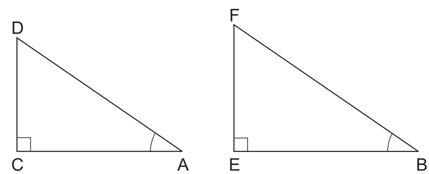
$\therefore \quad (\sin A \cdot \cos B + \cos A \cdot \sin B) = 1.$

(ii) $(\cos A \cos B - \sin A \sin B) = \left(\frac{\sqrt{3}}{2} \times \frac{1}{2} - \frac{1}{2} \times \frac{\sqrt{3}}{2}\right) = 0.$

$\therefore \quad (\cos A \cos B - \sin A \sin B) = 0.$

EXAMPLE 16 *If $\angle A$ and $\angle B$ are acute angles such that* $\cos A = \cos B$ *then prove that $\angle A = \angle B$.*

SOLUTION Let $\triangle ACD$ and $\triangle BEF$ be two right triangles given in such a way that $\cos A = \cos B$. Then,



$\cos A = \cos B$

$\Rightarrow \frac{AC}{AD} = \frac{BE}{BF}$

$\Rightarrow \frac{AC}{BE} = \frac{AD}{BF} = k$ (say)

$\Rightarrow AC = k(BE)$

and $AD = k(BF)$. … (i)

$\therefore \frac{CD}{EF} = \frac{\sqrt{AD^2 - AC^2}}{\sqrt{BF^2 - BE^2}}$ [using Pythagoras' theorem]

$= \frac{k\sqrt{BF^2 - BE^2}}{\sqrt{BF^2 - BE^2}} = k$ [using (i)].

Thus, we have: $\frac{AC}{BE} = \frac{AD}{BF} = \frac{CD}{EF}$.

$\therefore \quad \triangle ACD \sim \triangle BEF$ and hence, $\angle A = \angle B$.

EXERCISE 10

1. If $\sin\theta = \frac{\sqrt{3}}{2}$, find the value of all T-ratios of θ.
2. If $\cos\theta = \frac{7}{25}$, find the values of all T-ratios of θ.
3. If $\tan\theta = \frac{15}{8}$, find the values of all T-ratios of θ.
4. If $\cot\theta = 2$, find the values of all T-ratios of θ.
5. If $\operatorname{cosec}\theta = \sqrt{10}$, find the values of all T-ratios of θ.
6. If $\sin\theta = \frac{a^2 - b^2}{a^2 + b^2}$, find the values of all T-ratios of θ.
7. If $15 \cot A = 8$, find the values of $\sin A$ and $\sec A$.
8. If $\sin A = \frac{9}{41}$, find the values of $\cos A$ and $\tan A$.

9. If $\cos\theta = 0.6$, show that $(5\sin\theta - 3\tan\theta) = 0$.

10. If $\operatorname{cosec}\theta = 2$, show that $\left(\cot\theta + \dfrac{\sin\theta}{1+\cos\theta}\right) = 2$.

11. If $\tan\theta = \dfrac{1}{\sqrt{7}}$, show that $\dfrac{(\operatorname{cosec}^2\theta - \sec^2\theta)}{(\operatorname{cosec}^2\theta + \sec^2\theta)} = \dfrac{3}{4}$.

12. If $\tan\theta = \dfrac{20}{21}$, show that $\dfrac{(1-\sin\theta+\cos\theta)}{(1+\sin\theta+\cos\theta)} = \dfrac{3}{7}$.

13. If $\sec\theta = \dfrac{5}{4}$, show that $\dfrac{(\sin\theta - 2\cos\theta)}{(\tan\theta - \cot\theta)} = \dfrac{12}{7}$.

14. If $\cot\theta = \dfrac{3}{4}$, show that $\sqrt{\dfrac{\sec\theta - \operatorname{cosec}\theta}{\sec\theta + \operatorname{cosec}\theta}} = \dfrac{1}{\sqrt{7}}$.

15. If $\sin\theta = \dfrac{3}{4}$, show that $\sqrt{\dfrac{\operatorname{cosec}^2\theta - \cot^2\theta}{\sec^2\theta - 1}} = \dfrac{\sqrt{7}}{3}$.

16. If $\sin\theta = \dfrac{a}{b}$, show that $(\sec\theta + \tan\theta) = \sqrt{\dfrac{b+a}{b-a}}$.

17. If $\cos\theta = \dfrac{3}{5}$, show that $\dfrac{(\sin\theta - \cot\theta)}{2\tan\theta} = \dfrac{3}{160}$.

18. If $\tan\theta = \dfrac{4}{3}$, show that $(\sin\theta + \cos\theta) = \dfrac{7}{5}$.

19. If $\tan\theta = \dfrac{a}{b}$, show that $\dfrac{(a\sin\theta - b\cos\theta)}{(a\sin\theta + b\cos\theta)} = \dfrac{(a^2-b^2)}{(a^2+b^2)}$.

20. If $3\tan\theta = 4$, show that $\dfrac{(4\cos\theta - \sin\theta)}{(2\cos\theta + \sin\theta)} = \dfrac{4}{5}$.

21. If $3\cot\theta = 2$, show that $\dfrac{(4\sin\theta - 3\cos\theta)}{(2\sin\theta + 6\cos\theta)} = \dfrac{1}{3}$.

22. If $3\cot\theta = 4$, show that $\dfrac{(1-\tan^2\theta)}{(1+\tan^2\theta)} = (\cos^2\theta - \sin^2\theta)$.

23. If $\sec\theta = \dfrac{17}{8}$, verify that $\dfrac{(3-4\sin^2\theta)}{(4\cos^2\theta - 3)} = \dfrac{(3-\tan^2\theta)}{(1-3\tan^2\theta)}$.

24. In the adjoining figure, $\angle B = 90°$, $\angle BAC = \theta°$, $BC = CD = 4$ cm and $AD = 10$ cm. Find (i) $\sin\theta$ and (ii) $\cos\theta$.

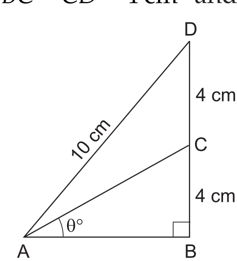
HINT $AB^2 = (AD^2 - BD^2) = 36\text{ cm}^2$

$\therefore\quad AB = 6$ cm.

$AC^2 = (AB^2 + BC^2) = 52\text{ cm}^2$

$\therefore\quad AC = 2\sqrt{13}$ cm.

Thus, $AB = 6$ cm and $AC = 2\sqrt{13}$ cm.



25. In a $\triangle ABC$, $\angle B = 90°$, $AB = 24$ cm and $BC = 7$ cm.

Find (i) $\sin A$ (ii) $\cos A$ (iii) $\sin C$ (iv) $\cos C$.

26. In a $\triangle ABC$, $\angle C = 90°$, $\angle ABC = \theta°$, $BC = 21$ units and $AB = 29$ units.
Show that $(\cos^2\theta - \sin^2\theta) = \dfrac{41}{841}\cdot$

27. In a $\triangle ABC$, $\angle B = 90°$, $AB = 12$ cm and $BC = 5$ cm.
Find (i) $\cos A$ (ii) $\operatorname{cosec} A$ (iii) $\cos C$ (iv) $\operatorname{cosec} C$.

28. If $\sin\alpha = \dfrac{1}{2}$, prove that $(3\cos\alpha - 4\cos^3\alpha) = 0$.

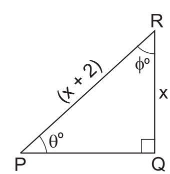
29. In a $\triangle ABC$, $\angle B = 90°$ and $\tan A = \dfrac{1}{\sqrt{3}}\cdot$ Prove that
(i) $\sin A\cdot\cos C + \cos A\cdot\sin C = 1$ (ii) $\cos A\cdot\cos C - \sin A\cdot\sin C = 0$

30. If $\angle A$ and $\angle B$ are acute angles such that $\sin A = \sin B$ then prove that $\angle A = \angle B$.

31. If $\angle A$ and $\angle B$ are acute angles such that $\tan A = \tan B$ then prove that $\angle A = \angle B$.

32. In a right $\triangle ABC$, right-angled at B, if $\tan A = 1$ then verify that $2\sin A\cdot\cos A = 1$.

33. In the figure of $\triangle PQR$, $\angle P = \theta°$ and $\angle R = \phi°$.
Find (i) $(\sqrt{x+1})\cot\phi$
(ii) $(\sqrt{x^3+x^2})\tan\theta$
(iii) $\cos\theta$



34. If $x = \operatorname{cosec} A + \cos A$ and $y = \operatorname{cosec} A - \cos A$ then prove that
$$\left(\frac{2}{x+y}\right)^2 + \left(\frac{x-y}{2}\right)^2 - 1 = 0.$$

35. If $x = \cot A + \cos A$ and $y = \cot A - \cos A$, prove that
$$\left(\frac{x-y}{x+y}\right)^2 + \left(\frac{x-y}{2}\right)^2 = 1.$$

ANSWERS (EXERCISE 10)

1. $\sin\theta = \dfrac{\sqrt{3}}{2}$, $\cos\theta = \dfrac{1}{2}$, $\tan\theta = \sqrt{3}$, $\cot\theta = \dfrac{1}{\sqrt{3}}$, $\operatorname{cosec}\theta = \dfrac{2}{\sqrt{3}}$, $\sec\theta = 2$

2. $\sin\theta = \dfrac{24}{25}$, $\cos\theta = \dfrac{7}{25}$, $\tan\theta = \dfrac{24}{7}$, $\cot\theta = \dfrac{7}{24}$, $\operatorname{cosec}\theta = \dfrac{25}{24}$, $\sec\theta = \dfrac{25}{7}$

3. $\sin\theta = \dfrac{15}{17}$, $\cos\theta = \dfrac{8}{17}$, $\tan\theta = \dfrac{15}{8}$, $\cot\theta = \dfrac{8}{15}$, $\operatorname{cosec}\theta = \dfrac{17}{15}$, $\sec\theta = \dfrac{17}{8}$

4. $\sin\theta = \dfrac{1}{\sqrt{5}}$, $\cos\theta = \dfrac{2}{\sqrt{5}}$, $\tan\theta = \dfrac{1}{2}$, $\cot\theta = 2$, $\operatorname{cosec}\theta = \sqrt{5}$, $\sec\theta = \dfrac{\sqrt{5}}{2}$

5. $\sin\theta = \dfrac{1}{\sqrt{10}}, \cos\theta = \dfrac{3}{\sqrt{10}}, \tan\theta = \dfrac{1}{3}, \cot\theta = 3, \text{cosec}\,\theta = \sqrt{10}, \sec\theta = \dfrac{\sqrt{10}}{3}$

6. $\sin\theta = \dfrac{a^2-b^2}{a^2+b^2}, \cos\theta = \dfrac{2ab}{a^2+b^2}, \tan\theta = \dfrac{a^2-b^2}{2ab},$

$\text{cosec}\,\theta = \dfrac{a^2+b^2}{a^2-b^2}, \sec\theta = \dfrac{a^2+b^2}{2ab}, \cot\theta = \dfrac{2ab}{a^2-b^2}$

7. $\sin A = \dfrac{15}{17}, \sec A = \dfrac{17}{8}$ **8.** $\cos A = \dfrac{40}{41}, \tan A = \dfrac{9}{40}$

24. (i) $\sin\theta = \dfrac{2\sqrt{13}}{13}$ (ii) $\cos\theta = \dfrac{3\sqrt{13}}{13}$

25. (i) $\dfrac{7}{25}$ (ii) $\dfrac{24}{25}$ (iii) $\dfrac{24}{25}$ (iv) $\dfrac{7}{25}$

27. (i) $\dfrac{12}{13}$ (ii) $\dfrac{13}{5}$ (iii) $\dfrac{5}{13}$ (iv) $\dfrac{13}{12}$

33. (i) $\dfrac{x}{2}$ (ii) $\dfrac{x^2}{2}$ (iii) $\dfrac{2\sqrt{x+1}}{(x+2)}$

HINTS TO SOME SELECTED QUESTIONS

6. In right $\triangle ABC$, $\angle B = 90^\circ$ and $\angle BAC = \theta$.

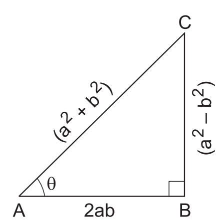
Given: $\sin\theta = \dfrac{a^2-b^2}{a^2+b^2} = \dfrac{BC}{AC}$.

Let $BC = (a^2-b^2)$ and $AC = (a^2+b^2)$. Then,

$AB^2 = (AC^2 - BC^2) = (a^2+b^2)^2 - (a^2-b^2)^2 = 4a^2b^2$

$\Rightarrow AB = 2ab.$

Now, we can find all the T-ratios of θ.

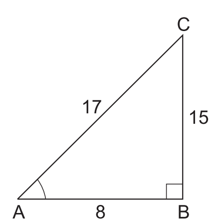


7. $\cot A = \dfrac{8}{15} \Rightarrow \tan A = \dfrac{15}{8} = \dfrac{BC}{AB}$.

$\therefore\ AC^2 = AB^2 + BC^2 = (8)^2 + (15)^2 = 64 + 225 = 289$

$\Rightarrow\ AC = \sqrt{289} = 17.$

$\therefore\ \sin A = \dfrac{BC}{AC} = \dfrac{15}{17}$ and $\sec A = \dfrac{AC}{AB} = \dfrac{17}{8}$.

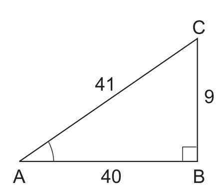


8. $\sin A = \dfrac{9}{41} = \dfrac{BC}{AC}$.

$\therefore\ AB^2 = AC^2 - BC^2 = (41)^2 - (9)^2 = 1681 - 81 = 1600$

$\Rightarrow\ AB = \sqrt{1600} = 40.$

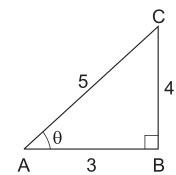
$\therefore\ \cos A = \dfrac{AB}{AC} = \dfrac{40}{41}$ and $\tan A = \dfrac{BC}{AB} = \dfrac{9}{40}$.



9. $\cos\theta = \frac{6}{10} = \frac{3}{5} = \frac{AB}{AC}$.

$BC^2 = AC^2 - AB^2 = (5)^2 - (3)^2 = 25 - 9 = 16 \Rightarrow BC = 4.$

$\therefore \quad \sin\theta = \frac{4}{5}$ and $\tan\theta = \frac{4}{3}$.

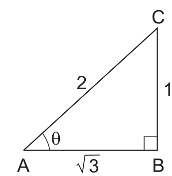


10. $\text{cosec}\,\theta = 2 \Rightarrow \sin\theta = \frac{1}{2} = \frac{BC}{AC}$.

$AB^2 = AC^2 - BC^2 = (2)^2 - (1)^2 = 4 - 1 = 3 \Rightarrow AB = \sqrt{3}.$

$\therefore \quad \sin\theta = \frac{1}{2}, \cos\theta = \frac{\sqrt{3}}{2}$ and $\cot\theta = \sqrt{3}$.

Given expression $= \left\{\sqrt{3} + \frac{1}{(2+\sqrt{3})} \times \frac{(2-\sqrt{3})}{(2-\sqrt{3})}\right\} = 2.$

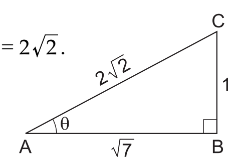


11. $\tan\theta = \frac{1}{\sqrt{7}} = \frac{BC}{AB}$.

$\therefore \quad AC^2 = AB^2 + BC^2 = (\sqrt{7})^2 + (1)^2 = (7+1) = 8 \Rightarrow AC = \sqrt{8} = 2\sqrt{2}.$

$\therefore \quad \text{cosec}\,\theta = \frac{AC}{BC} = \frac{2\sqrt{2}}{1} = 2\sqrt{2} \Rightarrow \text{cosec}^2\theta = 8$

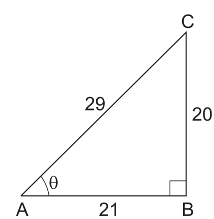
and $\sec\theta = \frac{AC}{AB} = \frac{2\sqrt{2}}{\sqrt{7}} \Rightarrow \sec^2\theta = \frac{8}{7}$.



12. $\tan\theta = \frac{20}{21} = \frac{BC}{AB}$.

$\therefore \quad AC^2 = AB^2 + BC^2 = (21)^2 + (20)^2 = 441 + 400 = 841$

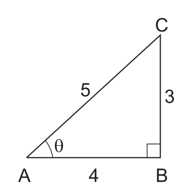
$\Rightarrow \quad AC = \sqrt{841} = 29.$



13. $\sec\theta = \frac{5}{4} \Rightarrow \cos\theta = \frac{4}{5} = \frac{AB}{AC}$.

$\therefore \quad BC^2 = AC^2 - AB^2 = (5)^2 - (4)^2 = 25 - 16 = 9 \Rightarrow BC = 3.$

$\therefore \quad \sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5}, \tan\theta = \frac{3}{4}, \cot\theta = \frac{4}{3}$.



16. $\sin\theta = \frac{a}{b} = \frac{BC}{AC}$.

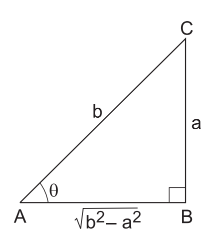
$\therefore \quad AB^2 = AC^2 - BC^2 = b^2 - a^2$

$\Rightarrow \quad AB = \sqrt{b^2 - a^2}.$

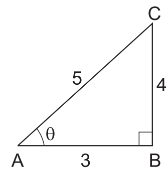
$\therefore \quad \sec\theta = \frac{b}{\sqrt{b^2 - a^2}}$ and $\tan\theta = \frac{a}{\sqrt{b^2 - a^2}}$

$\therefore \quad (\sec\theta + \tan\theta) = \frac{(b+a)}{\sqrt{b^2 - a^2}} = \frac{(b+a)}{\sqrt{(b-a)} \cdot \sqrt{(b+a)}}$

$= \frac{\sqrt{b+a} \cdot \sqrt{b+a}}{\sqrt{(b-a)} \cdot \sqrt{(b+a)}} = \sqrt{\frac{b+a}{b-a}}.$



18. $\tan\theta = \frac{4}{3} = \frac{BC}{AB}\cdot$



$\therefore \quad AC^2 = AB^2 + BC^2 = 9 + 16 = 25 \Rightarrow AC = 5.$

$\therefore \quad (\sin\theta + \cos\theta) = \left(\frac{4}{5} + \frac{3}{5}\right) = \frac{7}{5}\cdot$

19. $\frac{(a\sin\theta - b\cos\theta)}{(a\sin\theta + b\cos\theta)} = \frac{(a\tan\theta - b)}{(a\tan\theta + b)}$ [dividing num. and denom. by cos θ]

$$= \frac{\left(a \times \frac{a}{b} - b\right)}{\left(a \times \frac{a}{b} + b\right)} = \frac{(a^2 - b^2)}{(a^2 + b^2)}\cdot$$

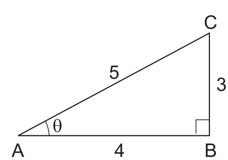
20. $\frac{(4\cos\theta - \sin\theta)}{(2\cos\theta + \sin\theta)} = \frac{(4 - \tan\theta)}{(2 + \tan\theta)}$ [dividing num. and denom. by cos θ]

$$= \frac{\left(4 - \frac{4}{3}\right)}{\left(2 + \frac{4}{3}\right)} = \frac{8}{10} = \frac{4}{5}\cdot$$

21. $\frac{(4\sin\theta - 3\cos\theta)}{(2\sin\theta + 6\cos\theta)} = \frac{(4 - 3\cot\theta)}{(2 + 6\cot\theta)}$ [dividing num. and denom. by sin θ]

$$= \frac{\left(4 - 3 \times \frac{2}{3}\right)}{\left(2 + 6 \times \frac{2}{3}\right)} = \frac{2}{6} = \frac{1}{3}\cdot$$

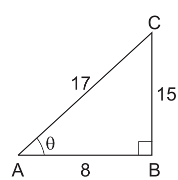
22. $\cot\theta = \frac{4}{3} = \frac{AB}{BC}\cdot$



$\therefore \quad AC^2 = AB^2 + BC^2 = (4)^2 + (3)^2 = 16 + 9 = 25 \Rightarrow AC = 5.$

$\therefore \quad \sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5}$ and $\tan\theta = \frac{3}{4}\cdot$

23. $\sec\theta = \frac{17}{8} \Rightarrow \cos\theta = \frac{8}{17} = \frac{AB}{AC}\cdot$



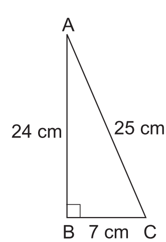
$\therefore \quad BC^2 = AC^2 - AB^2 = (17)^2 - (8)^2 = (289 - 64) = 225$

$\Rightarrow \quad BC = \sqrt{225} = 15.$

$\therefore \quad \sin\theta = \frac{15}{17}, \cos\theta = \frac{8}{17}$ and $\tan\theta = \frac{15}{8}\cdot$

Then, LHS = RHS = $\frac{33}{611}\cdot$

25. Given, $\triangle ABC$ in which $\angle B = 90^\circ$, $BC = 7$ cm and $AB = 24$ cm.



$\therefore \quad AC^2 = (24)^2 + (7)^2 = 576 + 49 = 625 \Rightarrow AC = 25$ cm.

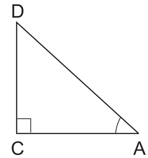
(i) $\sin A = \frac{BC}{AC} = \frac{7}{25}\cdot$ (ii) $\cos A = \frac{AB}{AC} = \frac{24}{25}\cdot$ (iii) $\sin C = \frac{AB}{AC} = \frac{24}{25}\cdot$

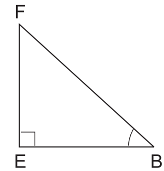
(iv) $\cos C = \frac{BC}{AC} = \frac{7}{25}\cdot$

28. $\cos^2\alpha = (1 - \sin^2\alpha) = \left(1 - \frac{1}{4}\right) = \frac{3}{4} \Rightarrow \cos\alpha = \frac{\sqrt{3}}{2}\cdot$

$\therefore \quad (3\cos\alpha - 4\cos^3\alpha) = \left(3 \times \frac{\sqrt{3}}{2} - 4 \times \frac{3\sqrt{3}}{8}\right) = \left(\frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{2}\right) = 0.$

30. Let $\triangle ACD$ and $\triangle BEF$ be two right triangles such that $\sin A = \sin B$.





Then, $\sin A = \sin B$

$\Rightarrow \quad \dfrac{CD}{AD} = \dfrac{EF}{BF}$

$\Rightarrow \quad \dfrac{CD}{EF} = \dfrac{AD}{BF} = k$ (say)

$\Rightarrow \quad CD = k(EF)$ and $AD = k(BF)$. … (i)

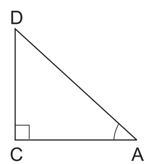
$\therefore \quad \dfrac{AC}{BE} = \dfrac{\sqrt{AD^2 - CD^2}}{\sqrt{BF^2 - EF^2}}$ [using Pythagoras' theorem]

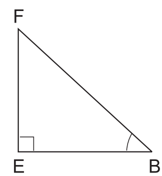
$= \dfrac{k\sqrt{BF^2 - EF^2}}{\sqrt{BF^2 - EF^2}} = k$ [using (i)].

$\therefore \quad \dfrac{CD}{EF} = \dfrac{AD}{BF} = \dfrac{AC}{BE}$.

$\Rightarrow \quad \triangle ACD \sim \triangle BEF$ and hence $\angle A = \angle B$.

31. Let $\triangle ACD$ and $\triangle BEF$ be two right triangles such that $\tan A = \tan B$.





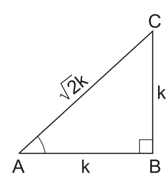
Then, $\tan A = \tan B$

$\Rightarrow \quad \dfrac{CD}{AC} = \dfrac{EF}{BE}$

$\Rightarrow \quad \dfrac{CD}{EF} = \dfrac{AC}{BE}$ and $\angle C = \angle E = 90°$.

$\therefore \quad \triangle ACD \sim \triangle BEF$ and hence, $\angle A = \angle B$.

32. $\tan A = 1 \Rightarrow \dfrac{BC}{AB} = 1 \Rightarrow BC = AB = k$ (say).



$\therefore \quad AC^2 = (AB^2 + BC^2) = (k^2 + k^2) = 2k^2 \Rightarrow AC = \sqrt{2}k$.

$\therefore \quad \sin A = \dfrac{BC}{AC} = \dfrac{k}{\sqrt{2}k} = \dfrac{1}{\sqrt{2}}$

and $\cos A = \dfrac{AB}{AC} = \dfrac{k}{\sqrt{2}k} = \dfrac{1}{\sqrt{2}}$.

$\therefore \quad 2\sin A \cos A = \left(2 \times \dfrac{1}{\sqrt{2}} \times \dfrac{1}{\sqrt{2}}\right) = 1$.

33. $PQ^2 = (PR^2 - QR^2) = (x+2)^2 - x^2 = 4(x+1)$

$\therefore \quad PQ = 2\sqrt{x+1}$.

(i) $\cot \phi = \dfrac{QR}{PQ} = \dfrac{x}{2\sqrt{x+1}} \Rightarrow (\sqrt{x+1})\cot \phi = \dfrac{x}{2}$.

(ii) $\tan \theta = \dfrac{QR}{PQ} = \dfrac{x}{2\sqrt{x+1}} = \dfrac{x^2}{2\sqrt{x^3+x^2}} \Rightarrow (\sqrt{x^3+x^2})\tan \theta = \dfrac{x^2}{2}$.

(iii) $\cos \theta = \dfrac{PQ}{PR} = \dfrac{2\sqrt{x+1}}{(x+2)}$.

34. Adding, we get, $\operatorname{cosec} A = \dfrac{(x+y)}{2} \Rightarrow \sin A = \dfrac{2}{(x+y)}$.

Subtracting, we get, $\cos A = \dfrac{x-y}{2}$.

$\therefore \quad \sin^2 A + \cos^2 A = 1 \Rightarrow \left(\dfrac{2}{x+y}\right)^2 + \left(\dfrac{x-y}{2}\right)^2 - 1 = 0$.

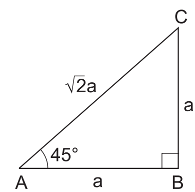
✿

11 T-Ratios of Some Particular Angles

TRIGONOMETRIC RATIOS OF 45°, 60° AND 30° (GEOMETRICALLY)

TRIGONOMETRIC RATIOS OF 45°

Let $\triangle ABC$ be a right-angled triangle in which $\angle B = 90°$ and $\angle A = 45°$. Then, clearly, $\angle C = 45°$.



$\angle A = \angle C \Rightarrow AB = BC$.

Let $AB = BC = a$ units. Then,

$$AC = \sqrt{AB^2 + BC^2} = \sqrt{a^2 + a^2} = \sqrt{2a^2} = \sqrt{2}\,a \text{ units.}$$

Base $AB = a$; perpendicular $BC = a$ and hypotenuse $AC = \sqrt{2}\,a$.

$$\therefore \quad \sin 45° = \frac{BC}{AC} = \frac{a}{\sqrt{2}\,a} = \frac{1}{\sqrt{2}}; \qquad \cos 45° = \frac{AB}{AC} = \frac{a}{\sqrt{2}\,a} = \frac{1}{\sqrt{2}};$$

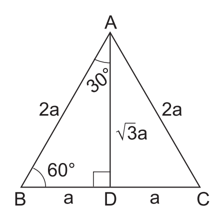
$$\tan 45° = \frac{BC}{AB} = \frac{a}{a} = 1; \qquad \operatorname{cosec} 45° = \frac{1}{\sin 45°} = \sqrt{2};$$

$$\sec 45° = \frac{1}{\cos 45°} = \sqrt{2}; \qquad \cot 45° = \frac{1}{\tan 45°} = 1.$$

TRIGONOMETRIC RATIOS OF 60° AND 30°

Consider an equilateral $\triangle ABC$ with each side equal to $2a$.

Then, each angle of $\triangle ABC$ is 60°.



From A, draw $AD \perp BC$.

Then, clearly, $BD = DC = a$.

Also, $\angle ADB = 90°$.

$\therefore \quad \angle BAD = 30°$.

From right-angled $\triangle ADB$, we have

$$AD = \sqrt{AB^2 - BD^2} = \sqrt{(2a)^2 - a^2} = \sqrt{4a^2 - a^2} = \sqrt{3a^2} = \sqrt{3}\,a.$$

T-RATIOS OF 60°

In right-angled $\triangle ADB$, we have

base $BD = a$, perpendicular $AD = \sqrt{3}\,a$ and hypotenuse $AB = 2a$.

$\therefore \quad \sin 60^\circ = \frac{AD}{AB} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$; $\quad \cos 60^\circ = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$;

$\tan 60^\circ = \frac{AD}{BD} = \frac{\sqrt{3}a}{a} = \sqrt{3}$; $\quad \operatorname{cosec} 60^\circ = \frac{1}{\sin 60^\circ} = \frac{2}{\sqrt{3}}$;

$\sec 60^\circ = \frac{1}{\cos 60^\circ} = 2$; $\quad \cot 60^\circ = \frac{1}{\tan 60^\circ} = \frac{1}{\sqrt{3}}$.

T-RATIOS OF 30°

In right-angled $\triangle ADB$, we have

base $AD = \sqrt{3}a$, perpendicular $BD = a$ and hypotenuse $AB = 2a$.

$\therefore \quad \sin 30^\circ = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$; $\quad \cos 30^\circ = \frac{AD}{AB} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$;

$\tan 30^\circ = \frac{BD}{AD} = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}}$; $\quad \operatorname{cosec} 30^\circ = \frac{1}{\sin 30^\circ} = 2$;

$\sec 30^\circ = \frac{1}{\cos 30^\circ} = \frac{2}{\sqrt{3}}$; $\quad \cot 30^\circ = \frac{1}{\tan 30^\circ} = \sqrt{3}$.

AXIOMS FOR T-RATIOS OF 0°

We define:

(i) $\sin 0^\circ = 0$ (ii) $\cos 0^\circ = 1$ (iii) $\tan 0^\circ = 0$ (iv) $\sec 0^\circ = 1$

NOTE cosec 0° and cot 0° are not defined.

AXIOMS FOR T-RATIOS OF 90°

We define:

(i) $\sin 90^\circ = 1$ (ii) $\cos 90^\circ = 0$ (iii) $\operatorname{cosec} 90^\circ = 1$ (iv) $\cot 90^\circ = 0$

NOTE tan 90° and sec 90° are not defined.

TABLE FOR T-RATIOS OF 0°, 30°, 45°, 60°, 90°

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\operatorname{cosec}\theta$	$\sec\theta$	$\cot\theta$
0°	0	1	0	not defined	1	not defined
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$
90°	1	0	not defined	1	not defined	0

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:*

(i) $\sin 60^\circ \cdot \cos 30^\circ - \cos 60^\circ \cdot \sin 30^\circ$

(ii) $\tan 30^\circ \cdot \operatorname{cosec} 60^\circ + \tan 60^\circ \cdot \sec 30^\circ$

SOLUTION On substituting the values of various T-ratios, we get

(i) $\sin 60^\circ \cdot \cos 30^\circ - \cos 60^\circ \cdot \sin 30^\circ$

$$= \left(\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} - \frac{1}{2} \times \frac{1}{2}\right) = \left(\frac{3}{4} - \frac{1}{4}\right) = \frac{2}{4} = \frac{1}{2}.$$

(ii) $\tan 30^\circ \cdot \operatorname{cosec} 60^\circ + \tan 60^\circ \cdot \sec 30^\circ$

$$= \left(\frac{1}{\sqrt{3}} \times \frac{2}{\sqrt{3}} + \sqrt{3} \times \frac{2}{\sqrt{3}}\right) = \left(\frac{2}{3} + 2\right) = 2\frac{2}{3}.$$

EXAMPLE 2 *Evaluate:*

(i) $\sin^2 30^\circ \cdot \cos^2 45^\circ + 4\tan^2 30^\circ + \frac{1}{2}\sin^2 90^\circ + \frac{1}{8}\cot^2 60^\circ$

(ii) $\dfrac{\tan^2 60^\circ + 4\sin^2 45^\circ + 3\sec^2 30^\circ + 5\cos^2 90^\circ}{\operatorname{cosec} 30^\circ + \sec 60^\circ - \cot^2 30^\circ}$

SOLUTION On substituting the values of various T-ratios, we get

(i) $\sin^2 30^\circ \cdot \cos^2 45^\circ + 4\tan^2 30^\circ + \frac{1}{2}\sin^2 90^\circ + \frac{1}{8}\cot^2 60^\circ$

$$= \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{\sqrt{2}}\right)^2 + 4 \times \left(\frac{1}{\sqrt{3}}\right)^2 + \frac{1}{2} \times (1)^2 + \frac{1}{8} \times \left(\frac{1}{\sqrt{3}}\right)^2$$

$$= \left(\frac{1}{4} \times \frac{1}{2} + 4 \times \frac{1}{3} + \frac{1}{2} \times 1 + \frac{1}{8} \times \frac{1}{3}\right)$$

$$= \left(\frac{1}{8} + \frac{4}{3} + \frac{1}{2} + \frac{1}{24}\right) = \frac{48}{24} = 2.$$

(ii) $\dfrac{\tan^2 60^\circ + 4\sin^2 45^\circ + 3\sec^2 30^\circ + 5\cos^2 90^\circ}{\operatorname{cosec} 30^\circ + \sec 60^\circ - \cot^2 30^\circ}$

$$= \frac{(\sqrt{3})^2 + 4 \times \left(\frac{1}{\sqrt{2}}\right)^2 + 3 \times \left(\frac{2}{\sqrt{3}}\right)^2 + 5 \times 0^2}{2 + 2 - (\sqrt{3})^2}$$

$$= \left(\frac{3 + 4 \times \frac{1}{2} + 3 \times \frac{4}{3} + 5 \times 0}{4 - 3}\right) = \left(\frac{3 + 2 + 4 + 0}{1}\right) = 9.$$

EXAMPLE 3 *Show that*

(i) $\cos 60^\circ \cdot \cos 30^\circ - \sin 60^\circ \cdot \sin 30^\circ = \cos 90^\circ$

(ii) $\cos 60^\circ = 1 - 2\sin^2 30^\circ = 2\cos^2 30^\circ - 1$

(iii) $\dfrac{\tan 60^\circ - \tan 30^\circ}{1 + \tan 60^\circ \cdot \tan 30^\circ} = \tan 30^\circ$

SOLUTION We have

(i) $\cos 60^\circ \cos 30^\circ - \sin 60^\circ \sin 30^\circ$

$= \left(\frac{1}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \times \frac{1}{2}\right) = \left(\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right) = 0.$

Also, $\cos 90^\circ = 0$.

$\therefore \quad \cos 60^\circ \cos 30^\circ - \sin 60^\circ \sin 30^\circ = \cos 90^\circ.$

(ii) $\cos 60^\circ = \frac{1}{2}$;

$1 - 2\sin^2 30^\circ = \left[1 - 2 \times \left(\frac{1}{2}\right)^2\right] = \left(1 - 2 \times \frac{1}{4}\right) = \left(1 - \frac{1}{2}\right) = \frac{1}{2};$

$2\cos^2 30^\circ - 1 = \left[2 \times \left(\frac{\sqrt{3}}{2}\right)^2 - 1\right] = \left(2 \times \frac{3}{4} - 1\right) = \left(\frac{3}{2} - 1\right) = \frac{1}{2}.$

$\therefore \quad \cos 60^\circ = 1 - 2\sin^2 30^\circ = 2\cos^2 30^\circ - 1.$

(iii) $\frac{\tan 60^\circ - \tan 30^\circ}{1 + \tan 60^\circ \tan 30^\circ} = \frac{\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)}{\left(1 + \sqrt{3} \times \frac{1}{\sqrt{3}}\right)} = \frac{(3-1)}{\sqrt{3} \times 2} = \frac{1}{\sqrt{3}}.$

Also, $\tan 30^\circ = \frac{1}{\sqrt{3}}$.

$\therefore \quad \frac{\tan 60^\circ - \tan 30^\circ}{1 + \tan 60^\circ \tan 30^\circ} = \tan 30^\circ.$

EXAMPLE 4 *Taking* $\theta = 30^\circ$, *verify each of the following:*

(i) $\sin 2\theta = 2\sin\theta\cos\theta$

(ii) $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

(iii) $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$

SOLUTION When $\theta = 30^\circ$, we have $2\theta = 60^\circ$.

(i) $\sin 2\theta = \sin 60^\circ = \frac{\sqrt{3}}{2}$.

$2\sin\theta\cos\theta = 2\sin 30^\circ \cos 30^\circ = \left(2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}.$

$\therefore \quad \sin 2\theta = 2\sin\theta\cos\theta.$

(ii) $\cos 2\theta = \cos 60^\circ = \frac{1}{2}$.

$2\cos^2\theta - 1 = 2 \times \cos^2 30^\circ - 1$

$= \left[2 \times \left(\frac{\sqrt{3}}{2}\right)^2 - 1\right] = \left(2 \times \frac{3}{4} - 1\right) = \left(\frac{3}{2} - 1\right) = \frac{1}{2}.$

$$1 - 2\sin^2\theta = 1 - 2\sin^2 30^\circ$$
$$= \left[1 - 2 \times \left(\frac{1}{2}\right)^2\right] = \left(1 - 2 \times \frac{1}{4}\right) = \left(1 - \frac{1}{2}\right) = \frac{1}{2}.$$

$\therefore \quad \cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta.$

(iii) $\tan 2\theta = \tan 60^\circ = \sqrt{3}.$

$$\frac{2\tan\theta}{1 - \tan^2\theta} = \frac{2\tan 30^\circ}{1 - \tan^2 30^\circ} = \frac{2 \times \frac{1}{\sqrt{3}}}{1 - \left(\frac{1}{\sqrt{3}}\right)^2} = \frac{\left(\frac{2}{\sqrt{3}}\right)}{\left(1 - \frac{1}{3}\right)}$$
$$= \left(\frac{2}{\sqrt{3}} \times \frac{3}{2}\right) = \sqrt{3}.$$

$\therefore \quad \tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}.$

EXAMPLE 5 *Taking* $\theta = 30^\circ$, *verify that:*

(i) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ *(ii)* $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

SOLUTION When $\theta = 30^\circ$, we have $3\theta = 90^\circ$.

(i) $\sin 3\theta = \sin 90^\circ = 1.$

And, $3\sin\theta - 4\sin^3\theta = 3\sin 30^\circ - 4\sin^3 30^\circ$
$$= \left[3 \times \frac{1}{2} - 4 \times \left(\frac{1}{2}\right)^3\right] = \left(\frac{3}{2} - 4 \times \frac{1}{8}\right)$$
$$= \left(\frac{3}{2} - \frac{1}{2}\right) = 1.$$

$\therefore \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta.$

(ii) $\cos 3\theta = \cos 90^\circ = 0.$

And, $4\cos^3\theta - 3\cos\theta = 4\cos^3 30^\circ - 3\cos 30^\circ$
$$= \left[4 \times \left(\frac{\sqrt{3}}{2}\right)^3 - 3 \times \frac{\sqrt{3}}{2}\right]$$
$$= \left(4 \times \frac{3\sqrt{3}}{8} - \frac{3\sqrt{3}}{2}\right)$$
$$= \left(\frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{2}\right) = 0.$$

$\therefore \quad \cos 3\theta = 4\cos^3\theta - 3\cos\theta.$

EXAMPLE 6 *Taking* $A = 60^\circ$ *and* $B = 30^\circ$, *verify that:*

$\sin(A - B) = \sin A\cos B - \cos A\sin B.$

SOLUTION $A = 60^\circ$ and $B = 30^\circ \Rightarrow A - B = (60^\circ - 30^\circ) = 30^\circ.$

$\therefore \quad \sin(A - B) = \sin 30^\circ = \dfrac{1}{2}.$

And, $\sin A \cdot \cos B - \cos A \cdot \sin B$

$$= \sin 60^\circ \cdot \cos 30^\circ - \cos 60^\circ \cdot \sin 30^\circ$$

$$= \left(\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} - \frac{1}{2} \times \frac{1}{2}\right)$$

$$= \left(\frac{3}{4} - \frac{1}{4}\right) = \frac{2}{4} = \frac{1}{2}.$$

$\therefore \quad \sin(A - B) = \sin A \cos B - \cos A \sin B.$

EXAMPLE 7 *If* $\sin(A + B) = 1$ *and* $\cos(A - B) = \frac{\sqrt{3}}{2}$, *find A and B.*

SOLUTION $\sin(A + B) = 1 \Rightarrow \sin(A + B) = \sin 90^\circ$ [$\because \sin 90^\circ = 1$]

$\Rightarrow A + B = 90^\circ.$ ... (i)

$\cos(A - B) = \frac{\sqrt{3}}{2} \Rightarrow \cos(A - B) = \cos 30^\circ$

$\Rightarrow A - B = 30^\circ.$ ... (ii)

Solving (i) and (ii), we get $A = 60^\circ$ and $B = 30^\circ$.

Hence, $A = 60^\circ$ and $B = 30^\circ$.

EXAMPLE 8 *If* $\tan(A + B) = \sqrt{3}$ *and* $\tan(A - B) = \frac{1}{\sqrt{3}}$, $0^\circ < A + B < 90^\circ$ *and* $A > B$, *find the values of A and B.* [CBSE 2008C]

SOLUTION $\tan(A + B) = \sqrt{3} \Rightarrow \tan(A + B) = \tan 60^\circ$ [$\because \tan 60^\circ = \sqrt{3}$]

$\Rightarrow A + B = 60^\circ.$ ... (i)

$\tan(A - B) = \frac{1}{\sqrt{3}} \Rightarrow \tan(A - B) = \tan 30^\circ$ $\left[\because \tan 30^\circ = \frac{1}{\sqrt{3}}\right]$

$\Rightarrow A - B = 30^\circ.$ ... (ii)

Solving (i) and (ii), we get $A = 45^\circ$ and $B = 15^\circ$.

Hence, $A = 45^\circ$ and $B = 15^\circ$.

EXAMPLE 9 *If* $\tan(A + B) = \sqrt{3}$ *and* $\tan(A - B) = 1$, $0^\circ < (A + B) < 90^\circ$ *and* $A > B$ *then find A and B.* [CBSE 2009C]

SOLUTION $\tan(A + B) = \sqrt{3} \Rightarrow \tan(A + B) = \tan 60^\circ$ [$\because \sqrt{3} = \tan 60^\circ$]

$\Rightarrow A + B = 60^\circ.$ ... (i)

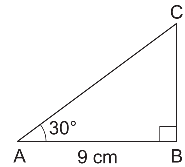
$\tan(A - B) = 1 \Rightarrow \tan(A - B) = \tan 45^\circ$ [$\because 1 = \tan 45^\circ$]

$\Rightarrow A - B = 45^\circ.$ ... (ii)

Solving (i) and (ii), we get $A = (52.5)^\circ$ and $B = (7.5)^\circ$.

Hence, $A = (52.5)^\circ$ and $B = (7.5)^\circ$.

EXAMPLE 10 *In the adjoining figure, $\triangle ABC$ is right-angled at B. If $\angle A = 30^\circ$ and $AB = 9$ cm, find (i) BC and (ii) AC.*



SOLUTION From right-angled $\triangle ABC$, we have

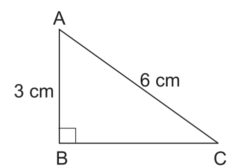
(i) $\frac{BC}{AB} = \tan 30^\circ \Rightarrow \frac{BC}{9\text{ cm}} = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad BC = \left(\frac{9}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}\right)\text{cm} = \frac{9\sqrt{3}}{3}\text{ cm} = 3\sqrt{3}\text{ cm}.$

(ii) $\frac{AC}{BC} = \text{cosec } 30^\circ = 2 \Rightarrow \frac{AC}{3\sqrt{3}\text{ cm}} = 2 \Rightarrow AC = 6\sqrt{3}\text{ cm}.$

Hence, $BC = 3\sqrt{3}$ cm and $AC = 6\sqrt{3}$ cm.

EXAMPLE 11 *In right $\triangle ABC$, $\angle B = 90^\circ$, $AB = 3$ cm and $AC = 6$ cm. Find $\angle C$ and $\angle A$.*



SOLUTION $\frac{AB}{AC} = \sin C \Rightarrow \sin C = \frac{3}{6} = \frac{1}{2}$

$\Rightarrow \quad \sin C = \sin 30^\circ \Rightarrow \angle C = 30^\circ$

$\therefore \quad \angle A = 180^\circ - (90^\circ + 30^\circ) = (180^\circ - 120^\circ) = 60^\circ.$

EXAMPLE 12 *If $6x = \sec\theta$ and $\frac{6}{x} = \tan\theta$, find the value of $9\left(x^2 - \frac{1}{x^2}\right)$.* [CBSE 2010]

SOLUTION We have

$6x = \sec\theta \quad \ldots \text{(i)} \qquad \text{and} \qquad \frac{6}{x} = \tan\theta \quad \ldots \text{(ii)}$

Adding (i) and (ii), we get

$$6\left(x + \frac{1}{x}\right) = (\sec\theta + \tan\theta). \quad \ldots \text{(iii)}$$

Subtracting (ii) from (i), we get

$$6\left(x - \frac{1}{x}\right) = (\sec\theta - \tan\theta). \quad \ldots \text{(iv)}$$

Multiplying the corresponding sides of (iii) and (iv), we get

$$36\left(x^2 - \frac{1}{x^2}\right) = (\sec^2\theta - \tan^2\theta) = 1 \Rightarrow 9\left(x^2 - \frac{1}{x^2}\right) = \frac{1}{4}.$$

Hence, $9\left(x^2 - \frac{1}{x^2}\right) = \frac{1}{4}$.

EXERCISE 11

Evaluate each of the following:

1. $\sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ$
2. $\cos 60^\circ \cos 30^\circ - \sin 60^\circ \sin 30^\circ$
3. $\cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$
4. $\frac{\sin 30^\circ}{\cos 45^\circ} + \frac{\cot 45^\circ}{\sec 60^\circ} - \frac{\sin 60^\circ}{\tan 45^\circ} + \frac{\cos 30^\circ}{\sin 90^\circ}$

5. $\dfrac{5\cos^2 60^\circ + 4\sec^2 30^\circ - \tan^2 45^\circ}{\sin^2 30^\circ + \cos^2 30^\circ}$

6. $2\cos^2 60^\circ + 3\sin^2 45^\circ - 3\sin^2 30^\circ + 2\cos^2 90^\circ$

7. $\cot^2 30^\circ - 2\cos^2 30^\circ - \dfrac{3}{4}\sec^2 45^\circ + \dfrac{1}{4}\operatorname{cosec}^2 30^\circ$

8. $(\sin^2 30^\circ + 4\cot^2 45^\circ - \sec^2 60^\circ)(\operatorname{cosec}^2 45^\circ \sec^2 30^\circ)$

9. $\dfrac{4}{\cot^2 30^\circ} + \dfrac{1}{\sin^2 30^\circ} - 2\cos^2 45^\circ - \sin^2 0^\circ$

10. Show that:

(i) $\dfrac{1-\sin 60^\circ}{\cos 60^\circ} = \dfrac{\tan 60^\circ - 1}{\tan 60^\circ + 1}$ (ii) $\dfrac{\cos 30^\circ + \sin 60^\circ}{1 + \sin 30^\circ + \cos 60^\circ} = \cos 30^\circ$

11. Verify each of the following:

(i) $\sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ = \sin 30^\circ$

(ii) $\cos 60^\circ \cos 30^\circ + \sin 60^\circ \sin 30^\circ = \cos 30^\circ$

(iii) $2\sin 30^\circ \cos 30^\circ = \sin 60^\circ$

(iv) $2\sin 45^\circ \cos 45^\circ = \sin 90^\circ$

12. If $A = 45^\circ$, verify that:

(i) $\sin 2A = 2\sin A\cos A$ (ii) $\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

13. If $A = 30^\circ$, verify that:

(i) $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$ (ii) $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$ (iii) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

14. If $A = 60^\circ$ and $B = 30^\circ$, verify that:

(i) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

(ii) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

15. If $A = 60^\circ$ and $B = 30^\circ$, verify that:

(i) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

(ii) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

(iii) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

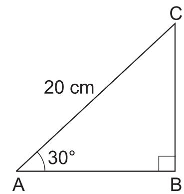
16. If A and B are acute angles such that $\tan A = \dfrac{1}{3}$, $\tan B = \dfrac{1}{2}$ and $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$, show that $A + B = 45^\circ$.

17. Using the formula, $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$, find the value of $\tan 60^\circ$, it being given that $\tan 30^\circ = \dfrac{1}{\sqrt{3}}$.

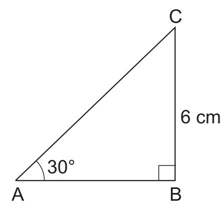
18. Using the formula, $\cos A = \sqrt{\dfrac{1+\cos 2A}{2}}$, find the value of $\cos 30^\circ$, it being given that $\cos 60^\circ = \dfrac{1}{2}$.

19. Using the formula, $\sin A = \sqrt{\frac{1-\cos 2A}{2}}$, find the value of $\sin 30°$, it being given that $\cos 60° = \frac{1}{2}$.

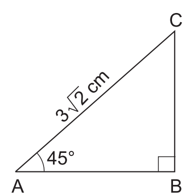
20. In the adjoining figure, $\triangle ABC$ is a right-angled triangle in which $\angle B = 90°$, $\angle A = 30°$ and $AC = 20$ cm. Find (i) BC, (ii) AB.



21. In the adjoining figure, $\triangle ABC$ is right-angled at B and $\angle A = 30°$. If $BC = 6$ cm, find (i) AB, (ii) AC.



22. In the adjoining figure, $\triangle ABC$ is right-angled at B and $\angle A = 45°$. If $AC = 3\sqrt{2}$ cm, find (i) BC, (ii) AB.



23. If $\sin(A+B) = 1$ and $\cos(A-B) = 1$, $0° \leq (A+B) \leq 90°$ and $A > B$ then find A and B.

24. If $\sin(A-B) = \frac{1}{2}$ and $\cos(A+B) = \frac{1}{2}$, $0° < (A+B) < 90°$ and $A > B$ then find A and B.

25. If $\tan(A-B) = \frac{1}{\sqrt{3}}$ and $\tan(A+B) = \sqrt{3}$, $0° < (A+B) < 90°$ and $A > B$ then find A and B.

26. If $3x = \operatorname{cosec}\theta$ and $\frac{3}{x} = \cot\theta$, find the value of $3\left(x^2 - \frac{1}{x^2}\right)$. [CBSE 2010]

27. If $\sin(A+B) = \sin A\cos B + \cos A\sin B$ and $\cos(A-B) = \cos A\cos B + \sin A\sin B$, find the values of (i) $\sin 75°$ and (ii) $\cos 15°$.

ANSWERS (EXERCISE 11)

1. 1 **2.** 0 **3.** $\frac{(\sqrt{3}+1)}{2\sqrt{2}}$ **4.** $\frac{(\sqrt{2}+1)}{2}$ **5.** $\frac{67}{12}$ **6.** $\frac{5}{4}$

7. 1 **8.** $\frac{2}{3}$ **9.** $\frac{13}{3}$ **17.** $\sqrt{3}$ **18.** $\frac{\sqrt{3}}{2}$ **19.** $\frac{1}{2}$

20. (i) $BC = 10$ cm (ii) $AB = 10\sqrt{3}$ cm

21. (i) $AB = 6\sqrt{3}$ cm (ii) $AC = 12$ cm

22. (i) $BC = 3$ cm (ii) $AB = 3$ cm **23.** $\angle A = \angle B = 45^\circ$

24. $\angle A = 45^\circ, \angle B = 15^\circ$ **25.** $\angle A = 45^\circ, \angle B = 15^\circ$ **26.** $\frac{1}{3}$

27. (i) $\dfrac{(\sqrt{3}+1)}{2\sqrt{2}}$ (ii) $\dfrac{(\sqrt{3}+1)}{2\sqrt{2}}$

HINTS TO SOME SELECTED QUESTIONS

20. Let $BC = x$ cm. Then, $\frac{x}{20} = \sin 30^\circ = \frac{1}{2} \Rightarrow x = 10.$

$AB^2 = (AC^2 - BC^2) = \{(20)^2 - (10)^2\}\text{ cm}^2 = 300\text{ cm}^2 \Rightarrow AB = \sqrt{300}\text{ cm} = 10\sqrt{3}$ cm.

$\therefore$ $BC = 10$ cm and $AB = 10\sqrt{3}$ cm.

21. Let $AB = x$ cm. Then, $\frac{x}{6} = \cot 30^\circ = \sqrt{3} \Rightarrow x = 6\sqrt{3}.$

$AC^2 = (AB^2 + BC^2) = \{(6\sqrt{3})^2 + (6)^2\}\text{ cm}^2 = (108+36)\text{ cm}^2 = 144\text{ cm}^2 \Rightarrow AC = 12$ cm.

22. Let $BC = x$ cm. Then, $\frac{x}{3\sqrt{2}} = \sin 45^\circ = \frac{1}{\sqrt{2}} \Rightarrow x = 3.$

$\frac{AB}{BC} = \tan 45^\circ = 1 \Rightarrow \frac{AB}{3\text{ cm}} = 1 \Rightarrow AB = 3$ cm.

23. $\sin(A+B) = 1 \Rightarrow \sin(A+B) = \sin 90^\circ.$

$\cos(A-B) = 1 \Rightarrow \cos(A-B) = \cos 0^\circ.$

Solve $A + B = 90^\circ$, $A - B = 0^\circ$ for A and B.

24. $\sin(A-B) = \frac{1}{2} \Rightarrow \sin(A-B) = \sin 30^\circ.$

$\cos(A+B) = \frac{1}{2} \Rightarrow \cos(A+B) = \cos 60^\circ.$

Solve $A - B = 30^\circ$ and $A + B = 60^\circ$ for A and B.

25. $\tan(A-B) = \frac{1}{\sqrt{3}} \Rightarrow \tan(A-B) = \tan 30^\circ.$

$\tan(A+B) = \sqrt{3} \Rightarrow \tan(A+B) = \tan 60^\circ.$

Solve $A - B = 30^\circ$ and $A + B = 60^\circ$ for A and B.

26. Adding, we get $3\left(x + \frac{1}{x}\right) = (\text{cosec}\,\theta + \cot\theta).$

Subtracting, we get $3\left(x - \frac{1}{x}\right) = (\text{cosec}\,\theta - \cot\theta).$

$\therefore$ $9\left(x^2 - \frac{1}{x^2}\right) = (\text{cosec}^2\theta - \cot^2\theta) = 1 \Rightarrow 3\left(x^2 - \frac{1}{x^2}\right) = \frac{1}{3}.$

27. Take $A = 45^\circ$ and $B = 30^\circ$.

✿

12 Trigonometric Ratios of Complementary Angles

TRIGONOMETRIC RATIOS OF COMPLEMENTARY ANGLES

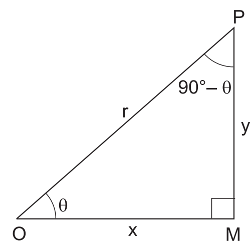
COMPLEMENTARY ANGLES *Two angles are said to be complementary if their sum is* $90°$.

Thus, θ and $(90° - \theta)$ are complementary angles.

TRIGONOMETRIC RATIOS OF COMPLEMENTARY ANGLES

Consider a $\triangle OMP$, right-angled at M in which

$\angle MOP = \theta$ and therefore, $\angle OPM = (90° - \theta)$.



Let $OM = x$, $MP = y$ and $OP = r$. Then,

$$\sin\theta = \frac{y}{r},\ \cos\theta = \frac{x}{r},\ \tan\theta = \frac{y}{x},$$

$$\operatorname{cosec}\theta = \frac{r}{y},\ \sec\theta = \frac{r}{x},\ \cot\theta = \frac{x}{y}.$$

For considering the trigonometric ratios of $(90° - \theta)$, we have

base = MP, perpendicular = OM and hypotenuse = OP.

$$\therefore \quad \sin(90° - \theta) = \frac{OM}{OP} = \frac{x}{r} = \cos\theta$$

$$\cos(90° - \theta) = \frac{PM}{OP} = \frac{y}{r} = \sin\theta$$

$$\tan(90° - \theta) = \frac{OM}{PM} = \frac{x}{y} = \cot\theta$$

$$\therefore \quad \operatorname{cosec}(90° - \theta) = \frac{1}{\sin(90° - \theta)} = \frac{1}{\cos\theta} = \sec\theta$$

$$\sec(90° - \theta) = \frac{1}{\cos(90° - \theta)} = \frac{1}{\sin\theta} = \operatorname{cosec}\theta$$

$$\cot(90° - \theta) = \frac{1}{\tan(90° - \theta)} = \frac{1}{\cot\theta} = \tan\theta$$

SUMMARY

(i) $\sin(90° - \theta) = \cos\theta$

(ii) $\cos(90° - \theta) = \sin\theta$

(iii) $\tan(90° - \theta) = \cot\theta$

(iv) $\cot(90° - \theta) = \tan\theta$

(v) $\sec(90° - \theta) = \operatorname{cosec}\theta$

(vi) $\operatorname{cosec}(90° - \theta) = \sec\theta$

SOLVED EXAMPLES

EXAMPLE 1 *Without using trigonometric tables, evaluate:*

(i) $\frac{\cos 53^\circ}{\sin 37^\circ}$ *(ii)* $\frac{\tan 68^\circ}{\cot 22^\circ}$ *(iii)* $\frac{\sec 49^\circ}{\operatorname{cosec} 41^\circ}$ *(iv)* $\frac{\sin 30^\circ 17'}{\cos 59^\circ 43'}$

SOLUTION (i) $\frac{\cos 53^\circ}{\sin 37^\circ} = \frac{\cos(90^\circ - 37^\circ)}{\sin 37^\circ} = \frac{\sin 37^\circ}{\sin 37^\circ} = 1$

$[\because \cos(90^\circ - \theta) = \sin\theta]$.

(ii) $\frac{\tan 68^\circ}{\cot 22^\circ} = \frac{\tan(90^\circ - 22^\circ)}{\cot 22^\circ} = \frac{\cot 22^\circ}{\cot 22^\circ} = 1$

$[\because \tan(90^\circ - \theta) = \cot\theta]$.

(iii) $\frac{\sec 49^\circ}{\operatorname{cosec} 41^\circ} = \frac{\sec(90^\circ - 41^\circ)}{\operatorname{cosec} 41^\circ} = \frac{\operatorname{cosec} 41^\circ}{\operatorname{cosec} 41^\circ} = 1$

$[\because \sec(90^\circ - \theta) = \operatorname{cosec}\theta]$.

(iv) $\frac{\sin 30^\circ 17'}{\cos 59^\circ 43'} = \frac{\sin[90^\circ - (59^\circ 43')]}{\cos 59^\circ 43'} = \frac{\cos 59^\circ 43'}{\cos 59^\circ 43'} = 1.$

EXAMPLE 2 *Without using trigonometric tables, evaluate:*

(i) $\cos 48^\circ - \sin 42^\circ$ *(ii)* $\operatorname{cosec} 31^\circ - \sec 59^\circ$

(iii) $\cot 34^\circ - \tan 56^\circ$ *(iv)* $\cos^2 13^\circ - \sin^2 77^\circ$

SOLUTION We have

(i) $\cos 48^\circ - \sin 42^\circ = \cos(90^\circ - 42^\circ) - \sin 42^\circ$

$= \sin 42^\circ - \sin 42^\circ = 0$

$[\because \cos(90^\circ - \theta) = \sin\theta]$.

(ii) $\operatorname{cosec} 31^\circ - \sec 59^\circ = \operatorname{cosec}(90^\circ - 59^\circ) - \sec 59^\circ$

$= \sec 59^\circ - \sec 59^\circ = 0$

$[\because \operatorname{cosec}(90^\circ - \theta) = \sec\theta]$.

(iii) $\cot 34^\circ - \tan 56^\circ = \cot(90^\circ - 56^\circ) - \tan 56^\circ$

$= \tan 56^\circ - \tan 56^\circ = 0$

$[\because \cot(90^\circ - \theta) = \tan\theta]$.

(iv) $\cos^2 13^\circ - \sin^2 77^\circ = \cos^2(90^\circ - 77^\circ) - \sin^2 77^\circ$

$= \sin^2 77^\circ - \sin^2 77^\circ = 0$

$[\because \cos(90^\circ - \theta) = \sin\theta]$.

EXAMPLE 3 *Without using trigonometric tables, prove that:*

(i) $\sin 43^\circ \cos 47^\circ + \cos 43^\circ \sin 47^\circ = 1$

(ii) $\cos 38^\circ \cos 52^\circ - \sin 38^\circ \sin 52^\circ = 0$

(iii) $\sec 50^\circ \sin 40^\circ + \cos 40^\circ \operatorname{cosec} 50^\circ = 2$

(iv) $\sec 70^\circ \sin 20^\circ - \cos 20^\circ \operatorname{cosec} 70^\circ = 0$

SOLUTION We have

(i) $\sin 43^\circ \cos 47^\circ + \cos 43^\circ \sin 47^\circ$

$= \sin (90^\circ - 47^\circ) \cos 47^\circ + \cos (90^\circ - 47^\circ) \sin 47^\circ$

$= \cos^2 47^\circ + \sin^2 47^\circ = 1$

$[\because \sin (90^\circ - \theta) = \cos \theta, \cos (90^\circ - \theta) = \sin \theta].$

(ii) $\cos 38^\circ \cos 52^\circ - \sin 38^\circ \sin 52^\circ$

$= \cos (90^\circ - 52^\circ) \cos 52^\circ - \sin (90^\circ - 52^\circ) \sin 52^\circ$

$= \sin 52^\circ \cos 52^\circ - \sin 52^\circ \cos 52^\circ = 0$

$[\because \cos (90^\circ - \theta) = \sin \theta \text{ and } \sin (90^\circ - \theta) = \cos \theta].$

(iii) $\sec 50^\circ \sin 40^\circ + \cos 40^\circ \operatorname{cosec} 50^\circ$

$= \sec (90^\circ - 40^\circ) \sin 40^\circ + \cos 40^\circ \operatorname{cosec} (90^\circ - 40^\circ)$

$= \operatorname{cosec} 40^\circ \sin 40^\circ + \cos 40^\circ \sec 40^\circ$

$\left[\because \begin{array}{l} \sec (90^\circ - \theta) = \operatorname{cosec} \theta, \\ \operatorname{cosec} (90^\circ - \theta) = \sec \theta \end{array}\right]$

$= \dfrac{\sin 40^\circ}{\sin 40^\circ} + \dfrac{\cos 40^\circ}{\cos 40^\circ} = 1 + 1 = 2.$

(iv) $\sec 70^\circ \sin 20^\circ - \cos 20^\circ \operatorname{cosec} 70^\circ$

$= \sec (90^\circ - 20^\circ) \sin 20^\circ - \cos 20^\circ \operatorname{cosec} (90^\circ - 20^\circ)$

$= \operatorname{cosec} 20^\circ \sin 20^\circ - \cos 20^\circ \sec 20^\circ$

$\left[\because \begin{array}{l} \sec (90^\circ - \theta) = \operatorname{cosec} \theta, \\ \operatorname{cosec} (90^\circ - \theta) = \sec \theta \end{array}\right]$

$= \dfrac{\sin 20^\circ}{\sin 20^\circ} - \dfrac{\cos 20^\circ}{\cos 20^\circ} = 1 - 1 = 0.$

EXAMPLE 4 *Without using trigonometric tables, prove that:*

(i) $\tan 48^\circ \tan 23^\circ \tan 42^\circ \tan 67^\circ = 1$

(ii) $\tan 7^\circ \tan 23^\circ \tan 60^\circ \tan 67^\circ \tan 83^\circ = \sqrt{3}$

(iii) $\cot 12^\circ \cot 38^\circ \cot 52^\circ \cot 60^\circ \cot 78^\circ = \dfrac{1}{\sqrt{3}}$

SOLUTION We have

(i) $\tan 48^\circ \tan 23^\circ \tan 42^\circ \tan 67^\circ$

$= (\tan 48^\circ \tan 42^\circ)(\tan 23^\circ \tan 67^\circ)$

$= \{\tan 48^\circ \cdot \tan (90^\circ - 48^\circ)\}\{\tan 23^\circ \cdot \tan (90^\circ - 23^\circ)\}$

$= (\tan 48^\circ \cot 48^\circ)(\tan 23^\circ \cot 23^\circ)$ [$\because$ $\tan (90^\circ - \theta) = \cot \theta$]

$= 1 \times 1 = 1.$

(ii) $\tan 7^\circ \tan 23^\circ \tan 60^\circ \tan 67^\circ \tan 83^\circ$

$= \tan 7^\circ \cdot \tan 83^\circ \cdot \tan 23^\circ \cdot \tan 67^\circ \cdot \tan 60^\circ$

$= \tan 7^\circ \cdot \tan (90^\circ - 7^\circ) \cdot \tan 23^\circ \cdot \tan (90^\circ - 23^\circ) \cdot \tan 60^\circ$

$= (\tan 7^\circ \cdot \cot 7^\circ) \cdot (\tan 23^\circ \cdot \cot 23^\circ) \cdot \tan 60^\circ$

[$\because$ $\tan (90^\circ - \theta) = \cot \theta$]

$= 1 \times 1 \times \sqrt{3} = \sqrt{3}$ [$\because$ $\tan \theta \cdot \cot \theta = 1$ and $\tan 60^\circ = \sqrt{3}$].

(iii) $\cot 12^\circ \cdot \cot 38^\circ \cdot \cot 52^\circ \cdot \cot 60^\circ \cdot \cot 78^\circ$

$= (\cot 12^\circ \cdot \cot 78^\circ) \cdot (\cot 38^\circ \cdot \cot 52^\circ) \cdot \cot 60^\circ$

$= \{\cot 12^\circ \cdot \cot (90^\circ - 12^\circ)\}\{\cot 38^\circ \cdot \cot (90^\circ - 38^\circ)\} \cdot \cot 60^\circ$

$= (\cot 12^\circ \cdot \tan 12^\circ) \cdot (\cot 38^\circ \cdot \tan 38^\circ) \cdot \cot 60^\circ$

[$\because$ $\cot (90^\circ - \theta) = \tan \theta$]

$= 1 \times 1 \times \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$ $\left[\because \cot \theta \cdot \tan \theta = 1 \text{ and } \cot 60^\circ = \frac{1}{\sqrt{3}}\right]$.

EXAMPLE 5 *Prove that*

$\tan 1^\circ \tan 2^\circ \tan 3^\circ \ldots \tan 89^\circ = 1.$

SOLUTION We have

$\tan 1^\circ \tan 2^\circ \tan 3^\circ \ldots \tan 89^\circ$

$= \tan 1^\circ \tan 2^\circ \ldots \tan 44^\circ \tan 45^\circ \tan 46^\circ \ldots \tan 88^\circ \tan 89^\circ$

$= (\tan 1^\circ \cdot \tan 89^\circ)(\tan 2^\circ \cdot \tan 88^\circ) \ldots (\tan 44^\circ \cdot \tan 46^\circ) \cdot \tan 45^\circ$

$= \{\tan 1^\circ \cdot \tan (90^\circ - 1^\circ)\} \cdot \{\tan 2^\circ \cdot \tan (90^\circ - 2^\circ)\}$

$\ldots \{\tan 44^\circ \cdot \tan (90^\circ - 44^\circ)\} \cdot \tan 45^\circ$

$= (\tan 1^\circ \cdot \cot 1^\circ)(\tan 2^\circ \cdot \cot 2^\circ) \ldots (\tan 44^\circ \cot 44^\circ) \cdot 1$

[$\because$ $\tan (90^\circ - \theta) = \cot \theta$ and $\tan 45^\circ = 1$]

$= 1 \times 1 \times \ldots \times 1 \times 1 = 1.$

EXAMPLE 6 *Evaluate:*

$$\frac{\cos 58^\circ}{\sin 32^\circ} + \frac{\sin 22^\circ}{\cos 68^\circ} - \frac{\cos 38^\circ \operatorname{cosec} 52^\circ}{\tan 18^\circ \tan 35^\circ \tan 60^\circ \tan 72^\circ \tan 55^\circ}.$$

[CBSE 2009]

SOLUTION Given expression

$$= \frac{\cos 58^\circ}{\sin 32^\circ} + \frac{\sin 22^\circ}{\cos 68^\circ} - \frac{\cos 38^\circ \operatorname{cosec} 52^\circ}{\tan 18^\circ \tan 35^\circ \tan 60^\circ \tan 72^\circ \tan 55^\circ}$$

$$= \frac{\cos 58^\circ}{\sin (90^\circ - 58^\circ)} + \frac{\sin 22^\circ}{\cos (90^\circ - 22^\circ)} - \frac{\cos 38^\circ \operatorname{cosec} (90^\circ - 38^\circ)}{(\tan 18^\circ \tan 72^\circ)(\tan 35^\circ \tan 55^\circ) \tan 60^\circ}$$

$$= \frac{\cos 58^\circ}{\cos 58^\circ} + \frac{\sin 22^\circ}{\sin 22^\circ} - \frac{\cos 38^\circ \sec 38^\circ}{\{\tan 18^\circ \tan (90^\circ - 18^\circ)\}\{\tan 35^\circ \tan (90^\circ - 35^\circ)\} \tan 60^\circ}$$

$$= (1 + 1) - \frac{1}{(\tan 18^\circ \cot 18^\circ)(\tan 35^\circ \cot 35^\circ)\sqrt{3}}$$

$$= \left(2 - \frac{1}{\sqrt{3}}\right) = \frac{2\sqrt{3} - 1}{\sqrt{3}} = \frac{2\sqrt{3} - 1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{1}{3}(6 - \sqrt{3}).$$

EXAMPLE 7 *Without using trigonometric tables, evaluate each of the following:*

(i) $\sin^2 65^\circ + \sin^2 25^\circ$ *(ii)* $\cos^2 17^\circ - \sin^2 73^\circ$

(iii) $\operatorname{cosec}^2 67^\circ - \tan^2 23^\circ$ *(iv)* $\sec^2 36^\circ - \cot^2 54^\circ$

SOLUTION We have

(i) $\sin^2 65^\circ + \sin^2 25^\circ = \{\sin (90^\circ - 25^\circ)\}^2 + \sin^2 25^\circ$

$= (\cos 25^\circ)^2 + \sin^2 25^\circ$

$[\because \sin (90^\circ - \theta) = \cos \theta]$

$= (\cos^2 25^\circ + \sin^2 25^\circ) = 1.$

(ii) $\cos^2 17^\circ - \sin^2 73^\circ = [\cos (90^\circ - 73^\circ)]^2 - \sin^2 73^\circ$

$= (\sin 73^\circ)^2 - \sin^2 73^\circ$

$[\because \cos (90^\circ - \theta) = \sin \theta]$

$= (\sin^2 73^\circ - \sin^2 73^\circ) = 0.$

(iii) $\operatorname{cosec}^2 67^\circ - \tan^2 23^\circ = [\operatorname{cosec} (90^\circ - 23^\circ)]^2 - \tan^2 23^\circ$

$= (\sec 23^\circ)^2 - \tan^2 23^\circ$

$[\because \operatorname{cosec} (90^\circ - \theta) = \sec \theta]$

$= (\sec^2 23^\circ - \tan^2 23^\circ) = 1$

$[\because \sec^2 \theta - \tan^2 \theta = 1].$

(iv) $\sec^2 36^\circ - \cot^2 54^\circ = [\sec (90^\circ - 54^\circ)]^2 - \cot^2 54^\circ$

$= (\operatorname{cosec} 54^\circ)^2 - \cot^2 54^\circ$

$[\because \sec (90^\circ - \theta) = \operatorname{cosec} \theta]$

$$= (\text{cosec}^2 54° - \cot^2 54°) = 1$$

$$[\because \ \text{cosec}^2\theta - \cot^2\theta = 1].$$

EXAMPLE 8 *Evaluate* $\dfrac{2\sin^2 63° + 1 + 2\sin^2 27°}{3\cos^2 17° - 2 + 3\cos^2 73°}$. [CBSE 2009C]

SOLUTION Given expression

$$= \frac{2\sin^2 63° + 1 + 2\sin^2 27°}{3\cos^2 17° - 2 + 3\cos^2 73°} = \frac{2(\sin^2 63° + \sin^2 27°) + 1}{3(\cos^2 17° + \cos^2 73°) - 2}$$

$$= \frac{2[\sin^2 63° + \sin^2(90° - 63°)] + 1}{3[\cos^2 17° + \cos^2(90° - 17°)] - 2}$$

$$= \frac{2[\sin^2 63° + \cos^2 63°] + 1}{3[\cos^2 17° + \sin^2 17°) - 2}$$

$$[\because \ \sin(90° - \theta) = \cos\theta \ \text{ and } \ \cos(90° - \theta) = \sin\theta]$$

$$= \frac{(2 \times 1) + 1}{(3 \times 1) - 2} = \frac{2 + 1}{3 - 2} = \frac{3}{1} = 3.$$

EXAMPLE 9 *Without using trigonometric tables, evaluate the following:*

$$\frac{\cos^2 20° + \cos^2 70°}{\sec^2 50° - \cot^2 40°} + 2\text{cosec}^2 58° - 2\cot 58° \tan 32°$$

$$-4\tan 13° \tan 37° \tan 45° \tan 53° \tan 77°.$$ [CBSE 2006C]

SOLUTION Given expression

$$= \frac{\cos^2 20° + \cos^2 70°}{\sec^2 50° - \cot^2 40°} + 2\,\text{cosec}^2 58° - 2\cot 58° \tan 32°$$

$$-4\tan 13° \tan 37° \tan 45° \tan 53° \tan 77°$$

$$= \frac{\cos^2 20° + [\cos(90° - 20°)]^2}{\sec^2 50° - [\cot(90° - 50°)]^2}$$

$$+2[\text{cosec}^2 58° - \cot 58° \tan(90° - 58°)]$$

$$-4(\tan 13° \tan 77°)(\tan 37° \tan 53°)\tan 45°$$

$$= \frac{\cos^2 20° + \sin^2 20°}{\sec^2 50° - \tan^2 50°} + 2[\text{cosec}^2 58° - \cot^2 58°]$$

$$-4\tan 13° \tan(90° - 13°) \cdot \tan 37° \tan(90° - 37°) \cdot \tan 45°$$

$$[\because \ \cos(90° - \theta) = \sin\theta, \cot(90° - \theta) = \tan\theta \text{ and } \tan(90° - \theta) = \cot\theta]$$

$$= \frac{1}{1} + (2 \times 1) - 4(\tan 13° \cot 13°) \cdot (\tan 37° \cot 37°) \cdot 1$$

$$[\because \cos^2\theta + \sin^2\theta = 1, \sec^2\theta - \tan^2\theta = 1, \text{cosec}^2\theta - \cot^2\theta = 1]$$

$$= (3 - 4 \times 1 \times 1 \times 1) = (3 - 4) = -1.$$

EXAMPLE 10 *If* $\sin 3A = \cos(A - 10^\circ)$, *where* $3A$ *is an acute angle then find the value of* A. [CBSE 2009C]

SOLUTION $\sin 3A = \cos(A - 10^\circ)$

$\Rightarrow \cos(90^\circ - 3A) = \cos(A - 10^\circ)$ $[\because \sin\theta = \cos(90^\circ - \theta)]$

$\Rightarrow 90^\circ - 3A = A - 10^\circ \Rightarrow 4A = 100^\circ \Rightarrow A = 25^\circ$.

Hence, $A = 25^\circ$.

EXAMPLE 11 *If* $\tan 2A = \cot(A - 21^\circ)$, *where* $2A$ *is an acute angle then find the value of* A. [CBSE 2009C]

SOLUTION $\tan 2A = \cot(A - 21^\circ)$

$\Rightarrow \cot(90^\circ - 2A) = \cot(A - 21^\circ)$ $[\because \tan 2A = \cot(90^\circ - 2A)]$

$\Rightarrow 90^\circ - 2A = A - 21^\circ \Rightarrow 3A = 111^\circ \Rightarrow A = 37^\circ$.

Hence, $A = 37^\circ$.

EXAMPLE 12 *If* $\sec 5A = \operatorname{cosec}(A - 30^\circ)$, *where* $5A$ *is an acute angle then find the value of* A.

SOLUTION $\sec 5A = \operatorname{cosec}(A - 30^\circ)$

$\Rightarrow \operatorname{cosec}(90^\circ - 5A) = \operatorname{cosec}(A - 30^\circ)$

$[\because \sec\theta = \operatorname{cosec}(90^\circ - \theta)]$

$\Rightarrow 90^\circ - 5A = A - 30^\circ \Rightarrow 6A = 120^\circ \Rightarrow A = 20^\circ$.

Hence, $A = 20^\circ$.

EXAMPLE 13 *If* $\sin(\theta + 34^\circ) = \cos\theta$ *and* $(\theta + 34^\circ)$ *is acute show that* $\theta = 28^\circ$.

SOLUTION $\sin(\theta + 34^\circ) = \cos\theta$

$\Rightarrow \sin(\theta + 34^\circ) = \sin(90^\circ - \theta)$ $[\because \cos\theta = \sin(90^\circ - \theta)]$

$\Rightarrow \theta + 34^\circ = 90^\circ - \theta \Rightarrow 2\theta = (90^\circ - 34^\circ) = 56^\circ \Rightarrow \theta = 28^\circ$.

Hence, $\theta = 28^\circ$.

EXAMPLE 14 *Express each of the following in terms of trigonometric ratios of angles between* 0° *and* 45°.

(i) $\sin 75^\circ + \operatorname{cosec} 75^\circ$ *(ii)* $\tan 65^\circ + \cot 49^\circ$

(iii) $\sec 67^\circ + \operatorname{cosec} 58^\circ$ *(iv)* $\cos 83^\circ - \sec 76^\circ$

SOLUTION (i) $\sin 75^\circ + \operatorname{cosec} 75^\circ = \sin(90^\circ - 15^\circ) + \operatorname{cosec}(90^\circ - 15^\circ)$

$= (\cos 15^\circ + \sec 15^\circ)$.

(ii) $\tan 65^\circ + \cot 49^\circ = \tan(90^\circ - 25^\circ) + \cot(90^\circ - 41^\circ)$

$= (\cot 25^\circ + \tan 41^\circ)$.

(iii) $\sec 67^\circ + \text{cosec } 58^\circ = \sec(90^\circ - 23^\circ) + \text{cosec}(90^\circ - 32^\circ)$

$= (\text{cosec } 23^\circ + \sec 32^\circ)$.

(iv) $\cos 83^\circ - \sec 76^\circ = \cos(90^\circ - 7^\circ) - \sec(90^\circ - 14^\circ)$

$= (\sin 7^\circ - \text{cosec } 14^\circ)$.

EXAMPLE 15 *If A and B are acute angles such that* $\sin A = \cos B$, *prove that* $(A + B) = 90^\circ$.

SOLUTION $\sin A = \cos B \Rightarrow \sin A = \sin(90^\circ - B)$

$\Rightarrow A = 90^\circ - B$ [$\because$ A and $(90^\circ - B)$ are acute]

$\Rightarrow A + B = 90^\circ$.

EXAMPLE 16 *If A, B, C are the angles of a* $\triangle ABC$, *show that* $\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}$.

[CBSE 2008C]

SOLUTION We know that the sum of the angles of a triangle is 180°.

$\therefore \quad A + B + C = 180^\circ \Rightarrow B + C = 180^\circ - A$

$\Rightarrow \quad \frac{B+C}{2} = 90^\circ - \frac{A}{2}$

$\Rightarrow \quad \sin\left(\frac{B+C}{2}\right) = \sin\left(90^\circ - \frac{A}{2}\right)$

$\Rightarrow \quad \sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}$ [$\because$ $\sin(90^\circ - \theta) = \cos\theta$].

Hence, $\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}$.

EXERCISE 12

1. Without using trigonometric tables, evaluate:

(i) $\frac{\sin 16^\circ}{\cos 74^\circ}$ (ii) $\frac{\sec 11^\circ}{\text{cosec } 79^\circ}$ (iii) $\frac{\tan 27^\circ}{\cot 63^\circ}$

(iv) $\frac{\cos 35^\circ}{\sin 55^\circ}$ (v) $\frac{\text{cosec } 42^\circ}{\sec 48^\circ}$ (vi) $\frac{\cot 38^\circ}{\tan 52^\circ}$

2. Without using trigonometric tables, prove that:

(i) $\cos 81^\circ - \sin 9^\circ = 0$ (ii) $\tan 71^\circ - \cot 19^\circ = 0$

(iii) $\text{cosec } 80^\circ - \sec 10^\circ = 0$ (iv) $\text{cosec}^2 72^\circ - \tan^2 18^\circ = 1$

(v) $\cos^2 75^\circ + \cos^2 15^\circ = 1$ (vi) $\tan^2 66^\circ - \cot^2 24^\circ = 0$

(vii) $\sin^2 48^\circ + \sin^2 42^\circ = 1$ (viii) $\cos^2 57^\circ - \sin^2 33^\circ = 0$

(ix) $(\sin 65^\circ + \cos 25^\circ)(\sin 65^\circ - \cos 25^\circ) = 0$

3. Without using trigonometric tables, prove that:
   (i) $\sin 53^\circ \cos 37^\circ + \cos 53^\circ \sin 37^\circ = 1$
   (ii) $\cos 54^\circ \cos 36^\circ - \sin 54^\circ \sin 36^\circ = 0$
   (iii) $\sec 70^\circ \sin 20^\circ + \cos 20^\circ \operatorname{cosec} 70^\circ = 2$
   (iv) $\sin 35^\circ \sin 55^\circ - \cos 35^\circ \cos 55^\circ = 0$
   (v) $(\sin 72^\circ + \cos 18^\circ)(\sin 72^\circ - \cos 18^\circ) = 0$
   (vi) $\tan 48^\circ \tan 23^\circ \tan 42^\circ \tan 67^\circ = 1$
4. Prove that:
   (i) $\dfrac{\sin 70^\circ}{\cos 20^\circ} + \dfrac{\operatorname{cosec} 20^\circ}{\sec 70^\circ} - 2\cos 70^\circ \operatorname{cosec} 20^\circ = 0$
   (ii) $\dfrac{\cos 80^\circ}{\sin 10^\circ} + \cos 59^\circ \operatorname{cosec} 31^\circ = 2$
   (iii) $\dfrac{2\sin 68^\circ}{\cos 22^\circ} - \dfrac{2\cot 15^\circ}{5\tan 75^\circ} - \dfrac{3\tan 45^\circ \tan 20^\circ \tan 40^\circ \tan 50^\circ \tan 70^\circ}{5} = 1$
   (iv) $\dfrac{\sin 18^\circ}{\cos 72^\circ} + \sqrt{3}(\tan 10^\circ \tan 30^\circ \tan 40^\circ \tan 50^\circ \tan 80^\circ) = 2$ [CBSE 2008]
   (v) $\dfrac{7\cos 55^\circ}{3\sin 35^\circ} - \dfrac{4(\cos 70^\circ \operatorname{cosec} 20^\circ)}{3(\tan 5^\circ \tan 25^\circ \tan 45^\circ \tan 65^\circ \tan 85^\circ)} = 1$
5. Prove that:
   (i) $\sin\theta\cos(90^\circ-\theta) + \sin(90^\circ-\theta)\cos\theta = 1$
   (ii) $\dfrac{\sin\theta}{\cos(90^\circ-\theta)} + \dfrac{\cos\theta}{\sin(90^\circ-\theta)} = 2$
   (iii) $\dfrac{\sin\theta\cos(90^\circ-\theta)\cos\theta}{\sin(90^\circ-\theta)} + \dfrac{\cos\theta\sin(90^\circ-\theta)\sin\theta}{\cos(90^\circ-\theta)} = 1$
   (iv) $\dfrac{\cos(90^\circ-\theta)\sec(90^\circ-\theta)\tan\theta}{\operatorname{cosec}(90^\circ-\theta)\sin(90^\circ-\theta)\cot(90^\circ-\theta)} + \dfrac{\tan(90^\circ-\theta)}{\cot\theta} = 2$
   (v) $\dfrac{\cos(90^\circ-\theta)}{1+\sin(90^\circ-\theta)} + \dfrac{1+\sin(90^\circ-\theta)}{\cos(90^\circ-\theta)} = 2\operatorname{cosec}\theta$
   (vi) $\dfrac{\sec(90^\circ-\theta)\operatorname{cosec}\theta - \tan(90^\circ-\theta)\cot\theta + \cos^2 25^\circ + \cos^2 65^\circ}{3\tan 27^\circ \tan 63^\circ} = \dfrac{2}{3}$
   [CBSE 2010]
   (vii) $\cot\theta\tan(90^\circ-\theta) - \sec(90^\circ-\theta)\operatorname{cosec}\theta + \sqrt{3}\tan 12^\circ \tan 60^\circ \tan 78^\circ = 2$
   [CBSE 2010]
6. Prove that:
   (i) $\tan 5^\circ \tan 25^\circ \tan 30^\circ \tan 65^\circ \tan 85^\circ = \dfrac{1}{\sqrt{3}}$
   (ii) $\cot 12^\circ \cot 38^\circ \cot 52^\circ \cot 60^\circ \cot 78^\circ = \dfrac{1}{\sqrt{3}}$

(iii) $\cos 15^\circ \cos 35^\circ \operatorname{cosec} 55^\circ \cos 60^\circ \operatorname{cosec} 75^\circ = \frac{1}{2}$

(iv) $\cos 1^\circ \cos 2^\circ \cos 3^\circ \ldots \cos 180^\circ = 0$

(v) $\left(\frac{\sin 49^\circ}{\cos 41^\circ}\right)^2 + \left(\frac{\cos 41^\circ}{\sin 49^\circ}\right)^2 = 2$

7. Prove that:

(i) $\sin(70^\circ + \theta) - \cos(20^\circ - \theta) = 0$

(ii) $\tan(55^\circ - \theta) - \cot(35^\circ + \theta) = 0$

(iii) $\operatorname{cosec}(67^\circ + \theta) - \sec(23^\circ - \theta) = 0$

(iv) $\operatorname{cosec}(65^\circ + \theta) - \sec(25^\circ - \theta) - \tan(55^\circ - \theta) + \cot(35^\circ + \theta) = 0$

(v) $\sin(50^\circ + \theta) - \cos(40^\circ - \theta) + \tan 1^\circ \tan 10^\circ \tan 80^\circ \tan 89^\circ = 1$

8. Express each of the following in terms of T-ratios of angles lying between 0° and 45°:

(i) $\sin 67^\circ + \cos 75^\circ$ (ii) $\cot 65^\circ + \tan 49^\circ$

(iii) $\sec 78^\circ + \operatorname{cosec} 56^\circ$ (iv) $\operatorname{cosec} 54^\circ + \sin 72^\circ$

9. If A, B and C are the angles of a $\triangle ABC$, prove that $\tan\left(\frac{C+A}{2}\right) = \cot\frac{B}{2}$.

10. If $\cos 2\theta = \sin 4\theta$, where 2θ and 4θ are acute angles, find the value of θ.

11. If $\sec 2A = \operatorname{cosec}(A - 42^\circ)$, where $2A$ is an acute angle, find the value of A. [CBSE 2008]

12. If $\sin 3A = \cos(A - 26^\circ)$, where $3A$ is an acute angle, find the value of A.

13. If $\tan 2A = \cot(A - 12^\circ)$, where $2A$ is an acute angle, find the value of A.

14. If $\sec 4A = \operatorname{cosec}(A - 15^\circ)$, where $4A$ is an acute angle, find the value of A.

15. Prove that:

$$\frac{2}{3}\operatorname{cosec}^2 58^\circ - \frac{2}{3}\cot 58^\circ \tan 32^\circ - \frac{5}{3}\tan 13^\circ \tan 37^\circ \tan 45^\circ \tan 53^\circ \tan 77^\circ = -1.$$ [CBSE 2009]

ANSWERS (EXERCISE 12)

1. (i) 1 (ii) 1 (iii) 1 (iv) 1 (v) 1 (vi) 1

8. (i) $\cos 23^\circ + \sin 15^\circ$ (ii) $\tan 25^\circ + \cot 41^\circ$ (iii) $\operatorname{cosec} 12^\circ + \sec 34^\circ$ (iv) $\sec 36^\circ + \cos 18^\circ$

10. $\theta = 15^\circ$ **11.** $A = 44^\circ$ **12.** $A = 29^\circ$ **13.** $A = 34^\circ$ **14.** $A = 21^\circ$

HINTS TO SOME SELECTED QUESTIONS

2. (iv) LHS $= \text{cosec}^2 72° - \tan^2(90° - 72°) = \text{cosec}^2 72° - \cot^2 72° = 1.$

(viii) LHS $= \cos^2 57° - \sin^2(90° - 57°) = \cos^2 57° - \cos^2 57° = 0.$

(ix) LHS $= \sin^2 65° - \cos^2 25° = \sin^2 65° - \cos^2(90° - 65°)$

$= \sin^2 65° - \sin^2 65° = 0.$

3. (iv) LHS $= \sin 35° \sin(90° - 35°) - \cos 35° \cos(90° - 35°)$

$= \sin 35° \cos 35° - \sin 35° \cos 35° = 0.$

(v) LHS $= \sin^2 72° - \cos^2 18° = \sin^2 72° - \cos^2(90° - 72°)$

$= \sin^2 72° - \sin^2 72° = 0.$

6. (i) LHS $= (\tan 5° \tan 85°)(\tan 25° \tan 65°)\tan 30°.$

(ii) LHS $= (\cot 12° \cot 78°)(\cot 38° \cot 52°)\cot 60°.$

(iii) LHS $= (\cos 15° \text{ cosec } 75°)(\cos 35° \text{ cosec } 55°)\cos 60°.$

(iv) LHS $= \cos 1° \cos 2° \cos 3° \ldots \cos 89° \cos 90° \cos 91° \ldots \cos 180° = 0$, since $\cos 90° = 0$.

(v) LHS $= \left\{\dfrac{\sin 49°}{\cos(90° - 49°)}\right\}^2 + \left\{\dfrac{\cos 41°}{\sin(90° - 41°)}\right\}^2 = \left(\dfrac{\sin 49°}{\sin 49°}\right)^2 + \left(\dfrac{\cos 41°}{\cos 41°}\right)^2 = 1 + 1 = 2.$

7. (i) LHS $= \sin\{90° - (20° - \theta)\} - \cos(20° - \theta) = \cos(20° - \theta) - \cos(20° - \theta) = 0.$

(v) LHS $= \sin\{90° - (40° - \theta)\} - \cos(40° - \theta) + (\tan 1° \tan 89°)(\tan 10° \tan 80°)$

$= \cos(40° - \theta) - \cos(40° - \theta) + (1 \times 1) = 1.$

8. (i) $\sin 67° + \cos 75° = \sin(90° - 23°) + \cos(90° - 15°) = \cos 23° + \sin 15°.$

(ii) $\cot 65° + \tan 49° = \cot(90° - 25°) + \tan(90° - 41°) = \tan 25° + \cot 41°.$

9. $A + B + C = 180° \Rightarrow C + A = 180° - B \Rightarrow \dfrac{C + A}{2} = \left(90° - \dfrac{B}{2}\right)$

$\Rightarrow \tan\left(\dfrac{C + A}{2}\right) = \tan\left(90° - \dfrac{B}{2}\right) = \cot \dfrac{B}{2}.$

10. $\cos 2\theta = \sin 4\theta \Rightarrow \sin(90° - 2\theta) = \sin 4\theta \Rightarrow 90° - 2\theta = 4\theta \Rightarrow 6\theta = 90° \Rightarrow \theta = 15°.$

11. $\sec 2A = \text{cosec}(A - 42°) \Rightarrow \text{cosec}(90° - 2A) = \text{cosec}(A - 42°) \Rightarrow 90° - 2A = A - 42°.$

12. $\sin 3A = \cos(A - 26°) \Rightarrow \cos(90° - 3A) = \cos(A - 26°) \Rightarrow 90° - 3A = A - 26°.$

13. $\tan 2A = \cot(A - 12°) \Rightarrow \cot(90° - 2A) = \cot(A - 12°) \Rightarrow 90° - 2A = A - 12°.$

14. $\sec 4A = \text{cosec}(A - 15°) \Rightarrow \text{cosec}(90° - 4A) = \text{cosec}(A - 15°) \Rightarrow 90° - 4A = A - 15°.$

✿

13 Trigonometric Identities

TRIGONOMETRIC IDENTITY

An equation involving trigonometric ratios of an angle is called a trigonometric identity if it is true for all values of the angle.

SOME TRIGONOMETRIC IDENTITIES In Chapter 10, we proved some identities, summarised below.

SUMMARY	
(i) $\sin^2\theta + \cos^2\theta = 1$	(ii) $1 + \tan^2\theta = \sec^2\theta$
(iii) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$	(iv) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$
(v) $\tan\theta \times \cot\theta = 1$	

SOLVED EXAMPLES

EXAMPLE 1 *Prove that*

(i) $(1 - \sin^2\theta)\sec^2\theta = 1$ *(ii)* $(1 + \tan^2\theta)\cos^2\theta = 1$

(iii) $(1 + \tan^2\theta)(1 - \sin\theta)(1 + \sin\theta) = 1$

(iv) $2\cos^2\theta + \dfrac{2}{(1 + \cot^2\theta)} = 2$ [CBSE 2009C]

SOLUTION We have

(i) LHS $= (1 - \sin^2\theta)\sec^2\theta$

$= \cos^2\theta \cdot \sec^2\theta$ $[\because 1 - \sin^2\theta = \cos^2\theta]$

$= \cos^2\theta \cdot \dfrac{1}{\cos^2\theta} = 1 =$ RHS.

$\therefore$ LHS = RHS.

(ii) LHS $= (1 + \tan^2\theta)\cos^2\theta$

$= \sec^2\theta \cdot \cos^2\theta$ $[\because 1 + \tan^2\theta = \sec^2\theta]$

$= \dfrac{1}{\cos^2\theta} \cdot \cos^2\theta = 1 =$ RHS.

$\therefore$ LHS = RHS.

(iii) LHS $= (1+\tan^2\theta)(1-\sin\theta)(1+\sin\theta)$

$= (1+\tan^2\theta)(1-\sin^2\theta) = \sec^2\theta\cdot\cos^2\theta$

$[\because\ (1+\tan^2\theta) = \sec^2\theta, (1-\sin^2\theta) = \cos^2\theta]$

$= \dfrac{1}{\cos^2\theta}\cdot\cos^2\theta$ $\quad\left[\because\ \sec^2\theta = \dfrac{1}{\cos^2\theta}\right]$

$= 1 =$ RHS.

$\therefore$ LHS = RHS.

(iv) LHS $= 2\cos^2\theta + \dfrac{2}{(1+\cot^2\theta)}$

$= 2\cos^2\theta + \dfrac{2}{\text{cosec}^2\theta}$ $\quad[\because\ 1+\cot^2\theta = \text{cosec}^2\theta]$

$= 2\cos^2\theta + 2\sin^2\theta$ $\quad\left[\because\ \dfrac{1}{\text{cosec}^2\theta} = \sin^2\theta\right]$

$= 2(\cos^2\theta+\sin^2\theta) = 2\times 1$ $\quad[\because\ \cos^2\theta+\sin^2\theta = 1]$

$= 2 =$ RHS.

$\therefore$ LHS = RHS.

EXAMPLE 2 *Prove that*

(i) $(\text{cosec}^2\theta - 1)\tan^2\theta = 1$ *(ii)* $(\sec^2\theta-1)(1-\text{cosec}^2\theta) = -1$

SOLUTION We have

(i) LHS $= (\text{cosec}^2\theta-1)\tan^2\theta$

$= (1+\cot^2\theta-1)\tan^2\theta$ $\quad[\because\ \text{cosec}^2\theta = 1+\cot^2\theta]$

$= \cot^2\theta\cdot\tan^2\theta$

$= \dfrac{1}{\tan^2\theta}\cdot\tan^2\theta$ $\quad\left[\because\ \cot\theta = \dfrac{1}{\tan\theta}\right]$

$= 1 =$ RHS.

(ii) LHS $= (\sec^2\theta-1)(1-\text{cosec}^2\theta)$

$= (1+\tan^2\theta-1)[1-(1+\cot^2\theta)]$

$[\because\ \sec^2\theta = 1+\tan^2\theta \text{ and } \text{cosec}^2\theta = 1+\cot^2\theta]$

$= \tan^2\theta\cdot(-\cot^2\theta)$

$= \tan^2\theta\cdot\left(\dfrac{-1}{\tan^2\theta}\right)$ $\quad\left[\because\ \cot\theta = \dfrac{1}{\tan\theta}\right]$

$= -1 =$ RHS.

EXAMPLE 3 *Prove that*

$$\frac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} + \sin\theta\cos\theta = 1. \quad \text{[CBSE 2009]}$$

SOLUTION We have

$$\text{LHS} = \frac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} + \sin\theta\cos\theta$$

$$= \frac{(\sin\theta + \cos\theta)(\sin^2\theta + \cos^2\theta - \sin\theta\cos\theta)}{(\sin\theta + \cos\theta)} + \sin\theta\cos\theta$$

$$[\because\ (a^3 + b^3) = (a+b)(a^2 + b^2 - ab)]$$

$$= (1 - \sin\theta\cos\theta) + \sin\theta\cos\theta = 1 \quad [\because\ \sin^2\theta + \cos^2\theta = 1]$$

$$= \text{RHS}.$$

$\therefore$ LHS = RHS.

EXAMPLE 4 *Prove that*

$$(\sin\theta - \operatorname{cosec}\theta)(\cos\theta - \sec\theta) = \frac{1}{\tan\theta + \cot\theta}. \quad \text{[CBSE 2009C, '10]}$$

SOLUTION We have

$$\text{LHS} = (\sin\theta - \operatorname{cosec}\theta)(\cos\theta - \sec\theta)$$

$$= \left(\sin\theta - \frac{1}{\sin\theta}\right)\left(\cos\theta - \frac{1}{\cos\theta}\right) = \frac{(\sin^2\theta - 1)}{\sin\theta} \cdot \frac{(\cos^2\theta - 1)}{\cos\theta}$$

$$= \frac{\{-(1 - \sin^2\theta)\} \cdot [-(1 - \cos^2\theta)\}}{\sin\theta\cos\theta}$$

$$= \frac{(-\cos^2\theta)(-\sin^2\theta)}{\sin\theta\cos\theta}$$

$$[\because\ 1 - \sin^2\theta = \cos^2\theta \text{ and } 1 - \cos^2\theta = \sin^2\theta]$$

$$= \frac{(\cos^2\theta\sin^2\theta)}{\sin\theta\cos\theta} = \cos\theta\sin\theta.$$

$$\text{RHS} = \frac{1}{\tan\theta + \cot\theta}$$

$$= \frac{1}{\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}} = \frac{1}{\dfrac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}} = \frac{\cos\theta\sin\theta}{(\sin^2\theta + \cos^2\theta)}$$

$$= \cos\theta\sin\theta \quad [\because\ \sin^2\theta + \cos^2\theta = 1].$$

$\therefore$ LHS = RHS.

EXAMPLE 5 *Prove that*

$$(1 + \cot\theta - \operatorname{cosec}\theta)(1 + \tan\theta + \sec\theta) = 2. \quad \text{[CBSE 2005, '07, '08]}$$

SOLUTION We have

$$\text{LHS} = (1 + \cot\theta - \text{cosec}\,\theta)(1 + \tan\theta + \sec\theta)$$

$$= \left(1 + \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}\right)\left(1 + \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)$$

$$= \frac{(\sin\theta + \cos\theta - 1)}{\sin\theta} \cdot \frac{(\cos\theta + \sin\theta + 1)}{\cos\theta}$$

$$= \frac{[(\sin\theta + \cos\theta) - 1] \cdot [(\sin\theta + \cos\theta) + 1]}{\sin\theta\cos\theta}$$

$$= \frac{(\sin\theta + \cos\theta)^2 - 1}{\sin\theta\cos\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 1}{\sin\theta\cos\theta}$$

$$= \frac{1 + 2\sin\theta\cos\theta - 1}{\sin\theta\cos\theta} \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$= \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} = 2 = \text{RHS}.$$

$\therefore$ LHS = RHS.

EXAMPLE 6 *Prove that*

$$\frac{\cos\theta}{(1 - \tan\theta)} + \frac{\sin\theta}{(1 - \cot\theta)} = (\cos\theta + \sin\theta).$$ [CBSE 2007]

SOLUTION We have

$$\text{LHS} = \frac{\cos\theta}{(1 - \tan\theta)} + \frac{\sin\theta}{(1 - \cot\theta)} = \frac{\cos\theta}{\left(1 - \frac{\sin\theta}{\cos\theta}\right)} + \frac{\sin\theta}{\left(1 - \frac{\cos\theta}{\sin\theta}\right)}$$

$$= \frac{\cos^2\theta}{(\cos\theta - \sin\theta)} + \frac{\sin^2\theta}{(\sin\theta - \cos\theta)}$$

$$= \frac{\cos^2\theta}{\cos\theta - \sin\theta} - \frac{\sin^2\theta}{\cos\theta - \sin\theta}$$

$$= \frac{\cos^2\theta - \sin^2\theta}{(\cos\theta - \sin\theta)} = \frac{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}{(\cos\theta - \sin\theta)}$$

$$= (\cos\theta + \sin\theta) = \text{RHS}.$$

$\therefore$ LHS = RHS.

EXAMPLE 7 *Prove that*

$$\frac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \tan\theta.$$ [CBSE 2010]

SOLUTION We have

$$\text{LHS} = \frac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \frac{\sin\theta(1 - 2\sin^2\theta)}{\cos\theta(2\cos^2\theta - 1)}$$

$$= \frac{\sin\theta(1-2\sin^2\theta)}{\cos\theta(2\cos^2\theta-1)}$$

$$= \tan\theta\cdot\frac{[1-2(1-\cos^2\theta)]}{(2\cos^2\theta-1)} \qquad [\because\ \sin^2\theta = 1-\cos^2\theta]$$

$$= \tan\theta\cdot\frac{(2\cos^2\theta-1)}{(2\cos^2\theta-1)} = \tan\theta = \text{RHS.}$$

$\therefore$ LHS = RHS.

EXAMPLE 8 *Prove that*

$$\frac{\tan A}{(1-\cot A)} + \frac{\cot A}{(1-\tan A)} = (1+\tan A+\cot A). \qquad \text{[CBSE 2010]}$$

SOLUTION We have

$$\text{LHS} = \frac{\tan A}{(1-\cot A)} + \frac{\cot A}{(1-\tan A)} = \frac{\frac{\sin A}{\cos A}}{\left(1-\frac{\cos A}{\sin A}\right)} + \frac{\frac{\cos A}{\sin A}}{\left(1-\frac{\sin A}{\cos A}\right)}$$

$$= \frac{\sin^2 A}{\cos A(\sin A-\cos A)} + \frac{\cos^2 A}{\sin A(\cos A-\sin A)}$$

$$= \frac{\sin^2 A}{\cos A(\sin A-\cos A)} - \frac{\cos^2 A}{\sin A(\sin A-\cos A)}$$

$$= \frac{\sin^3 A-\cos^3 A}{\sin A\cos A(\sin A-\cos A)}$$

$$= \frac{(\sin A-\cos A)(\sin^2 A+\cos^2 A+\sin A\cos A)}{\sin A\cos A(\sin A-\cos A)}$$

$$[\because\ (a^3-b^3) = (a-b)(a^2+b^2+ab)]$$

$$= \frac{1+\sin A\cos A}{\sin A\cos A}.$$

$$\text{RHS} = (1+\tan A+\cot A)$$

$$= \left(1+\frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}\right) = \frac{\sin A\cos A+\sin^2 A+\cos^2 A}{\sin A\cos A}$$

$$= \frac{(1+\sin A\cos A)}{\sin A\cos A}.$$

$\therefore$ LHS = RHS.

EXAMPLE 9 *Prove that*

$$(\text{cosec}\, A-\sin A)(\sec A-\cos A) = \frac{1}{(\tan A+\cot A)}.$$

[CBSE 2010]

SOLUTION We have

$$\text{LHS} = (\text{cosec } A - \sin A)(\sec A - \cos A)$$

$$= \left(\frac{1}{\sin A} - \sin A\right)\left(\frac{1}{\cos A} - \cos A\right) = \left(\frac{1-\sin^2 A}{\sin A}\right)\cdot\left(\frac{1-\cos^2 A}{\cos A}\right)$$

$$= \frac{\cos^2 A \sin^2 A}{\cos A \sin A} = \cos A \sin A.$$

$$\text{RHS} = \frac{1}{(\tan A + \cot A)} = \frac{1}{\left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right)}$$

$$= \frac{\cos A \sin A}{(\sin^2 A + \cos^2 A)} = \cos A \sin A \quad [\because \sin^2 A + \cos^2 A = 1].$$

Hence, LHS = RHS.

EXAMPLE 10 *Prove that*

$$\left(\frac{1+\tan^2 A}{1+\cot^2 A}\right) = \left(\frac{1-\tan A}{1-\cot A}\right)^2 = \tan^2 A.$$ [CBSE 2008C]

SOLUTION We have

$$\left(\frac{1+\tan^2 A}{1+\cot^2 A}\right) = \frac{\sec^2 A}{\text{cosec}^2 A}$$

$$[\because 1 + \tan^2 A = \sec^2 A \text{ and } 1 + \cot^2 A = \text{cosec}^2 A]$$

$$= \frac{1}{\cos^2 A}\cdot\sin^2 A = \frac{\sin^2 A}{\cos^2 A} = \tan^2 A.$$

And, $$\left(\frac{1-\tan A}{1-\cot A}\right)^2 = \frac{\left(1-\frac{\sin A}{\cos A}\right)^2}{\left(1-\frac{\cos A}{\sin A}\right)^2}$$

$$= \frac{\frac{(\cos A - \sin A)^2}{\cos^2 A}}{\frac{(\sin A - \cos A)^2}{\sin^2 A}} = \frac{\sin^2 A}{\cos^2 A} = \tan^2 A.$$

$$\therefore \quad \left(\frac{1+\tan^2 A}{1+\cot^2 A}\right) = \left(\frac{1-\tan A}{1-\cot A}\right)^2 = \tan^2 A.$$

EXAMPLE 11 *Prove that*

(i) $\dfrac{\tan A + \sin A}{\tan A - \sin A} = \dfrac{\sec A + 1}{\sec A - 1}$

(ii) $\dfrac{\cot A - \cos A}{\cot A + \cos A} = \dfrac{\text{cosec } A - 1}{\text{cosec } A + 1}$ [CBSE 2008]

SOLUTION We have

(i) $\text{LHS} = \dfrac{\tan A + \sin A}{\tan A - \sin A}$

$$= \frac{\dfrac{\sin A}{\cos A} + \sin A}{\dfrac{\sin A}{\cos A} - \sin A} = \frac{\sin A\left(\dfrac{1}{\cos A} + 1\right)}{\sin A\left(\dfrac{1}{\cos A} - 1\right)}$$

$$= \frac{\left(\dfrac{1}{\cos A} + 1\right)}{\left(\dfrac{1}{\cos A} - 1\right)} = \frac{\sec A + 1}{\sec A - 1} = \text{RHS.}$$

$\therefore$ LHS = RHS.

(ii) $\text{LHS} = \dfrac{\cot A - \cos A}{\cot A + \cos A}$

$$= \frac{\dfrac{\cos A}{\sin A} - \cos A}{\dfrac{\cos A}{\sin A} + \cos A} = \frac{\cos A\left(\dfrac{1}{\sin A} - 1\right)}{\cos A\left(\dfrac{1}{\sin A} + 1\right)}$$

$$= \frac{\left(\dfrac{1}{\sin A} - 1\right)}{\left(\dfrac{1}{\sin A} + 1\right)} = \frac{(\text{cosec } A - 1)}{(\text{cosec } A + 1)} = \text{RHS.}$$

$\therefore$ LHS = RHS.

EXAMPLE 12 *Prove that*

(i) $\sec^2\theta + \text{cosec}^2\theta = \sec^2\theta\,\text{cosec}^2\theta$

(ii) $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$

(iii) $\tan^2\theta + \cot^2\theta + 2 = \sec^2\theta\,\text{cosec}^2\theta$

SOLUTION We have

(i) $\text{LHS} = \sec^2\theta + \text{cosec}^2\theta$

$$= \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin^2\theta}$$

$$= \frac{1}{\cos^2\theta\sin^2\theta} \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

$= \sec^2\theta\,\text{cosec}^2\theta = \text{RHS.}$

$\therefore$ LHS = RHS.

(ii) $\text{LHS} = \tan^2\theta - \sin^2\theta$

$$= \frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta = \frac{\sin^2\theta - \sin^2\theta\cos^2\theta}{\cos^2\theta}$$

$$= \frac{\sin^2\theta(1-\cos^2\theta)}{\cos^2\theta} = \frac{\sin^2\theta}{\cos^2\theta}\cdot\sin^2\theta$$

$$= \tan^2\theta\sin^2\theta = \text{RHS}.$$

$\therefore$ LHS = RHS.

(iii) $\text{LHS} = \tan^2\theta + \cot^2\theta + 2$

$$= (1+\tan^2\theta) + (1+\cot^2\theta) = \sec^2\theta + \text{cosec}^2\theta$$

$$= \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin^2\theta}$$

$$= \frac{1}{\cos^2\theta\sin^2\theta} = \sec^2\theta\,\text{cosec}^2\theta = \text{RHS}.$$

$\therefore$ LHS = RHS.

EXAMPLE 13 *Prove that*

$$(\text{cosec}\,\theta - \cot\theta)^2 = \frac{1-\cos\theta}{1+\cos\theta}.$$ [CBSE 2000C]

SOLUTION $\text{LHS} = (\text{cosec}\,\theta - \cot\theta)^2$

$$= \left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)^2 = \left(\frac{1-\cos\theta}{\sin\theta}\right)^2$$

$$= \frac{(1-\cos\theta)^2}{\sin^2\theta} = \frac{(1-\cos\theta)(1-\cos\theta)}{(1-\cos^2\theta)} \quad [\because \sin^2\theta = 1-\cos^2\theta]$$

$$= \frac{(1-\cos\theta)(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)} = \frac{(1-\cos\theta)}{(1+\cos\theta)} = \text{RHS}.$$

$\therefore$ LHS = RHS.

EXAMPLE 14 *Prove that*

$$(\sec^4\theta - \sec^2\theta) = (\tan^2\theta + \tan^4\theta).$$

SOLUTION We have

$$\text{LHS} = (\sec^4\theta - \sec^2\theta) = \sec^2\theta(\sec^2\theta - 1)$$

$$= (1+\tan^2\theta)(1+\tan^2\theta-1) = (1+\tan^2\theta)\tan^2\theta$$

$$= (\tan^2\theta + \tan^4\theta) = \text{RHS}.$$

$\therefore$ LHS = RHS.

EXAMPLE 15 *Prove that*

$$\left(1+\frac{1}{\tan^2 A}\right)\left(1+\frac{1}{\cot^2 A}\right) = \frac{1}{(\sin^2 A - \sin^4 A)}.$$ [CBSE 2006C]

SOLUTION We have

$$\text{LHS} = \left(1+\frac{1}{\tan^2 A}\right)\left(1+\frac{1}{\cot^2 A}\right)$$
$$= (1+\cot^2 A)(1+\tan^2 A) = \operatorname{cosec}^2 A\cdot \sec^2 A$$
$$= \frac{1}{\sin^2 A}\cdot\frac{1}{\cos^2 A} = \frac{1}{\sin^2 A\cos^2 A}$$
$$= \frac{1}{\sin^2 A(1-\sin^2 A)} = \frac{1}{(\sin^2 A - \sin^4 A)} = \text{RHS}.$$

$\therefore$ LHS = RHS.

EXAMPLE 16 *Prove that*

$$(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 = (7+\tan^2\theta + \cot^2\theta).$$

[CBSE 2008, '09C]

SOLUTION We have

$$\text{LHS} = (\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta+\sec\theta)^2$$
$$= (\sin^2\theta + \operatorname{cosec}^2\theta + 2\sin\theta\operatorname{cosec}\theta) + (\cos^2\theta + \sec^2\theta + 2\cos\theta\sec\theta)$$
$$= (\sin^2\theta + \operatorname{cosec}^2\theta + 2) + (\cos^2\theta + \sec^2\theta + 2)$$
$$[\because \sin\theta\operatorname{cosec}\theta = 1 \text{ and } \cos\theta\sec\theta = 1]$$
$$= (\sin^2\theta + \cos^2\theta) + 4 + (\operatorname{cosec}^2\theta + \sec^2\theta)$$
$$= 1 + 4 + (1+\cot^2\theta) + (1+\tan^2\theta)$$
$$[\because \sin^2\theta + \cos^2\theta = 1, \operatorname{cosec}^2\theta = 1+\cot^2\theta \text{ and } \sec^2\theta = 1+\tan^2\theta]$$
$$= (7+\tan^2\theta+\cot^2\theta) = \text{RHS}.$$

$\therefore$ LHS = RHS.

EXAMPLE 17 *Prove that*

(i) $\dfrac{1-\sin\theta}{1+\sin\theta} = (\sec\theta - \tan\theta)^2$ [CBSE 2008C]

(ii) $\dfrac{(1+\cos\theta)}{(1-\cos\theta)} = (\operatorname{cosec}\theta + \cot\theta)^2$ [CBSE 2007C]

SOLUTION We have

(i) $\text{RHS} = (\sec\theta - \tan\theta)^2$

$$= \left(\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right)^2 = \left(\frac{1-\sin\theta}{\cos\theta}\right)^2 = \frac{(1-\sin\theta)^2}{\cos^2\theta}$$
$$= \frac{(1-\sin\theta)^2}{(1-\sin^2\theta)} = \frac{(1-\sin\theta)(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}$$

$$= \frac{(1 - \sin\theta)}{(1 + \sin\theta)} = \text{LHS.}$$

$\therefore$ RHS = LHS.

(ii) $\text{RHS} = (\text{cosec}\,\theta + \cot\theta)^2 = \left(\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\right)^2$

$$= \left(\frac{1 + \cos\theta}{\sin\theta}\right)^2 = \frac{(1 + \cos\theta)^2}{\sin^2\theta} = \frac{(1 + \cos\theta)(1 + \cos\theta)}{1 - \cos^2\theta}$$

$$= \frac{(1 + \cos\theta)(1 + \cos\theta)}{(1 + \cos\theta)(1 - \cos\theta)} = \frac{(1 + \cos\theta)}{(1 - \cos\theta)} = \text{LHS.}$$

$\therefore$ LHS = RHS.

EXAMPLE 18 *Prove that*

(i) $\frac{1}{(\text{cosec}\,\theta - \cot\theta)} = (\text{cosec}\,\theta + \cot\theta)$

(ii) $\frac{(\sec\theta - \tan\theta)}{(\sec\theta + \tan\theta)} = (1 + 2\tan^2\theta - 2\sec\theta\tan\theta)$

SOLUTION We have

(i) $\text{LHS} = \frac{1}{(\text{cosec}\,\theta - \cot\theta)} = \frac{1}{(\text{cosec}\,\theta - \cot\theta)} \times \frac{(\text{cosec}\,\theta + \cot\theta)}{(\text{cosec}\,\theta + \cot\theta)}$

$$= \frac{(\text{cosec}\,\theta + \cot\theta)}{(\text{cosec}^2\theta - \cot^2\theta)} = \text{cosec}\,\theta + \cot\theta = \text{RHS}$$

$[\because \text{cosec}^2\theta - \cot^2\theta = 1]$.

$\therefore$ LHS = RHS.

(ii) $\text{LHS} = \frac{(\sec\theta - \tan\theta)}{(\sec\theta + \tan\theta)} = \frac{(\sec\theta - \tan\theta)}{(\sec\theta + \tan\theta)} \times \frac{(\sec\theta - \tan\theta)}{(\sec\theta - \tan\theta)}$

$$= \frac{(\sec\theta - \tan\theta)^2}{(\sec^2\theta - \tan^2\theta)} = (\sec\theta - \tan\theta)^2$$

$[\because \sec^2\theta - \tan^2\theta = 1]$

$$= \sec^2\theta + \tan^2\theta - 2\sec\theta\tan\theta$$

$$= (1 + \tan^2\theta) + \tan^2\theta - 2\sec\theta\tan\theta$$

$$= 1 + 2\tan^2\theta - 2\sec\theta\tan\theta = \text{RHS.}$$

$\therefore$ LHS = RHS.

EXAMPLE 19 *Prove that*

$$\frac{1}{(\text{cosec}\,\theta - \cot\theta)} - \frac{1}{\sin\theta} = \frac{1}{\sin\theta} - \frac{1}{(\text{cosec}\,\theta + \cot\theta)}.$$

[CBSE 2002C, '06]

SOLUTION LHS $= \frac{1}{(\text{cosec}\,\theta - \cot\theta)} - \frac{1}{\sin\theta}$

$= \frac{1}{(\text{cosec}\,\theta - \cot\theta)} \times \frac{(\text{cosec}\,\theta + \cot\theta)}{(\text{cosec}\,\theta + \cot\theta)} - \frac{1}{\sin\theta}$

$= \frac{(\text{cosec}\,\theta + \cot\theta)}{(\text{cosec}^2\theta - \cot^2\theta)} - \text{cosec}\,\theta$ $\left[\because \frac{1}{\sin\theta} = \text{cosec}\,\theta\right]$

$= (\text{cosec}\,\theta + \cot\theta) - \text{cosec}\,\theta$ $[\because \text{cosec}^2\theta - \cot^2\theta = 1]$

$= \cot\theta.$

RHS $= \frac{1}{\sin\theta} - \frac{1}{(\text{cosec}\,\theta + \cot\theta)}$

$= \text{cosec}\,\theta - \frac{1}{(\text{cosec}\,\theta + \cot\theta)} \times \frac{(\text{cosec}\,\theta - \cot\theta)}{(\text{cosec}\,\theta - \cot\theta)}$

$= \text{cosec}\,\theta - \frac{(\text{cosec}\,\theta - \cot\theta)}{(\text{cosec}^2\theta - \cot^2\theta)}$

$= \text{cosec}\,\theta - (\text{cosec}\,\theta - \cot\theta)$ $[\because \text{cosec}^2\theta - \cot^2\theta = 1]$

$= \cot\theta.$

$\therefore$ LHS = RHS.

EXAMPLE 20 *Prove that*

$$\frac{1}{(\sec\theta - \tan\theta)} - \frac{1}{\cos\theta} = \frac{1}{\cos\theta} - \frac{1}{(\sec\theta + \tan\theta)}.$$ [CBSE 2005, '08]

SOLUTION We have

LHS $= \frac{1}{(\sec\theta - \tan\theta)} - \frac{1}{\cos\theta}$

$= \frac{1}{(\sec\theta - \tan\theta)} \times \frac{(\sec\theta + \tan\theta)}{(\sec\theta + \tan\theta)} - \sec\theta$

$= \frac{(\sec\theta + \tan\theta)}{(\sec^2\theta - \tan^2\theta)} - \sec\theta$

$= (\sec\theta + \tan\theta) - \sec\theta$ $[\because \sec^2\theta - \tan^2\theta = 1]$

$= \tan\theta.$

RHS $= \frac{1}{\cos\theta} - \frac{1}{(\sec\theta + \tan\theta)}$

$= \sec\theta - \frac{1}{(\sec\theta + \tan\theta)} \times \frac{(\sec\theta - \tan\theta)}{(\sec\theta - \tan\theta)}$

$= \sec\theta - \frac{(\sec\theta - \tan\theta)}{(\sec^2\theta - \tan^2\theta)}$

$= \sec\theta - (\sec\theta - \tan\theta)$ $[\because \sec^2\theta - \tan^2\theta = 1]$

$= \tan\theta.$

$\therefore$ LHS = RHS.

EXAMPLE 21 *Prove that*

(i) $\dfrac{\sin\theta}{(1-\cos\theta)} = (\operatorname{cosec}\theta + \cot\theta)$

(ii) $\dfrac{1}{(\sec\theta - \tan\theta)} = (\sec\theta + \tan\theta)$

SOLUTION We have

(i) $\text{LHS} = \dfrac{\sin\theta}{(1-\cos\theta)}$

$= \dfrac{\sin\theta}{(1-\cos\theta)} \times \dfrac{(1+\cos\theta)}{(1+\cos\theta)}$

[multiplying num. and denom. by $(1+\cos\theta)$]

$= \dfrac{\sin\theta(1+\cos\theta)}{(1-\cos^2\theta)} = \dfrac{\sin\theta(1+\cos\theta)}{\sin^2\theta} = \dfrac{(1+\cos\theta)}{\sin\theta}$

$= \left(\dfrac{1}{\sin\theta} + \dfrac{\cos\theta}{\sin\theta}\right) = (\operatorname{cosec}\theta + \cot\theta) = \text{RHS}.$

$\therefore$ LHS = RHS.

(ii) $\text{LHS} = \dfrac{1}{(\sec\theta - \tan\theta)}$

$= \dfrac{1}{(\sec\theta - \tan\theta)} \times \dfrac{(\sec\theta + \tan\theta)}{(\sec\theta + \tan\theta)}$

[multiplying num. and denom. by $(\sec\theta + \tan\theta)$]

$= \dfrac{(\sec\theta + \tan\theta)}{(\sec^2\theta - \tan^2\theta)}$

$= (\sec\theta + \tan\theta)$ $\quad [\because \sec^2\theta - \tan^2\theta = 1]$

= RHS.

$\therefore$ LHS = RHS.

EXAMPLE 22 *Prove that*

$\dfrac{\sec\theta + \tan\theta - 1}{\tan\theta - \sec\theta + 1} = \dfrac{\cos\theta}{(1-\sin\theta)}.$ [CBSE 2002, '03, '05C]

SOLUTION $\text{LHS} = \dfrac{\sec\theta + \tan\theta - 1}{\tan\theta - \sec\theta + 1}$

$= \dfrac{(\sec\theta + \tan\theta) - (\sec^2\theta - \tan^2\theta)}{(\tan\theta - \sec\theta + 1)}$ $\quad [\because \sec^2\theta - \tan^2\theta = 1]$

$= \dfrac{(\sec\theta + \tan\theta)[1 - (\sec\theta - \tan\theta)]}{(\tan\theta - \sec\theta + 1)}$

$= \dfrac{(\sec\theta + \tan\theta)(\tan\theta - \sec\theta + 1)}{(\tan\theta - \sec\theta + 1)} = (\sec\theta + \tan\theta)$

$$= \left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right) = \frac{(1+\sin\theta)}{\cos\theta} = \frac{(1+\sin\theta)}{\cos\theta} \times \frac{(1-\sin\theta)}{(1-\sin\theta)}$$

$$= \frac{(1-\sin^2\theta)}{\cos\theta(1-\sin\theta)} = \frac{\cos^2\theta}{\cos\theta(1-\sin\theta)} = \frac{\cos\theta}{(1-\sin\theta)} = \text{RHS}.$$

Hence, $\dfrac{\sec\theta+\tan\theta-1}{\tan\theta-\sec\theta+1} = \dfrac{\cos\theta}{(1-\sin\theta)}$.

EXAMPLE 23 *Prove that*

$$\frac{\sin\theta-\cos\theta+1}{\sin\theta+\cos\theta-1} = \frac{1}{(\sec\theta-\tan\theta)}.$$

SOLUTION $\text{LHS} = \dfrac{\sin\theta-\cos\theta+1}{\sin\theta+\cos\theta-1}$

$$= \frac{\dfrac{\sin\theta}{\cos\theta} - 1 + \dfrac{1}{\cos\theta}}{\dfrac{\sin\theta}{\cos\theta} + 1 - \dfrac{1}{\cos\theta}}$$

[on dividing num. and denom. by $\cos\theta$]

$$= \frac{\tan\theta-1+\sec\theta}{\tan\theta+1-\sec\theta} = \frac{(\sec\theta+\tan\theta-1)}{(\tan\theta-\sec\theta+1)}$$

$$= \frac{(\sec\theta+\tan\theta)-(\sec^2\theta-\tan^2\theta)}{(\tan\theta-\sec\theta+1)} \qquad [\because\ 1 = \sec^2\theta-\tan^2\theta]$$

$$= \frac{(\sec\theta+\tan\theta)[1-(\sec\theta-\tan\theta)]}{(\tan\theta-\sec\theta+1)}$$

$$= \frac{(\sec\theta+\tan\theta)(\tan\theta-\sec\theta+1)}{(\tan\theta-\sec\theta+1)} = (\sec\theta+\tan\theta).$$

$\text{RHS} = \dfrac{1}{(\sec\theta-\tan\theta)}$

$$= \frac{1}{(\sec\theta-\tan\theta)} \times \frac{(\sec\theta+\tan\theta)}{(\sec\theta+\tan\theta)} = \frac{(\sec\theta+\tan\theta)}{(\sec^2\theta-\tan^2\theta)}$$

$$= (\sec\theta+\tan\theta) \qquad [\because\ \sec^2\theta-\tan^2\theta = 1].$$

Hence, LHS = RHS.

EXAMPLE 24 *Prove that*

$$\frac{\cot\theta+\operatorname{cosec}\theta-1}{\cot\theta-\operatorname{cosec}\theta+1} = \frac{1+\cos\theta}{\sin\theta}.$$

SOLUTION We have

$$\text{LHS} = \frac{(\cot\theta+\operatorname{cosec}\theta)-1}{(\cot\theta-\operatorname{cosec}\theta+1)}$$

$$= \frac{(\text{cosec }\theta + \cot\theta) - (\text{cosec}^2\theta - \cot^2\theta)}{(\cot\theta - \text{cosec }\theta + 1)}$$

$[\because\ 1 = \text{cosec}^2\theta - \cot^2\theta]$

$$= \frac{(\text{cosec }\theta + \cot\theta)[1 - (\text{cosec }\theta - \cot\theta)]}{(\cot\theta - \text{cosec }\theta + 1)}$$

$$= \frac{(\text{cosec }\theta + \cot\theta)(\cot\theta - \text{cosec }\theta + 1)}{(\cot\theta - \text{cosec }\theta + 1)}$$

$$= (\text{cosec }\theta + \cot\theta) = \left(\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\right) = \frac{1+\cos\theta}{\sin\theta} = \text{RHS.}$$

Hence, $\dfrac{\cot\theta + \text{cosec }\theta - 1}{\cot\theta - \text{cosec }\theta + 1} = \dfrac{1+\cos\theta}{\sin\theta}$.

EXAMPLE 25 *Prove that*

$$\sqrt{\sec^2\theta + \text{cosec}^2\theta} = \tan\theta + \cot\theta.$$

SOLUTION We have

$$\text{LHS} = \sqrt{\sec^2\theta + \text{cosec}^2\theta}$$

$$= \sqrt{(1+\tan^2\theta) + (1+\cot^2\theta)} = \sqrt{\tan^2\theta + \cot^2\theta + 2}$$

$$= \sqrt{\tan^2\theta + \cot^2\theta + 2\tan\theta\cdot\cot\theta} \qquad [\because\ \tan\theta\cdot\cot\theta = 1]$$

$$= \sqrt{(\tan\theta + \cot\theta)^2} = \tan\theta + \cot\theta = \text{RHS.}$$

$\therefore$ LHS = RHS.

EXAMPLE 26 *Prove that*

(i) $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$

(ii) $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} = \text{cosec }\theta + \cot\theta$

SOLUTION We have

(i) $$\text{LHS} = \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \frac{\sqrt{1-\sin\theta}}{\sqrt{1+\sin\theta}} \times \frac{\sqrt{1-\sin\theta}}{\sqrt{1-\sin\theta}}$$

$$= \frac{1-\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{1-\sin\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta} = \sec\theta - \tan\theta = \text{RHS.}$$

$\therefore$ LHS = RHS.

(ii) $$\text{LHS} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \frac{\sqrt{1+\cos\theta}}{\sqrt{1-\cos\theta}} \times \frac{\sqrt{1+\cos\theta}}{\sqrt{1+\cos\theta}}$$

$$= \frac{1+\cos\theta}{\sqrt{1-\cos^2\theta}} = \frac{1+\cos\theta}{\sin\theta}$$

$$= \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} = \operatorname{cosec}\theta + \cot\theta = \text{RHS}.$$

$\therefore$ LHS = RHS.

EXAMPLE 27 *Prove that*

$$\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} + \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = 2\sec\theta.$$ [CBSE 2001C]

SOLUTION We have

$$\text{LHS} = \sqrt{\frac{1+\sin\theta}{1-\sin\theta}} + \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$$

$$= \frac{\sqrt{1+\sin\theta}}{\sqrt{1-\sin\theta}} + \frac{\sqrt{1-\sin\theta}}{\sqrt{1+\sin\theta}} = \frac{1+\sin\theta+1-\sin\theta}{\sqrt{(1-\sin\theta)(1+\sin\theta)}}$$

$$= \frac{2}{\sqrt{(1-\sin^2\theta)}} = \frac{2}{\sqrt{\cos^2\theta}} = \frac{2}{\cos\theta} = 2\sec\theta = \text{RHS}.$$

$\therefore$ LHS = RHS.

EXAMPLE 28 *Prove that*

$$\sqrt{\frac{\sec\theta-1}{\sec\theta+1}} + \sqrt{\frac{\sec\theta+1}{\sec\theta-1}} = 2\operatorname{cosec}\theta.$$ [CBSE 2006C]

SOLUTION We have

$$\text{LHS} = \sqrt{\frac{\sec\theta-1}{\sec\theta+1}} + \sqrt{\frac{\sec\theta+1}{\sec\theta-1}} = \frac{\sqrt{\sec\theta-1}}{\sqrt{\sec\theta+1}} + \frac{\sqrt{\sec\theta+1}}{\sqrt{\sec\theta-1}}$$

$$= \frac{\sec\theta-1+\sec\theta+1}{\sqrt{(\sec\theta+1)(\sec\theta-1)}} = \frac{2\sec\theta}{\sqrt{\sec^2\theta-1}} = \frac{2\sec\theta}{\tan\theta}$$

$[\because \sec^2\theta - 1 = \tan^2\theta]$

$$= 2\sec\theta\cot\theta = \frac{2}{\cos\theta} \times \frac{\cos\theta}{\sin\theta} = \frac{2}{\sin\theta} = 2\operatorname{cosec}\theta = \text{RHS}.$$

$\therefore$ LHS = RHS.

EXAMPLE 29 *Prove that*

$$\frac{\operatorname{cosec}\theta}{(\operatorname{cosec}\theta-1)} + \frac{\operatorname{cosec}\theta}{(\operatorname{cosec}\theta+1)} = 2\sec^2\theta.$$ [CBSE 2004C, '09]

SOLUTION We have

$$\text{LHS} = \frac{\operatorname{cosec}\theta}{(\operatorname{cosec}\theta-1)} + \frac{\operatorname{cosec}\theta}{(\operatorname{cosec}\theta+1)}$$

$$= \frac{\operatorname{cosec}\theta(\operatorname{cosec}\theta+1) + \operatorname{cosec}\theta(\operatorname{cosec}\theta-1)}{(\operatorname{cosec}^2\theta-1)}$$

$$= \frac{2\operatorname{cosec}^2\theta}{(1+\cot^2\theta-1)}$$ $[\because \operatorname{cosec}^2\theta = 1+\cot^2\theta]$

$$= \frac{2\operatorname{cosec}^2\theta}{\cot^2\theta} = 2\operatorname{cosec}^2\theta\tan^2\theta$$

$$= 2 \times \frac{1}{\sin^2\theta} \times \frac{\sin^2\theta}{\cos^2\theta} = \frac{2}{\cos^2\theta} = 2\sec^2\theta = \text{RHS}.$$

AN IMPORTANT REMARK

In order to show that a given trigonometric equation is not an identity, it is sufficient to find by hit and trial, a single value of θ which does not satisfy it.

EXAMPLE 30 *Show that* $(\cos^2\theta - \sin^2\theta) = \dfrac{2\tan\theta}{(1-\tan^2\theta)}$ *is not an identity.*

SOLUTION Putting $\theta = 30^\circ$, we find:

$$\text{LHS} = (\cos^2 30^\circ - \sin^2 30^\circ)$$

$$= \left\{\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right\} = \left(\frac{3}{4} - \frac{1}{4}\right) = \frac{2}{4} = \frac{1}{2}, \text{ and}$$

$$\text{RHS} = \frac{2\tan 30^\circ}{(1-\tan^2 30^\circ)} = \frac{2 \times \frac{1}{\sqrt{3}}}{\left[1 - \left(\frac{1}{\sqrt{3}}\right)^2\right]} = \frac{\frac{2}{\sqrt{3}}}{\left(1 - \frac{1}{3}\right)} = \left(\frac{2}{\sqrt{3}} \times \frac{3}{2}\right) = \sqrt{3}.$$

$\therefore$ LHS $\neq$ RHS.

Hence, the given equation is not an identity.

EXAMPLE 31 *Prove that*

(i) $(\sin^2 A\cos^2 B - \cos^2 A\sin^2 B) = (\sin^2 A - \sin^2 B)$

(ii) $(\tan^2 A\sec^2 B - \sec^2 A\tan^2 B) = (\tan^2 A - \tan^2 B)$

SOLUTION We have

(i) $\text{LHS} = (\sin^2 A\cos^2 B - \cos^2 A\sin^2 B)$

$$= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B$$

$$= \sin^2 A - \sin^2 B = \text{RHS}.$$

(ii) $\text{LHS} = (\tan^2 A\sec^2 B - \sec^2 A\tan^2 B)$

$$= \tan^2 A(1+\tan^2 B) - (1+\tan^2 A)\tan^2 B$$

$$= (\tan^2 A - \tan^2 B) = \text{RHS}.$$

EXAMPLE 32 *Prove that*

$$(\tan^2 A - \tan^2 B) = \frac{(\sin^2 A - \sin^2 B)}{\cos^2 A\cos^2 B} = \frac{(\cos^2 B - \cos^2 A)}{\cos^2 B\cos^2 A}.$$

[CBSE 2005]

SOLUTION We have

$$(\tan^2 A - \tan^2 B) = \frac{\sin^2 A}{\cos^2 A} - \frac{\sin^2 B}{\cos^2 B}$$

$$= \frac{\sin^2 A \cos^2 B - \cos^2 A \sin^2 B}{\cos^2 A \cos^2 B}$$

$$= \frac{\sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A)\sin^2 B}{\cos^2 A \cos^2 B}$$

$$= \frac{(\sin^2 A - \sin^2 B)}{\cos^2 A \cos^2 B}.$$

Also, $(\tan^2 A - \tan^2 B) = \dfrac{\sin^2 A}{\cos^2 A} - \dfrac{\sin^2 B}{\cos^2 B}$

$$= \frac{\sin^2 A \cos^2 B - \cos^2 A \sin^2 B}{\cos^2 A \cos^2 B}$$

$$= \frac{(1 - \cos^2 A)\cos^2 B - \cos^2 A (1 - \cos^2 B)}{\cos^2 B \cos^2 A}$$

$$= \frac{(\cos^2 B - \cos^2 A)}{\cos^2 B \cos^2 A}.$$

Hence, $(\tan^2 A - \tan^2 B) = \dfrac{(\sin^2 A - \sin^2 B)}{\cos^2 A \cos^2 B} = \dfrac{(\cos^2 B - \cos^2 A)}{\cos^2 B \cos^2 A}$.

EXERCISE 13A

Prove each of the following identities:

1. (i) $(1 - \cos^2\theta)\operatorname{cosec}^2\theta = 1$ (ii) $(1 + \cot^2\theta)\sin^2\theta = 1$

2. (i) $(\sec^2\theta - 1)\cot^2\theta = 1$ (ii) $(\sec^2\theta - 1)(\operatorname{cosec}^2\theta - 1) = 1$

   (iii) $(1 - \cos^2\theta)\sec^2\theta = \tan^2\theta$

3. (i) $\sin^2\theta + \dfrac{1}{(1 + \tan^2\theta)} = 1$ (ii) $\dfrac{1}{(1 + \tan^2\theta)} + \dfrac{1}{(1 + \cot^2\theta)} = 1$

4. (i) $(1 + \cos\theta)(1 - \cos\theta)(1 + \cot^2\theta) = 1$

   (ii) $\operatorname{cosec}\theta(1 + \cos\theta)(\operatorname{cosec}\theta - \cot\theta) = 1$

5. (i) $\cot^2\theta - \dfrac{1}{\sin^2\theta} = -1$ (ii) $\tan^2\theta - \dfrac{1}{\cos^2\theta} = -1$

   (iii) $\cos^2\theta + \dfrac{1}{(1 + \cot^2\theta)} = 1$

6. $\dfrac{1}{(1 + \sin\theta)} + \dfrac{1}{(1 - \sin\theta)} = 2\sec^2\theta$

7. (i) $\sec\theta(1-\sin\theta)(\sec\theta+\tan\theta)=1$

(ii) $\sin\theta(1+\tan\theta)+\cos\theta(1+\cot\theta)=(\sec\theta+\operatorname{cosec}\theta)$ [CBSE 2008]

8. (i) $1+\dfrac{\cot^2\theta}{(1+\operatorname{cosec}\theta)}=\operatorname{cosec}\theta$

(ii) $1+\dfrac{\tan^2\theta}{(1+\sec\theta)}=\sec\theta$

9. $\dfrac{(1+\tan^2\theta)\cot\theta}{\operatorname{cosec}^2\theta}=\tan\theta$

10. $\dfrac{\tan^2\theta}{(1+\tan^2\theta)}+\dfrac{\cot^2\theta}{(1+\cot^2\theta)}=1$

11. $\dfrac{\sin\theta}{(1+\cos\theta)}+\dfrac{(1+\cos\theta)}{\sin\theta}=2\operatorname{cosec}\theta$

12. $\dfrac{\tan\theta}{(1-\cot\theta)}+\dfrac{\cot\theta}{(1-\tan\theta)}=(1+\sec\theta\operatorname{cosec}\theta)$ [CBSE 2008C]

13. $\dfrac{\cos^2\theta}{(1-\tan\theta)}+\dfrac{\sin^3\theta}{(\sin\theta-\cos\theta)}=(1+\sin\theta\cos\theta)$

14. $\dfrac{\cos\theta}{(1-\tan\theta)}-\dfrac{\sin^2\theta}{(\cos\theta-\sin\theta)}=(\cos\theta+\sin\theta)$

15. $(1+\tan^2\theta)(1+\cot^2\theta)=\dfrac{1}{(\sin^2\theta-\sin^4\theta)}$

16. $\dfrac{\tan\theta}{(1+\tan^2\theta)^2}+\dfrac{\cot\theta}{(1+\cot^2\theta)^2}=\sin\theta\cos\theta$

17. (i) $\sin^6\theta+\cos^6\theta=1-3\sin^2\theta\cos^2\theta$ [CBSE 2007]

(ii) $\sin^2\theta+\cos^4\theta=\cos^2\theta+\sin^4\theta$

(iii) $\operatorname{cosec}^4\theta-\operatorname{cosec}^2\theta=\cot^4\theta+\cot^2\theta$

18. (i) $\dfrac{1-\tan^2\theta}{1+\tan^2\theta}=(\cos^2\theta-\sin^2\theta)$

(ii) $\dfrac{1-\tan^2\theta}{\cot^2-1}=\tan^2\theta$

19. (i) $\dfrac{\tan\theta}{(\sec\theta-1)}+\dfrac{\tan\theta}{(\sec\theta+1)}=2\operatorname{cosec}\theta$

(ii) $\dfrac{\cot\theta}{(\operatorname{cosec}\theta+1)}+\dfrac{(\operatorname{cosec}\theta+1)}{\cot\theta}=2\sec\theta$

20. (i) $\dfrac{\sec\theta-1}{\sec\theta+1}=\dfrac{\sin^2\theta}{(1+\cos\theta)^2}$

(ii) $\dfrac{\sec\theta-\tan\theta}{\sec\theta+\tan\theta}=\dfrac{\cos^2\theta}{(1+\sin\theta)^2}$

21. (i) $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} = (\sec\theta + \tan\theta)$

(ii) $\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = (\text{cosec}\,\theta - \cot\theta)$

(iii) $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} + \sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = 2\,\text{cosec}\,\theta$

22. $\dfrac{\cos^3\theta + \sin^3\theta}{\cos\theta + \sin\theta} + \dfrac{\cos^3\theta - \sin^3\theta}{\cos\theta - \sin\theta} = 2$ [CBSE 2009C]

23. $\dfrac{\sin\theta}{(\cot\theta + \text{cosec}\,\theta)} - \dfrac{\sin\theta}{(\cot\theta - \text{cosec}\,\theta)} = 2$

24. (i) $\dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta} + \dfrac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta} = \dfrac{2}{(2\sin^2\theta - 1)}$

(ii) $\dfrac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta} + \dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta} = \dfrac{2}{(1 - 2\cos^2\theta)}$

25. $\dfrac{1 + \cos\theta - \sin^2\theta}{\sin\theta(1+\cos\theta)} = \cot\theta$ [CBSE 2003]

26. (i) $\dfrac{\text{cosec}\,\theta + \cot\theta}{\text{cosec}\,\theta - \cot\theta} = (\text{cosec}\,\theta + \cot\theta)^2 = 1 + 2\cot^2\theta + 2\,\text{cosec}\,\theta\cot\theta$

[CBSE 2003]

(ii) $\dfrac{\sec\theta + \tan\theta}{\sec\theta - \tan\theta} = (\sec\theta + \tan\theta)^2 = 1 + 2\tan^2\theta + 2\sec\theta\tan\theta$

27. (i) $\dfrac{1 + \cos\theta + \sin\theta}{1 + \cos\theta - \sin\theta} = \dfrac{1 + \sin\theta}{\cos\theta}$

(ii) $\dfrac{\sin\theta + 1 - \cos\theta}{\cos\theta - 1 + \sin\theta} = \dfrac{1 + \sin\theta}{\cos\theta}$ [CBSE 2001C]

28. $\dfrac{\sin\theta}{(\sec\theta + \tan\theta - 1)} + \dfrac{\cos\theta}{(\text{cosec}\,\theta + \cot\theta - 1)} = 1$

29. $\dfrac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta} + \dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta} = \dfrac{2}{(\sin^2\theta - \cos^2\theta)} = \dfrac{2}{(2\sin^2\theta - 1)}$ [CBSE 2007]

30. $\dfrac{\cos\theta\,\text{cosec}\,\theta - \sin\theta\sec\theta}{\cos\theta + \sin\theta} = \text{cosec}\,\theta - \sec\theta$

31. $(1 + \tan\theta + \cot\theta)(\sin\theta - \cos\theta) = \left(\dfrac{\sec\theta}{\text{cosec}^2\theta} - \dfrac{\text{cosec}\,\theta}{\sec^2\theta}\right)$

32. $\dfrac{\cot^2\theta(\sec\theta - 1)}{(1 + \sin\theta)} + \dfrac{\sec^2\theta(\sin\theta - 1)}{(1 + \sec\theta)} = 0$

33. $\left\{\dfrac{1}{(\sec^2\theta - \cos^2\theta)} + \dfrac{1}{(\text{cosec}^2\theta - \sin^2\theta)}\right\}(\sin^2\theta\cos^2\theta) = \dfrac{1 - \sin^2\theta\cos^2\theta}{2 + \sin^2\theta\cos^2\theta}$

34. $\dfrac{(\sin A - \sin B)}{(\cos A + \cos B)} + \dfrac{(\cos A - \cos B)}{(\sin A + \sin B)} = 0$

35. $\dfrac{\tan A+\tan B}{\cot A+\cot B}=\tan A\tan B$

36. Show that none of the following is an identity:

(i) $\cos^2\theta+\cos\theta=1$ (ii) $\sin^2\theta+\sin\theta=2$ (iii) $\tan^2\theta+\sin\theta=\cos^2\theta$

37. Prove that: $(\sin\theta-2\sin^3\theta)=(2\cos^3\theta-\cos\theta)\tan\theta$.

38. If $1+\sin^2\theta=3\sin\theta\cos\theta$ then prove that $\tan\theta=1$ or $\frac{1}{2}\cdot$

HINTS TO SOME SELECTED QUESTIONS

8. (i) Write $\cot^2\theta=(\operatorname{cosec}^2\theta-1)$.

(ii) Write $\tan^2\theta=(\sec^2\theta-1)$.

10. LHS $=\dfrac{\tan^2\theta}{\sec^2\theta}+\dfrac{\cot^2\theta}{\operatorname{cosec}^2\theta}=\left(\dfrac{\sin^2\theta}{\cos^2\theta}\times\cos^2\theta\right)+\left(\dfrac{\cos^2\theta}{\sin^2\theta}\times\sin^2\theta\right)=(\sin^2\theta+\cos^2\theta)=1.$

12. Write $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta=\dfrac{\cos\theta}{\sin\theta}\cdot$

13. Write $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$ and simplify.

14. Write $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$ and simplify.

15. LHS $=\sec^2\theta\cdot\operatorname{cosec}^2\theta=\dfrac{1}{\cos^2\theta\sin^2\theta}$

$=\dfrac{1}{\sin^2\theta(1-\sin^2\theta)}=\dfrac{1}{(\sin^2\theta-\sin^4\theta)}\cdot$

16. LHS $=\dfrac{\tan\theta}{\sec^4\theta}+\dfrac{\cot\theta}{\operatorname{cosec}^4\theta}=\left(\dfrac{\sin\theta}{\cos\theta}\times\cos^4\theta\right)+\left(\dfrac{\cos\theta}{\sin\theta}\times\sin^4\theta\right)$

$=(\sin\theta\cos^3\theta+\sin^3\theta\cos\theta)=\sin\theta\cos\theta(\cos^2\theta+\sin^2\theta)=\sin\theta\cos\theta.$

17. (i) $\sin^6\theta+\cos^6\theta=(\sin^2\theta)^3+(\cos^2\theta)^3$

$=(\sin^2\theta+\cos^2\theta)(\sin^4\theta+\cos^4\theta-\sin^2\theta\cos^2\theta)$

$=(\sin^2\theta+\cos^2\theta)^2-3\sin^2\theta\cos^2\theta=(1-3\sin^2\theta\cos^2\theta).$

(ii) $(\cos^4\theta-\sin^4\theta)=(\cos^2\theta-\sin^2\theta)(\cos^2\theta+\sin^2\theta)$

$\Rightarrow\quad(\cos^4\theta-\sin^4\theta)=(\cos^2\theta-\sin^2\theta)$

$\Rightarrow\quad(\sin^2\theta+\cos^4\theta)=(\cos^2\theta+\sin^4\theta).$

(iii) $(\operatorname{cosec}^4\theta-\cot^4\theta)=(\operatorname{cosec}^2\theta+\cot^2\theta)(\operatorname{cosec}^2\theta-\cot^2\theta)$

$\Rightarrow\quad(\operatorname{cosec}^4\theta-\cot^4\theta)=(\operatorname{cosec}^2\theta+\cot^2\theta)$ $[\because\ \operatorname{cosec}^2\theta-\cot^2\theta=1]$

$\Rightarrow\quad(\operatorname{cosec}^4\theta-\operatorname{cosec}^2\theta)=(\cot^4\theta+\cot^2\theta).$

20. (i) LHS $=\dfrac{\left(\dfrac{1}{\cos\theta}-1\right)}{\left(\dfrac{1}{\cos\theta}+1\right)}=\dfrac{(1-\cos\theta)}{(1+\cos\theta)}\times\dfrac{(1+\cos\theta)}{(1+\cos\theta)}=\dfrac{(1-\cos^2\theta)}{(1+\cos\theta)^2}=\dfrac{\sin^2\theta}{(1+\cos\theta)^2}\cdot$

(ii) LHS $=\dfrac{\left(\dfrac{1}{\cos\theta}-\dfrac{\sin\theta}{\cos\theta}\right)}{\left(\dfrac{1}{\cos\theta}+\dfrac{\sin\theta}{\cos\theta}\right)}=\dfrac{(1-\sin\theta)}{(1+\sin\theta)}\times\dfrac{(1+\sin\theta)}{(1+\sin\theta)}=\dfrac{1-\sin^2\theta}{(1+\sin\theta)^2}=\dfrac{\cos^2\theta}{(1+\sin\theta)^2}\cdot$

21. (i) LHS $= \frac{\sqrt{1+\sin\theta}}{\sqrt{1-\sin\theta}} \times \frac{\sqrt{1+\sin\theta}}{\sqrt{1+\sin\theta}} = \frac{(1+\sin\theta)}{\sqrt{1-\sin^2\theta}} = \frac{(1+\sin\theta)}{\cos\theta} = \left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)$

$= (\sec\theta + \tan\theta)$.

22. $(\cos^3\theta + \sin^3\theta) = (\cos\theta + \sin\theta)(\cos^2\theta + \sin^2\theta - \cos\theta\sin\theta)$.

And, $(\cos^3\theta - \sin^3\theta) = (\cos\theta - \sin\theta)(\cos^2\theta + \sin^2\theta + \cos\theta\sin\theta)$.

23. Write $\cot\theta = \frac{\cos\theta}{\sin\theta}$ and $\text{cosec}\,\theta = \frac{1}{\sin\theta}$.

25. Write $\sin^2\theta = (1 - \cos^2\theta)$.

26. (i) LHS $= \frac{(\text{cosec}\,\theta + \cot\theta)}{(\text{cosec}\,\theta - \cot\theta)} \times \frac{(\text{cosec}\,\theta + \cot\theta)}{(\text{cosec}\,\theta + \cot\theta)} = (\text{cosec}\,\theta + \cot\theta)^2$.

And, $(\text{cosec}\,\theta + \cot\theta)^2 = \text{cosec}^2\theta + \cot^2\theta + 2\,\text{cosec}\,\theta\cot\theta$

$= (1 + \cot^2\theta) + \cot^2\theta + 2\,\text{cosec}\,\theta\cot\theta$.

(ii) LHS $= \frac{(\sec\theta + \tan\theta)}{(\sec\theta - \tan\theta)} \times \frac{(\sec\theta + \tan\theta)}{(\sec\theta + \tan\theta)}$.

27. (i) On dividing num. and denom. by $\cos\theta$, we get

$$\text{LHS} = \frac{(\sec\theta + \tan\theta) + 1}{(\sec\theta + 1 - \tan\theta)} = \frac{(\sec\theta + \tan\theta) + (\sec^2\theta - \tan^2\theta)}{(\sec\theta + 1 - \tan\theta)}$$

$$= \frac{(\sec\theta + \tan\theta)(1 + \sec\theta - \tan\theta)}{(1 + \sec\theta - \tan\theta)} = (\sec\theta + \tan\theta) = \left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right).$$

(ii) On dividing num. and denom. by $\cos\theta$, we get

$$\text{LHS} = \frac{\tan\theta + \sec\theta - 1}{1 - \sec\theta + \tan\theta} = \frac{(\sec\theta + \tan\theta) - (\sec^2\theta - \tan^2\theta)}{(\tan\theta - \sec\theta + 1)}$$

$$= \frac{(\sec\theta + \tan\theta)[1 - (\sec\theta - \tan\theta)]}{(\tan\theta - \sec\theta + 1)} = (\sec\theta + \tan\theta).$$

28. LHS $= \frac{\sin\theta}{\left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} - 1\right)} + \frac{\cos\theta}{\left(\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} - 1\right)}$

$= \left\{\frac{\sin\theta\cos\theta}{1 + \sin\theta - \cos\theta} + \frac{\sin\theta\cos\theta}{1 + \cos\theta - \sin\theta}\right\}$

$= (\sin\theta\cos\theta) \cdot \left\{\frac{1}{1 + (\sin\theta - \cos\theta)} + \frac{1}{1 - (\sin\theta - \cos\theta)}\right\}$

$= (\sin\theta\cos\theta) \cdot \left\{\frac{(1 - \sin\theta + \cos\theta) + (1 + \sin\theta - \cos\theta)}{1 - (\sin\theta - \cos\theta)^2}\right\}$

$= (\sin\theta\cos\theta) \times \frac{2}{2\sin\theta\cos\theta} = 1$.

31. LHS $= \left(1 + \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)(\sin\theta - \cos\theta) = \frac{(\sin\theta\cos\theta + \sin^2\theta + \cos^2\theta)(\sin\theta - \cos\theta)}{\sin\theta\cos\theta}$

$= \frac{(\sin^3\theta - \cos^3\theta)}{\sin\theta\cos\theta} = \left(\frac{\sin^3\theta}{\sin\theta\cos\theta} - \frac{\cos^3\theta}{\sin\theta\cos\theta}\right) = \left(\frac{\sin^2\theta}{\cos\theta} - \frac{\cos^2\theta}{\sin\theta}\right)$

$= \left(\frac{\sec\theta}{\text{cosec}^2\theta} - \frac{\text{cosec}\,\theta}{\sec^2\theta}\right) = \text{RHS}$.

32. LHS $= \frac{\cot^2\theta(\sec^2\theta - 1) + \sec^2\theta(\sin^2\theta - 1)}{(1 + \sin\theta)(1 + \sec\theta)} = \frac{(\cot^2\theta\tan^2\theta) - (\sec^2\theta\cos^2\theta)}{(1 + \sin\theta)(1 + \sec\theta)} = 0$.

33. $\text{LHS} = \left\{ \dfrac{1}{\left(\dfrac{1}{\cos^2\theta} - \cos^2\theta\right)} + \dfrac{1}{\left(\dfrac{1}{\sin^2\theta} - \sin^2\theta\right)} \right\} (\sin^2\theta \cos^2\theta)$

$= \left\{ \dfrac{\cos^2\theta}{(1-\cos^4\theta)} + \dfrac{\sin^2\theta}{(1-\sin^4\theta)} \right\} \sin^2\theta \cos^2\theta$

$= \left\{ \dfrac{\cos^2\theta}{\sin^2\theta(1+\cos^2\theta)} + \dfrac{\sin^2\theta}{\cos^2\theta(1+\sin^2\theta)} \right\} \sin^2\theta \cos^2\theta$

$= \dfrac{\cos^4\theta(1+\sin^2\theta) + \sin^4\theta(1+\cos^2\theta)}{(1+\cos^2\theta)(1+\sin^2\theta)}$

$= \dfrac{\cos^4\theta + \sin^4\theta + \cos^2\theta \sin^2\theta(\cos^2\theta + \sin^2\theta)}{1 + (\sin^2\theta + \cos^2\theta) + \sin^2\theta\cos^2\theta}$

$= \dfrac{(\cos^2\theta + \sin^2\theta)^2 - \sin^2\theta\cos^2\theta}{2 + \sin^2\theta\cos^2\theta} = \dfrac{1 - \sin^2\theta\cos^2\theta}{2 + \sin^2\theta\cos^2\theta}.$

34. $\text{LHS} = \dfrac{(\sin A - \sin B)(\sin A + \sin B) + (\cos A - \cos B)(\cos A + \cos B)}{(\cos A + \cos B)(\sin A + \sin B)}$

$= \dfrac{(\sin^2 A - \sin^2 B) + (\cos^2 A - \cos^2 B)}{(\cos A + \cos B)(\sin A + \sin B)} = \dfrac{(\sin^2 A + \cos^2 A) - (\sin^2 B + \cos^2 B)}{(\cos A + \cos B)(\sin A + \sin B)}$

$= \dfrac{(1-1)}{(\cos A + \cos B)(\sin A + \sin B)} = 0.$

35. $\text{LHS} = \dfrac{\tan A + \tan B}{\dfrac{1}{\tan A} + \dfrac{1}{\tan B}} = \dfrac{\tan A \tan B(\tan A + \tan B)}{(\tan A + \tan B)} = \tan A \tan B.$

36. (i) $\cos^2 30^\circ + \cos 30^\circ = \left(\dfrac{\sqrt{3}}{2}\right)^2 + \dfrac{\sqrt{3}}{2} = \dfrac{3}{4} + \dfrac{\sqrt{3}}{2} = \dfrac{3 + 2\sqrt{3}}{4} \neq 1.$

(ii) $\sin^2 30^\circ + \sin 30^\circ = \left(\dfrac{1}{2}\right)^2 + \dfrac{1}{2} = \dfrac{1}{4} + \dfrac{1}{2} = \dfrac{3}{4} \neq 2.$

(iii) $\text{LHS} = \tan^2 45^\circ + \sin 45^\circ = (1)^2 + \dfrac{1}{\sqrt{2}} = 1 + \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}+1}{\sqrt{2}}.$

$\text{RHS} = \cos^2 45^\circ = \left(\dfrac{1}{\sqrt{2}}\right)^2 = \dfrac{1}{2}.$

$\therefore$ LHS $\neq$ RHS.

38. $1 + \sin^2\theta = 3\sin\theta\cos\theta \Rightarrow \sec^2\theta + \tan^2\theta = 3\tan\theta$ [dividing throughout by $\cos^2\theta$]

$\Rightarrow (1 + \tan^2\theta) + \tan^2\theta = 3\tan\theta$

$\Rightarrow 2\tan^2\theta - 3\tan\theta + 1 = 0$

$\Rightarrow 2\tan^2\theta - 2\tan\theta - \tan\theta + 1 = 0$

$\Rightarrow 2\tan\theta(\tan\theta - 1) - (\tan\theta - 1) = 0$

$\Rightarrow (\tan\theta - 1)(2\tan\theta - 1) = 0$

$\Rightarrow \tan\theta = 1 \text{ or } \dfrac{1}{2}.$

ELIMINATION OF TRIGONOMETRIC RATIOS

In order to eliminate the T-ratios from given relations, we make use of the fundamental trigonometrical identities, as shown in the examples given below.

SOLVED EXAMPLES

EXAMPLE 1 *If* $x = a\sin\theta + b\cos\theta$ *and* $y = a\cos\theta - b\sin\theta$, *prove that* $x^2 + y^2 = a^2 + b^2$.

SOLUTION We have

$$x^2 + y^2 = (a\sin\theta + b\cos\theta)^2 + (a\cos\theta - b\sin\theta)^2$$
$$= a^2(\sin^2\theta + \cos^2\theta) + b^2(\cos^2\theta + \sin^2\theta)$$
$$= a^2 + b^2 \quad [\because \sin^2\theta + \cos^2\theta = 1].$$

Hence, $x^2 + y^2 = a^2 + b^2$.

EXAMPLE 2 *If* $x = a\sin\theta$ *and* $y = b\tan\theta$, *prove that* $\left(\frac{a^2}{x^2} - \frac{b^2}{y^2}\right) = 1$.

SOLUTION We have

$$x = a\sin\theta \Rightarrow \frac{a}{x} = \frac{1}{\sin\theta} \Rightarrow \frac{a}{x} = \operatorname{cosec}\theta \quad \ldots \text{(i)}$$

and $y = b\tan\theta \Rightarrow \frac{b}{y} = \frac{1}{\tan\theta} \Rightarrow \frac{b}{y} = \cot\theta.$ … (ii)

Squaring (i) and (ii) and subtracting, we get

$$\left(\frac{a^2}{x^2} - \frac{b^2}{y^2}\right) = (\operatorname{cosec}^2\theta - \cot^2\theta) = 1.$$

Hence, $\left(\frac{a^2}{x^2} - \frac{b^2}{y^2}\right) = 1$.

EXAMPLE 3 *If* $\tan\theta + \sin\theta = m$ *and* $\tan\theta - \sin\theta = n$, *prove that*

$$(m^2 - n^2) = 4\sqrt{mn}.$$ [CBSE 2010]

SOLUTION We have

$$\text{LHS} = (m^2 - n^2)$$
$$= (\tan\theta + \sin\theta)^2 - (\tan\theta - \sin\theta)^2 = 4\tan\theta\sin\theta$$
$$[\because (a+b)^2 - (a-b)^2 = 4ab].$$

$$\text{RHS} = 4\sqrt{mn} = 4\sqrt{(\tan\theta + \sin\theta)(\tan\theta - \sin\theta)}$$
$$= 4\sqrt{(\tan^2\theta - \sin^2\theta)} = 4\cdot\sqrt{\left(\frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta\right)}$$

$$= 4 \cdot \frac{\sqrt{\sin^2\theta - \sin^2\theta\cos^2\theta}}{\cos\theta} = 4 \cdot \frac{\sin\theta}{\cos\theta} \cdot \sqrt{1-\cos^2\theta}$$

$$= 4\tan\theta \cdot \sqrt{\sin^2\theta} = 4\tan\theta\sin\theta.$$

Thus, LHS = RHS.

Hence, $(m^2 - n^2) = 4\sqrt{mn}$.

EXAMPLE 4 *If* $\sec\theta + \tan\theta = m$, *show that* $\dfrac{(m^2-1)}{(m^2+1)} = \sin\theta$. [CBSE 2004]

SOLUTION We have

$$\begin{aligned}(m^2-1) &= (\sec\theta+\tan\theta)^2 - 1\\ &= \sec^2\theta + \tan^2\theta + 2\sec\theta\tan\theta - 1\\ &= (\sec^2\theta - 1) + \tan^2\theta + 2\sec\theta\tan\theta\\ &= 2\tan^2\theta + 2\sec\theta\tan\theta \qquad [\because \sec^2\theta - 1 = \tan^2\theta]\\ &= 2\tan\theta(\tan\theta + \sec\theta). \qquad \text{... (i)}\end{aligned}$$

$$\begin{aligned}(m^2+1) &= (\sec\theta+\tan\theta)^2 + 1\\ &= \sec^2\theta + \tan^2\theta + 2\sec\theta\tan\theta + 1\\ &= (1+\tan^2\theta) + \sec^2\theta + 2\sec\theta\tan\theta\\ &= 2\sec^2\theta + 2\sec\theta\tan\theta \qquad [\because 1+\tan^2\theta = \sec^2\theta]\\ &= 2\sec\theta(\sec\theta + \tan\theta). \qquad \text{... (ii)}\end{aligned}$$

From (i) and (ii), we get

$$\frac{(m^2-1)}{(m^2+1)} = \frac{\tan\theta}{\sec\theta} = \left(\frac{\sin\theta}{\cos\theta} \times \cos\theta\right) = \sin\theta.$$

Hence, $\dfrac{(m^2-1)}{(m^2+1)} = \sin\theta$.

EXAMPLE 5 *If* $\sin\theta + \cos\theta = m$ *and* $\sec\theta + \operatorname{cosec}\theta = n$, *prove that*

$$n(m^2-1) = 2m.$$

SOLUTION We have

$$\begin{aligned}n(m^2-1) &= (\sec\theta + \operatorname{cosec}\theta)[(\sin\theta+\cos\theta)^2 - 1]\\ &= \left(\frac{1}{\cos\theta} + \frac{1}{\sin\theta}\right)[(\sin^2\theta + \cos^2\theta) + 2\sin\theta\cos\theta - 1]\\ &= \frac{(\sin\theta+\cos\theta)}{\sin\theta\cos\theta} \cdot 2\sin\theta\cos\theta\\ &= 2(\sin\theta + \cos\theta) = 2m.\end{aligned}$$

Hence, $n(m^2-1) = 2m$.

EXAMPLE 6 *If* $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, *show that* $(\cos\theta - \sin\theta) = \sqrt{2}\sin\theta$.

[CBSE 2008C]

SOLUTION We have

$$\cos\theta + \sin\theta = \sqrt{2}\cos\theta$$

$\Rightarrow$ $(\cos\theta + \sin\theta)^2 = 2\cos^2\theta$

$\Rightarrow$ $\cos^2\theta + \sin^2\theta + 2\cos\theta\sin\theta = 2\cos^2\theta$

$\Rightarrow$ $\cos^2\theta - 2\cos\theta\sin\theta = \sin^2\theta$

$\Rightarrow$ $\cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta = \sin^2\theta + \sin^2\theta$

[adding $\sin^2\theta$ on both sides]

$\Rightarrow$ $(\cos\theta - \sin\theta)^2 = 2\sin^2\theta$

$\Rightarrow$ $(\cos\theta - \sin\theta) = \sqrt{2}\sin\theta.$

Hence, $(\cos\theta - \sin\theta) = \sqrt{2}\sin\theta.$

EXAMPLE 7 *If* $\sin\theta + \sin^2\theta = 1$, *prove that* $\cos^2\theta + \cos^4\theta = 1.$

SOLUTION $\sin\theta + \sin^2\theta = 1$

$\Rightarrow$ $\sin\theta = 1 - \sin^2\theta$

$\Rightarrow$ $\sin\theta = \cos^2\theta$ $\quad$ [$\because$ $1 - \sin^2\theta = \cos^2\theta$]

$\Rightarrow$ $\sin^2\theta = \cos^4\theta$

$\Rightarrow$ $1 - \cos^2\theta = \cos^4\theta$ $\quad$ [$\because$ $\sin^2\theta = 1 - \cos^2\theta$]

$\Rightarrow$ $\cos^2\theta + \cos^4\theta = 1.$

Hence, $\cos^2\theta + \cos^4\theta = 1.$

EXAMPLE 8 *If* $\cos\theta + \sin\theta = 1$, *prove that* $\cos\theta - \sin\theta = \pm 1.$

SOLUTION We have

$$(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2 = 2(\cos^2\theta + \sin^2\theta)$$

$\Rightarrow$ $(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2 = 2$ $\quad$ [$\because$ $(\cos^2\theta + \sin^2\theta) = 1$]

$\Rightarrow$ $1 + (\cos\theta - \sin\theta)^2 = 2$ $\quad$ [$\because$ $\cos\theta + \sin\theta = 1$ (given)]

$\Rightarrow$ $(\cos\theta - \sin\theta)^2 = 1$

$\Rightarrow$ $(\cos\theta - \sin\theta) = \pm 1$ $\quad$ [taking square root on both sides].

Hence, $(\cos\theta - \sin\theta) = \pm 1.$

EXAMPLE 9 *If* $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$ *and* $x\sin\theta = y\cos\theta$, *prove that* $x^2 + y^2 = 1.$

SOLUTION $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$

$\Rightarrow$ $(x\sin\theta)\sin^2\theta + (y\cos\theta)\cos^2\theta = \sin\theta\cos\theta$

$\Rightarrow$ $(x\sin\theta)\sin^2\theta + (x\sin\theta)\cos^2\theta = \sin\theta\cos\theta$

[$\because$ $y\cos\theta = x\sin\theta$]

$\Rightarrow \quad (x \sin \theta)(\sin^2\theta + \cos^2\theta) = \sin \theta \cos \theta$

$\Rightarrow \quad x \sin \theta = \sin \theta \cos \theta \qquad [\because \sin^2\theta + \cos^2\theta = 1]$

$\Rightarrow \quad x = \cos \theta. \qquad \ldots \text{(i)}$

Now, $x \sin \theta = y \cos \theta$

$\Rightarrow \quad \cos \theta \sin \theta = y \cos \theta \qquad [\because x = \cos \theta]$

$\Rightarrow \quad y = \sin \theta. \qquad \ldots \text{(ii)}$

On squaring (i) and (ii) and adding, we get $x^2 + y^2 = 1$.

Hence, $x^2 + y^2 = 1$.

EXAMPLE 10 *If* $x = a \sec \theta \cos \phi$, $y = b \sec \theta \sin \phi$ *and* $z = c \tan \theta$ *then prove that* $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = \left(1 + \frac{z^2}{c^2}\right)$.

SOLUTION We have

$$\frac{x}{a} = \sec \theta \cos \phi \quad \ldots \text{(i)} \quad \text{and} \quad \frac{y}{b} = \sec \theta \sin \phi \quad \ldots \text{(ii)}.$$

Squaring (i) and (ii) and adding, we get

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = \sec^2\theta (\cos^2\phi + \sin^2\phi)$$

$$= \sec^2\theta \qquad [\because \cos^2\phi + \sin^2\phi = 1]$$

$$= (1 + \tan^2\theta) = \left(1 + \frac{z^2}{c^2}\right) \qquad \left[\because \tan \theta = \frac{z}{c}\right]$$

Hence, $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = \left(1 + \frac{z^2}{c^2}\right)$.

EXAMPLE 11 *If* $x = r \sin \alpha \cos \beta$, $y = r \sin \alpha \sin \beta$ *and* $z = r \cos \alpha$, *prove that*

$$x^2 + y^2 + z^2 = r^2.$$

SOLUTION We have

$$x^2 + y^2 + z^2 = r^2 \sin^2\alpha \cos^2\beta + r^2 \sin^2\alpha \sin^2\beta + r^2 \cos^2\alpha$$

$$= r^2 \sin^2\alpha (\cos^2\beta + \sin^2\beta) + r^2 \cos^2\alpha$$

$$= r^2 \sin^2\alpha + r^2 \cos^2\alpha \qquad [\because \cos^2\beta + \sin^2\beta = 1]$$

$$= r^2 (\sin^2\alpha + \cos^2\alpha) = r^2 \quad [\because \sin^2\alpha + \cos^2\alpha = 1].$$

Hence, $(x^2 + y^2 + z^2) = r^2$.

EXAMPLE 12 *If* $\operatorname{cosec} \theta - \sin \theta = m$ *and* $\sec \theta - \cos \theta = n$, *prove that*

$$(m^2 n)^{2/3} + (mn^2)^{2/3} = 1.$$

SOLUTION We have

$$m^2 n = (\text{cosec}\,\theta - \sin\theta)^2 \cdot (\sec\theta - \cos\theta)$$

$$= \left(\frac{1}{\sin\theta} - \sin\theta\right)^2 \cdot \left(\frac{1}{\cos\theta} - \cos\theta\right)$$

$$= \frac{(1-\sin^2\theta)^2}{\sin^2\theta} \cdot \frac{(1-\cos^2\theta)}{\cos\theta} = \frac{\cos^4\theta}{\sin^2\theta} \times \frac{\sin^2\theta}{\cos\theta} = \cos^3\theta$$

$\therefore \quad (m^2 n)^{1/3} = \cos\theta.$ … (i)

Again, $mn^2 = (\text{cosec}\,\theta - \sin\theta) \cdot (\sec\theta - \cos\theta)^2$

$$= \left(\frac{1}{\sin\theta} - \sin\theta\right) \cdot \left(\frac{1}{\cos\theta} - \cos\theta\right)^2$$

$$= \frac{(1-\sin^2\theta)}{\sin\theta} \cdot \frac{(1-\cos^2\theta)^2}{\cos^2\theta}$$

$$= \left(\frac{\cos^2\theta}{\sin\theta} \times \frac{\sin^4\theta}{\cos^2\theta}\right) = \sin^3\theta$$

$\therefore \quad (mn^2)^{1/3} = \sin\theta.$ … (ii)

On squaring (i) and (ii) and adding the results, we get

$$(m^2 n)^{2/3} + (mn^2)^{2/3} = 1 \qquad [\because \cos^2\theta + \sin^2\theta = 1].$$

Hence, $(m^2 n)^{2/3} + (mn^2)^{2/3} = 1.$

EXAMPLE 13 *If* $a\cos\theta - b\sin\theta = c$, *prove that*

$$(a\sin\theta + b\cos\theta) = \pm\sqrt{a^2 + b^2 - c^2}.$$ [CBSE 2006C]

SOLUTION Given, $a\cos\theta - b\sin\theta = c.$ … (i)

Now, $(a\cos\theta - b\sin\theta)^2 + (a\sin\theta + b\cos\theta)^2$

$$= a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta) = (a^2 + b^2).$$

Thus, $(a\cos\theta - b\sin\theta)^2 + (a\sin\theta + b\cos\theta)^2 = (a^2 + b^2)$

$\Rightarrow \quad c^2 + (a\sin\theta + b\cos\theta)^2 = (a^2 + b^2)$

$\Rightarrow \quad (a\sin\theta + b\cos\theta)^2 = (a^2 + b^2 - c^2)$

$\Rightarrow \quad (a\sin\theta + b\cos\theta) = \pm\sqrt{a^2 + b^2 - c^2}.$

Hence, $(a\sin\theta + b\cos\theta) = \pm\sqrt{a^2 + b^2 - c^2}.$

EXAMPLE 14 *If* $(3\sin\theta + 5\cos\theta) = 5$, *prove that* $(5\sin\theta - 3\cos\theta) = \pm 3.$

SOLUTION We have

$$(3\sin\theta + 5\cos\theta)^2 + (5\sin\theta - 3\cos\theta)^2$$
$$= 9(\sin^2\theta + \cos^2\theta) + 25(\sin^2\theta + \cos^2\theta)$$
$$= (9 + 25) = 34.$$

$\therefore\quad (3\sin\theta + 5\cos\theta)^2 + (5\sin\theta - 3\cos\theta)^2 = 34$

$\Rightarrow\quad 5^2 + (5\sin\theta - 3\cos\theta)^2 = 34$ [$\because$ $3\sin\theta + 5\cos\theta = 5$]

$\Rightarrow\quad (5\sin\theta - 3\cos\theta) = \pm 3$ [taking square root on each side]

Hence, $(5\sin\theta - 3\cos\theta) = \pm 3$.

EXERCISE 13B

1. If $a\cos\theta + b\sin\theta = m$ and $a\sin\theta - b\cos\theta = n$, prove that
$$(m^2 + n^2) = (a^2 + b^2).$$
2. If $x = a\sec\theta + b\tan\theta$ and $y = a\tan\theta + b\sec\theta$, prove that
$$(x^2 - y^2) = (a^2 - b^2).$$
3. If $\left(\frac{x}{a}\sin\theta - \frac{y}{b}\cos\theta\right) = 1$ and $\left(\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta\right) = 1$, prove that
$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = 2.$$
4. If $(\sec\theta + \tan\theta) = m$ and $(\sec\theta - \tan\theta) = n$, show that $mn = 1$.
5. If $(\text{cosec}\,\theta + \cot\theta) = m$ and $(\text{cosec}\,\theta - \cot\theta) = n$, show that $mn = 1$.
6. If $x = a\cos^3\theta$ and $y = b\sin^3\theta$, prove that $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$.
7. If $(\tan\theta + \sin\theta) = m$ and $(\tan\theta - \sin\theta) = n$, prove that
$$(m^2 - n^2)^2 = 16mn.$$
8. If $(\cot\theta + \tan\theta) = m$ and $(\sec\theta - \cos\theta) = n$, prove that
$$(m^2 n)^{2/3} - (mn^2)^{2/3} = 1.$$
9. If $(\text{cosec}\,\theta - \sin\theta) = a^3$ and $(\sec\theta - \cos\theta) = b^3$, prove that
$$a^2 b^2 (a^2 + b^2) = 1.$$
10. If $(2\sin\theta + 3\cos\theta) = 2$, prove that $(3\sin\theta - 2\cos\theta) = \pm 3$.
11. If $(\sin\theta + \cos\theta) = \sqrt{2}\cos\theta$, show that $\cot\theta = (\sqrt{2} + 1)$.
12. If $(\cos\theta + \sin\theta) = \sqrt{2}\sin\theta$, prove that $(\sin\theta - \cos\theta) = \sqrt{2}\cos\theta$.
13. If $\sec\theta + \tan\theta = p$, prove that

(i) $\sec\theta = \frac{1}{2}\left(p + \frac{1}{p}\right)$ (ii) $\tan\theta = \frac{1}{2}\left(p - \frac{1}{p}\right)$ (iii) $\sin\theta = \frac{p^2 - 1}{p^2 + 1}$.

14. If $\tan A = n \tan B$ and $\sin A = m \sin B$, prove that $\cos^2 A = \dfrac{(m^2-1)}{(n^2-1)}$.

15. If $m = (\cos\theta - \sin\theta)$ and $n = (\cos\theta + \sin\theta)$ then show that

$$\sqrt{\frac{m}{n}} + \sqrt{\frac{n}{m}} = \frac{2}{\sqrt{1-\tan^2\theta}}.$$

HINTS TO SOME SELECTED QUESTIONS

7. See Example 3.

8. $m = \left(\dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta}\right) = \left(\dfrac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}\right) = \dfrac{1}{\sin\theta\cos\theta}$.

$n = \left(\dfrac{1}{\cos\theta} - \cos\theta\right) = \left(\dfrac{1-\cos^2\theta}{\cos\theta}\right) = \dfrac{\sin^2\theta}{\cos\theta}$.

$\therefore \quad m^2 n = \left(\dfrac{1}{\sin^2\theta\cos^2\theta} \times \dfrac{\sin^2\theta}{\cos\theta}\right) = \dfrac{1}{\cos^3\theta} = \sec^3\theta$

and $mn^2 = \left(\dfrac{1}{\sin\theta\cos\theta} \times \dfrac{\sin^4\theta}{\cos^2\theta}\right) = \dfrac{\sin^3\theta}{\cos^3\theta} = \tan^3\theta$.

$\therefore \quad (m^2n)^{2/3} - (mn^2)^{2/3} = (\sec^3\theta)^{2/3} - (\tan^3\theta)^{2/3} = (\sec^2\theta - \tan^2\theta) = 1.$

9. $a^3 = \left(\dfrac{1}{\sin\theta} - \sin\theta\right) = \left(\dfrac{1-\sin^2\theta}{\sin\theta}\right) = \dfrac{\cos^2\theta}{\sin\theta} \Rightarrow a = \dfrac{\cos^{2/3}\theta}{\sin^{1/3}\theta}$.

$b^3 = \left(\dfrac{1}{\cos\theta} - \cos\theta\right) = \left(\dfrac{1-\cos^2\theta}{\cos\theta}\right) = \dfrac{\sin^2\theta}{\cos\theta} \Rightarrow b = \dfrac{\sin^{2/3}\theta}{\cos^{1/3}\theta}$.

$\therefore \quad a^2b^2(a^2+b^2) = a^4b^2 + a^2b^4 = a^3(ab^2) + (a^2b)b^3$

$$= \frac{\cos^2\theta}{\sin\theta} \cdot \left[\frac{\cos^{2/3}\theta}{\sin^{1/3}\theta} \cdot \frac{\sin^{4/3}\theta}{\cos^{2/3}\theta}\right] + \left[\frac{\cos^{4/3}\theta}{\sin^{2/3}\theta} \cdot \frac{\sin^{2/3}\theta}{\cos^{1/3}\theta}\right] \cdot \frac{\sin^2\theta}{\cos\theta}$$

$$= \frac{\cos^2\theta}{\sin\theta} \cdot \sin\theta + \cos\theta \cdot \frac{\sin^2\theta}{\cos\theta} = (\cos^2\theta + \sin^2\theta) = 1.$$

10. $(2\sin\theta + 3\cos\theta)^2 + (3\sin\theta - 2\cos\theta)^2 = 13(\sin^2\theta + \cos^2\theta) = 13$

$\Rightarrow \quad 2^2 + (3\sin\theta - 2\cos\theta)^2 = 13 \Rightarrow (3\sin\theta - 2\cos\theta)^2 = 9$

$\Rightarrow \quad (3\sin\theta - 2\cos\theta) = \pm 3.$

11. $\sin\theta + \cos\theta = \sqrt{2}\cos\theta$

$\Rightarrow \quad 1 + \cot\theta = \sqrt{2}\cot\theta \quad$ [dividing both side by $\sin\theta$]

$\Rightarrow \quad (\sqrt{2}-1)\cot\theta = 1 \Rightarrow \cot\theta = \dfrac{1}{(\sqrt{2}-1)} = \dfrac{1}{(\sqrt{2}-1)} \times \dfrac{(\sqrt{2}+1)}{(\sqrt{2}+1)} = (\sqrt{2}+1).$

12. $(\cos\theta + \sin\theta)^2 = 2\sin^2\theta \Rightarrow \cos^2\theta + \sin^2\theta + 2\cos\theta\sin\theta = 2\sin^2\theta$

$\Rightarrow \quad \sin^2\theta - 2\cos\theta\sin\theta = \cos^2\theta \Rightarrow \sin^2\theta - 2\cos\theta\sin\theta + \cos^2\theta = 2\cos^2\theta$

$\Rightarrow \quad (\sin\theta - \cos\theta)^2 = 2\cos^2\theta \Rightarrow (\sin\theta - \cos\theta) = \sqrt{2}\cos\theta.$

13. $\sec\theta + \tan\theta = p.$ ... (i)

$\sec^2\theta - \tan^2\theta = 1.$ ... (ii)

On dividing (ii) by (i), we get

$$\sec\theta - \tan\theta = \frac{1}{p}. \quad \text{... (iii)}$$

$\therefore$ from (i) and (iii), we get $\sec\theta = \frac{1}{2}\left(p + \frac{1}{p}\right)$, $\tan\theta = \frac{1}{2}\left(p - \frac{1}{p}\right)$.

Also, $\sin\theta = \frac{\tan\theta}{\sec\theta} = \frac{\frac{1}{2}\left(p - \frac{1}{p}\right)}{\frac{1}{2}\left(p + \frac{1}{p}\right)} = \frac{(p^2 - 1)}{(p^2 + 1)}.$

14. We have to eliminate B. So, we find cosec B and cot B from the given relations and use the identity $\operatorname{cosec}^2 B - \cot^2 B = 1$.

$$\sin A = m \sin B \Rightarrow \operatorname{cosec} B = \frac{m}{\sin A}. \quad \text{... (i)}$$

$$\tan A = n \tan B \Rightarrow \cot B = \frac{n}{\tan A}. \quad \text{... (ii)}$$

Squaring (i) and (ii), and subtracting the results, we get

$$\frac{m^2}{\sin^2 A} - \frac{n^2}{\tan^2 A} = 1 \Rightarrow m^2 - n^2\cos^2 A = \sin^2 A \Rightarrow m^2 - n^2\cos^2 A = 1 - \cos^2 A$$

$$\Rightarrow (n^2 - 1)\cos^2 A = (m^2 - 1) \Rightarrow \cos^2 A = \frac{(m^2 - 1)}{(n^2 - 1)}.$$

15. LHS $= \frac{(m+n)}{\sqrt{mn}}.$

Now, $(m+n) = 2\cos\theta$ and $mn = (\cos^2\theta - \sin^2\theta) = \cos^2\theta(1 - \tan^2\theta)$.

$\therefore$ $\sqrt{mn} = \cos\theta\sqrt{1 - \tan^2\theta}$.

Hence, LHS $= \frac{(m+n)}{\sqrt{mn}} = \frac{2\cos\theta}{\cos\theta\sqrt{1 - \tan^2\theta}} = \frac{2}{\sqrt{1 - \tan^2\theta}} =$ RHS.

EXERCISE 13C

Very-Short and Short-Answer Questions

1. Write the value of $(1 - \sin^2\theta)\sec^2\theta$.
2. Write the value of $(1 - \cos^2\theta)\operatorname{cosec}^2\theta$.
3. Write the value of $(1 + \tan^2\theta)\cos^2\theta$.
4. Write the value of $(1 + \cot^2\theta)\sin^2\theta$.
5. Write the value of $\left(\sin^2\theta + \frac{1}{1 + \tan^2\theta}\right)$.
6. Write the value of $\left(\cot^2\theta - \frac{1}{\sin^2\theta}\right)$.
7. Write the value of $\sin\theta\cos(90° - \theta) + \cos\theta\sin(90° - \theta)$.
8. Write the value of $\operatorname{cosec}^2(90° - \theta) - \tan^2\theta$.

9. Write the value of $\sec^2\theta(1+\sin\theta)(1-\sin\theta)$. [CBSE 2009]
10. Write the value of $\operatorname{cosec}^2\theta(1+\cos\theta)(1-\sin\theta)$.
11. Write the value of $\sin^2\theta\cos^2\theta(1+\tan^2\theta)(1+\cot^2\theta)$.
12. Write the value of $(1+\tan^2\theta)(1+\sin\theta)(1-\sin\theta)$. [CBSE 2008]
13. Write the value of $3\cot^2\theta - 3\operatorname{cosec}^2\theta$.
14. Write the value of $4\tan^2\theta - \dfrac{4}{\cos^2\theta}$.
15. Write the value of $\dfrac{\tan^2\theta-\sec^2\theta}{\cot^2\theta-\operatorname{cosec}^2\theta}$.
16. If $\sin\theta = \dfrac{1}{2}$, write the value of $(3\cot^2\theta+3)$. [CBSE 2009]
17. If $\cos\theta = \dfrac{2}{3}$, write the value of $(4+4\tan^2\theta)$.
18. If $\cos\theta = \dfrac{7}{25}$, write the value of $(\tan\theta+\cot\theta)$. [CBSE 2008]
19. If $\cos\theta = \dfrac{2}{3}$, write the value of $\dfrac{(\sec\theta-1)}{(\sec\theta+1)}$.
20. If $5\tan\theta = 4$, write the value of $\dfrac{(\cos\theta-\sin\theta)}{(\cos\theta+\sin\theta)}$.
21. If $3\cot\theta = 4$, write the value of $\dfrac{(2\cos\theta+\sin\theta)}{(4\cos\theta-\sin\theta)}$.
22. If $\cot\theta = \dfrac{1}{\sqrt{3}}$, write the value of $\dfrac{(1-\cos^2\theta)}{(2-\sin^2\theta)}$.
23. If $\tan\theta = \dfrac{1}{\sqrt{5}}$, write the value of $\dfrac{(\operatorname{cosec}^2\theta-\sec^2\theta)}{(\operatorname{cosec}^2\theta+\sec^2\theta)}$.
24. If $\cot A = \dfrac{4}{3}$ and $(A+B) = 90^\circ$, what is the value of $\tan B$?
25. If $\cos B = \dfrac{3}{5}$ and $(A+B) = 90^\circ$, find the value of $\sin A$.
26. If $\sqrt{3}\sin\theta = \cos\theta$ and θ is an acute angle, find the value of θ.
27. Write the value of $\tan 10^\circ \tan 20^\circ \tan 70^\circ \tan 80^\circ$.
28. Write the value of $\tan 1^\circ \tan 2^\circ \ldots \tan 89^\circ$.
29. Write the value of $\cos 1^\circ \cos 2^\circ \ldots \cos 180^\circ$.
30. If $\tan A = \dfrac{5}{12}$, find the value of $(\sin A + \cos A)\sec A$. [CBSE 2008]
31. If $\sin\theta = \cos(\theta - 45^\circ)$, where θ is acute, find the value of θ.
32. Find the value of $\dfrac{\sin 50^\circ}{\cos 40^\circ} + \dfrac{\operatorname{cosec} 40^\circ}{\sec 50^\circ} - 4\cos 50^\circ \operatorname{cosec} 40^\circ$.
33. Find the value of $\sin 48^\circ \sec 42^\circ + \cos 48^\circ \operatorname{cosec} 42^\circ$.

34. If $x = a\sin\theta$ and $y = b\cos\theta$, write the value of $(b^2x^2 + a^2y^2)$.

35. If $5x = \sec\theta$ and $\frac{5}{x} = \tan\theta$, find the value of $5\left(x^2 - \frac{1}{x^2}\right)$. [CBSE 2010]

36. If $\operatorname{cosec}\theta = 2x$ and $\cot\theta = \frac{2}{x}$, find the value of $2\left(x^2 - \frac{1}{x^2}\right)$. [CBSE 2010]

37. If $\sec\theta + \tan\theta = x$, find the value of $\sec\theta$.

38. Find the value of $\frac{\cos 38^\circ \operatorname{cosec} 52^\circ}{\tan 18^\circ \tan 35^\circ \tan 60^\circ \tan 72^\circ \tan 55^\circ}$.

39. If $\sin\theta = x$, write the value of $\cot\theta$.

40. If $\sec\theta = x$, write the value of $\tan\theta$.

ANSWERS (EXERCISE 13C)

1. 1 **2.** 1 **3.** 1 **4.** 1 **5.** 1 **6.** –1 **7.** 1 **8.** 1 **9.** 1

10. 1 **11.** 1 **12.** 1 **13.** –3 **14.** –4 **15.** 1 **16.** 12 **17.** 9 **18.** $\frac{625}{168}$

19. $\frac{1}{5}$ **20.** $\frac{1}{9}$ **21.** $\frac{11}{13}$ **22.** $\frac{3}{5}$ **23.** $\frac{2}{3}$ **24.** $\frac{4}{3}$ **25.** $\frac{3}{5}$ **26.** 30° **27.** 1

28. 1 **29.** 0 **30.** $\frac{17}{12}$ **31.** $(67.5)^\circ$ **32.** –2 **33.** 2 **34.** a^2b^2 **35.** $\frac{1}{5}$

36. $\frac{1}{2}$ **37.** $\frac{x^2+1}{2x}$ **38.** $\frac{1}{\sqrt{3}}$ **39.** $\frac{\sqrt{1-x^2}}{x}$ **40.** $\sqrt{x^2-1}$

HINTS TO SOME SELECTED QUESTIONS

2. $(1 - \cos^2\theta)\operatorname{cosec}^2\theta = \sin^2\theta \cdot \operatorname{cosec}^2\theta = 1$.

4. $(1 + \cot^2\theta)\sin^2\theta = \operatorname{cosec}^2\theta \sin^2\theta = 1$.

5. $\left(\sin^2\theta + \frac{1}{1+\tan^2\theta}\right) = \left(\sin^2\theta + \frac{1}{\sec^2\theta}\right) = (\sin^2\theta + \cos^2\theta) = 1$.

6. $\left(\cot^2\theta - \frac{1}{\sin^2\theta}\right) = (\cot^2\theta - \operatorname{cosec}^2\theta) = \cot^2\theta - (1 + \cot^2\theta) = -1$.

7. $\sin\theta\cos(90^\circ - \theta) + \cos\theta\sin(90^\circ - \theta) = \sin\theta \cdot \sin\theta + \cos\theta \cdot \cos\theta = \sin^2\theta + \cos^2\theta = 1$.

8. $\operatorname{cosec}^2(90^\circ - \theta) - \tan^2\theta = \sec^2\theta - \tan^2\theta = 1$.

9. Given expression $= \sec^2\theta(1 - \sin^2\theta) = \sec^2\theta\cos^2\theta = 1$.

10. Given expression $= \operatorname{cosec}^2\theta(1 - \cos^2\theta) = \operatorname{cosec}^2\theta\sin^2\theta = 1$.

11. Given expression $= \sin^2\theta(1 + \cot^2\theta)\cos^2\theta(1 + \tan^2\theta)$

$= (\sin^2\theta\operatorname{cosec}^2\theta)(\cos^2\theta\sec^2\theta) = (1 \times 1) = 1$.

12. Given expression $= (1 + \tan^2\theta)(1 - \sin^2\theta) = \sec^2\theta\cos^2\theta = 1$.

13. Given expression $= 3\cot^2\theta - 3(1+\cot^2\theta) = -3.$

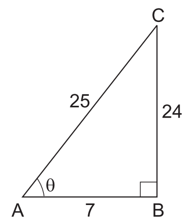
14. Given expression $= 4\tan^2\theta - 4\sec^2\theta = 4\tan^2\theta - 4(1+\tan^2\theta) = -4.$

15. Given expression $= \dfrac{(\sec^2\theta - \tan^2\theta)}{(\text{cosec}^2\theta - \cot^2\theta)} = \dfrac{1}{1} = 1.$

16. $3\cot^2\theta + 3 = 3(\cot^2\theta + 1) = 3\,\text{cosec}^2\theta = (3\times 4) = 12 \quad [\because\ \text{cosec}\,\theta = 2].$

17. $(4 + 4\tan^2\theta) = 4(1+\tan^2\theta) = 4\sec^2\theta = \left(4\times\dfrac{9}{4}\right) = 9 \quad \left[\because\ \sec\theta = \dfrac{3}{2}\right].$

18. Consider a right $\triangle ABC$ in which $\angle B = 90°$, $AB = 7$ and $AC = 25$.



$BC^2 = AC^2 - AB^2 = (25)^2 - (7)^2 = (625 - 49) = 576.$

$\therefore\quad BC = \sqrt{576} = 24.$

$\therefore\quad (\tan\theta + \cot\theta) = \left(\dfrac{24}{7} + \dfrac{7}{24}\right) = \dfrac{576+49}{168} = \dfrac{625}{168}.$

19. $\dfrac{(\sec\theta - 1)}{\sec\theta + 1)} = \dfrac{\left(\frac{3}{2} - 1\right)}{\left(\frac{3}{2} + 1\right)} = \left(\dfrac{1}{2}\times\dfrac{2}{5}\right) = \dfrac{1}{5}.$

20. $\dfrac{(\cos\theta - \sin\theta)}{(\cos\theta + \sin\theta)} = \dfrac{(1-\tan\theta)}{(1+\tan\theta)}$ [dividing num. and denom. by $\cos\theta$]

$= \dfrac{\left(1-\frac{4}{5}\right)}{\left(1+\frac{4}{5}\right)} = \dfrac{1}{9} \quad \left[\because\ \tan\theta = \dfrac{4}{5}\right].$

21. $\dfrac{(2\cos\theta + \sin\theta)}{(4\cos\theta - \sin\theta)} = \dfrac{(2\cot\theta + 1)}{(4\cot\theta - 1)}$ [dividing num. and denom. by $\sin\theta$]

$= \left(\dfrac{2\times\frac{4}{3}+1}{4\times\frac{4}{3}-1}\right) = \dfrac{11}{13}.$

22. $\dfrac{(1-\cos^2\theta)}{(2-\sin^2\theta)} = \dfrac{\sin^2\theta}{(2-\sin^2\theta)} = \dfrac{\frac{3}{4}}{\left(2-\frac{3}{4}\right)} \qquad \left[\text{cosec}^2\theta = (1+\cot^2\theta) = \left(1+\dfrac{1}{3}\right) = \dfrac{4}{3} \Rightarrow \sin^2\theta = \dfrac{3}{4}\right]$

$= \left(\dfrac{3}{4}\times\dfrac{4}{5}\right) = \dfrac{3}{5}.$

23. $\text{cosec}^2\theta = (1+\cot^2\theta) = (1+5) = 6,\ \sec^2\theta = (1+\tan^2\theta) = \left(1+\dfrac{1}{5}\right) = \dfrac{6}{5}.$

$\therefore\quad \dfrac{(\text{cosec}^2\theta - \sec^2\theta)}{(\text{cosec}^2\theta + \sec^2\theta)} = \dfrac{\left(6-\frac{6}{5}\right)}{\left(6+\frac{6}{5}\right)} = \dfrac{24}{36} = \dfrac{2}{3}.$

24. $A = 90° - B \Rightarrow \cot A = \cot(90° - B) = \tan B.$

$\therefore\quad \tan B = \cot A = \dfrac{4}{3}.$

25. $B = 90° - A \Rightarrow \cos B = \cos(90° - A) = \sin A$

$\therefore\quad \sin A = \cos B = \dfrac{3}{5}.$

26. $\sqrt{3}\sin\theta = \cos\theta \Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = 30^\circ$.

27. Given expression $= (\tan 10^\circ \tan 80^\circ)(\tan 20^\circ \tan 70^\circ)$
$= \{\tan 10^\circ \tan(90^\circ - 10^\circ)\}\{\tan 20^\circ \cdot \tan(90^\circ - 20^\circ)\}$
$= (\tan 10^\circ \cot 10^\circ)(\tan 20^\circ \cot 20^\circ) = (1 \times 1) = 1.$

28. Given expression
$= (\tan 1^\circ \tan 89^\circ)(\tan 2^\circ \tan 88^\circ)\ldots(\tan 44^\circ \tan 46^\circ)\tan 45^\circ$
$= \{\tan 1^\circ \tan(90^\circ - 1^\circ)\}\{\tan 2^\circ \tan(90^\circ - 2^\circ)\}\ldots\{\tan 44^\circ \tan(90^\circ - 44^\circ)\}\tan 45^\circ$
$= (\tan 1^\circ \cot 1^\circ)(\tan 2^\circ \cot 2^\circ)\ldots(\tan 44^\circ \cot 44^\circ) \times 1$
$= (1 \times 1 \times 1 \times \ldots \times 1 \times 1) = 1.$

29. Given expression
$= (\cos 1^\circ \cos 2^\circ \ldots \cos 89^\circ \cos 90^\circ \cos 91^\circ \ldots \cos 180^\circ) = 0$, since $\cos 90^\circ = 0$.

30. $(\sin A + \cos A)\sec A = (\sin A + \cos A) \times \frac{1}{\cos A} = \left(\frac{\sin A}{\cos A} + \frac{\cos A}{\cos A}\right)$
$= (\tan A + 1) = \left(\frac{5}{12} + 1\right) = \frac{17}{12}.$

31. $\sin\theta = \cos(\theta - 45^\circ) \Rightarrow \cos(90^\circ - \theta) = \cos(\theta - 45^\circ)$.
$\Rightarrow \quad 90^\circ - \theta = \theta - 45^\circ \Rightarrow 2\theta = 135^\circ \Rightarrow \theta = (67.5)^\circ$.

32. Given expression
$= \frac{\sin 50^\circ}{\cos(90^\circ - 50^\circ)} + \frac{\operatorname{cosec} 40^\circ}{\sec(90^\circ - 40^\circ)} - 4\cos 50^\circ \operatorname{cosec}(90^\circ - 50^\circ)$
$= \frac{\sin 50^\circ}{\sin 50^\circ} + \frac{\operatorname{cosec} 40^\circ}{\operatorname{cosec} 40^\circ} - 4\cos 50^\circ \sec 50^\circ = 1 + 1 - 4 = -2.$

33. $\sin 48^\circ \sec 42^\circ + \cos 48^\circ \operatorname{cosec} 42^\circ$
$= \sin 48^\circ \sec(90^\circ - 48^\circ) + \cos 48^\circ \operatorname{cosec}(90^\circ - 48^\circ)$
$= \sin 48^\circ \operatorname{cosec} 48^\circ + \cos 48^\circ \sec 48^\circ = 1 + 1 = 2.$

34. $\sin\theta = \frac{x}{a}$ and $\cos\theta = \frac{y}{b}$.
Squaring and adding, we get $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow b^2x^2 + a^2y^2 = a^2b^2$.

35. Squaring and subtracting, we get
$25\left(x^2 - \frac{1}{x^2}\right) = (\sec^2\theta - \tan^2\theta) = 1 \Rightarrow 5\left(x^2 - \frac{1}{x^2}\right) = \frac{1}{5}.$

36. Squaring and subtracting, we get
$4\left(x^2 - \frac{1}{x^2}\right) = (\operatorname{cosec}^2\theta - \cot^2\theta) = 1 \Rightarrow 2\left(x^2 - \frac{1}{x^2}\right) = \frac{1}{2}.$

37. $\sec\theta + \tan\theta = x \Rightarrow \sec\theta - \tan\theta = \frac{1}{x}$.
Adding, we get $2\sec\theta = \left(x + \frac{1}{x}\right) \Rightarrow \sec\theta = \left(\frac{x^2 + 1}{2x}\right).$

38. Given expression
$= \frac{\cos 38^\circ \operatorname{cosec}(90^\circ - 38^\circ)}{(\tan 18^\circ \tan 72^\circ)(\tan 35^\circ \tan 55^\circ)\tan 60^\circ}$

$$= \frac{\cos 38^\circ \sec 38^\circ}{\{\tan 18^\circ \tan(90^\circ - 18^\circ)\}\{\tan 35^\circ \tan(90^\circ - 35^\circ)\} \cdot \sqrt{3}}$$

$$= \frac{1}{(\tan 18^\circ \cot 18^\circ)(\tan 35^\circ \cot 35^\circ)\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

39. $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}.$

$\therefore \quad \cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{\sqrt{1-x^2}}{x}.$

40. $\sec^2\theta = (1+\tan^2\theta) \Rightarrow \tan^2\theta = (\sec^2\theta - 1)$

$\Rightarrow \tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{x^2-1}.$

....

MULTIPLE-CHOICE QUESTIONS (MCQ)

Choose the correct answer in each of the following questions:

1. $\dfrac{\sec 30^\circ}{\operatorname{cosec} 60^\circ} = ?$

(a) $\dfrac{2}{\sqrt{3}}$ (b) $\dfrac{\sqrt{3}}{2}$ (c) $\sqrt{3}$ (d) 1

2. $\dfrac{\tan 35^\circ}{\cot 55^\circ} + \dfrac{\cot 78^\circ}{\tan 12^\circ} = ?$

(a) 0 (b) 1 (c) 2 (d) none of these

3. $\tan 10^\circ \tan 15^\circ \tan 75^\circ \tan 80^\circ = ?$

(a) $\sqrt{3}$ (b) $\dfrac{1}{\sqrt{3}}$ (c) –1 (d) 1

4. $\tan 5^\circ \tan 25^\circ \tan 30^\circ \tan 65^\circ \tan 85^\circ = ?$

(a) $\sqrt{3}$ (b) $\dfrac{1}{\sqrt{3}}$ (c) 1 (d) none of these

5. $\cos 1^\circ \cos 2^\circ \cos 3^\circ \ldots \cos 180^\circ = ?$

(a) –1 (b) 1 (c) 0 (d) $\dfrac{1}{2}$

6. $\dfrac{2\sin^2 63^\circ + 1 + 2\sin^2 27^\circ}{3\cos^2 17^\circ - 2 + 3\cos^2 73^\circ} = ?$

(a) $\dfrac{3}{2}$ (b) $\dfrac{2}{3}$ (c) 2 (d) 3

7. $\sin 47^\circ \cos 43^\circ + \cos 47^\circ \sin 43^\circ = ?$

(a) $\sin 4^\circ$ (b) $\cos 4^\circ$ (c) 1 (d) 0

8. $\sec 70^\circ \sin 20^\circ + \cos 20^\circ \operatorname{cosec} 70^\circ = ?$

(a) 0 (b) 1 (c) –1 (d) 2

9. If $\sin 3A = \cos (A - 10^\circ)$ and $3A$ is acute then $\angle A = ?$

(a) 35° (b) 25° (c) 20° (d) 45°

10. If $\sec 4A = \operatorname{cosec} (A - 10^\circ)$ and $4A$ is acute then $\angle A = ?$

(a) 20° (b) 30° (c) 40° (d) 50°

11. If A and B are acute angles such that $\sin A = \cos B$ then $(A + B) = ?$

(a) 45° (b) 60° (c) 90° (d) 180°

12. If $\cos (\alpha + \beta) = 0$ then $\sin (\alpha - \beta) = ?$

(a) $\sin \alpha$ (b) $\cos \beta$ (c) $\sin 2\alpha$ (d) $\cos 2\beta$

13. $\sin (45^\circ + \theta) - \cos (45^\circ - \theta) = ?$

(a) $2 \sin \theta$ (b) $2 \cos \theta$ (c) 0 (d) 1

14. $\sec^2 10^\circ - \cot^2 80^\circ = ?$

(a) 1 (b) 0 (c) $\frac{3}{2}$ (d) $\frac{1}{2}$

15. $\operatorname{cosec}^2 57^\circ - \tan^2 33^\circ = ?$

(a) 0 (b) 1 (c) –1 (d) 2

16. $\dfrac{2 \tan^2 30^\circ \sec^2 52^\circ \sin^2 38^\circ}{\operatorname{cosec}^2 70^\circ - \tan^2 20^\circ} = ?$

(a) 2 (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $\frac{3}{2}$

17. $\left\{ \dfrac{(\sin^2 22^\circ + \sin^2 68^\circ)}{(\cos^2 22^\circ + \cos^2 68^\circ)} + \sin^2 63^\circ + \cos 63^\circ \sin 27^\circ \right\} = ?$

(a) 0 (b) 1 (c) 2 (d) 3

18. $\dfrac{\cot (90^\circ - \theta) \cdot \sin (90^\circ - \theta)}{\sin \theta} + \dfrac{\cot 40^\circ}{\tan 50^\circ} - (\cos^2 20^\circ + \cos^2 70^\circ) = ?$

(a) 0 (b) 1 (c) –1 (d) none of these

19. $\dfrac{\cos 38^\circ \operatorname{cosec} 52^\circ}{\tan 18^\circ \tan 35^\circ \tan 60^\circ \tan 72^\circ \tan 55^\circ} = ?$

(a) $\sqrt{3}$ (b) $\frac{1}{3}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{2}{\sqrt{3}}$

20. If $2 \sin 2\theta = \sqrt{3}$ then $\theta = ?$

(a) 30° (b) 45° (c) 60° (d) 90°

21. If $2 \cos 3\theta = 1$ then $\theta = ?$

(a) 10° (b) 15° (c) 20° (d) 30°

22. If $\sqrt{3}\tan 2\theta - 3 = 0$ then $\theta = ?$
(a) 15° (b) 30° (c) 45° (d) 60°

23. If $\tan x = 3\cot x$ then $x = ?$
(a) 45° (b) 60° (c) 30° (d) 15°

24. If $x\tan 45^\circ \cos 60^\circ = \sin 60^\circ \cot 60^\circ$ then $x = ?$
(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\sqrt{3}$

25. If $\tan^2 45^\circ - \cos^2 30^\circ = x\sin 45^\circ \cos 45^\circ$ then $x = ?$
(a) 2 (b) –2 (c) $\frac{1}{2}$ (d) $\frac{-1}{2}$

26. $\sec^2 60^\circ - 1 = ?$
(a) 2 (b) 3 (c) 4 (d) 0

27. $(\cos 0^\circ + \sin 30^\circ + \sin 45^\circ)(\sin 90^\circ + \cos 60^\circ - \cos 45^\circ) = ?$
(a) $\frac{5}{6}$ (b) $\frac{5}{8}$ (c) $\frac{3}{5}$ (d) $\frac{7}{4}$

28. $\sin^2 30^\circ + 4\cot^2 45^\circ - \sec^2 60^\circ = ?$
(a) 0 (b) $\frac{1}{4}$ (c) 4 (d) 1

29. $3\cos^2 60^\circ + 2\cot^2 30^\circ - 5\sin^2 45^\circ = ?$
(a) $\frac{13}{6}$ (b) $\frac{17}{4}$ (c) 1 (d) 4

30. $\cos^2 30^\circ \cos^2 45^\circ + 4\sec^2 60^\circ + \frac{1}{2}\cos^2 90^\circ - 2\tan^2 60^\circ = ?$
(a) $\frac{73}{8}$ (b) $\frac{75}{8}$ (c) $\frac{81}{8}$ (d) $\frac{83}{8}$

31. If $\operatorname{cosec}\theta = \sqrt{10}$ then $\sec\theta = ?$
(a) $\frac{3}{\sqrt{10}}$ (b) $\frac{\sqrt{10}}{3}$ (c) $\frac{1}{\sqrt{10}}$ (d) $\frac{2}{\sqrt{10}}$

32. If $\tan\theta = \frac{8}{15}$ then $\operatorname{cosec}\theta = ?$
(a) $\frac{17}{8}$ (b) $\frac{8}{17}$ (c) $\frac{17}{15}$ (d) $\frac{15}{17}$

33. If $\sin\theta = \frac{a}{b}$ then $\cos\theta = ?$
(a) $\frac{b}{\sqrt{b^2 - a^2}}$ (b) $\frac{\sqrt{b^2 - a^2}}{b}$ (c) $\frac{a}{\sqrt{b^2 - a^2}}$ (d) $\frac{b}{a}$

34. If $\tan\theta = \sqrt{3}$ then $\sec\theta = ?$
(a) $\frac{2}{\sqrt{3}}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) 2

35. If $\sec\theta = \frac{25}{7}$ then $\sin\theta = ?$

(a) $\frac{7}{24}$ (b) $\frac{24}{7}$ (c) $\frac{24}{25}$ (d) none of these

36. If $\sin\theta = \frac{1}{2}$ then $\cot\theta = ?$

(a) $\frac{1}{\sqrt{3}}$ (b) $\sqrt{3}$ (c) $\frac{\sqrt{3}}{2}$ (d) 1

37. If $\cos\theta = \frac{4}{5}$ then $\tan\theta = ?$

(a) $\frac{3}{4}$ (b) $\frac{4}{3}$ (c) $\frac{3}{5}$ (d) $\frac{5}{3}$

38. If $3x = \operatorname{cosec}\theta$ and $\frac{3}{x} = \cot\theta$ then $3\left(x^2 - \frac{1}{x^2}\right) = ?$

(a) $\frac{1}{27}$ (b) $\frac{1}{81}$ (c) $\frac{1}{3}$ (d) $\frac{1}{9}$

39. If $2x = \sec A$ and $\frac{2}{x} = \tan A$ then $2\left(x^2 - \frac{1}{x^2}\right) = ?$

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $\frac{1}{16}$

40. If $\tan\theta = \frac{4}{3}$ then $(\sin\theta + \cos\theta) = ?$

(a) $\frac{7}{3}$ (b) $\frac{7}{4}$ (c) $\frac{7}{5}$ (c) $\frac{5}{7}$

41. If $(\tan\theta + \cot\theta) = 5$ then $(\tan^2\theta + \cot^2\theta) = ?$

(a) 27 (b) 25 (c) 24 (d) 23

42. If $(\cos\theta + \sec\theta) = \frac{5}{2}$ then $(\cos^2\theta + \sec^2\theta) = ?$

(a) $\frac{21}{4}$ (b) $\frac{17}{4}$ (c) $\frac{29}{4}$ (d) $\frac{33}{4}$

43. If $\tan\theta = \frac{1}{\sqrt{7}}$ then $\frac{(\operatorname{cosec}^2\theta - \sec^2\theta)}{(\operatorname{cosec}^2\theta + \sec^2\theta)} = ?$

(a) $\frac{-2}{3}$ (b) $\frac{-3}{4}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

44. If $7\tan\theta = 4$ then $\frac{(7\sin\theta - 3\cos\theta)}{(7\sin\theta + 3\cos\theta)} = ?$

(a) $\frac{1}{7}$ (b) $\frac{5}{7}$ (c) $\frac{3}{7}$ (d) $\frac{5}{14}$

45. If $3\cot\theta = 4$ then $\dfrac{(5\sin\theta + 3\cos\theta)}{(5\sin\theta - 3\cos\theta)} = ?$

(a) $\frac{1}{3}$ (b) 3 (c) $\frac{1}{9}$ (d) 9

46. If $\tan\theta = \dfrac{a}{b}$ then $\dfrac{(a\sin\theta - b\cos\theta)}{(a\sin\theta + b\cos\theta)} = ?$

(a) $\dfrac{(a^2+b^2)}{(a^2-b^2)}$ (b) $\dfrac{(a^2-b^2)}{(a^2+b^2)}$ (c) $\dfrac{a^2}{(a^2+b^2)}$ (d) $\dfrac{b^2}{(a^2+b^2)}$

47. If $\sin A + \sin^2 A = 1$ then $\cos^2 A + \cos^4 A = ?$

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) 3

48. If $\cos A + \cos^2 A = 1$ then $\sin^2 A + \sin^4 A = ?$

(a) 1 (b) 2 (c) 4 (d) 3

49. $\sqrt{\dfrac{1-\sin A}{1+\sin A}} = ?$

(a) $\sec A + \tan A$ (b) $\sec A - \tan A$

(c) $\sec A \tan A$ (d) none of these

50. $\sqrt{\dfrac{1+\cos A}{1-\cos A}} = ?$

(a) $\operatorname{cosec} A - \cot A$ (b) $\operatorname{cosec} A + \cot A$

(c) $\operatorname{cosec} A \cot A$ (d) none of these

51. If $\tan\theta = \dfrac{a}{b}$ then $\dfrac{(\cos\theta + \sin\theta)}{(\cos\theta - \sin\theta)} = ?$

(a) $\dfrac{a+b}{a-b}$ (b) $\dfrac{a-b}{a+b}$ (c) $\dfrac{b+a}{b-a}$ (d) $\dfrac{b-a}{b+a}$

52. $(\operatorname{cosec}\theta - \cot\theta)^2 = ?$

(a) $\dfrac{1+\cos\theta}{1-\cos\theta}$ (b) $\dfrac{1-\cos\theta}{1+\cos\theta}$ (c) $\dfrac{1+\sin\theta}{1-\sin\theta}$ (d) none of these

53. $(\sec A + \tan A)(1-\sin A) = ?$

(a) $\sin A$ (b) $\cos A$ (c) $\sec A$ (d) $\operatorname{cosec} A$

ANSWERS (MCQ)

1. (d)	**2.** (c)	**3.** (d)	**4.** (b)	**5.** (c)	**6.** (d)	**7.** (c)	**8.** (d)	**9.** (b)
10. (a)	**11.** (c)	**12.** (d)	**13.** (c)	**14.** (a)	**15.** (b)	**16.** (c)	**17.** (c)	**18.** (b)
19. (c)	**20.** (a)	**21.** (c)	**22.** (b)	**23.** (b)	**24.** (a)	**25.** (c)	**26.** (b)	**27.** (d)

28. (b) **29.** (b) **30.** (d) **31.** (b) **32.** (a) **33.** (b) **34.** (d) **35.** (c) **36.** (b)
37. (a) **38.** (c) **39.** (a) **40.** (c) **41.** (d) **42.** (b) **43.** (d) **44.** (a) **45.** (d)
46. (b) **47.** (b) **48.** (a) **49.** (b) **50.** (b) **51.** (c) **52.** (b) **53.** (b)

HINTS TO SOME SELECTED QUESTIONS

2. Given expression $= \dfrac{\tan 35^\circ}{\cot (90^\circ - 35^\circ)} + \dfrac{\cot 78^\circ}{\tan (90^\circ - 78^\circ)} = \dfrac{\tan 35^\circ}{\tan 35^\circ} + \dfrac{\cot 78^\circ}{\cot 78^\circ} = (1+1) = 2.$

3. Given expression $= (\tan 10^\circ \tan 80^\circ)(\tan 15^\circ \tan 75^\circ)$
$= \{\tan 10^\circ \tan (90^\circ - 10^\circ)\}\{\tan 15^\circ \tan (90^\circ - 15^\circ)\}$
$= (\tan 10^\circ \cdot \cot 10^\circ)(\tan 15^\circ \cdot \cot 15^\circ) = 1 \times 1 = 1.$

4. Given expression $= (\tan 5^\circ \tan 85^\circ)(\tan 25^\circ \tan 65^\circ) \tan 30^\circ$
$= \{\tan 5^\circ \cdot \tan (90^\circ - 5^\circ)\}\{\tan 25^\circ \cdot \tan (90^\circ - 25^\circ)\} \cdot \tan 30^\circ$
$= (\tan 5^\circ \cdot \cot 5^\circ)(\tan 25^\circ \cdot \cot 25^\circ) \cdot \dfrac{1}{\sqrt{3}} = 1 \times 1 \times \dfrac{1}{\sqrt{3}} = \dfrac{1}{\sqrt{3}}.$

6. Given expression $= \dfrac{2(\sin^2 63^\circ + \sin^2 27^\circ) + 1}{3(\cos^2 17^\circ + \cos^2 73^\circ) - 2} = \dfrac{2(\sin^2 63^\circ + \cos^2 63^\circ) + 1}{3(\cos^2 17^\circ + \sin^2 17^\circ) - 2}$
$= \dfrac{(2 \times 1) + 1}{(3 \times 1) - 2} = \dfrac{3}{1} = 3.$ $\left[\because \begin{array}{l} \sin 27^\circ = \sin (90^\circ - 63^\circ) = \cos 63^\circ \\ \cos 73^\circ = \cos (90^\circ - 17^\circ) = \sin 17^\circ \end{array}\right]$

7. Given expression $= \sin 47^\circ \cos (90^\circ - 47^\circ) + \cos 47^\circ \sin (90^\circ - 47^\circ)$
$= (\sin^2 47^\circ + \cos^2 47^\circ) = 1.$

8. Given expression $= \sec 70^\circ \sin (90^\circ - 70^\circ) + \cos 20^\circ \operatorname{cosec} (90^\circ - 20^\circ)$
$= \sec 70^\circ \cos 70^\circ + \cos 20^\circ \sec 20^\circ = 1 + 1 = 2.$

9. $\sin 3A = \cos (A - 10^\circ) \Rightarrow \cos (90^\circ - 3A) = \cos (A - 10^\circ)$
$\Rightarrow \quad 90^\circ - 3A = A - 10^\circ \Rightarrow 4A = 100^\circ \Rightarrow A = 25^\circ.$

10. $\sec 4A = \operatorname{cosec} (A - 10^\circ) \Rightarrow \operatorname{cosec} (90^\circ - 4A) = \operatorname{cosec} (A - 10^\circ)$
$\Rightarrow \quad 90^\circ - 4A = A - 10^\circ \Rightarrow 5A = 100^\circ \Rightarrow A = 20^\circ.$

11. $\sin A = \cos B \Rightarrow \cos (90^\circ - A) = \cos B \Rightarrow 90^\circ - A = B \Rightarrow A + B = 90^\circ.$

12. $\cos (\alpha + \beta) = 0 \Rightarrow \alpha + \beta = 90^\circ \Rightarrow \alpha = 90^\circ - \beta$
$\therefore \quad (\alpha - \beta) = (90^\circ - 2\beta) \Rightarrow \sin (\alpha - \beta) = \sin (90^\circ - 2\beta) = \cos 2\beta.$

13. $\sin (45^\circ + \theta) - \cos (45^\circ - \theta) = \sin (45^\circ + \theta) - \sin \{90^\circ - (45^\circ - \theta)\}$
$= \sin (45^\circ + \theta) - \sin (45^\circ + \theta) = 0.$

14. $\sec^2 10^\circ - \cot^2 80^\circ = \sec^2 10^\circ - \cot^2 (90^\circ - 10^\circ) = \sec^2 10^\circ - \tan^2 10^\circ = 1.$

15. $\operatorname{cosec}^2 57^\circ - \tan^2 33^\circ = \operatorname{cosec}^2 57^\circ - \tan^2 (90^\circ - 57^\circ) = \operatorname{cosec}^2 57^\circ - \cot^2 57^\circ = 1.$

16. Given expression $= \dfrac{2 \times \left(\dfrac{1}{\sqrt{3}}\right)^2 \cdot \sec^2 52^\circ \sin^2 (90^\circ - 52^\circ)}{\operatorname{cosec}^2 70^\circ - \tan^2 (90^\circ - 70^\circ)} = \dfrac{2}{3} \cdot \dfrac{\sec^2 52^\circ \cos^2 52^\circ}{(\operatorname{cosec}^2 70^\circ - \cot^2 70^\circ)}$
$= \dfrac{2}{3} \times \dfrac{1}{1} = \dfrac{2}{3}.$

17. Given expression $= \dfrac{\sin^2 22^\circ + \sin^2(90^\circ - 22^\circ)}{\cos^2 22^\circ + \cos^2(90^\circ - 22^\circ)} + \sin^2 63^\circ + \cos 63^\circ \sin(90^\circ - 63^\circ)$

$$= \frac{\sin^2 22^\circ + \cos^2 22^\circ}{\cos^2 22^\circ + \sin^2 22^\circ} + \sin^2 63^\circ + \cos^2 63^\circ = \frac{1}{1} + 1 = 1 + 1 = 2.$$

18. Given expression $= \dfrac{\tan\theta \cdot \cos\theta}{\sin\theta} + \dfrac{\cot 40^\circ}{\tan(90^\circ - 40^\circ)} - \{\cos^2 20^\circ + \cos^2(90^\circ - 20^\circ)\}$

$$= \frac{\sin\theta}{\cos\theta} \cdot \frac{\cos\theta}{\sin\theta} + \frac{\cot 40^\circ}{\cot 40^\circ} - \{\cos^2 20^\circ + \sin^2 20^\circ\} = 1 + 1 - 1 = 1.$$

19. $\dfrac{\cos 38^\circ \operatorname{cosec} 52^\circ}{\tan 18^\circ \tan 35^\circ \tan 60^\circ \tan 72^\circ \tan 55^\circ} = \dfrac{\cos 38^\circ \operatorname{cosec}(90^\circ - 38^\circ)}{(\tan 18^\circ \tan 72^\circ)(\tan 35^\circ \tan 55^\circ)\tan 60^\circ}$

$$= \frac{\cos 38^\circ \sec 38^\circ}{\tan 18^\circ \tan(90^\circ - 18^\circ) \tan 35^\circ \tan(90^\circ - 35^\circ)\sqrt{3}} \qquad [\because \tan 60^\circ = \sqrt{3}]$$

$$= \frac{1}{(\tan 18^\circ \cot 18^\circ)(\tan 35^\circ \cot 35^\circ)\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

20. $2\sin 2\theta = \sqrt{3} \Rightarrow \sin 2\theta = \dfrac{\sqrt{3}}{2} = \sin 60^\circ \Rightarrow 2\theta = 60^\circ \Rightarrow \theta = 30^\circ.$

21. $2\cos 3\theta = 1 \Rightarrow \cos 3\theta = \dfrac{1}{2} = \cos 60^\circ \Rightarrow 3\theta = 60^\circ \Rightarrow \theta = 20^\circ.$

22. $\sqrt{3}\tan 2\theta = 3 \Rightarrow \tan 2\theta = \dfrac{3}{\sqrt{3}} = \sqrt{3} = \tan 60^\circ \Rightarrow 2\theta = 60^\circ \Rightarrow \theta = 30^\circ.$

23. $\tan x = 3\cot x \Rightarrow \tan^2 x = 3 \Rightarrow \tan x = \sqrt{3} = \tan 60^\circ \Rightarrow x = 60^\circ.$

24. $x = \dfrac{\sin 60^\circ \cot 60^\circ}{\tan 45^\circ \cos 60^\circ} = \dfrac{\tan 60^\circ \cot 60^\circ}{\tan 45^\circ} = 1.$

25. $x = \dfrac{\tan^2 45^\circ - \cos^2 30^\circ}{\sin 45^\circ \cos 45^\circ} = \dfrac{1 - \left(\frac{\sqrt{3}}{2}\right)^2}{\left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right)} = \dfrac{1 - \frac{3}{4}}{\frac{1}{2}} = \dfrac{1}{4} \times \dfrac{2}{1} = \dfrac{1}{2}.$

26. $\sec^2 60^\circ - 1 = 2^2 - 1 = 4 - 1 = 3.$

27. $(\cos 0^\circ + \sin 30^\circ + \sin 45^\circ)(\sin 90^\circ + \cos 60^\circ - \cos 45^\circ)$

$$= \left(1 + \frac{1}{2} + \frac{1}{\sqrt{2}}\right)\left(1 + \frac{1}{2} - \frac{1}{\sqrt{2}}\right) = \left(\frac{3}{2} + \frac{1}{\sqrt{2}}\right)\left(\frac{3}{2} - \frac{1}{\sqrt{2}}\right) = \left(\frac{9}{4} - \frac{1}{2}\right) = \frac{7}{4}.$$

28. $\sin^2 30^\circ + 4\cot^2 45^\circ - \sec^2 60^\circ = \left(\dfrac{1}{2}\right)^2 + 4 \times (1)^2 - 2^2 = \dfrac{1}{4} + 4 - 4 = \dfrac{1}{4}.$

29. $3\cos^2 60^\circ + 2\cot^2 30^\circ - 5\sin^2 45^\circ$

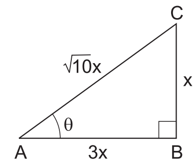
$$= 3 \times \left(\frac{1}{2}\right)^2 + 2 \times (\sqrt{3})^2 - 5 \times \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{3}{4} + 6 - \frac{5}{2} = \frac{17}{4}.$$

30. $\cos^2 30^\circ \cos^2 45^\circ + 4\sec^2 60^\circ + \dfrac{1}{2}\cos^2 90^\circ - 2\tan^2 60^\circ$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 \cdot \left(\frac{1}{\sqrt{2}}\right)^2 + (4 \times 2^2) + \left(\frac{1}{2} \times 0^2\right) - 2 \times (\sqrt{3})^2$$

$$= \left(\frac{3}{4} \times \frac{1}{2}\right) + 16 + 0 - 6 = \frac{3}{8} + 10 = \frac{83}{8}.$$

31. $\text{cosec}\,\theta = \frac{AC}{BC} = \frac{\sqrt{10}}{1} = \frac{\sqrt{10}\,x}{x} \Rightarrow AC = \sqrt{10x}$ and $BC = x$.

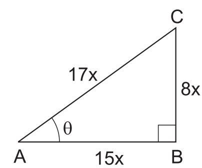


$\therefore \quad AB^2 = AC^2 - BC^2 = (\sqrt{10}\,x)^2 - (x^2) = 10x^2 - x^2 = 9x^2$

$\Rightarrow \quad AB = \sqrt{9x^2} = 3x.$

$\therefore \quad \sec\theta = \frac{AC}{AB} = \frac{\sqrt{10}\,x}{3x} = \frac{\sqrt{10}}{3}.$

32. $\tan\theta = \frac{BC}{AB} = \frac{8}{15} = \frac{8x}{15x} \Rightarrow BC = 8x$ and $AB = 15x$.



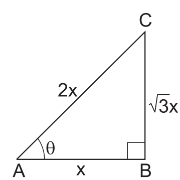
$\therefore \quad AC^2 = AB^2 + BC^2 = (225x^2 + 64x^2) = 289x^2$

$\Rightarrow \quad AC = \sqrt{289x^2} = 17x.$

$\therefore \quad \text{cosec}\,\theta = \frac{AC}{BC} = \frac{17x}{8x} = \frac{17}{8}.$

33. $\cos^2\theta = (1 - \sin^2\theta) = \left(1 - \frac{a^2}{b^2}\right) = \frac{b^2 - a^2}{b^2} \Rightarrow \cos\theta = \frac{\sqrt{b^2 - a^2}}{b}.$

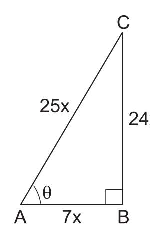
34. $\tan\theta = \frac{BC}{AB} = \frac{\sqrt{3}}{1} = \frac{\sqrt{3}\,x}{x} \Rightarrow BC = \sqrt{3}\,x$ and $AB = x$.



$\therefore \quad AC^2 = AB^2 + BC^2 = x^2 + 3x^2 = 4x^2 \Rightarrow AC = 2x.$

$\therefore \quad \sec\theta = \frac{AC}{AB} = \frac{2x}{x} = 2.$

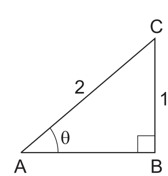
35. $\sec\theta = \frac{AC}{AB} = \frac{25}{7} = \frac{25x}{7x} \Rightarrow AC = 25x$ and $AB = 7x$.



$\therefore \quad BC^2 = AC^2 - AB^2 = 625x^2 - 49x^2 = 576x^2 \Rightarrow BC = 24x.$

$\therefore \quad \sin\theta = \frac{BC}{AC} = \frac{24x}{25x} = \frac{24}{25}.$

36. $\sin\theta = \frac{1}{2} = \frac{BC}{AC}.$

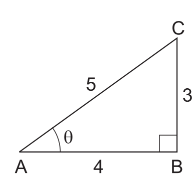


$\therefore \quad AB^2 = AC^2 - BC^2 = 2^2 - 1^2 = 3$

$\therefore \quad AB = \sqrt{3}.$

$\cot\theta = \frac{AB}{BC} = \frac{\sqrt{3}}{1} = \sqrt{3}.$

37. $\cos\theta = \frac{4}{5} = \frac{AB}{AC}.$



$\therefore \quad BC^2 = AC^2 - AB^2 = 25 - 16 = 9$

$\Rightarrow \quad BC = 3.$

$\therefore \quad \tan\theta = \frac{BC}{AB} = \frac{3}{4}.$

38. We know that $\operatorname{cosec}^2\theta - \cot^2\theta = 1$.

$\therefore \quad (3x)^2 - \left(\frac{3}{x}\right)^2 = 1 \Rightarrow 9x^2 - \frac{9}{x^2} = 1 \Rightarrow 9\left(x^2 - \frac{1}{x^2}\right) = 1$

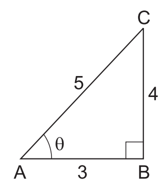
$\Rightarrow \quad \left(x^2 - \frac{1}{x^2}\right) = \frac{1}{9} \Rightarrow 3\left(x^2 - \frac{1}{x^2}\right) = 3 \times \frac{1}{9} = \frac{1}{3}.$

39. We know that $\sec^2 A - \tan^2 A = 1$.

$\therefore \quad (2x)^2 - \left(\frac{2}{x}\right)^2 = 1 \Rightarrow 4x^2 - \frac{4}{x^2} = 1 \Rightarrow 4\left(x^2 - \frac{1}{x^2}\right) = 1$

$\Rightarrow \left(x^2 - \frac{1}{x^2}\right) = \frac{1}{4} \Rightarrow 2\left(x^2 - \frac{1}{x^2}\right) = 2 \times \frac{1}{4} = \frac{1}{2}.$

40. $\tan\theta = \frac{4}{3} = \frac{BC}{AB}.$



$\therefore \quad AC^2 = AB^2 + BC^2 = (3)^2 + (4)^2 = 25$

$\Rightarrow \quad AC = \sqrt{25} = 5.$

$\therefore \quad (\sin\theta + \cos\theta) = \left(\frac{4}{5} + \frac{3}{5}\right) = \frac{7}{5}.$

41. $(\tan\theta + \cot\theta)^2 = 5^2 \Rightarrow \tan^2\theta + \cot^2\theta + 2 = 25$

$\therefore \quad (\tan^2\theta + \cot^2\theta) = 23.$

42. $(\cos\theta + \sec\theta)^2 = \frac{25}{4} \Rightarrow \cos^2\theta + \sec^2\theta + 2 = \frac{25}{4} \Rightarrow \cos^2\theta + \sec^2\theta = \frac{25}{4} - 2 = \frac{17}{4}.$

43. $\tan\theta = \frac{1}{\sqrt{7}} \Rightarrow \cot\theta = \sqrt{7}.$

$\sec^2\theta = (1 + \tan^2\theta) = \left(1 + \frac{1}{7}\right) = \frac{8}{7}$ and $\operatorname{cosec}^2\theta = (1 + \cot^2\theta) = (1 + 7) = 8.$

$$\therefore \quad \frac{\operatorname{cosec}^2\theta - \sec^2\theta}{\operatorname{cosec}^2\theta + \sec^2\theta} = \frac{\left(8 - \frac{8}{7}\right)}{\left(8 + \frac{8}{7}\right)} = \frac{48}{64} = \frac{3}{4}.$$

44. Given, $\tan\theta = \frac{4}{7}.$

$$\therefore \quad \frac{(7\sin\theta - 3\cos\theta)}{(7\sin\theta + 3\cos\theta)} = \frac{(7\tan\theta - 3)}{(7\tan\theta + 3)} \quad \text{[dividing num. and denom. by } \cos\theta]$$

$$= \frac{\left(7 \times \frac{4}{7} - 3\right)}{\left(7 \times \frac{4}{7} + 3\right)} = \frac{(4 - 3)}{(4 + 3)} = \frac{1}{7}.$$

45. Given, $\cot\theta = \frac{4}{3}.$

$$\therefore \quad \left(\frac{5\sin\theta + 3\cos\theta}{5\sin\theta - 3\cos\theta}\right) = \frac{5 + 3\cot\theta}{5 - 3\cot\theta} \quad \text{[dividing num. and denom. by } \sin\theta]$$

$$= \frac{\left(5 + 3 \times \frac{4}{3}\right)}{\left(5 - 3 \times \frac{4}{3}\right)} = \frac{(5 + 4)}{(5 - 4)} = \frac{9}{1} = 9.$$

46. Dividing num. and denom. by $\cos\theta$, we get

$$\frac{(a\sin\theta - b\cos\theta)}{(a\sin\theta + b\cos\theta)} = \frac{(a\tan\theta - b)}{(a\tan\theta + b)} = \frac{\left(a \times \frac{a}{b} - b\right)}{\left(a \times \frac{a}{b} + b\right)} = \frac{(a^2 - b^2)}{(a^2 + b^2)}.$$

47. $\sin A + \sin^2 A = 1 \Rightarrow \sin A = (1 - \sin^2 A) \Rightarrow \sin A = \cos^2 A.$

$\therefore \quad (\cos^2 A + \cos^4 A) = (\sin A + \sin^2 A) = 1.$

48. $\cos A + \cos^2 A = 1 \Rightarrow \cos A = (1 - \cos^2 A) \Rightarrow \cos A = \sin^2 A.$

$\therefore \quad (\sin^2 A + \sin^4 A) = (\cos A + \cos^2 A) = 1.$

49. $$\sqrt{\frac{1-\sin A}{1+\sin A}} = \sqrt{\frac{(1-\sin A)}{(1+\sin A)} \times \frac{(1-\sin A)}{(1-\sin A)}} = \frac{(1-\sin A)}{\sqrt{1-\sin^2 A}} = \frac{(1-\sin A)}{\sqrt{\cos^2 A}}$$
$$= \frac{(1-\sin A)}{\cos A} = \left(\frac{1}{\cos A} - \frac{\sin A}{\cos A}\right) = (\sec A - \tan A).$$

50. $$\sqrt{\frac{1+\cos A}{1-\cos A}} = \sqrt{\frac{(1+\cos A)}{(1-\cos A)} \times \frac{(1+\cos A)}{(1+\cos A)}} = \frac{(1+\cos A)}{\sqrt{1-\cos^2 A}} = \frac{(1+\cos A)}{\sqrt{\sin^2 A}}$$
$$= \frac{(1+\cos A)}{\sin A} = \left(\frac{1}{\sin A} + \frac{\cos A}{\sin A}\right) = (\operatorname{cosec} A + \cot A).$$

51. $$\frac{(\cos\theta + \sin\theta)}{(\cos\theta - \sin\theta)} = \frac{(1+\tan\theta)}{(1-\tan\theta)} \quad \text{[dividing num. and denom. by } \cos\theta]$$
$$= \frac{\left(1+\frac{a}{b}\right)}{\left(1-\frac{a}{b}\right)} = \frac{(b+a)}{(b-a)}.$$

52. $$(\operatorname{cosec}\theta - \cot\theta)^2 = \left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)^2 = \frac{(1-\cos\theta)^2}{\sin^2\theta} = \frac{(1-\cos\theta)^2}{(1-\cos^2\theta)} = \frac{(1-\cos\theta)}{(1+\cos\theta)}.$$

53. $$(\sec A + \tan A)(1-\sin A) = \left(\frac{1}{\cos A} + \frac{\sin A}{\cos A}\right)(1-\sin A)$$
$$= \frac{(1+\sin A)(1-\sin A)}{\cos A} = \frac{1-\sin^2 A}{\cos A} = \frac{\cos^2 A}{\cos A} = \cos A.$$

....

TEST YOURSELF

MCQ

1. $\dfrac{\cos^2 56^\circ + \cos^2 34^\circ}{\sin^2 56^\circ + \sin^2 34^\circ} + 3\tan^2 56^\circ \tan^2 34^\circ = ?$

(a) $3\frac{1}{2}$ (b) 4 (c) 6 (d) 5

2. The value of $\left(\sin^2 30^\circ \cos^2 45^\circ + 4\tan^2 30^\circ + \frac{1}{2}\sin^2 90^\circ + \frac{1}{8}\cot^2 60^\circ\right) = ?$

(a) $\frac{3}{8}$ (b) $\frac{5}{8}$ (c) 6 (d) 2

3. If $\cos A + \cos^2 A = 1$ then $(\sin^2 A + \sin^4 A) = ?$

(a) $\frac{1}{2}$ (b) 2 (c) 1 (d) 4

4. If $\sin\theta = \frac{\sqrt{3}}{2}$ then $(\text{cosec}\,\theta + \cot\theta) = ?$

(a) $(2+\sqrt{3})$ (b) $2\sqrt{3}$ (c) $\sqrt{2}$ (d) $\sqrt{3}$

Short-Answer Questions

5. If $\cot A = \frac{4}{5}$, prove that $\frac{(\sin A + \cos A)}{(\sin A - \cos A)} = 9$.

6. If $2x = \sec A$ and $\frac{2}{x} = \tan A$, prove that $\left(x^2 - \frac{1}{x^2}\right) = \frac{1}{4}$.

7. If $\sqrt{3}\tan\theta = 3\sin\theta$, prove that $(\sin^2\theta - \cos^2\theta) = \frac{1}{3}$.

8. Prove that $\frac{(\sin^2 73^\circ + \sin^2 17^\circ)}{(\cos^2 28^\circ + \cos^2 62^\circ)} = 1$.

9. If $2\sin 2\theta = \sqrt{3}$, prove that $\theta = 30^\circ$.

10. Prove that $\sqrt{\frac{1+\cos A}{1-\cos A}} = (\text{cosec}\,A + \cot A)$.

11. If $\text{cosec}\,\theta + \cot\theta = p$, prove that $\cos\theta = \frac{(p^2-1)}{(p^2+1)}$.

12. Prove that $(\text{cosec}\,A - \cot A)^2 = \frac{(1-\cos A)}{(1+\cos A)}$.

13. If $5\cot\theta = 3$, show that the value of $\left(\frac{5\sin\theta - 3\cos\theta}{4\sin\theta + 3\cos\theta}\right)$ is $\frac{16}{29}$.

14. Prove that $(\sin 32^\circ \cos 58^\circ + \cos 32^\circ \sin 58^\circ) = 1$.

15. If $x = a\sin\theta + b\cos\theta$ and $y = a\cos\theta - b\sin\theta$, prove that $x^2 + y^2 = a^2 + b^2$.

16. Prove that $\frac{(1+\sin\theta)}{(1-\sin\theta)} = (\sec\theta + \tan\theta)^2$.

Long-Answer Questions

17. Prove that $\frac{1}{(\sec\theta - \tan\theta)} - \frac{1}{\cos\theta} = \frac{1}{\cos\theta} - \frac{1}{(\sec\theta + \tan\theta)}$.

18. Prove that $\frac{(\sin A - 2\sin^3 A)}{(2\cos^3 A - \cos A)} = \tan A$.

19. Prove that $\frac{\tan A}{(1-\cot A)} + \frac{\cot A}{(1-\tan A)} = (1 + \tan A + \cot A)$.

20. If $\sec 5A = \text{cosec}\,(A - 36^\circ)$ and $5A$ is an acute angle, show that $A = 21^\circ$.

ANSWERS (TEST YOURSELF)

1. (b) **2.** (d) **3.** (c) **4.** (d)

✿